Scattering States of Plektons (Particles with Braid Group Statistics) in 2 + 1 Dimensional Quantum Field Theory

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Abstract: A Haag–Ruelle scattering theory for particles with braid group statistics is developed, and the arising structure of the Hilbert space of multiparticle states is analyzed.

1. Introduction

Particles in 2 + 1 dimensional spacetime are not necessarily bosons or fermions; in general, their statistics may be described by a unitary representation of Artin's braid group [1]. Such particles will be called plektons, in the following. In a quantum mechanical framework the possible existence of plektons in 2 space dimensions was first observed by Leinaas and Myrheim [17] in their analysis of the principle of indistinguishability of identical particles. In the framework of quantum field theory the presupposed correspondence between particles and local fields seemed to forbid exotic statistics in more than 1 + 1 dimensions. But adopting the point of view of algebraic quantum field theory that locality has to be assumed only for observables, Buchholz and one of us showed [2] that even in purely massive theories particles might correspond to non-local fields with the consequence that ordinary statistics could be derived only in 3 + 1 (or more) dimensions.

Models for particles with a one dimensional representation of the braid group ("anyons") were first invented by Wilczek [30]. Non-abelian gauge theories with a Chern–Simons term in the action are candidates for models with non-abelian braid group statistics. Anyons are considered to be the excitations which are responsible for the fractional Quantum Hall Effect [16].

In order to predict phenomena caused by plektons a multiparticle formalism is desirable. In the case of permutation group statistics the multiparticle space (as a representation space of the Poincaré group) is obtained by the choice of a Poincaré invariant metric (determined by the statistics) on the tensor product of Poincaré group representations on single particle spaces (see [3]). This is no longer true in the plektonic case because the sum rules for spins involve the statistics [7]. A

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multiplekton space which satisfies these requirements was recently constructed by Mund and Schrader [20]: it is determined by the Poincaré group representation in the single particle spaces and a representation of the braid group B_n.

In this paper we show that in quantum field theory with braid group statistics scattering states exist, and we compute the structure of the "plektonic Fock space" of scattering states. This space turns out to be a direct sum of the n-particle spaces of Mund and Schrader where the braid group representations are induced by a Markov trace on the braid group B_∞. Some work in this direction has already been done by Fröhlich and Marchetti [11], who concentrated on the abelian case, and by Schroer [26], who pointed out problems and made some prospective remarks.

This work is based in part on the diploma theses of two of us [25, 12]. In Sect. 2 we describe the general set-up as it was developed in the framework of algebraic quantum field theory [3, 15]. In order to obtain a coherent description of the non-local operators which one has to add to the algebra of local observables we use the formalism of [10, II] (see also [8, 13]). That is, we extend the algebra of local observables (on Minkowski space) to an algebra $\mathscr{A}_0(\bar{\mathscr{M}})$ which may be considered as the algebra of local observables on the union of Minkowski space with the hyperboloid at spacelike infinity. We then define the associated field bundle [3], an intrinsic structure equivalent to the exchange algebra of vertex operators [22] which is known from conformal field theory. In the next step we develop a new geometrical description for the (statistics) intertwiners in which the (geometrical) role of the braid group becomes apparent. Using this formulation we can then give a compact formula for the commutation relation of generalized fields.

In Sect. 3 we construct Haag–Ruelle approximants to scattering states. This can be done as in [2] (see e.g. [11]). The computation of scalar products is somewhat complicated and some formulae guessed previously could not be confirmed. There is a subtle point concerning the dependence of the scattering states on the choice of the Lorentz frame. As pointed out to us by Schrader some years ago, such a difficulty had to be expected in view of the absence of a Lorentz invariant ordering on a mass hyperboloid in 3 dimensional Minkowski space, and actually, the proof in [2] that the construction is Lorentz invariant, breaks down in 3 dimensions. We therefore reformulate the Haag–Ruelle theory in a manifestly Lorentz invariant way. The scattering vectors depend then only on the spacelike directions which characterize the localization of the stringlike localized fields by which the vectors were generated.

The main result of the paper is the analysis of this rather complicated dependence. We show that the scattering vectors are locally independent of the spacelike directions, and we find an explicit transformation formula (in terms of the statistics intertwiner) relating scattering vectors corresponding to different sets of spacelike directions. This result enables us to unveil the global structure of the space of scattering vectors: the set of directions can be regarded as parametrizing local trivializations of the universal covering space $\tilde{C}_n$ of the configuration space C_n of n non-coincident velocities in 3-dimensional Minkowski space. And the space of scattering vectors has the structure of the Hilbert space of square integrable sections of a vector bundle which is associated to this covering space [27].

This structure (for the case of single particle spaces with an irreducible representation of the Poincaré group) was anticipated by Schrader [28] (see also [29]) and further elaborated in [20].

2. Commutation Relations of Generalized Local Fields

Let us start with describing the algebraic framework of quantum field theory.[1] We are given a family of von Neumann algebras $\mathscr{A}(\mathscr{O})$ (the algebras of observables measurable within $\mathscr{O}$) on some Hilbert space $\mathscr{H}_0$ indexed by the open double-cones $\mathscr{O}$ in Minkowski space $\mathscr{M}$ which satisfy isotony

$$\mathscr{A}(\mathscr{O}_1) \subset \mathscr{A}(\mathscr{O}_2) \quad \text{if } \mathscr{O}_1 \subset \mathscr{O}_2$$

and locality

$$\mathscr{A}(\mathscr{O}_1) \subset \mathscr{A}(\mathscr{O}_2)' \quad \text{if } \mathscr{O}_1 \subset \mathscr{O}_2' ,$$

where $\mathscr{A}(\mathscr{O}_2)'$ is the commutant of $\mathscr{A}(\mathscr{O}_2)$ and $\mathscr{O}_2'$ denotes the spacelike complement of $\mathscr{O}_2$ in Minkowski space. Given isotony, we can define the algebra of observables $\mathscr{A}_0(\mathscr{M})$ to be the norm closure of the union $\cup \mathscr{A}(\mathscr{O})$ of all local algebras. Moreover, for an arbitrary region $\mathscr{R} \subset \mathscr{M}$, we define $\mathscr{A}_0(\mathscr{R})$ to be the C^*-subalgebra of $\mathscr{A}_0(\mathscr{M})$ generated by all algebras $\mathscr{A}(\mathscr{O})$ with double-cones $\mathscr{O} \subset \mathscr{R}$, and $\mathscr{A}(\mathscr{R})$ to be its weak closure.

In order to describe a Poincaré covariant theory, we assume that there is a representation α of the identity component $\mathscr{P}_+^\uparrow$ of the Poincaré group by automorphisms of $\mathscr{A}_0(\mathscr{M})$ such that

$$\alpha_{(x,\Lambda)}(\mathscr{A}(\mathscr{O})) = \mathscr{A}((x,\Lambda)(\mathscr{O})) . \tag{2.1}$$

Furthermore, we want to assume that the action of $\mathscr{P}_+^\uparrow$ is represented on $\mathscr{H}_0$ by a strongly continuous unitary representation U_0,

$$U_0(L)AU_0(L)^* = \alpha_L(A) ,$$

where $L \in \mathscr{P}_+^\uparrow$ and $A \in \mathscr{A}_0(\mathscr{M})$. The generators of the translations P_ν satisfy the spectrum condition

$$\operatorname{sp} P \subset \{0\} \cup \{p \in \mathscr{M} \,|\, p^2 > \mu^2, p_0 > 0\} \tag{2.2}$$

for some $\mu \geqq 0$, and there is a cyclic unit vector $\Omega \in \mathscr{H}_0$, unique up to a phase, such that $U_0(x,\Lambda)\Omega = \Omega$. Ω represents the vacuum. The embedding of $\mathscr{A}_0(\mathscr{M})$ into the algebra $\mathscr{B}(\mathscr{H}_0)$ of all bounded operators on $\mathscr{H}_0$ is called the vacuum representation π_0.

