Commun. Math. Phys. 175, 161 – 176 (1996)

Communications in
Mathematical
Physics

Free q-Schrödinger Equation from Homogeneous Spaces of the 2-dim Euclidean Quantum Group

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Received: 13 August 1994/in revised form: 30 March 1995

Abstract: After a preliminary review of the definition and the general properties of the homogeneous spaces of quantum groups, the quantum hyperboloid qH and the quantum plane qP are determined as homogeneous spaces of $\mathscr{F}_q(E(2))$. The canonical action of $E_q(2)$ is used to define a natural q-analog of the free Schrödinger equation, that is studied in the momentum and angular momentum bases. In the first case the eigenfunctions are factorized in terms of products of two q-exponentials. In the second case we determine the eigenstates of the unitary representation, which, in the qP case, are given in terms of Hahn–Exton functions. Introducing the universal T-matrix for $E_q(2)$ we prove that the Hahn–Exton as well as Jackson q-Bessel functions are also obtained as matrix elements of T, thus giving the correct extension to quantum groups of well known methods in harmonic analysis.

1. Introduction

The fundamental role played by homogeneous spaces in harmonic analysis and in applications to physical theories cannot be overestimated. Apparently different mathematical problems, like the definition of special functions and integral transforms, from the one side, and the classification of elementary Hamiltonian systems by means of coadjoint orbits and their quantization according to the Kirillov theory [1], from the other, find their unifying *leitmotiv* in homogeneous spaces. Also the fundamental wave equations of mathematical physics have their natural origin in the study of homogeneous spaces of groups with kinematical or dynamical meaning, such as the Euclidean or the Poincaré group: more specifically, they are determined by the canonical action of the Casimir of the corresponding Lie algebra on spaces of functions on these homogeneous manifolds.

With the development of the theory of quantum groups and just after the first steps in the study of their structure, it seemed extremely natural to investigate the analogs of homogeneous spaces in this new quantum framework. As the notion of manifold underlying the algebraic structure is obviously lacking, the right approach starts from the injection of the algebra of the quantum functions of the homogeneous space into the algebra of the quantum functions of the group. After the pioneering work of Podleś on quantum spheres [2] and the generalization to the quantum framework of some relationships between groups and special functions [3–7], the program of extending harmonic analysis to quantum homogeneous spaces has been undertaken and results have been found for quantum spheres [8, Ch. 14] and $U_q(n-1)\backslash U_q(n)$ [9].

Still more recently, the geometry of quantum homogeneous spaces has received increasing attention: fibered structures on them have been coherently defined and have made possible the geometrical setting for gauge theories, leading to the study of the quantum counterpart of the Dirac monopole [10]; the duality aspects between q-functions and q-deformed universal enveloping algebra have been introduced into the subject and have led to an efficient way of analyzing and determining quantum homogeneous spaces [11, 12].

The explicit calculations have been made, almost always, starting from compact groups and especially for quantum spheres. However, for the purpose of physical applications, we should as well consider quantum groups arising from deformations of kinematical symmetries, as for instance the Heisenberg or the Euclidean groups [13–16]. This paper deals with the two dimensional Euclidean group, whose q-deformations have been deeply analyzed by many authors [17–25]: the novelty of the present approach is that we shall show how different aspects previously considered can be unified by an appropriate use of quantum homogeneous spaces of $E_q(2)$, which are recognized as "quantum planes" [26, 27] and "quantum hyperboloids" [28]. Since the Euclidean quantum algebra acts canonically on the latter, the action can be used to recover a quantum analog of many results of the classical theory and, in a certain sense, to establish the defining automorphisms of a concrete model of noncommutative geometry. It has also been shown [18, 19] that the notion of Haar functional makes sense for $E_q(2)$ and can thus be transported to homogeneous spaces, so one could mimic the construction of induced representations and, tentatively, look also for some kind of physical interpretation along the usual lines of wave mechanics.

The plan of this paper is as follows. In Sect. 2 we give very shortly some preliminary notions on classical and quantum homogeneous spaces. The method used is very "didactic": we start in a Lie group context and we express the relevant definitions on homogeneous spaces only by means of algebraic properties of the functions on them. In this form they are easily realized to be independent of the commutativity of the algebra and can therefore be extended to a quantum group framework. Some duality aspects will prove to have a practical use for doing explicit computations. In Sect. 3 we apply the general theory to the determination of the homogeneous spaces of the two-dimensional Euclidean quantum group, namely quantum planes and quantum hyperboloids. We also specify a canonical action of $E_q(2)$ on these spaces: this will be used in Sect. 4 to define an eigenvalue equation for the Casimir of $E_q(2)$ that constitutes the q-version of the free Schrödinger equation. The diagonalization of the action on linear and angular momentum bases can be defined in a canonical way. In particular, the angular momentum states are determined by series reducing to Bessel functions in the classical limit $q \to 1$. In the case of the quantum plane the Hahn–Exton functions are recovered. The last section recalls the construction of the "universal T-matrix" [31, 23] and provides a new perspective for studying q-special functions. We calculate the matrix elements of T, recovering the Hahn–Exton functions and we propose a definition for the zonal and associated spherical functions that in the case of $E_q(2)$ are written in terms of Jackson q-Bessel. We thus extend to quantum groups those well known methods that have proved to be so fruitful for Lie groups.

2. Preliminaries on Classical and Quantum Homogeneous Spaces

In order to make the treatment reasonably self-consistent, in this section we give a short discussion of the principal definitions and properties concerning quantum homogeneous spaces according to the main lines developed in [11, 12]. For the sake of clarity we shall first recall the classical definitions and then, by means of a pullback on the algebra of functions, we shall express them in a form which maintains its validity also in the quantum case, being independent of commutativity.

Let G be a Lie group and $m : G \times G \to G$ its composition law. Let M be a (left) G-space, with action $a : G \times M \to M$. Let then $\Delta : \mathscr{F}(G) \to \mathscr{F}(G) \otimes \mathscr{F}(G)$ be the comultiplication and $\delta : \mathscr{F}(M) \to \mathscr{F}(G) \otimes \mathscr{F}(M)$ the pullback of the action, or coaction. From the associativity of the action, $a \circ (id \times a) = a \circ (m \times id)$, we get at once the coassociativity for the coaction,

$$(id \otimes \delta) \circ \delta = (\Delta \otimes id) \circ \delta . \tag{2.1}$$

If $\{e\} \subseteq G$ is the trivial subgroup and $j_e : \{e\} \hookrightarrow G$ the canonical inclusion, then $j_e^* : \mathscr{F}(G) \to \mathscr{F}(\{e\}) \equiv \mathbf{C}$ is explicitly given by $j_e^* f = f(e)$, so that j_e^* is the counit ε of the Hopf algebra $\mathscr{F}(G)$. By identifying $\{e\} \times M$ with M, the unital property of the action reads $a \circ (j_e \times id) = id : M \to M$ and its pullback gives the equality

$$(\varepsilon \otimes id) \circ \delta = id \tag{2.2}$$

of maps of $\mathscr{F}(M)$ into itself, after the obvious identification of $\mathbf{C} \otimes \mathscr{F}(M)$ with $\mathscr{F}(M)$.

