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Vertex Operators of Quantum Affine Lie Algebras $U_q(D_n^{(1)})$

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Abstract: We give an explicit formula for the vertex operators related to the level 1 representations of the quantum affine Lie algebras $U_q(D_n^{(1)})$ in terms of bosons. As an application, we derive an integral formula for the correlation functions of the vertex models with $U_q(D_n^{(1)})$-symmetry.

1. Introduction

In [FR], Frenkel and Reshetikhin constructed a q-analogue of the WZW model on the sphere based on the representation theory of the quantum affine Lie algebras. They defined q-deformed chiral vertex operators as intertwining operators between representations of certain types and derived a system of difference equations called the *quantum Knizhnik–Zamolodchikov equations*, which is satisfied by the vacuum expectation value of compositions of q-vertex operators. They also observed that the connection matrices between the solutions of quantum Knizhnik–Zamolodchikov equations with different asymptotics provide elliptic solutions of the Yang–Baxter equations in the face formulation. It shows that the above theory is very closely related to the solvable lattice model theory. The q-vertex operators are characterized by the intertwining conditions, however, it is difficult to know explicit forms for them in general. In [JMMN], the bosonization of the level 1 vertex operators for $U_q(\widehat{sl}_2)$ was constructed using the Frenkel–Jing construction of level 1 irreducible highest weight modules. Following [JMMN], the level 1 case for $U_q(\widehat{sl}_n)$ was done in [Ko]. For general levels for $U_q(\widehat{sl}_2)$, the bosonization of vertex operators was constructed in [KSQ] and in [M] using a q-deformation of Wakimoto modules. The main purpose of this article is to give an explicit formula for the level 1 vertex operators related to $U_q(D_n^{(1)})$.

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On the other hand, it was shown in [DFJMN] that the q-vertex operators related to $U_q(\widehat{sl}_2)$ appear as the dynamical symmetries of the XXZ-model in the thermodynamic limit. The XXZ model is a one-dimensional quantum spin chain model with the Hamiltonian

$$\mathscr{H}_{XXZ} = -\frac{1}{2}\sum_{k\in\mathbb{Z}} \left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta\sigma_k^z\sigma_{k+1}^z\right) ,$$

where σ_k^x, σ_k^y, σ_k^z are the Pauli matrices acting on the k^{th} component of the infinite tensor product

$$V^{\otimes\infty} = \cdots \otimes V \otimes V \otimes V \otimes \cdots .$$

It was observed in [DFJMN] that the quantum affine Lie algebra $U_q(\widehat{sl}_2)$ acts on the above space formally via the iterated comultiplication. When $\Delta = (q+q^{-1})/2$, a formal manipulation shows that

$$[\mathscr{H}_{XXZ}, U_q'(\widehat{sl}_2)] = 0 ,$$

where $U_q'(\widehat{sl}(2))$ denotes the subalgebra of $U_q(\widehat{sl}(2))$ without the grading operator d. Thus the algebra $U_q(\widehat{sl}_2)$ provides an exact symmetry of the Hamiltonian $\mathscr{H}_{XXZ}$, while d plays the role of the boost operator. The new method proposed in [DFJMN] for studying the model is based on the hypothesis that the space of physical states for the above model in the anti-ferromagnetic regime (i.e. $-1 < q < 0$) can be regarded as a $U_q(\widehat{sl}_2)$-module. More precisely, they postulated that the space of physical states is the level 0 $U_q(\widehat{sl}_2)$-module:

$$\mathscr{F}_{\lambda\mu} = V(\lambda)\widehat{\otimes} V(\mu)^* = \mathrm{Hom}_{\mathbb{C}}(V(\mu), V(\lambda)) ,$$