In the present paper we want to analyze multiparticle scattering states in a purely massive theory. We are therefore interested in representations of the algebra of observables which describe massive particles, i.e. representations π of $\mathscr{A}_0(\mathscr{M})$ by bounded operators on a Hilbert space $\mathscr{H}_\pi$ together with a strongly continuous representation U_π of the covering group $\widetilde{P_+^\uparrow}$ of $P_+^\uparrow$ satisfying

$$\operatorname{Ad} U_\pi(g) \circ \pi = \pi \circ \alpha_{L(g)} \tag{2.3}$$

for any $g \in \widetilde{P_+^\uparrow}$, where $g \mapsto L(g)$ is the covering homomorphism and the generators of the translations are required to fulfill the spectrum condition

$$H_m \subset \operatorname{sp} P \subset H_m \cup \{p \in \mathscr{M} \,|\, p^2 > M^2, p_0 > 0\} \tag{2.4}$$

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[1] For more details see [3, 15] and references therein.

with $0 < m < M$. Here H_m is the mass shell $H_m = \{p \in \mathcal{M} \,|\, p^2 = m^2, p_0 > 0\}$, m is interpreted as the mass of the particle described by π and π is called "massive single particle representation."

It was shown in [2] that for any irreducible massive single particle representation π there is a unique vacuum representation π_0, i.e. a representation satisfying (2.2) (with $\mu \geqq M - m$) such that π and π_0 are unitarily equivalent when restricted to the algebra of the causal complement of any spacelike cone S,

$$\pi|_{\mathcal{A}_0(S')} \cong \pi_0|_{\mathcal{A}_0(S')} \,. \tag{2.5}$$

A spacelike cone S is the convex set

$$S := a + \bigcup_{\lambda > 0} \lambda \mathcal{O} \,, \tag{2.6}$$

where $a \in \mathcal{M}$ is the apex and $\mathcal{O}$ is a double-cone of spacelike directions

$$\mathcal{O} = \{r \in \mathcal{M} \,|\, r^2 = -1 \text{ and } r_+ - r, r - r_- \in V_+\} \tag{2.7}$$

with $r_+^2 = r_-^2 = -1$, $r_+ - r_- \in V_+$, V_+ denoting the interior of the forward light cone. We denote the set of spacelike cones by $\mathcal{S}$.

In view of this result we shall from now on fix the vacuum representation π_0 and identify it with the defining (identical) representation of $\mathcal{A}_0(\mathcal{M})$ on $\mathcal{H}_0$. We consider only those massive single particle representations π which satisfy the "selection criterion" (2.5) with respect to π_0. Furthermore, we shall assume that the fixed vacuum representation fulfills Haag duality for spacelike cones, i.e.

$$\mathcal{A}(S') = \mathcal{A}(S)' \quad \text{for all } S \in \mathcal{S} \,. \tag{2.8}$$

One now proceeds as in the DHR-analysis [3] and identifies the representation spaces $\mathcal{H}_0$ and $\mathcal{H}_\pi$. To this end one exploits (2.5) to define for any representation π, satisfying (2.5), corresponding homomorphisms $\varrho : \mathcal{A}_0(\mathcal{M}) \to B(\mathcal{H}_0)$ which are unitarily equivalent to π. In the present context these ϱ are in general not endomorphisms, i.e. $\varrho(\mathcal{A}_0(\mathcal{M})) \not\subset \mathcal{A}_0(\mathcal{M})$, and thus the usual definition for the composition of sectors is ill-defined [2].

To overcome this difficulty one can pass from the algebra of quasi-local observables $\mathcal{A}_0(\mathcal{M})$ to a larger C^*-algebra B^r depending on a forbidden spacelike direction r,

$$B^r := \overline{\bigcup_{S \in \mathcal{S}(r)} \mathcal{A}(S')} \,. \tag{2.9}$$

Here r is a spacelike unit vector (i.e. $r^2 = -1$), and $\mathcal{S}(r)$ is the set of spacelike cones, which "contain the direction r"

$$\mathcal{S}(r) := \{S \in \mathcal{S} \,|\, \bar{S} + r \subset S\} \,. \tag{2.10}$$

One can subsequently show (see [2] resp. [25] for details) that if ϱ is a morphism localized in a cone S spacelike to r (i.e. there exists a $\hat{S} \in \mathcal{S}(r)$ such that $\hat{S} \subset S'$) then ϱ extends uniquely to an endomorphism ϱ^r of B^r which is weakly continuous on all $\mathcal{A}(S')$ for $S \in \mathcal{S}(r)$. Thus one can proceed in very much the same way as in the DHR-analysis; in particular, one defines the composition of sectors via the composition of the corresponding morphisms on B^r. However, one has to check that all structural properties are independent of the choice of the direction r.

To avoid singling out this "auxiliary direction" one can embed all B^r into an even larger C^*-algebra, the *universal algebra* $\mathcal{A}_0(\bar{\mathcal{M}})$, which can be uniquely characterized by the following universality conditions (this construction was proposed in [8] and further developed in [13] and [10, II]):

- There are unital embeddings $i^I : \mathcal{A}(I) \to \mathcal{A}_0(\bar{\mathcal{M}})$ such that for all $I, J \in \mathcal{K} := \{S, S' | S \in \mathcal{S}\}$,

$$i^J|_{\mathcal{A}(I)} = i^I \quad \text{if } I \subset J \tag{2.11}$$

and $\mathcal{A}_0(\bar{\mathcal{M}})$ is generated by the algebras $i^I(\mathcal{A}(I))$.
- For every family of normal representations $(\pi^I)_{I \in \mathcal{K}}$, $\pi^I : \mathcal{A}(I) \to B(\mathcal{H}_\pi)$ which satisfies the compatibility condition

$$\pi^J|_{\mathcal{A}(I)} = \pi^I \quad \text{if } I \subset J , \tag{2.12}$$

there is a unique representation π of $\mathcal{A}_0(\bar{\mathcal{M}})$ in $\mathcal{H}_\pi$ such that

$$\pi \circ i^I = \pi^I . \tag{2.13}$$

The usefulness of this definition is due to the fact that the endomorphisms ϱ^r have a common extension to an endomorphism ϱ of $\mathcal{A}_0(\bar{\mathcal{M}})$ such that the unique extensions of π and π_0 to $\mathcal{A}_0(\bar{\mathcal{M}})$ (which shall be denoted by the same symbols) satisfy[2]

$$\pi = \pi_0 \circ \varrho . \tag{2.14}$$

However, in general the vacuum representation π_0 is no longer faithful on $\mathcal{A}_0(\bar{\mathcal{M}})$ (see [10, II] for details).

The endomorphisms ϱ one obtains are localized in some $I \in K$ in the following sense [9].

Definition 2.1. *An endomorphism ϱ of $\mathcal{A}_0(\bar{\mathcal{M}})$ is called localizable within $I \in \mathcal{K}$ if for all $I_0 \subset I, I_0 \in \mathcal{K}$ there exists a unitary $U \in \mathcal{A}(I)$ such that*

$$\varrho(A) = \text{Ad } U(A), \quad A \in \mathcal{A}(I_0') ,$$

$$\text{Ad } U^* \circ \varrho(\mathcal{A}(I_1)) \subset \mathcal{A}(I_1), \quad I_1 \supset I_0, I_1 \in \mathcal{K} .$$

An endomorphism ϱ is called transportable if for every $I \in \mathcal{K}$ there exists an endomorphism ϱ' of $\mathcal{A}_0(\bar{\mathcal{M}})$ which is localizable within I and is inner equivalent to ϱ, i.e. there exists a unitary $U \in \mathcal{A}_0(\bar{\mathcal{M}})$ such that $\varrho' = \text{Ad } U \circ \varrho$.

Note that endomorphisms which are localizable within I are not necessarily localizable within $J \supset I$. However transportable endomorphisms which are localizable within some region are automatically localized in every larger region.

Let Δ denote the set of transportable endomorphisms and $\Delta(I)$ the subset of transportable endomorphisms which are localizable within I.