Consider next a point $p \in M$ and the inclusion $j_p : \{p\} \hookrightarrow M$, with pullback $f \mapsto j_p^* f = f(p) : \mathscr{F}(M) \to \mathscr{F}(\{p\}) \equiv \mathbf{C}$. It appears that $j_p^* \equiv \tilde{\varepsilon}$ is an evaluation and therefore a character of the algebra $\mathscr{F}(M)$: $\tilde{\varepsilon}(fg) = \tilde{\varepsilon}(f)\tilde{\varepsilon}(g)$. Since we are willing to study homogeneous spaces, we first study the action of G on the orbit $\mathcal{O}_p$ through the point p, which gives rise to a map $a \circ (id \times j_p) : G \to M$ under the natural identification of $G \times \{p\}$ with G. The pullback reads then $(id \otimes \tilde{\varepsilon}) \circ \delta : \mathscr{F}(M) \to \mathscr{F}(G)$. It is immediately checked that the associativity of the action is written as $a \circ (m \times j_p) = a \circ (id \times a) \circ (id \times id \times j_p)$, so that, if we define $\Psi = (id \otimes \tilde{\varepsilon}) \circ \delta$, we get the relationship

$$\Delta \circ \Psi = (id \otimes \Psi) \circ \delta , \tag{2.3}$$

showing that, on the orbit $\mathcal{O}_p$, Ψ intertwines the coaction δ with the comultiplication Δ. For what concerns the unital property, the map $(e, p) \mapsto p : \{e\} \times \{p\} \to M$ gives $(\varepsilon \otimes \tilde{\varepsilon}) \circ \delta : \mathscr{F}(M) \to \mathbf{C}$ which satisfies $(\varepsilon \otimes \tilde{\varepsilon}) \circ \delta = \tilde{\varepsilon}$, or, equivalently,

$$\tilde{\varepsilon} = \varepsilon \circ \Psi . \tag{2.4}$$

Moreover, if the coaction δ is assigned, it is clear that (2.3) and (2.4) provide a bijective correspondence between the characters $\tilde{\varepsilon}$ of $\mathscr{F}(M)$ and the $*$-algebra homomorphisms $\Psi : \mathscr{F}(M) \to \mathscr{F}(G)$ intertwining Δ with δ.

Once these properties of the action on an orbit have been established, they can be used as a model for a general definition. A manifold M is a homogeneous G-space whenever the action a is transitive, i.e. whenever the map $a \circ (id \times j_p) : G \to M$ is surjective for some $p \in M$. Similarly, we shall say that a coaction δ is *transitive* if there exists a character $\tilde{\varepsilon}$ of $\mathscr{F}(M)$ for which the corresponding $\Psi : \mathscr{F}(M) \to \mathscr{F}(G)$ is injective. In fact the main properties of homogeneous spaces can be very briefly summarized by observing that, according to the definition, the image $\Psi(\mathscr{F}(M))$ is a subalgebra of $\mathscr{F}(G)$ and that, by Eq. (2.3), $\Psi(\mathscr{F}(M))$ is a left coideal in $\mathscr{F}(G)$, since $\Delta(\Psi(f)) \in \mathscr{F}(G) \otimes \Psi(\mathscr{F}(M))$ for any $f \in \mathscr{F}(M)$. The explicit requirement of the existence of at least one "point" $\tilde{\varepsilon}$ makes the present definition somewhat less general with respect to those of other authors, [2, 9]: the former, however, is connected with the intertwining character of Ψ, which makes possible the use of the duality we present below. We refer to [11] for further details.

Let us give a more precise definition of the subalgebra $\mathscr{F}(M)$ in terms of invariance properties. It is well known that a homogeneous (left) G-manifold M is diffeomorphic to the quotient G/H, where H is the isotropy subgroup of any point $p \in M$. We can then identify the functions on M with the functions on G which are constant on the H-cosets, namely with those $f \in \mathscr{F}(G)$ such that $f(xy) = f(x)$, for $x \in G$, $y \in H$. Denoting by $j_H : H \hookrightarrow G$ the canonical inclusion and by $p_G : G \times H \to G$ the projection onto the first factor, the H-invariance of f reads: $\Delta f \circ (id \times j_H) = f \circ p_G$. Therefore, letting $f \mapsto p_G^*(f) = f \otimes 1 : \mathscr{F}(G) \to \mathscr{F}(G) \otimes \mathscr{F}(H)$ and $\pi_H = j_H^* : \mathscr{F}(G) \to \mathscr{F}(H)$, we see that

$$\mathscr{M} = \{ f \mid (id \otimes \pi_H) \circ \Delta f = f \otimes 1 \} \tag{2.5}$$

coincides with $\mathscr{F}(M)$. Moreover, by applying $(\varepsilon \otimes id)$ to both sides of the above condition, we also find

$$\pi_H f = \varepsilon(f)\, 1\,, \quad f \in \mathscr{M}\,. \tag{2.6}$$

Some observations are here in order, although all of them are rather obvious. In the first place, it is easily seen that π_H is actually a morphism of Hopf algebras, so that its kernel, $\mathscr{K}_H$, is a Hopf ideal, i.e. an ideal and two-sided coideal that is invariant under the antipode map S of $\mathscr{F}(G)$ into itself, $Sf(x) = f(x^{-1})$, $(x \in G)$. Secondly, $\mathscr{F}(M)$ *is not* a Hopf ideal of $\mathscr{F}(G)$: however, as already observed, it is a subalgebra and, besides that, $\mathscr{F}(M)$ is also a left coideal of $\mathscr{F}(G)$. Indeed an easy calculation shows that, for an f satisfying (2.5), one gets $(id \otimes id \otimes \pi_H) \circ (id \otimes \Delta) \circ \Delta f = \Delta f \otimes 1$, which is the very same condition that defines the elements of $\mathscr{F}(G) \otimes \mathscr{F}(M)$, thus proving the statement. A third observation we want to make concerns the possible existence of an involution endowing $\mathscr{F}(G)$ with a $*$-Hopf algebra structure. By this expression we mean that the involution $*$ is an antilinear, antimultiplicative mapping, compatible with Δ and ε: this implies, in particular, that $(S \circ *)^2 = id$. If then $\mathscr{K}_H$ is a $*$-invariant Hopf ideal and π_H the projection it determines, a straightforward calculation shows that $\mathscr{M}$, as defined by (2.5), is a $*$-subalgebra and an S^2-invariant right coideal. Finally, as a last point on the subject, we shall remark that, once the fundamental relations of the theory have been cast in the form (2.1–2.6), they do not depend any more on the commutativity of the initial Hopf algebra, so that they can *bona fide* be assumed as the defining relations also for homogeneous spaces of quantum groups.