where $V(\lambda)$ is the level 1 highest weight $U_q(\widehat{sl}_2)$-module and $V(\mu)^*$ is the (restricted) dual module of the level 1 highest weight $U_q(\widehat{sl}_2)$-module $V(\mu)$. The space $V(\lambda)$ can be embedded into the half infinite tensor product $\cdots \otimes V \otimes V \otimes V$ by iterating the vertex operator (called type I in [DFJMN])

$$\Phi_\lambda^{\mu V} : V(\lambda) \to V(\mu) \otimes V .$$

It was conjectured in [DFJMN] that there is a unique normalization of the above which makes the infinite iteration convergent. This conjecture was proved in [E], and the unique normalization of vertex operators under which the convergence holds was explicitly computed by M. Jimbo ([E, Sect. 4]). Similarly, $V(\mu)^*$ can be embedded into the other half infinite tensor product $V \otimes V \otimes V \otimes \cdots$. Altogether, we get the embedding

$$\mathscr{F}_{\lambda,\mu} \to \cdots \otimes V \otimes V \otimes V \otimes \cdots .$$

The above embedding relates the naive picture on $V^{\otimes\infty}$ with the representation theory of $U_q(\widehat{sl}_2)$. The shift operator on $V^{\otimes\infty}$ and the Hamiltonian $\mathscr{H}_{XXZ}$ are interpreted as operators on $\mathscr{F}_{\lambda,\mu}$, and the correlation functions are evaluated by the trace of the vertex operators (see [DFJMN] for the details). The method can be applied to the other model associated to any quantum affine Lie algebra. The other purpose of this article is to give an integral formula for the correlation functions of the vertex model associated with the vector representation of $U_q(D_n^{(1)})$ using the technique developed in [JMMN].

This paper is organized as follows. In Sect. 2, we summarize some notations for representation theory of $U_q(D_n^{(1)})$ and give a comultiplication formula for Drinfeld generators. The exact formula for comultiplication of Drinfeld generators was obtained in [Be]. Our formula does not contain all the terms, but it is enough for the bosonization of the vertex operators. In Sect. 3, we recall the construction of level 1 irreducible highest weight modules (Frenkel–Jing constructions) and construct level 1 vertex operators explicitly in terms of bosons. In Sect. 4, as an application of the expression of the vertex operators, we derive an integral formula for the correlation functions of the vertex models with $U_q(D_n^{(1)})$-symmetry.

2. The Quantum Affine Lie Algebra $U_q(D_n^{(1)})$

2.1. The algebra $U_q(D_n^{(1)})$. Let $I = \{0, 1, \ldots, n\}$ be an index set, and let $A = (a_{ij})_{i,j\in I}$ be an affine generalized Cartan matrix of type $D_n^{(1)}$:

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & & & & \\ 0 & 2 & -1 & 0 & \cdots & & & & \\ -1 & -1 & 2 & -1 & \cdots & & & & \\ 0 & 0 & -1 & 2 & \cdots & & & & \\ & \vdots & \vdots & & \ddots & & \vdots & \vdots & \\ & & & & \cdots & 2 & -1 & 0 & 0 \\ & & & & \cdots & -1 & 2 & -1 & -1 \\ & & & & \cdots & 0 & -1 & 2 & 0 \\ & & & & \cdots & 0 & -1 & 0 & 2 \end{pmatrix} \tag{2.1}$$

Let $\mathfrak{h}$ be a complex vector space with a basis $\{h_0, h_1, \ldots, h_n, d\}$ and define the linear functionals $\alpha_i \in \mathfrak{h}^*(i \in I)$ by

$$\alpha_i(h_j) = a_{ji}, \qquad \alpha_i(d) = \delta_{i,0} \quad \text{for } j \in I . \tag{2.2}$$

Then the triple $(\mathfrak{h}, \Pi = \{\alpha_i | i \in I\}, \Pi^\vee = \{h_i | i \in I\})$ is the realization of the matrix A. The Kac-Moody algebra $\mathfrak{g}$ associated with the matrix A is called the *affine Kac–Moody algebra of type* $D_n^{(1)}(n \geqq 4)$ (cf. [K]). We denote by e_i, f_i, $h_i(i \in I)$ and d the generators of the Kac–Moody algebra $\mathfrak{g}$. The elements of Π (resp. $\Pi^\vee$) are called the *simple roots* (resp. *simple coroots*) of $\mathfrak{g}$.