In the $s + 1$-dimensional situation, $s \geq 3$, it is possible to embed $\mathcal{A}_0(\mathcal{M})$ into a net of field algebras $\mathcal{F}$ which transform covariantly under some compact group of internal symmetries and satisfy graded locality [4]. These fields may generate single particle states from the vacuum, and one can use them for the construction of multiparticle scattering states by the standard recipe of the Haag–Ruelle theory [14, 15]. At the time when [3] and [2] were written the existence of field algebras was not

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[2] From now on we shall consider $\mathcal{A}(I), I \in K$ as abstract subalgebras of $\mathcal{A}_0(\bar{\mathcal{M}})$ and only $\pi(A)$ (resp. $\pi_0(A)$) as operators on the vacuum Hilbert space $\mathcal{H}_0$.

established, so a different method for the construction of scattering states had to be used. This method is based on the fact that the partial intertwiners which exist between representations satisfying a localizability condition of the type (2.5) behave in many respects in the same way as field operators. A convenient formalism for the description of these partial intertwiners is the so-called field bundle introduced in [3, II]. In cases where nontrivial braid group statistics occurs a general construction of field algebras is difficult (see, however, [19,23]). Interestingly enough, the exchange algebra formalism of chiral conformal field theory [22] is equivalent to the field bundle formalism [10, I].

To define the field bundle, we first fix a spacelike cone S_0. We then describe vectors Ψ in some representation $\pi_0 \circ \varrho$ by a pair $\mathbf{\Psi} = \{\varrho; \Psi\}$ and consider $\Delta(S_0) \times \mathscr{H}_0 = \mathscr{H}$ as a hermitian vector bundle over $\Delta(S_0)$, where on every fiber $\mathscr{H}_\varrho = \{\varrho\} \times \mathscr{H}_0$ the scalar product is that of $\mathscr{H}_0$.

In the next step, we define *generalized field operators* as pairs $\mathbf{B} = \{\varrho; B\} \in \Delta(S_0) \times \mathscr{A}_0(\bar{\mathscr{M}})$ which act on $\mathscr{H}$ by

$$\{\hat{\varrho}; B\}\{\varrho; \Psi\} = \{\varrho\hat{\varrho}; \pi_0 \circ \varrho(B)\Psi\} \tag{2.15}$$

and possess the norm $\|\{\hat{\varrho}; B\}\| := \|B\|$. The fibers $\{\varrho\} \times \mathscr{A}_0(\bar{\mathscr{M}})$ are linear spaces isomorphic to $\mathscr{A}_0(\bar{\mathscr{M}})$. In addition, there is an associative multiplication law of field operators (consistent with the above action on $\mathscr{H}$)

$$\{\varrho_1; B_1\}\{\varrho_2; B_2\} := \{\varrho_2\varrho_1; \varrho_2(B_1)B_2\} . \tag{2.16}$$

Observables correspond to field operators of the form $\{id; A\}, A \in \mathscr{A}_0(\bar{\mathscr{M}})$.

The formalism contains a large redundancy which can be described by the action of intertwiners $T \in \mathscr{A}_0(\bar{\mathscr{M}})$ satisfying $\varrho_1(A)T = T\varrho(A)$, $A \in \mathscr{A}_0(\bar{\mathscr{M}})$. Here T is an *intertwiner* from ϱ to ϱ_1 and it induces the actions

$$(\varrho_1|T|\varrho)\{\varrho; \Psi\} = \{\varrho_1; \pi_0(T)\Psi\}, \qquad (\varrho_1|T|\varrho) \circ \{\varrho; B\} = \{\varrho_1; TB\} . \tag{2.17}$$

To implement the Poincaré covariance of the theory into the field bundle, we define

$$U(g)\{\varrho; \Psi\} = \{\varrho; U_\varrho(g)\Psi\} , \tag{2.18}$$

$$\alpha(g)\{\varrho; B\} = \{\varrho; Y_\varrho(g)\alpha_{L(g)}B\} , \tag{2.19}$$

where $U_\varrho(g)$ is the representation of $\widetilde{P_+^\uparrow}$ corresponding to $\pi = \pi_0 \circ \varrho$ (see (2.3)), α_L denotes the extension of α_L from $\mathscr{A}_0(\mathscr{M})$ to $\mathscr{A}_0(\bar{\mathscr{M}})$ and

$$Y_\varrho(g) = \pi_0^{-1}(U_\varrho(g) U_0(L(g))^{-1}) . \tag{2.20}$$

Equation (2.19) is to be understood only for g sufficiently close to the identity in order to make sure that there is a path $L(t)$ in the homotopy class g such that $\bigcup_t L(t)S_0$ is spacelike to some spacelike direction r. Then $U_\varrho(g) U_0(L(g))^{-1} \in \pi_0(B^r)$. This guarantees that the preimage under π_0 is well-defined since π_0 is faithful on B^r. The general case is obtained by successive use of (2.19).

Poincaré transformations commute with intertwiners, i.e. if T is an intertwiner from ϱ to ϱ_1, then

$$\pi_0(T) U_\varrho(g) = U_{\varrho_1}(g) \pi_0(T) . \tag{2.21}$$

Finally, the representation in the fiber $\varrho_1\varrho_2$ is related to the ones in the fibers ϱ_1 and ϱ_2 by

$$U_{\varrho_1\varrho_2}(g) = \pi_0 \circ \varrho_1(Y_{\varrho_2}(g)) \, U_{\varrho_2}(g) \tag{2.22}$$

(see [10, II] for details).

A necessary condition for a generalized field operator $\mathbf{B} = \{\varrho; B\}$ to be localized in $I \in \mathscr{K}$ is that B intertwines the identity with ϱ on $\mathscr{A}(I')$. But due to the existence of global self-intertwiners, the intertwining property of B is too weak for a derivation of commutation relations between spacelike separated generalized fields. Instead one characterizes the localization by a path in $\mathscr{K}$, i.e. a finite sequence $I_i \in \mathscr{K}$, $i = 0, \ldots, n$ with $I_0 = S_0$ and such that either $I_i \subset I_{i-1}$ or $I_i \supset I_{i-1}$, $i = 1, \ldots, n$. For each i there is some unitary $U_i \in \mathscr{A}(I_i \cup I_{i-1})$ such that $\mathrm{Ad}\, U_i \cdots U_1 \circ \varrho \in \Delta(I_i)$. Then $\{\varrho, B\}$ is called localized in $(I_0, \ldots, I_n)$ if

$$U_n \cdots U_1 B \in \mathscr{A}(I_n) \,. \tag{2.23}$$

The concept of localization described above is an extension of the corresponding notion in [3] following ideas of [10, II]. Clearly, the localization depends only on the homotopy class $\tilde{I}$ of a path $(I_0, \ldots, I_n)$ where homotopy is defined in the obvious way. The set of these classes shall be denoted by $\tilde{\mathscr{K}}$ and the set of field operators localized in $\tilde{I}$ by $\mathscr{F}(\tilde{I})$.

Let us now consider paths with the same endpoint. They differ (up to homotopy) by a closed path $\gamma = (I_0, \ldots, I_k)$ with $I_k = I_0$. We choose associated intertwiners $U_1, \ldots, U_k$ with $\pi_0(U_k \cdots U_1) = 1$. Then $\gamma \mapsto U(\gamma) = U_k \cdots U_1$ is a representation of the homotopy group by unitary elements of $\mathscr{A}_0(\tilde{\mathscr{M}})$.