In the remaining part of this section we shall give an "infinitesimal" version of the arguments so far treated, starting once again from the classical situation and assuming that $\mathscr{F}(G)$ is the algebra of the representative functions.

If $f \in \mathscr{F}(G)$ satisfies $f(xy) = f(x)$ for $x \in G$, $y \in H$, then it satisfies also $Y \cdot f \equiv D_t f(x\, e^{tY})|_{t=0} = 0$ for any $Y \in \mathrm{Lie}\, H$. Now, using the standard "(f)"

notation for the comultiplication, [29], we see that we can write the action of any $X \in \mathrm{Lie}\, G$ as $D_t f(x\, e^{tX})|_{t=0} = D_t \Delta f(x, e^{tX})|_{t=0} = \sum_{(f)} f_{(1)}(x) D_t f_{(2)}(e^{tX})|_{t=0}$, namely

$$X \cdot f = \sum_{(f)} f_{(1)} \langle X \,,\, f_{(2)} \rangle \,, \tag{2.7}$$

where the map $(X, f) \mapsto \langle X, f \rangle = D_t f(e^{tX})|_{t=0}$ can be extended to a canonical and nondegenerate duality pairing $\mathscr{U}(\mathrm{Lie}\, G) \times \mathscr{F}(G) \to \mathbf{C}$. Conversely, once we are given a nondegenerate duality pairing of Hopf algebras $\mathscr{H}_1 \times \mathscr{H}_2 \to \mathbf{C}$, it is a simple matter of computation to verify that (2.7) defines an action of $\mathscr{H}_1$ on $\mathscr{H}_2$, independently of the commutativity of these algebras. It is therefore natural to call an element $f \in \mathscr{H}_2$ "*infinitesimally invariant*" with respect to an element $X \in \mathscr{H}_1$ if $X \cdot f = 0$.

Let us now return to the $*$-subalgebra and left coideal $\mathscr{M} \subseteq \mathscr{F}(G)$ that we have previously defined and consider the subset $K_{\mathscr{M}} \subseteq \mathscr{U}(\mathrm{Lie}\, G)$ of those elements for which $\mathscr{M}$ is infinitesimally invariant. Letting $\tau = * \circ S$, it follows from the definitions that $K_{\mathscr{M}}$ is a τ-invariant two-sided coideal and a left ideal in $\mathscr{U}(\mathrm{Lie}\, G)$. The converse of this statement is relevant for applications. Observe that if K is a τ-invariant two-sided coideal,

$$\mathscr{M}_K = \{ f \in \mathscr{F}(G) \,|\, K \cdot f = 0 \} \tag{2.8}$$

is a $*$-subalgebra and left coideal, and hence it defines a homogeneous space for the group G. As observed above, this formulation does not depend upon the commutativity of $\mathscr{F}(G)$ and will explicitly be used in the next section to produce the homogeneous spaces of $\mathscr{F}_q(E(2))$.

3. Quantum Homogeneous Spaces of $E_q(2)$

Let us begin by reviewing the principal facts about the quantizations of the functions and of the universal enveloping algebra of the Euclidean group $E(2)$, [22].

(3.1) Definition. *The Hopf algebra generated by v, $\bar{v}$, n, $\bar{n}$, with relations*

$$vn = q^2 nv \,, \qquad v\bar{n} = q^2 \bar{n} v \,, \qquad n\bar{n} = q^2 \bar{n} n \,,$$

$$\bar{n}\bar{v} = q^2 \bar{v}\bar{n} \,, \qquad n\bar{v} = q^2 \bar{v} n \,, \qquad v\bar{v} = \bar{v} v = 1 \,,$$

coalgebra operations

$$\Delta v = v \otimes v \,, \qquad \Delta \bar{v} = \bar{v} \otimes \bar{v} \,, \qquad \Delta n = n \otimes 1 + v \otimes n \,, \qquad \Delta \bar{n} = \bar{n} \otimes 1 + \bar{v} \otimes \bar{n} \,,$$

$$\varepsilon(v) = \varepsilon(\bar{v}) = 1 \,, \qquad \varepsilon(n) = \varepsilon(\bar{n}) = 0$$

and antipode map

$$S(v) = \bar{v} \,, \qquad S(\bar{v}) = v \,, \qquad S(n) = -\bar{v} n \,, \qquad S(\bar{n}) = -v\bar{n} \,,$$

will be called the **algebra of the quantized functions on $E(2)$** *and denoted by $\mathscr{F}_q(E(2))$. Assuming from now on a real q, a compatible involution is given by*

$$v^* = \bar{v} \,, \qquad n^* = \bar{n} \,.$$

The **quantized enveloping algebra** $\mathcal{U}_q(E(2)) \equiv E_q(2)$ *is generated by the unity and the three elements* $P_\pm$, J, *such that*

$$[P_+, P_-] = 0\,, \qquad [J, P_\pm] = \pm P_\pm$$

and

$$\Delta J = J \otimes 1 + 1 \otimes J\,, \qquad \Delta P_\pm = q^{-J} \otimes P_\pm + P_\pm \otimes q^J$$
$$S(J) = -J\,, \qquad S(P_\pm) = -q^{\pm 1} P_\pm\,,$$

with vanishing counit and involution

$$J^* = J\,, \qquad P_\pm^* = P_\mp\,.$$

Since J is primitive in $E_q(2)$, $q^{\pm J}$ are group-like and $P_\pm$ are twisted-primitive with respect to q^{-J}.