Let $\Lambda_i \in \mathfrak{h}^*(i \in I)$ and $\delta \in \mathfrak{h}^*$ be the linear functionals on $\mathfrak{h}$ defined by

$$\Lambda_i(h_j) = \delta_{ij}, \qquad \Lambda_i(d) = 0 ,$$
$$\delta(h_j) = 0, \qquad \delta(d) = 1 \quad \text{for } j \in I . \tag{2.3}$$

We define the *affine weight lattice* of $\mathfrak{g}$ to be $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta$. The lattice $P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$ is called the *dual affine weight lattice*. The *derived weight lattice* $\overline{P}$ and the *dual derived weight lattice* $\overline{P}^\vee$ are defined to be $\overline{P} = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n$ and $\overline{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$, respectively, where the Λ_i are regarded as the linear functionals on $\mathfrak{h}' = \bigoplus_{i=0}^n \mathbb{C}h_i$. Since the matrix A is symmetric, there is a nondegenerate symmetric bilinear form $(\,|\,)$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i|\alpha_j) = a_{ij}, \qquad (\delta|\alpha_i) = (\delta|\delta) = 0 \quad \text{for all } i, j \in I . \tag{2.4}$$

The *quantum affine Lie algebra* $U_q(D_n^{(1)})$ is the associative algebra with 1 over $\mathbb{C}(q^{1/2})$ generated by the elements e_i, $f_i (i \in I)$ and $q^h (h \in P^\vee)$ with the following defining relations:

$$q^0 = 1, \qquad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee ,$$
$$q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \qquad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \text{for } h \in P^\vee (i \in I) ,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \quad \text{where } t_i = q^{h_i} \quad \text{and} \quad i, j \in I ,$$
$$\sum_{m+k=1-a_{ij}} (-1)^m e_i^{(m)} e_j e_i^{(n)} = 0 ,$$
$$\sum_{m+n=1-a_{ij}} (-1)^m f_i^{(m)} f_j f_i^{(n)} = 0 \quad \text{for } i \neq j , \tag{2.5}$$

where $e_i^{(k)} = e_i^k/[k]!$, $f_i^{(k)} = f_i^k/[k]!$, $[m]! = \prod_{k=1}^m [k]$, and $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$. We denote by $U_q'(D_n^{(1)})$ the subalgebra of $U_q(D_n^{(1)})$ generated by e_i, f_i, $t_i (i \in I)$.

The algebra $U_q(D_n^{(1)})$ has a Hopf algebra structure with comultiplication Δ, counit ε, and antipode S defined by

$$\Delta(q^h) = q^h \otimes q^h \quad \text{for } h \in P^\vee ,$$
$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i ,$$
$$\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \text{for } i \in I , \tag{2.6}$$

$$\varepsilon(q^h) = 1 \quad \text{for } h \in P^\vee ,$$
$$\varepsilon(e_i) = \varepsilon(f_i) = 0 \quad \text{for } i \in I , \tag{2.7}$$

$$S(q^h) = q^{-h} \quad \text{for } h \in P^\vee ,$$
$$S(e_i) = -t_i^{-1} e_i, \qquad S(f_i) = -f_i t_i \quad \text{for } i \in I . \tag{2.8}$$

The Hopf algebra structure of $U_q(D_n^{(1)})$ enables us to define a $U_q(D_n^{(1)})$-module structure on the tensor product of $U_q(D_n^{(1)})$-modules and the (restricted) dual space of a $U_q(D_n^{(1)})$-module. More precisely, if V, W are $U_q(D_n^{(1)})$-modules and V^* is the (restricted) dual space of V, then we define

$$x \cdot (v \otimes w) = \Delta(x)(v \otimes w) , \tag{2.9}$$

and

$$(x \cdot v^*)(u) = v^*(S(x) \cdot u) \tag{2.10}$$

for $x \in U_q(\mathfrak{g})$, u, $v \in V$, $w \in W$, and $v^* \in V^*$.