In a next step we look at products of field operators with mutually spacelike localization. Here $(I_0, \ldots, I_n) = \tilde{I}$ and $(J_0, \ldots, J_k) = \tilde{J}$, $I_0 = J_0 = S_0$, are called mutually spacelike if the endpoint of $\tilde{I}$, $e(\tilde{I}) = I_n$, is spacelike separated from $e(\tilde{J}) = J_k$. Let $\mathbf{B}_i = \{\varrho_i, B_i\}$ be localized in $\tilde{I}_i$, $i = 1, \ldots, n$. Then

$$\mathbf{B}_{\sigma(n)} \cdots \mathbf{B}_{\sigma(1)} = \varepsilon \circ \mathbf{B}_n \cdots \mathbf{B}_1 \,, \tag{2.24}$$

where ε is an intertwiner from $\varrho_1 \ldots \varrho_n$ to $\varrho_{\sigma(1)} \ldots \varrho_{\sigma(n)}$. ε depends on the endomorphisms $\varrho_i \in \Delta(S_0)$, on the localizations $\tilde{I}_i$ and on the permutation σ. It is described in terms of a unitary representation of the groupoid of colored braids on the cylinder [10, II]. The associated braid can be obtained by the following geometrical construction.

Mutually spacelike paths $\tilde{I}_i$ are continuously deformed to paths γ_i on the set of spatial directions in some Lorentz frame, i.e. to paths on the circle S^1 with a fixed initial point z_0 corresponding to I_0 and disjoint endpoints z_i corresponding to the endpoints $e(\tilde{I}_i)$ of $\tilde{I}_i$. On the cylinder $S^1 \times \mathbb{R}$ we choose points (z_0, i), $i = 1, \ldots, n$ and paths Γ_i from (z_0, i) to $(z_0, \sigma(i))$,

$$\Gamma_i = (\gamma_i^{-1}, \sigma(i)) \circ (z_i, i \to \sigma(i)) \circ (\gamma_i, i) \,. \tag{2.25}$$

The braid is now the usual equivalence class of the family of strands Γ_i, $i = 1, \ldots, n$ (see for example Fig. 1, where the 3rd dimension is introduced for visualizing the parameter of the paths $(z_i, i \to \sigma(i))$).

By the standard techniques of algebraic field theory (see [3, 10] for more details) it follows that ε is invariant under small deformations of $\tilde{I}_1, \ldots, \tilde{I}_n$ – so equivalent families Γ_i, $i = 1, \ldots, n$ give the same intertwiner ε – and that the braid relations are respected.
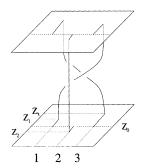
Fig. 1. The braid corresponding to the permutation $\sigma = \tau_1 \tau_2 \tau_1$ in the special case where all three paths γ_i have trivial winding number and thus can be represented as paths in the plane.

For the calculation of scalar products of scattering state vectors in the next section we need the notion of a *left inverse* of an endomorphism $\varrho \in \Delta(S_0)$ (see [10, 5, 18] for more details): A left inverse ϕ of a ϱ is a positive mapping of $\mathscr{A}_0(\bar{\mathscr{M}})$, mapping $\mathscr{A}(S_0)$ into itself such that $\phi \circ \varrho = id$ and such that $\varrho \circ \phi$ is a conditional expectation from $\mathscr{A}_0(\bar{\mathscr{M}})$ to $\varrho(\mathscr{A}_0(\bar{\mathscr{M}}))$. If ϱ is irreducible, i.e. $\pi \circ \varrho$ is irreducible, and has finite statistics, a property which is automatically satisfied for irreducible single particle representations [5], the left inverse is unique. If $\varrho_1, \ldots, \varrho_n \in \Delta(S_0)$ are irreducible with finite statistics, the product $\varrho = \varrho_1 \cdots \varrho_n$ is not irreducible in general, and there is no unique left inverse. But there is a so-called standard left inverse which is given by

$$\phi = \phi_n \cdots \phi_1 \tag{2.26}$$

with ϕ_i the unique left inverse of ϱ_i, $i = 1, \ldots, n$. The standard left inverse is a trace on the algebra of local self-intertwiners of ϱ, i.e. if $S, T \in \mathscr{A}(S_0)$ commute with $\varrho(\mathscr{A}_0(\bar{\mathscr{M}}))$, then

$$\phi(ST) = \phi(TS). \tag{2.27}$$

Some of the following formulae are more easily expressed in terms of *right inverses* of endomorphisms which have been recently introduced by Roberts [24]. The right inverse of ϱ is only defined on the class of intertwiners of the form $(\varrho'' \varrho | T | \varrho' \varrho)$. For such an intertwiner, the right inverse, $\chi_\varrho(T)$, is an intertwiner from ϱ' to ϱ''. If ϱ has a conjugate representation, a right inverse of ϱ can be defined as

$$\chi_\varrho(T) = \varrho''(\bar{R})^* T \varrho'(\bar{R}), \tag{2.28}$$

where $\bar{R}$ is an isometric intertwiner from the vacuum representation to $\varrho\bar{\varrho}$. Roberts has shown that there is a unique right inverse, the *standard right inverse*, which agrees with the standard left inverse on local self-intertwiners. The standard right inverse is unique for irreducible ϱ with finite statistics. Furthermore, the product of standard right inverses is the standard right inverse of the composite endomorphism.

We also need a version of the cluster theorem [3] which is adapted to the present situation. Let $\mathbf{B}_i = \{\varrho_i, B_i\} \in \mathscr{F}(\tilde{I}_i)$, $i = 2, 4$ with $\tilde{I}_2 = \tilde{I}_4$ and let $\mathbf{B}_j = \{\varrho_j, B_j\}$, $j = 1, 3$ be products of field operators. For fixed $e, e^2 = 1$, let τ be the supremum of $|t|$ for all t for which all the field operators in $\mathbf{B}_1$ and $\mathbf{B}_3$ are spacelike localized with respect to $I_2 + te$. Furthermore, let T be an intertwiner from $\varrho_1 \varrho_2$ to $\varrho_3 \varrho_4$. We are interested in the leading behavior of $(\mathbf{B}_4 \mathbf{B}_3 \Omega, T \mathbf{B}_2 \mathbf{B}_1 \Omega)$ for large τ. Let us assume that ϱ_4 is irreducible with finite statistics with right inverse χ_4, and

denote by $\{W_j\}$ a (possibly empty) orthonormal basis of the Hilbert space of local intertwiners from ϱ_4 to ϱ_2. Then we have the following

Lemma 2.2.

$$\left| (\mathbf{B}_4\mathbf{B}_3\mathbf{\Omega}, T\mathbf{B}_2\mathbf{B}_1\mathbf{\Omega}) - \sum_j (\mathbf{B}_3\mathbf{\Omega}, \chi_4(T\varrho_1(W_j))\mathbf{B}_1\mathbf{\Omega})(\mathbf{B}_4\mathbf{\Omega}, W_j^*\mathbf{B}_2\mathbf{\Omega}) \right| \leqq e^{-\mu\tau}\prod_i \|\mathbf{B}_i\| .$$

$$(2.29)$$

Proof. The proof of this lemma is similar to the proof of Lemma 7.3 in [3, II]. However, introducing the concept of a right inverse makes the present proof in our opinion conceptually much clearer. In particular, it demonstrates that the proof extends directly to the case of non-trivial statistics.

The field bundle is invariant (up to isomorphism) under changing the location of S_0 locally. Therefore the same is true globally, where, however, the isomorphism depends on the homotopy class of the path connecting the two different localizations of S_0. Using this property, we can assume without loss of generality that the localization of $\tilde{I}_2 = \tilde{I}_4$ coincides with S_0. We calculate

$$(\mathbf{B}_4\mathbf{B}_3\mathbf{\Omega}, T\mathbf{B}_2\mathbf{B}_1\mathbf{\Omega}) = ((\mathbb{1}_{\varrho_3} \times \bar{R}_4)\mathbf{B}_3\mathbf{\Omega}, \mathbf{B}_4^\dagger T\mathbf{B}_2\mathbf{B}_1\mathbf{\Omega})$$

$$= (\mathbf{B}_3\mathbf{\Omega}, (\mathbb{1}_{\varrho_3} \times \bar{R}_4^*)(T \times \mathbb{1}_{\bar{\varrho}_4})\mathbf{B}_4^\dagger\mathbf{B}_2\mathbf{B}_1\mathbf{\Omega})$$

$$= (\mathbf{B}_3\mathbf{\Omega}, (\mathbb{1}_{\varrho_3} \times \bar{R}_4^*)(T \times \mathbb{1}_{\bar{\varrho}_4})\varepsilon(\varrho_2\bar{\varrho}_4, \varrho_1)\mathbf{B}_1\mathbf{B}_4^\dagger\mathbf{B}_2\mathbf{\Omega}) , \qquad (2.30)$$

where $\bar{R}_4$ is the local isometric intertwiner from the vacuum to $\varrho_4\bar{\varrho}_4$ which appears in the definition of the adjoint operator in the field bundle formalism and $\varepsilon(\varrho_2\bar{\varrho}_4, \varrho_1)$ depends on the localization of $\mathbf{B}_1$ relative to S_0. (Here we use the usual notation for the intertwiner calculus in the field bundle, see e.g. [3] for definitions.)