We finally recall the duality pairing between $E_q(2)$ and $\mathcal{F}_q(E(2))$, whose explicit form is given by [21–23]

$$\langle J, v^r n^s \bar{n}^t \rangle = -r\, \delta_{s,0}\delta_{t,0}\,, \qquad \langle P_-, v^r n^s \bar{n}^t \rangle = -q^{r-1}\delta_{s,1}\delta_{t,0}\,,$$
$$\langle P_+, v^r n^s \bar{n}^t \rangle = q^r\, \delta_{s,0}\delta_{t,1}\,. \tag{3.2}$$

We shall now consider the following two different left actions of $E_q(2)$ on $\mathcal{F}_q(E(2))$ [6, 7, 32, 25]: for $X \in E_q(2)$ and $f \in \mathcal{F}_q(E(2))$, we let

$$\ell(X)f = (id \otimes X) \circ \Delta f = \sum_{(f)} f_{(1)} \langle X, f_{(2)} \rangle\,,$$
$$\lambda(X)f = (S(X) \otimes id) \circ \Delta f = \sum_{(f)} \langle S(X), f_{(1)} \rangle f_{(2)}\,, \tag{3.3}$$

where it is evident that in the classical case and for a group-like element $X = x \in E(2)$, ℓ is the multiplication to the right of the argument of f by x and λ the multiplication to the left of the argument by x^{-1}.

For future convenience, we also recall the most usual definitions of some q-combinatorial quantities:

$$[\alpha]_q = (q^\alpha - q^{-\alpha})/(q - q^{-1})\,, \qquad [s]_q! = [s]_q\,[s-1]_q \cdots [1]_q\,,$$
$$(\alpha; q)_s = \prod_{j=1}^{s} (1 - q^{j-1}\alpha)\,, \tag{3.4}$$

with $s \in \mathbf{N}$, while α can be chosen in $E_q(2)$.

(3.5) Lemma. *We have the relations* [25]

$$\ell(q^{\pm J})\, v^r n^s \bar{n}^t = q^{\mp r}\, v^r n^s \bar{n}^t\,,$$

$$\ell(P_-)\, v^r n^s \bar{n}^t = -[s]_q\, q^{r-s}\, v^{r+1} n^{s-1} \bar{n}^t\,,$$

$$\ell(P_+)\, v^r n^s \bar{n}^t = [t]_q\, q^{r+2s+t-1}\, v^{r-1} n^s \bar{n}^{t-1}\,.$$

and

$$\lambda(q^{\pm J})\, v^r n^s \bar{n}^t = q^{\pm(r+s-t)}\, v^r n^s \bar{n}^t ,$$

$$\lambda(P_-)\, v^r n^s \bar{n}^t = [s]_q\; q^{r+t-2}\, v^r n^{s-1} \bar{n}^t ,$$

$$\lambda(P_+)\, v^r n^s \bar{n}^t = -[t]_q\; q^{r+s+1}\, v^r n^s \bar{n}^{t-1} .$$

Proof. Taking into account that, for $X, Y \in E_q(2)$, $f, g \in \mathscr{F}_q(E(2))$ we have

$$\ell(XY)f = \ell(X)\ell(Y)f , \qquad \lambda(XY)f = \lambda(X)\lambda(Y)f$$

and

$$\ell(X)fg = \sum_{(X)} \ell(X_{(1)})f\ell(X_{(2)})g , \qquad \lambda(X)fg = \sum_{(X)} \lambda(X_{(2)})f\lambda(X_{(1)})g ,$$

the result follows by using the duality relations (3.2). □

Let us now study some homogeneous spaces of $\mathscr{F}_q(E(2))$ by the procedure based on "infinitesimal invariance," as explained at the end of Sect. 2.

(3.6) Lemma. *Let $\rho \in (0, \infty)$. Define*

$$X_\rho = \rho\,[J]_q + P_+ + qP_- , \qquad X_{i\rho} = i\rho\,[J]_q + P_+ - qP_- .$$

Define also

$$X_\infty = [J]_q .$$

For each $\rho \in (0, \infty]$ the linear span of the element X_ρ or $X_{i\rho}$ constitutes a τ-invariant two-sided coideal of $E_q(2)$, twisted-primitive with respect to q^{-J}.

Proof. A straightforward calculation shows that the elements of the form $A_1(q^{-J} - q^{J}) + A_2P_+ + A_3P_-$ form a two-sided coideal twisted-primitive with respect to q^{-J}. After an obvious rescaling that allows to eliminate inessential parameters, the stated form of X_ρ and $X_{i\rho}$ is obtained by the τ-invariance. □

We now present the main result of this section, which consists in determining the algebras of the functions on the quantum homogenous spaces of $\mathscr{F}_q(E(2))$ solving (2.8), namely $\ell(X_\rho)f = 0$ and $\ell(X_{i\rho})f = 0$. For a swifter exposition we define a parameter μ that can assume the values $-\rho$ and $i\rho$. We define then

$$X_\mu = -\bar{\mu}\,[J]_q + P_+ + q\,(\bar{\mu}/\mu)\,P_- .$$

(3.7) Proposition. *For $|\mu| \in (0, \infty]$ consider the pair of elements z and $\bar{z}$ defined as follows:*

$$z = v + \mu\, n , \qquad \bar{z} = \bar{v} + \bar{\mu}\,\bar{n} , \qquad (|\mu| < \infty),$$

$$z = n , \qquad \bar{z} = \bar{n} , \qquad (|\mu| = \infty).$$

Then $(z, \bar{z})$ satisfy the relations

$$(qH) : z\bar{z} = q^2\,\bar{z}z + (1 - q^2),$$

$$(qP) : z\bar{z} = q^2\,\bar{z}z$$

for $|\mu| < \infty$ and $|\mu| = \infty$ respectively. Moreover they are connected by the involution $*$ and generate the $*$-invariant subalgebra and left coideal

$$B = \{f \in \mathcal{F}_q(E(2)) \mid \ell(X_\mu)f = 0\}$$

of $\mathcal{F}_q(E(2))$. They thus define quantum homogeneous spaces respectively called quantum hyperboloid and quantum plane (see [28]). The explicit forms of the coactions read

$$(qH) : \delta z = v \otimes z + \mu\, n \otimes 1\,, \qquad \delta\bar{z} = \bar{v} \otimes \bar{z} + \bar{\mu}\,\bar{n} \otimes 1\,,$$

$$(qP) : \delta z = v \otimes z + n \otimes 1\,, \qquad \delta\bar{z} = \bar{v} \otimes \bar{z} + \bar{n} \otimes 1\,.$$

Proof. The proof is much easier for the case of the quantum plane. Indeed, if we look for polynomials $f = \sum M_{r,s,t}\, v^r n^s \bar{n}^t$ $(r \in \mathbf{Z};\ s, t \in \mathbf{N})$ that solve the equation $\ell(X_\infty)f = 0$, it is immediately realized that the space of solutions is formed by the polynomials in n and $\bar{n}$, thus reproducing a well known result (see, e.g., [26]).