The usual cluster theorem (see for example [6] for a proof) implies that the scalar product is dominated by the contribution of the vacuum sector in $\mathbf{B}_4^\dagger\mathbf{B}_2\mathbf{\Omega}$. Inserting the corresponding projector onto the vacuum

$$\sum_j |(W_j \times \mathbb{1}_{\bar{\varrho}_4})\bar{R}_4\mathbf{\Omega}\rangle \langle(W_j \times \mathbb{1}_{\bar{\varrho}_4})\bar{R}_4\mathbf{\Omega}| \qquad (2.31)$$

into the scalar product, we can conclude that the right-hand side of (2.30) is

$$\sum_j (\mathbf{B}_3\mathbf{\Omega}, (\mathbb{1}_{\varrho_3} \times \bar{R}_4^*)(T\varrho_1(W_j) \times \mathbb{1}_{\bar{\varrho}_4})\varepsilon(\varrho_4\bar{\varrho}_4, \varrho_1)\mathbf{B}_1\bar{R}_4\mathbf{\Omega}) \ (\bar{R}_4\mathbf{\Omega}, \mathbf{B}_4^\dagger W_j^*\mathbf{B}_2\mathbf{\Omega}) \quad (2.32)$$

up to a term which is smaller than the right-hand side of (2.29). We observe that the second term in the product (for each j) is just

$$(\bar{R}_4\mathbf{\Omega}, \mathbf{B}_4^\dagger W_j^*\mathbf{B}_2\mathbf{\Omega}) = (\mathbf{B}_4\mathbf{\Omega}, W_j^*\mathbf{B}_2\mathbf{\Omega}) . \qquad (2.33)$$

We can commute the intertwiner $\bar{R}_4$ past $\mathbf{B}_1$ in the first term in the product, and thus obtain (for each j)

$$(\mathbf{B}_3\mathbf{\Omega}, (\mathbb{1}_{\varrho_3} \times \bar{R}_4^*)(T\varrho_1(W_j) \times \mathbb{1}_{\bar{\varrho}_4})\varepsilon(\varrho_4\bar{\varrho}_4, \varrho_1)(\mathbb{1}_{\varrho_1} \times \bar{R}_4)\mathbf{B}_1\mathbf{\Omega}) . \qquad (2.34)$$

It remains to show that the product of intertwiners in this expression is $\chi_4(T_{\varrho_1}(W_j))$. As the localization of $\tilde{I}_2 = \tilde{I}_4$ is S_0, we can write the ε intertwiner as

$$\varepsilon(\varrho_4\bar{\varrho}_4, \varrho_1) = V_1^{-1}\varrho_4\bar{\varrho}_4(V_1), \tag{2.35}$$

where V_1 is an intertwiner from ϱ_1, localized in S_0, to $\hat{\varrho}_1$ whose localization corresponds to the localization of the different operators in the operator product $\mathbf{B}_1$. Thus (cf. [10, II(2.21)])

$$\varepsilon(\varrho_4\bar{\varrho}_4, \varrho_1)\bar{R}_4 = V_1^{-1}\varrho_4\bar{\varrho}_4(V_1)\bar{R}_4 = V_1^{-1}\bar{R}_4 V_1$$

$$= V_1^{-1}\hat{\varrho}_1(\bar{R}_4)V_1 = \varrho_1(\bar{R}_4), \tag{2.36}$$

where, we have used that $\bar{R}_4$ is local and thus that $\hat{\varrho}_1(\bar{R}_4) = \bar{R}_4$. By (2.28) the product of intertwiners is indeed just $\chi_4(T_{\varrho_1}(W_j))$. $\square$

3. The Structure of Scattering States

To construct multiparticle scattering states one wants to follow the general recipe of the Haag–Ruelle theory (for an introduction see [15]): one first constructs almost local one particle creation operators $\mathbf{B}_i$ (here almost localized in spacelike cones) and propagates them to other times by using the Klein Gordon equation. In this way one obtains operators $\mathbf{B}_i(t)$ which are essentially localized at time t and create one particle state vectors $\mathbf{\Psi}_i = \mathbf{B}_i(t)\mathbf{\Omega}$ independent of t. Then one proves convergence of

$$\mathbf{B}_n(t)\cdots\mathbf{B}_1(t)\mathbf{\Omega} \tag{3.1}$$

for $t \to \pm\infty$ and interprets the limits as the outgoing or incoming, respectively, scattering state vectors corresponding to single particle vectors $\mathbf{\Psi}_i$, $i = 1,\ldots,n$.

The scalar products of scattering state vectors can be computed by using the cluster theorem. In the case of finitely localized sectors one can finally show that the scattering vectors do not depend on the choice of the operators $\mathbf{B}_i(t)$, nor on the choice of the time direction, but only on the single particle vectors $\mathbf{\Psi}_i$, $i = 1,\ldots,n$. In the case of sectors localized in spacelike cones this remains true in $s + 1$ dimensional spacetime with $s > 2$ [2], but it definitely fails for $s = 2$ if the sectors have nontrivial braid group statistics.

This is the reason why the multiparticle space, considered as a representation space of the Poincaré group, is in general not isomorphic to a tensor product of one particle spaces [7]. A satisfactory description of the scattering space in terms of asymptotic fields does not yet exist. However, the structure of this space can be completely computed and turns out to be identical to the structure proposed by Mund and Schrader [20].

In a first step we show that a manifestly Lorentz invariant formulation of the Haag–Ruelle theory is possible. In this formulation each particle propagates in its own rest frame.

Let $\mathbf{B} \in \mathscr{F}(\tilde{I})$ for some localization $\tilde{I} \in \tilde{\mathscr{K}}$ be such that the energy momentum spectrum $\mathrm{sp}_{\mathbf{U}}\mathbf{B}\mathbf{\Omega}$ contains an isolated mass shell H_m. Let $f \in \mathscr{S}(\mathscr{M})$ have a Fourier transform $\tilde{f}$ with compact support in V_+ such that $\mathrm{supp}\,\tilde{f} \cap \mathrm{sp}_{\mathbf{U}}\mathbf{B}\mathbf{\Omega} \subset H_m$. Then we define

$$\mathbf{B}(t) := \int d^3x\, f_t(x)\, \alpha_x(\mathbf{B}), \tag{3.2}$$

where

$$f_t(x) = (2\pi)^{-\frac{3}{2}} \int d^3p \, e^{-ipx + i(\frac{p^2 - m^2}{2m})t} \, \tilde{f}(p) \,. \tag{3.3}$$

$\mathbf{B}(t)\Omega = \Psi$ is an eigenvector of the mass operator $M^2 = P^2$ with eigenvalue m^2 and it does not depend on t.

The functions f_t have the following properties:

Lemma 3.1. *Let $f \in \mathscr{S}(\mathscr{M})$ have a Fourier transform $\tilde{f}$ with compact support. Then the following two statements hold:*

(i) *There exists a constant $c > 0$ such that*

$$\int d^3x |f_t(x)| < c(1 + |t|^3) \,.$$

(ii) *For each $\varepsilon > 0$, $N \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$|f_t(x)| < C \left| \mathrm{dist}\left(x, \frac{p}{m} t\right) + 1 \right|^{-N}$$

for all $p \in \mathrm{supp}\tilde{f}$ and all $x \in \mathscr{M}, t \in \mathbb{R}$ such that $\mathrm{dist}(\frac{x}{t}, \frac{q}{m}) > \varepsilon \;\; \forall q \in \mathrm{supp}\tilde{f}$. Here dist means an Euclidean metric on $\mathscr{M}$.

The proof follows by standard techniques of the stationary phase approximation (see e.g. [21]).

Let $V_\varepsilon(f) = \{v \in \mathscr{M}, \mathrm{dist}(v, \frac{p}{m}) < \varepsilon$ for some $p \in \mathrm{supp}\tilde{f}\}$. It follows from the lemma that the operators $\mathbf{B}(t)$ can be approximated (for large t) by operators $\mathbf{B}_\varepsilon(t) \in \mathscr{F}(\tilde{I} + tV_\varepsilon(f))$,

$$\mathbf{B}_\varepsilon(t) = \int\limits_{tV_\varepsilon(f)} d^3x \, f_t(x) \, \alpha_x(\mathbf{B}) \,, \tag{3.4}$$

such that $\|\mathbf{B}(t) - \mathbf{B}_\varepsilon(t)\| < c_N |t|^{-N}$ for suitable constants c_N. Moreover, the norms of these operators are bounded by $\|\mathbf{B}(t)\| < c(1 + |t|^3)$.

We now consider a configuration $\tilde{I}_i \in K$, $\mathbf{B}_i \in \mathscr{F}(\tilde{I}_i)$, $\tilde{f}_i \in \mathscr{C}_0^\infty(V_+)$, $\varepsilon_i > 0$, $i = 1, \ldots, n$ such that the regions $\tilde{I}_i + tV_{\varepsilon_i}(f_i)$ are mutually spacelike for large t. Then the limit

$$\lim_{t \to \infty} \mathbf{B}_n(t) \cdots \mathbf{B}_1(t)\Omega \tag{3.5}$$

exists and may be interpreted as a vector describing an outgoing configuration of n particles with state vectors $\Psi_i = \mathbf{B}_i(t)\Omega$. As long as the localizations $\tilde{I}_i$ are kept fixed, the scattering vectors depend only on these 1-particle vectors and we may write

$$\lim_{t \to \infty} \mathbf{B}_n(t) \cdots \mathbf{B}_1(t)\Omega = (\Psi_n, \tilde{I}_n) \times \cdots \times (\Psi_1, \tilde{I}_1) \,. \tag{3.6}$$

It is also easy to see how the Poincaré group acts and how the scattering vectors depend on the order of the one particle vectors:

$$\mathbf{U}(L)(\Psi_n, \tilde{I}_n) \times \cdots \times (\Psi_1, \tilde{I}_1) = (\mathbf{U}(L)\Psi_n, L\tilde{I}_n) \times \cdots \times (\mathbf{U}(L)\Psi_1, L\tilde{I}_1) \,, \tag{3.7}$$

and

$$(\boldsymbol{\Psi}_{\sigma(n)}, \tilde{I}_{\sigma(n)}) \times \cdots \times (\boldsymbol{\Psi}_{\sigma(1)}, \tilde{I}_{\sigma(1)}) = \varepsilon(b)(\boldsymbol{\Psi}_n, \tilde{I}_n) \times \cdots \times (\boldsymbol{\Psi}_1, \tilde{I}_1) \,, \qquad (3.8)$$

where b is the cylinder braid defined in Sect. 2. Note that ε acts via the vacuum representation. Since the intertwiner describing the transition from $\tilde{I}_1$ to other sheets is trivially represented in the vacuum, $\pi_0 \circ \varepsilon$ is actually a representation of the groupoid of colored braids on the plane.

In order to understand the dependence of the scattering vectors on the localizations $\tilde{I}_i \in \tilde{\mathcal{K}}$, let us assume that there exists for some $j \in \{1, \ldots, n\}$ a localization $\tilde{J}_j \in \tilde{\mathcal{K}}$, a field operator $C_j \in \mathcal{F}(\tilde{J}_j)$ and a test function g_j with $\operatorname{supp} \tilde{g}_j \subset m_j V_\varepsilon(f_j)$ such that (with $C_j(t)$ defined in analogy to (3.2))

$$\boldsymbol{\Psi}_j = C_j(t)\boldsymbol{\Omega} \,, \qquad (3.9)$$

and $\tilde{J}_j + tV_\varepsilon(f_j)$ is spacelike to $\tilde{I}_i + tV_\varepsilon(f_i)$ for $i \neq j$ and for large t. If $j = 1$ the scattering vector does not change when $B_j(t)$ is replaced by $C_j(t)$. If $j \neq 1$ we first commute $B_j(t)$ to the right, then replace it by $C_j(t)$ and commute it back to the jth place. The whole procedure amounts to an application of an intertwiner $\varepsilon(b)$ to the scattering vector where b is a pure cylinder braid obtained by the prescription in Sect. 2.

Next, we turn to the computation of the scalar product of scattering vectors. Let $V_i \subset H_1$ be compact and $\tilde{I}_i \in \tilde{\mathcal{K}}, i = 1, \ldots, n$ such that for suitable neighborhoods V_i^ε of V_i in V_+ the regions $tV_i^\varepsilon + \tilde{I}_i$ are mutually spacelike for large t, and let f_i be test-functions with $\operatorname{supp} \tilde{f}_i \subset V_i^\varepsilon$. Let $B_i, C_i \in \mathcal{F}(\tilde{I}_i)$ with associated single particle representations ϱ_i and σ_i, respectively, $i = 1, \ldots, n$. Let $T \in \mathcal{A}(S_0)$ be an intertwiner from $\sigma_1 \cdots \sigma_n$ to $\varrho_1 \cdots \varrho_n$. Then, with $\boldsymbol{\Psi}_i = B_i(t)\boldsymbol{\Omega}, \boldsymbol{\Phi}_i = C_i(t)\boldsymbol{\Omega}$, we find the following

Theorem 3.2. *Let ϕ_i be the unique left inverse of ϱ_i, $i = 1, \ldots, n$.*

(i) *If ϱ_i is not equivalent to σ_i for some $i \in \{1, \ldots, n\}$, then*

$$((\boldsymbol{\Psi}_n, \tilde{I}_n) \times \cdots \times (\boldsymbol{\Psi}_1, \tilde{I}_1), T(\boldsymbol{\Phi}_n, \tilde{I}_n) \times \cdots \times (\boldsymbol{\Phi}_1, \tilde{I}_1)) = 0 \,. \qquad (3.10)$$

(ii) *If $\varrho_i = \sigma_i$, $i = 1, \ldots, n$, then*

$$((\boldsymbol{\Psi}_n, \tilde{I}_n) \times \cdots \times (\boldsymbol{\Psi}_1, \tilde{I}_1), T(\boldsymbol{\Phi}_n, \tilde{I}_n) \times \cdots \times (\boldsymbol{\Phi}_1, \tilde{I}_1)) = \phi_n \cdots \phi_1(T) \prod_i (\boldsymbol{\Psi}_i, \boldsymbol{\Phi}_i) \,.$$

$$\qquad (3.11)$$

Proof. We apply recursively Lemma 2.2 to the left-hand side of the above expression. If ϱ_i is not equivalent to σ_i for some i, we obtain zero. Otherwise, we have

$$\chi_1 \cdots \chi_n(T) \prod_i (\boldsymbol{\Psi}_i, \boldsymbol{\Phi}_i) \,. \qquad (3.12)$$

$\chi_1 \cdots \chi_n$ is the standard right inverse on $\varrho_1 \cdots \varrho_n$ and therefore agrees with the standard left inverse $\phi_n \cdots \phi_1$ on local intertwiners. $\quad\square$

Thus the scattering vectors depend in a continuous way on the one particle vectors. Since $\mathcal{F}(\tilde{I})$ is dense in $\mathcal{H}$ for all $\tilde{I} \in \tilde{\mathcal{K}}$, we find, by going to the closure, all scattering states corresponding to single particle states with prescribed momentum support.