Let us discuss the case $|\mu| < \infty$. We define $x = \bar{v}n$ and $y = v\bar{n}$, so that we can write $\mathcal{F}_q(E(2)) = \bigoplus_{d\in\mathbf{Z}} A_d$, where A_d is the linear space spanned by $\{v^d x^a y^b\}_{a,b\in\mathbf{N}}$. Since $\ell(X_\mu)A_d \subseteq A_d$, we also have $B = \bigoplus_{d\in\mathbf{Z}} B_d$, with $B_d = B \cap A_d$. We solve in B_d the equation $\ell(X_\mu)f_d = 0$ for polynomial elements of the form $f_d = v^d \sum_{s,t\in\mathbf{N}} g_{s,t}\, x^s y^t$. The recurrence equation we deduce results in

$$\bar{\mu}\,[d + t - s]_q\, g_{s,t} + [t + 1]_q\, q^{d-s+1}\, g_{s,t+1} - (\bar{\mu}/\mu)\,[s + 1]_q\, q^{d-t-1}\, g_{s+1,t} = 0\,. \quad (3.8)$$

The content of (3.8) will be discussed according to the following strategy. In the first place we determine the two spaces of the solutions depending on v and x only and on v and y only respectively. We then show that the space B is generated by these solutions.

In the former case the recurrence equation (3.8) simplifies to

$$\mu\,[d - s]_q\, g_{s,0} = [s + 1]_q\, q^{d-1}\, g_{s+1,0}\,.$$

If we choose, without loss of generality, $g_{0,0} = 1$, using the combinatorial identity

$$[s]_q! = (-1)^s\, \frac{(q^2;q^2)_s}{(q - q^{-1})^s}\, q^{-s(s+1)/2},$$

and the standard definition of the q-hypergeometric function ${}_1\phi_0$ [8], we find

$$f_d = v^d \sum_{s=0}^{\infty} \frac{(q^{-2d};q^2)_s}{(q^2;q^2)_s}\,(-\mu q^2 x)^s = v^d\, {}_1\phi_0(q^{-2d};q^2,-\mu\, q^2 x)\,.$$

Since $(q^{-2d};q^2)_{d+1}$ vanishes for $d \geqq 0$, we get

$$f_d = \sum_{s=0}^{d} \begin{bmatrix} d \\ s \end{bmatrix}_{q^2} (\mu n)^s\, v^{d-s} = (v + \mu n)^d := z^d\,,$$

where we have used the standard notation for the q-binomial coefficients (see, e.g., [33]). We therefore conclude that for $d \geqq 0$ the polynomial solutions independent of $\bar{n}$ are proportional to z^d. In the very same way it is found that for $d \leqq 0$ the polynomial solutions independent of n are proportional to

$$f_d = v^d \,{}_1\phi_0(q^{-2d}; q^{-2}, -\mu q^{-2} y) = (\bar{v} + \bar{\mu}\bar{n})^{-d} := \bar{z}^{|d|}.$$

We now observe that $z^{d_1}\bar{z}^{d_2} \in B_{d_1-d_2}$ and we define $\widetilde{B}_d$, ($d \geqq 0$), to be the linear subspace of B_d generated by $\{z^{d+m}\bar{z}^m\}_{m\geqq 0}$. We will show that indeed $\widetilde{B}_d = B_d$. For this consider again the general Eq. (3.8). As we are looking for polynomial solutions $f_d \in B_d$, we shall define $s_0 = \deg_x f_d$ and $t_0 = \deg_y f_d$ to be the highest degrees of f_d in x and y respectively. Take $s = s_0$. It is easily seen that there exists an integer $\hat{t} \leqq t_0$ such that $g_{s_0,t} \neq 0$ only for $t \leqq \hat{t}$ and such that $d + \hat{t} - s_0 = 0$. Analogously taking $t = t_0$ there exists $\hat{s} \leqq s_0$ such that $g_{s,t_0} \neq 0$ only for $s \leqq \hat{s}$ and $d + t_0 - \hat{s} = 0$. From this we deduce that $t_0 + s_0 = \hat{t} + \hat{s}$ and therefore $\hat{s} = s_0$ and $\hat{t} = t_0$. Hence g_{s_0,t_0} is nonvanishing and $s_0 = d + t_0$.

Take now in B_d the linear subspace $B_d^{t_0} = \{f_d \in B_d \,|\, \deg_y f_d \leqq t_0\}$. Since $\deg_x z^{d+t_0}\bar{z}^{t_0} = d + t_0$ and $\deg_y z^{d+t_0}\bar{z}^{t_0} = t_0$, for any $f_d \in B_d^{t_0}$ we can determine $\alpha \in \mathbf{C}(q)$ such that

$$f_d = \alpha z^{d+t_0}\bar{z}^{t_0} + \widetilde{f}, \quad \text{with } \widetilde{f} \in B_d^{t_0-1}.$$

Due to the fact that the solutions in B_d^0 are multiple of z^d, it turns out that $B_d = \widetilde{B}_d$. Repeating the same argument for $d < 0$, the result $B = \bigoplus_{d\in\mathbf{Z}} B_d$ follows.

It is now straightforward to calculate the relationships between z and $\bar{z}$ as well as the coactions on these generators and to check that they are of the form (qH) stated in the proposition. □

4. Free q-Schrödinger Equation

In this section we shall write the canonical action of the Casimir of $E_q(2)$ on the homogeneous spaces so far determined,

$$\lambda(P_+P_-)\psi = E\psi. \tag{4.1}$$

This will constitute the natural q-analog of the free Schrödinger equation for the functions on the plane and we shall discuss its solutions by diagonalizing the Casimir in two bases which are the natural q-counterparts of the plane wave and of the angular momentum bases.

By virtue of (3.5) the following relations hold for the (qH) case:

$$\lambda(q^{\pm J})\bar{z}^j z^m = q^{\mp(j-m)}\bar{z}^j z^m, \qquad \lambda(P_-)\bar{z}^j z^m = \mu\,[m]_q\, q^{-j-2}\bar{z}^j z^{m-1},$$
$$\lambda(P_+)\bar{z}^j z^m = -\bar{\mu}\,[j]_q\, q^{-m+1}\bar{z}^{j-1} z^m. \tag{4.2}$$

The analogous relations for (qP) are simply obtained from (4.2) by letting $\mu = 1$: we shall adopt this convention for what follows. The λ-action of the Casimir P_+P_- on a formal series ψ is therefore

$$\lambda(P_+P_-)\psi = \lambda(P_+P_-)\sum_{j,m} c_{j,m}\bar{z}^j z^m = \sum_{j,m} c_{j,m}(-\mu\bar{\mu})q^{-j-m}[j]_q[m]_q\bar{z}^{j-1}z^{m-1},$$

so that, when substituted in (4.1) with the position $\mathscr{E} = -E/(\mu\bar{\mu})$, yields the recurrence relation

$$q^{-j-m-2}\,[j+1]_q\,[m+1]_q\;c_{j+1,\,m+1} = \mathscr{E}\,c_{j,\,m}\,. \tag{4.3}$$

(*i*) *The plane wave states.* We begin by investigating solutions of (4.1) that are factorizable in the variables z and $\bar{z}$. This means that we look for coefficients of the form

$$c_{j,\,m} = \frac{k^j\,\tilde{k}^m}{[j]_q!\,[m]_q!}\;q^{-\vartheta(j,\,m)}\,, \tag{4.4}$$

where k and $\tilde{k}$ are quantities to be determined while $q^{\vartheta(j,\,m)}$ is required to factorize in its arguments.

A first relation for solving the problem can be written by substituting (4.4) in Eq. (4.3). We get

$$q^{\vartheta(j,\,m)-\vartheta(j+1,\,m+1)}\,k\tilde{k} = q^{j+m+2}\,\mathscr{E}\,. \tag{4.5}$$

In order to obtain more information we shall now choose a second element to be diagonalized in addition to the Casimir. This procedure is completely analogous to the classical one, where we take one component of the momentum or both, as they are mutually commuting. However, by making the apparently straightforward extension to the q-framework and diagonalizing P_+ and/or P_-, we do not find any result in the wished direction. The right choice is instead that of diagonalizing the two commuting operators $b_- = -q^{-J}P_-$ and $b_+ = P_+\,q^J$: the reason for this fact has to be searched in the duality properties that have been explained in [21–23] and that will be resumed in the next section, when dealing with the universal T-matrix. Consider therefore the eigenvalue equation $\lambda(b_-)\psi = \beta\psi$. Expanding ψ in powers of z and $\bar{z}$ and performing the usual computations, we find that the coefficients of the expansion satisfy the relation

$$-\,\mu\,q^{-m-2}\,[m+1]_q\;c_{j,\,m+1} = \beta\,c_{j,\,m}\,. \tag{4.6}$$

From (4.4–4.6) we deduce the system

$$\vartheta(j,m) - \vartheta(j+1,m+1) - j - m = const_1,$$

$$\vartheta(j,m) - \vartheta(j,m+1) - m = const_2\,,$$

where, up to an inessential rescaling, the two constants can be fixed to zero. The factorization requirement previously made, yields the solution

$$\vartheta(j,m) = -\frac{1}{2}\,j(j-1) - \frac{1}{2}\,m(m-1),$$

and finally, from (4.5),

$$\mathscr{E} = k\tilde{k}/q^2\,.$$

From $\mathscr{E} < 0$ we have $k\tilde{k} < 0$. We can therefore formalize these results as follows.

(4.7) Proposition. *The eigenvalue equation* (4.1) *is satisfied by the states*

$$\psi_{k\tilde{k}} = E_{q^2}[(1-q^2)\,\tilde{k}\bar{z}]\;E_{q^2}[(1-q^2)\,kz]\,,$$

where $E = -\mu\bar{\mu}k\tilde{k}/q^2$ *and* $E_q(x) = {}_0\phi_0(-;-;q,-x)$ *denotes a q-exponential* [30]. $\psi_{k\tilde{k}}$ *are also eigenstates of* $\lambda(b_-)$ *with eigenvalue* $-\mu\tilde{k}/q^2$ *and of* $\lambda(b_+)$ *with eigenvalue* $-\bar{\mu}k$.

Proof. The only thing we still have to prove is that the actual expression is given in terms of q-exponentials. However, collecting the results so far found, we have

$$\psi_{k\tilde{k}} = \sum_j \frac{q^{j(j-1)/2}}{[j]_q!} (\tilde{k}\bar{z})^j \sum_m \frac{q^{m(m-1)/2}}{[m]_q!} (kz)^m ,$$

which just gives the stated result once (3.8) is accounted for. □

(*ii*) *The angular momentum states.* Instead of diagonalizing $\lambda(b_-)$ as in item (*i*), we shall now discuss Eq. (4.1) with the additional diagonalization of q^J. From (4.2) it appears that a solution ψ of the eigenvalue equation for q^J has an expansion in terms of z and $\bar{z}$ whose coefficients $c_{j,m}$ satisfy the condition $m - j = \pm r = const$, $r > 0$, so that the eigenvalue is $q^{\pm r}$. Letting $c_{j,m} = \delta_{m-j,\pm r}\, d_j$, Eq. (4.3) reduces to

$$q^{-2(j+1)-r}\,[j+1]_q\,[j+r+1]_q\; d_{j+1} = \mathscr{E}\, d_j ,$$

and it is solved by

$$d_j = \frac{[r]_q!}{[j]_q!\,[j+r]_q!}\; q^{j(j+1)}\,(\mathscr{E}q^r)^j .$$

We therefore state the result as follows.

(4.8) Proposition. *The eigenvalue equation* (4.1) *is satisfied by the states*

$$\psi_r = \frac{q^{2r}\,\mathscr{E}^{r/2}}{[r]_q!}\, J_r^{(q)}\, z^r \quad \text{and} \quad \psi_{-r} = \frac{q^{2r}\,\mathscr{E}^{r/2}}{[r]_q!}\, \bar{z}^r\, J_r^{(q)} ,$$

where

$$J_r^{(q)} = \sum_{j=0}^{\infty} \frac{[r]_q!}{[j]_q!\,[j+r]_q!}\; q^{j(j+1)}\,(\mathscr{E}q^r)^j\,\bar{z}^j z^j . \tag{4.9}$$

Moreover $\psi_{\pm r}$ *are also eigenfunctions of* q^J *with eigenvalues* $q^{\pm r}$. *The series* (4.9) *reduces to the Bessel function* $J_r(\bar{z}z)$ *for* $q \to 1$.

(*4.10*) *Remarks.* (*i*) The expression $J_r^{(q)}$ can be written in terms of the variable $\bar{z}z$. For the case of the quantum plane, we observe that $\bar{z}^j z^j = q^{-j(j-1)}(\bar{z}z)^j$, so that

$$\begin{aligned} J_r^{(q)} &= \sum_{j=0}^{\infty} (-)^j \frac{q^{j(j-1)}}{(q^2;q^2)_j\,(q^{2(r+1)};q^2)_j}\; [q^{2r}(1-q^2)^2 E]^j\,(q^2\bar{z}z)^j \\ &= {}_1\phi_1(0; q^{2(r+1)}; q^2, q^{2(r+1)}(1-q^2)^2 E\,\bar{z}z) , \end{aligned}$$

namely the Hahn–Exton q-Bessel function [18, 30].