For a single particle representation ϱ with mass m_ϱ, let $\mathscr{H}_\varrho(V) = \{\Psi \in \mathscr{H}_\varrho, \mathrm{sp}_{\mathbf{U}}\Psi$
$\subset m_\varrho V\}$. Given an irreducible endomorphism $\sigma \in \varDelta(S_0)$, let $\mathscr{H}_{\mathbf{V},\varrho,\sigma} \subset \mathscr{H}_\sigma$ denote the
Hilbert space spanned by the scattering vectors

$$T(\Psi_n, \tilde{I}_n) \times \cdots \times (\Psi_1, \tilde{I}_1),\tag{3.13}$$

where $\Psi_i \in \mathscr{H}_{\varrho_i}(V_i)$, $\mathbf{V} = (V_1, \ldots, V_n)$, $\varrho = (\varrho_1, \ldots, \varrho_n)$ and $T \in \mathscr{A}(S_0)$ is an inter-
twiner from $\varrho_1 \cdots \varrho_n$ to σ. Finally, we define

$$\mathscr{H}_{\mathbf{V},\varrho,\sigma}^{(0)} = \left(\bigotimes_i \mathscr{H}_{\varrho_i}(V_i) \right) \otimes \mathscr{H}_{\varrho_1 \cdots \varrho_n, \sigma},\tag{3.14}$$

where $\mathscr{H}_{\varrho_1 \cdots \varrho_n, \sigma}$ is the space of intertwiners $T \in \mathscr{A}(S_0)$ from $\varrho_1 \cdots \varrho_n$ to σ, equipped
with the scalar product

$$(S, T)\mathbb{1} = \phi_n \cdots \phi_1(S^* T) = TS^* \frac{d_\sigma}{d_{\varrho_1} \cdots d_{\varrho_n}},\tag{3.15}$$

where d_ϱ is the statistical dimension of $\varrho \in \varDelta(S_0)$ and TS^* is a multiple of the
identity as a local self-intertwiner of the irreducible endomorphism $\sigma \in \varDelta(S_0)$.

Theorem 3.2 implies that for each configuration $\tilde{\mathbf{I}} = (\tilde{I}_1, \ldots, \tilde{I}_n), \tilde{I}_i \in \mathscr{K}$, such
that the sets $\tilde{I}_i + tV_i$ are mutually spacelike for large t and for each unitary local
self-intertwiner U of $\varrho_1 \cdots \varrho_n$, there exists an isometric embedding

$$i(\tilde{\mathbf{I}}, U): \mathscr{H}_{\mathbf{V},\varrho,\sigma}^{(0)} \to \mathscr{H}_\sigma\tag{3.16}$$

given by

$$(\Psi_1 \otimes \cdots \otimes \Psi_n) \otimes T \mapsto T U(\Psi_n, \tilde{I}_n) \times \cdots \times (\Psi_1, \tilde{I}_1).\tag{3.17}$$

Embeddings related to different choices of $\tilde{\mathbf{I}}$ are related by a pure braid which
acts from the right on the intertwiner U. This braid can be chosen to be a local
intertwiner, as it acts via the vacuum representation (cf. (2.17)). Thus the space of
scattering states $\mathscr{H}_{\mathbf{V},\varrho,\sigma}$ does not depend on $\tilde{\mathbf{I}}$ or U.

It is easy to see that the embeddings do not change when the localizations are
translated or made smaller. Hence we may label the configurations $\tilde{\mathbf{I}}$ by points $\tilde{r}_i$ in
the covering space of the spacelike hyperboloid $\{x \in \mathscr{M}, x^2 = -1\}$. Moreover, the
embeddings are locally constant in $\tilde{r}_2, \ldots, \tilde{r}_n$ and are globally constant in $\tilde{r}_1$.

To describe the global structure of the space of scattering states let us introduce
the following notation. Let $e_0 \in H_1$ be arbitrary. A configuration of disjoint particle
velocities $q_i = \frac{p_i}{m_i} \neq q_j = \frac{p_j}{m_j}$, $i \neq j$, is called regular (with respect to e_0) if there are
mutually spacelike cones S_i with apex e_0 such that $q_i \in S_i$ (in particular $q_i \neq e_0$).
To a regular configuration $\mathbf{q} = (q_1, \ldots, q_n)$ we associate a configuration of spacelike
directions $\mathbf{r} = (r_1, \ldots, r_n)$, with $r_i = q_i - e_0$.

We now pick a regular "reference" configuration $\mathbf{q}^0$ and choose n homotopy
classes of paths of spacelike cones $\tilde{I}_i$, $i = 1, \ldots, n$ whose endpoints $e(\tilde{I}_i)$ correspond
to the canonical directions r_i^0. We label these homotopy classes by $\tilde{\mathbf{r}}^0$. We also
choose a unitary local self-intertwiner U^0. These choices then determine a reference
embedding $i(\tilde{\mathbf{r}}^0, U^0)$ for a neighborhood $\mathbf{V}^0$ of $\mathbf{q}^0$. (Note that $\mathbf{V}^0$ may contain also
nonregular points.)

In the next step, we cover the configuration space of non-coinciding velocities
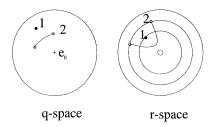by embeddings around regular configurations $\mathbf{q}$. (This is possible.) We label these

q-space   r-space

Fig. 2. A path $\mathbf{q}(t)$ and the corresponding path $\mathbf{r}(t)$.

embeddings by paths $\gamma_{\mathbf{q}}$ from $\mathbf{q}^0$ to $\mathbf{q}$ which have the property that none of the n velocities passes through e_0. The embedding corresponding to $\gamma_{\mathbf{q}}$ is then specified in the following way. As before, we describe the spacelike directions $\mathbf{r}^0$ by the n points $(r_1^0, 1), \ldots, (r_n^0, n)$ on the cylinder $S^1 \times \mathbb{R}_+$. As long as $\mathbf{q}(t)$ is regular, the corresponding path $\mathbf{r}(t)$ is canonically determined. At a critical point, where $\mathbf{q}(t)$ ceases to be regular, two directions r_i and r_j coincide.[3] In a neighborhood of this critical point we define $\mathbf{r}(t)$ by the following prescription: we move the direction r_k corresponding to the *smaller* velocity from (r_k, k) to $(r_k, 1/2)$, then change the two directions past each other and finally move $(r_k', 1/2)$ back to (r_k', k). Geometrically, this means that the points $\mathbf{r}(t)$ on the cylinder, viewed from the $(S^1, 0)$-end of the cylinder and looking in the long direction, perform the same motion as the velocities $\mathbf{q}(t)$ when viewed from e_0 (compare Fig. 2). We denote the so determined path $\mathbf{r}(t)$ by $\gamma^r = (\gamma_1^r, \ldots, \gamma_n^r)$.

Each path γ_i^r lifts to a unique path $\tilde{\gamma}_i^r$ from $\tilde{r}_i^0$ to $\tilde{r}_i$. We denote the corresponding configuration by $\tilde{\mathbf{r}} = (\tilde{r}_1, \ldots, \tilde{r}_n)$. Each path $\gamma_{\mathbf{q}}$ therefore determines a pure braid on the cylinder, namely the homotopy class

$$b(\gamma_{\mathbf{q}}) = (\tilde{\mathbf{r}})^{-1} \circ \gamma^r \circ (\tilde{\mathbf{r}}^0) \,. \tag{3.18}$$

Note that b defines a homomorphism of the groupoid of colored braids on the cylinder (corresponding to paths $\gamma_{\mathbf{q}}$, where $\mathbf{q}$ is a permutation of $\mathbf{q}^0$) to the pure braid group of the cylinder. This homomorphism is actually an automorphism of the pure braid group of the cylinder when restricted to closed paths $\gamma_{\mathbf{q}^0}$. (Both properties can be easily seen from the geometrical description.) Note also, that the image of a given path does not depend on the choice of e_0 locally.