(*ii*) For the (qH) case we have the following relation:

$$\bar{z}^j z^j = (1 - \bar{z}z; q^{-2})_j .$$

This can be proved solving the functional recurrence relation

$$\bar{z}z\, P_{n-1}(q^{-2}\bar{z}z + (1-q^{-2})) = P_n(\bar{z}z) ,$$

obtained by means of the normalization $P_n(\bar{z}z) = \bar{z}^n z^n$ and by the use of the identity $\bar{z}z(\bar{z}z)^i = \bar{z}(z\bar{z})^i z$. It is therefore immediate to give a formal expression for $J_r^{(q)}$ in terms of a q-hypergeometric function: however, contrary to the (qP) case, the expression $\bar{z}z$ appears now in a parameter rather than in the argument of the hypergeometric.

Let us refer to [18, 19] for a detailed discussion on the Haar functional for the quantum group $\mathscr{F}_q(E(2))$. Here we will be concerned with the standard orthonormality relations between the ψ_r.

(4.11) Proposition. *The states ψ_r give rise to the unitary representation of $E_q(2)$,*

$$\lambda(J)\,\psi_r = r\,\psi_r\,, \qquad \lambda(P_+)\,\psi_r = \bar{R}\,\psi_{r+1}\,, \qquad \lambda(P_-)\,\psi_r = R\,\psi_{r-1}\,,$$

where $R = \mu\mathscr{E}^{1/2}$.

Proof. The result is obtained by a direct calculation using (4.2) and (4.9). □

5. Universal T-Matrix and Special Functions

In this final section we want to connect the previous analysis with what is known as the "*universal T-matrix*" [31, 23]. To our knowledge the procedure we are going to present is a novelty and it has at least two interesting features. Indeed from a theoretical point of view it shows that important concepts in Lie group representations can be extended to a quantum group context, provided that the method used for the extension is "proper," namely it is *only* based on canonical objects of the theory. From a practical point of view our construction is very transparent, since it permits an explicit definition of most q-special functions and a study directly related to the quantum symmetry in a completely close analogy to the classical case.

Let us briefly recall the construction and the main properties of the universal T-matrix [31]. Consider two Hopf algebras in nondegenerate duality pairing that we shall assume as the quantization of the universal enveloping algebra of a Lie group G, $\mathscr{U}_q(\mathrm{Lie}\,G)$, and the quantization $\widetilde{\mathscr{F}}_q(G)$ of the algebra of the canonical coordinates of the second kind of $\widetilde{\mathscr{F}}(G)$. Let $\{X_B\}$ and $\{x^A\}$ respectively be two dual linear bases, with A and B running in an appropriate set of indices, so that $\langle x^A, X_B\rangle = \delta^A_B$. We define the element $T \in \widetilde{\mathscr{F}}_q(G) \otimes \mathscr{U}_q(\mathrm{Lie}\,G)$ as

$$T = \sum_A x^A \otimes X_A\,.$$

If we want to illustrate the construction on the explicit example of a compact Lie group G, denoting by X_k , $(k = 1, \ldots, n)$, the generators of $\mathrm{Lie}\,G$, a basis of the universal enveloping algebra is of the form $X_A = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$. The dual elements $x^A \in \widetilde{\mathscr{F}}(G)$ are then $x^A = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}/(a_1!a_2! \cdots a_n!)$, where x_i are the canonical coordinates of the second kind of G and $\langle x_k, X_j\rangle = \delta_{kj}$. Therefore the universal T matrix results in

$$T = \sum_{a_1} \frac{x_1^{a_1} \otimes X_1^{a_1}}{a_1!} \cdots \sum_{a_n} \frac{x_n^{a_n} \otimes X_n^{a_n}}{a_n!} = e^{\,x_1 \otimes X_1} \,\cdots\, e^{\,x_n \otimes X_n}\,.$$

It appears therefore that the evaluation of T on an element of a neighborhood of the identity of the group G reproduces that element expressed by means of the exponential mapping, so that the universal matrix can be regarded as a resolution of the identity mapping of G into itself. Moreover, if we choose a representation of the Lie algebra, we correspondingly obtain matrices whose entries are expressed in terms of special functions: this property extends to the quantum case, despite the fact that the x^A are now elements of a noncommutative algebra. Indeed if we consider a representation $\mathscr{R}$ of $\mathscr{U}_q(\mathrm{Lie}\, G)$ the elements $t^{\mathscr{R}}_{rs} = \big((1 \otimes \mathscr{R})T\big)_{rs} \in \mathscr{F}_q(G)$ satisfy the usual definition $\langle t^{\mathscr{R}}_{rs}, X\rangle = \mathscr{R}(X)_{rs}$ for every $X \in \mathscr{U}_q(\mathrm{Lie}\, G)$.

In the following we shall explicitly treat the case of $E_q(2)$. As already anticipated in Sect. 4, we define J, $b_- = -q^{-J}P_-$, $b_+ = P_+q^J$ as the generators of $\mathscr{U}_q(E(2))$ and π, $\pi_\pm$ the corresponding canonical coordinates of the second kind of $\widetilde{\mathscr{F}}_q(E(2))$. Introducing the q-exponential $e_q(x) = \sum_{j=0}^{\infty} x^j/(q;q)_j = {}_1\phi_0(0;-;q,x)$ as in [30], the following result is proved by a direct calculation.

(5.1) Proposition. *We have the duality relations*

$$\langle b_-^r J^s b_+^t\,,\, \pi_-^{r'} \pi^{s'} \pi_+^{t'}\rangle = \frac{(q^2;q^2)_r}{(1-q^2)^r}\; s!\; \frac{(q^{-2};q^{-2})_t}{(1-q^{-2})^t}\;\delta_{r,r'}\,\delta_{s,s'}\,\delta_{t,t'}\,,$$

where π, $\pi_\pm$ form the Hopf algebra specified as follows ($q = e^z$):

$$[\pi_+, \pi_-] = 0\,, \qquad [\pi, \pi_\pm] = -2z\,\pi_\pm$$

and

$$\Delta\pi_- = \pi_- \otimes 1 + e^{-\pi} \otimes \pi_-\,, \qquad \Delta\pi_+ = \pi_+ \otimes e^{-\pi} + 1 \otimes \pi_+\,,$$

$$S(\pi_-) = -\,e^{\pi}\,\pi_-\,, \qquad S(\pi_+) = -\,\pi_+ e^{\pi}\,,$$

with π a primitive element. We therefore find the universal T-matrix

$$T = e_{q^2}[(1-q^2)\,\pi_- \otimes b_-]\,e^{\,\pi\otimes J}\,e_{q^{-2}}[(1-q^{-2})\,\pi_+ \otimes b_+]\,.$$

The algebra $\mathscr{F}_q(E(2))$ is obtained as a subalgebra of $\widetilde{\mathscr{F}}_q(E(2))$ by letting

$$v = e^{-\pi}\,, \qquad n = \pi_-\,, \qquad \bar{n} = e^{\pi}\pi_+\,.$$

The matrix elements obtained in a natural way from the universal T-matrix using the representation (4.11) are precisely the matrix elements t^{λ}_{rs}: this will give a clear connection of our theory with the approach described in references [17, 18].

(5.2) Proposition. *The matrix elements $\big((1 \otimes \lambda)T\big)_{rs}$ are given in terms of Hahn–Exton q-Bessel functions.*

Proof. From the the expansion of $e_q(x)$ and using (4.11), after some lengthy but straightforward calculations we have, for $s > r$,

$$\big((1\otimes\lambda)T\big)_{rs} = \left(\frac{-R\,(1-q^2)}{q^{(s+r-1)/2}}\right)^{s-r} \cdot \frac{n^{s-r}\,\bar{v}^{s}}{(q^2;q^2)_{s-r}}\;{}_1\phi_1(0;q^{2(s-r+1)};q^2,|R|^2 q^{2s}(1-q^2)^2\,n\bar{n})\,.$$

An analogous result is obtained for $s < r$. $\square$

(5.3) Remarks. (*i*) We see a perfect agreement with the results of [18], provided that the identifications $v = \alpha^2$, $\bar{v} = \delta^2$, $n = -q^{-1/2}\beta\alpha$, $\bar{n} = q^{1/2}\delta\gamma$, $r = -i$ and $s = -j$ are done.

(*ii*) We can make a comparison of our results with those of ref. [34]. The q-exponentials appearing in the latter paper are defined in the universal enveloping algebra by a method based on appropriate choices which are not connected with any canonical construction. However they prove to be efficient tools to calculate the matrix elements of the representation.

We shall conclude the paper by proposing a definition of spherical elements based on the use of the T-matrix and in close analogy with the classical theory [35]. We shall then specify the result to the case of $E_q(2)$.

(5.4) Definition. *Given a τ invariant two sided coideal $K_{\mathcal{M}}$ in $\mathcal{U}_q(\mathrm{Lie}G)$, consider a unitary representation $\mathcal{R}$ of $\mathcal{U}_q(\mathrm{Lie}\,G)$ and suppose there exists an element ξ spanning a one dimensional kernel of $K_{\mathcal{M}}$ in the representation space of $\mathcal{R}$. Denoting by $(\ ,\ )$ the scalar product of $\mathcal{R}$, we define the* **zonal spherical function** *$t^{\mathcal{R}}_{\mathrm{zon}}$ of the representation $\mathcal{R}$ with respect to $K_{\mathcal{M}}$ as follows:*

$$t^{\mathcal{R}}_{\mathrm{zon}} = (\xi, (1 \otimes \mathcal{R})\, T\, \xi)\,.$$

We then call **associated spherical functions** *the elements*

$$t^{\mathcal{R}}_{k} = (\xi_k, (1 \otimes \mathcal{R})\, T\, \xi)\,,$$

where $\{\xi_k\}$ is a basis of the representation $\mathcal{R}$.

(5.5) Lemma. *For any $Y \in K_{\mathcal{M}}$ the zonal spherical function $t^{\mathcal{R}}_{\mathrm{zon}}$ satisfies*

$$\lambda(Y)\, t^{\mathcal{R}}_{\mathrm{zon}} = 0\,, \qquad \ell(Y)\, t^{\mathcal{R}}_{\mathrm{zon}} = 0\,.$$

For the associated spherical function $t^{\mathcal{R}}_k$ we have $\ell(Y)\, t^{\mathcal{R}}_k = 0$, for any $Y \in K_{\mathcal{M}}$.

Proof. We will prove the first relation only. The proofs of the other statements are similar.

From the definitions of the action λ and of the universal matrix T, we have:

$$\begin{aligned}\lambda(Y)\, t^{\mathcal{R}}_{\mathrm{zon}} &= \sum_A \lambda(Y) x^A\, (\xi\,, \mathcal{R}(X_A)\, \xi) = \sum_A x^A\, (\xi\,, \mathcal{R}(S(Y)\, X_A)\, \xi) \\ &= \sum_A x^A\, (\mathcal{R}(\tau(Y))\xi\,, \mathcal{R}(X_A)\, \xi) = 0\,,\end{aligned}$$

where we used the τ invariance of Y and the unitarity of the representation $\mathcal{R}$. □

In the case of $E_q(2)$ the kernel of X_μ in the representation (4.11) with $R = i\sqrt{E}\,(\mu/|\mu|)$ has been calculated in [18]: it is given by

$$\xi = \sum_r q^{r(r-1)/2}\, i^r\, J^{(2)}_r(\sigma; q^2)\psi_r\,,$$

where $J^{(2)}_r$ are the Jackson q-Bessel functions and $\sigma = 2q\,(q - q^{-1})\,\sqrt{E}/|\mu|$. Using this result and our definitions of the spherical functions it is straightforward to prove the proposition that follows.

(5.6) Proposition. *The zonal spherical function of the representation given in* (4.11) *with respect to* X_μ *defined in* (3.6) *is*

$$t^{\lambda}_{\text{zon}} = \sum_{r,s} q^{r(r-1)/2}\, q^{s(s-1)/2}\, i^{s-r}\, J^{(2)}_r(\sigma; q^2)\, J^{(2)}_s(\sigma; q^2)\, t^{\lambda}_{rs}\,,$$

and the associated spherical functions are

$$t^{\lambda}_k = \sum_s q^{s(s-1)/2}\, i^s\, J^{(2)}_s(\sigma; q^2)\, t^{\lambda}_{ks}\,.$$

(5.7) Remarks. In this final remark we shall compare our presentation with the one given in ref. [18]. In [18] the (s,t) – spherical elements are defined as the elements $a \in \mathscr{F}_q(E(2))$ such that $\ell(X_s)a = r(X_t)a = 0$, where $r(X)a = \sum_{(a)} \langle X\,,\, a_{(1)}\rangle\, a_{(2)}$. We can easily relate our definitions to the (s,s) – spherical function: indeed it can be observed that if a is ℓ and r invariant, then $r(q^{-J})\,a$ is ℓ and λ invariant. Therefore our definition implies the same calculations in order to determine the zonal spherical function: still we consider it useful for the more transparent analogy with the classical theory and for the simple extension to the case of the associated functions.

Acknowledgements. R.G. is indebted to M.A. Rieffel for critical reading of the manuscript and interesting discussions during his visit to the University of Bologna. All the authors would like to thank the Referee for the constructive suggestions given in his report.

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Communicated by M. Jimbo