The embedding corresponding to $\gamma_{\mathbf{q}}$ is then determined by

$$i(\gamma_{\mathbf{q}}) = i(\tilde{\mathbf{r}}, U^0 \varepsilon(b(\gamma_{\mathbf{q}}))^{-1}) \,. \tag{3.19}$$

Next we want to calculate transition functions between different embeddings. Let $\mathbf{q}' \in \mathbf{V}_1 \cap \mathbf{V}_2$, where $\mathbf{V}_i$ is the neighborhood of $\mathbf{q}_i$ for which the corresponding embedding is well defined. In a first step we want to calculate the transition function between embeddings $\gamma_1 = \gamma_{\mathbf{q}_1}$ and $\gamma_2 = \gamma_{\mathbf{q}_2}$, where γ_2 is obtained from γ_1

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[3] This is the generic case. The general case is covered by the geometrical description given below.

by composition with the path $\mathbf{q}_1 \to \mathbf{q}' \to \mathbf{q}_2$ in $\mathbf{V}_1 \cup \mathbf{V}_2$. In this case we have the following lemma

Lemma 3.3. *The transition function between two such embeddings is given as*

$$i_2^{-1} \circ i_1 = id \,. \tag{3.20}$$

Proof. We have to calculate how the vector

$$i_1(\boldsymbol{\Psi}_1 \otimes \cdots \otimes \boldsymbol{\Psi}_n \otimes T) = T \, U_1(\boldsymbol{\Psi}_n, \tilde{I}_n^1) \times \cdots \times (\boldsymbol{\Psi}_1, \tilde{I}_1^1) \tag{3.21}$$

transforms under changing the localization cones $\tilde{\mathbf{I}}^1$ to $\tilde{\mathbf{I}}^2$. Without loss of generality we can assume that $\tilde{\mathbf{I}}^2$ differs from $\tilde{\mathbf{I}}^1$ only by the relative orientation of two adjoining cones, as the general case can be reduced to this case recursively. To calculate the unitary intertwiner in this case, we use (3.8) to commute the single particle vector corresponding to the *smaller* velocity to the right. We then change its localization to the localization corresponding to $\tilde{\mathbf{I}}^2$ and commute it back, using (3.8) again. (We have to change the localization region of the vector with the smaller velocity in order to guarantee that the two localization regions become mutually spacelike for large t for both $\tilde{\mathbf{I}}^1$ and $\tilde{\mathbf{I}}^2$.) Thus the unitary intertwiner which relates the two localizations is just the pure braid obtained from the path from $\mathbf{q}_1$ to $\mathbf{q}_2$ in $\mathbf{V}_1 \cup \mathbf{V}_2$ via (3.18). As the two embeddings differ by this braid, the transition function (3.20) is trivial. $\quad\square$

Now let $\Phi \in \mathcal{H}_\sigma$ and denote the projector onto $\mathcal{H}_{\mathbf{V}, \varrho, \sigma}$ by $E_{\mathbf{V}}$. For each $\gamma_{\mathbf{q}}$ such that $i(\gamma_{\mathbf{q}})$ is an embedding $\mathcal{H}_{\mathbf{V}, \varrho, \sigma}^{(0)} \to \mathcal{H}_\sigma$, we can associate to $E_{\mathbf{V}} \Phi$

$$i(\gamma_{\mathbf{q}})^{-1}(E_{\mathbf{V}} \Phi) \in \mathcal{H}_{\mathbf{V}, \varrho, \sigma}^{(0)} \,. \tag{3.22}$$

Because of the above lemma,

$$i(\gamma_{\mathbf{q}})^{-1}(E_{\mathbf{V} \cap \mathbf{V}'} \Phi) = i(\gamma_{\mathbf{q}'})^{-1}(E_{\mathbf{V} \cap \mathbf{V}'} \Phi) \tag{3.23}$$

if $\gamma_{\mathbf{q}}$ and $\gamma_{\mathbf{q}'}$ differ by a path in $\mathbf{V} \cup \mathbf{V}'$. Thus we can extend the map

$$\Phi(\gamma_{\mathbf{q}}) = i(\gamma_{\mathbf{q}})^{-1}(E_{\mathbf{V}} \Phi) \in \mathcal{H}_{\mathbf{V}, \varrho, \sigma}^{(0)} \,, \tag{3.24}$$

where $\mathbf{V}$ is some neighborhood of $\mathbf{q}$, to arbitrary (not necessarily regular) configurations $\mathbf{q}$ of noncoinciding velocities. As we have given an explicit definition of the embeddings (3.19), we can calculate the covariance property of $\Phi(\gamma_{\mathbf{q}})$. We find

$$\Phi(\gamma_{\mathbf{q}} \circ \gamma_{\mathbf{q}^0}) = \Phi(\gamma_{\mathbf{q}}) \, U^0 \, \varepsilon(b(\gamma_{\mathbf{q}^0})) (U^0)^{-1} \,, \tag{3.25}$$

where $\gamma_{\mathbf{q}^0}$ is a pure braid in the homotopy class of $\mathbf{q}^0$, b is the automorphism of the pure braid group defined in (3.18), and the right action of $\gamma_{\mathbf{q}^0}$ is the global action of the structure group in the universal covering space. To prove (3.25), it is sufficient to consider product vectors

$$E_{\mathbf{V}} \Phi = T(\boldsymbol{\Psi}_n, \tilde{I}_n) \times \cdots \times (\boldsymbol{\Psi}_1, \tilde{I}_1) \,, \tag{3.26}$$

where $\tilde{\mathbf{I}}$ corresponds to $\gamma_{\mathbf{q}}$. Then the preimage under $i(\gamma_{\mathbf{q}} \circ \gamma_{\mathbf{q}^0})$ is given by

$$\boldsymbol{\Psi}_1 \otimes \cdots \otimes \boldsymbol{\Psi}_n \otimes T \, U_{\gamma_{\mathbf{q}} \circ \gamma_{\mathbf{q}^0}}^{-1} \,, \tag{3.27}$$

where we have

$$U_{\gamma_{\mathbf{q}} \circ \gamma_{\mathbf{q}^0}}^{-1} = (U^0 \varepsilon(b(\gamma_{\mathbf{q}} \circ \gamma_{\mathbf{q}^0}))^{-1})^{-1} = \varepsilon(b(\gamma_{\mathbf{q}})) \, \varepsilon(b(\gamma_{\mathbf{q}^0})) \, (U^0)^{-1}$$

$$= (U^0 \varepsilon(b(\gamma_{\mathbf{q}}))^{-1})^{-1} U^0 \varepsilon(b(\gamma_{\mathbf{q}^0})) \, (U^0)^{-1} \, . \qquad (3.28)$$

This proves (3.25).

As the braids (3.18) are locally independent of e_0, $\Phi(\gamma_{\mathbf{q}})$ is locally independent of e_0. To show that $\Phi(\gamma_{\mathbf{q}})$ is also globally independent of e_0, it is sufficient to prove that the covariance corresponding to the motion of any velocity around e_0 is trivially represented. To this end let us observe that we can extend the covariance (3.25) to the colored braid group of the cylinder (cf. the above remark about b). Then, the fact that the motion of the first velocity around e_0 is trivially represented already implies that the same holds for any velocity.

Now, if $\mathscr{H}_{V,\varrho,\sigma}$ has the structure of a function space over the configuration space of non-coinciding velocities (this can be shown, for example, under the assumption of Lorentz covariance as assumed here), $\Phi(\gamma_{\mathbf{q}})$ is a function on the universal covering space, possessing the covariance property (3.25). These functions are in one-to-one correspondence with sections in the associated vector bundle. We have therefore shown that the space of scattering vectors has the structure of the Hilbert space of square integrable sections of an associated vector bundle over the configuration space of non-coinciding velocities. This is precisely the structure proposed by Mund and Schrader in [20].

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