2.2. Drinfeld's Realization. In this section, we recall Drinfeld's realization of the quantum affine Lie algebra $U_q(D_n^{(1)})$ (and of $U_q'(D_n^{(1)})$) (cf. [Dr]). Let $\mathbf{U}$ be the associative algebra 1 over $\mathbb{C}(q^{1/2})$ generated by the elements $x_i^{\pm}(k)$, $a_i(l)$, $K_i^{\pm 1}$, $\gamma^{\pm 1/2}, q^{\pm d}(i = 1, 2, \dots, n, k \in \mathbb{Z}, l \in \mathbb{Z}\backslash\{0\})$ with the following defining relations:

$$[\gamma^{\pm 1/2}, u] = 0 \quad \text{for all } u \in \mathbf{U}\,,$$

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1\,,$$

$$[a_i(k), a_j(l)] = \delta_{k+l,0}\frac{[(\alpha_i|\alpha_j)k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}\,,$$

$$[a_i(k), K_j^{\pm 1}] = [q^{\pm d}, K_j^{\pm 1}] = 0\,,$$

$$q^d x_i^{\pm}(k) q^{-d} = q^k x_i^{\pm}(k), \quad q^d a_i(l) q^{-d} = q^l a_i(l)\,,$$

$$K_i x_j^{\pm}(k) K_i^{-1} = q^{\pm(\alpha_i|\alpha_j)} x_j^{\pm}(k)\,,$$

$$[a_i(k), x_j^{\pm}(l)] = \pm\frac{[(\alpha_i|\alpha_j)k]}{k}\gamma^{\mp|k|/2} x_j^{\pm}(k+l)\,,$$

$$\begin{aligned} &x_i^{\pm}(k+1)x_j^{\pm}(l) - q^{\pm(\alpha_i|\alpha_j)}x_j^{\pm}(l)x_i^{\pm}(k+1) \\ &\quad = q^{\pm(\alpha_i|\alpha_j)}x_i^{\pm}(k)x_j^{\pm}(l+1) - x_j^{\pm}(l+1)x_i^{\pm}(k)\,, \end{aligned}$$

$$[x_i^+(k), x_j^-(l)] = \frac{\delta_{ij}}{q - q^{-1}}\left(\gamma^{\frac{k-l}{2}}\psi_i(k+l) - \gamma^{\frac{l-k}{2}}\varphi_i(k+l)\right)\,,$$

where $\psi_i(m)$ and $\varphi_i(-m)(m \in \mathbb{Z}_{\geqq 0})$ are defined by

$$\sum_{m=0}^{\infty}\psi_i(m)z^{-m} = K_i \exp\left((q - q^{-1})\sum_{k=1}^{\infty} a_i(k)z^{-k}\right)\,,$$

$$\sum_{m=0}^{\infty}\varphi_i(-m)z^{m} = K_i^{-1} \exp\left(-(q - q^{-1})\sum_{k=1}^{\infty} a_i(-k)z^{k}\right)\,,$$

$$[x_i^{\pm}(k), x_j^{\pm}(l)] = 0 \quad \text{if } (\alpha_i|\alpha_j) = 0\,,$$

$$\begin{aligned} &(x_i^{\pm}(k)x_i^{\pm}(l)x_j^{\pm}(m) + x_i^{\pm}(l)x_i^{\pm}(k)x_j^{\pm}(m)) - (q+q^{-1})(x_i^{\pm}(k)x_j^{\pm}(m)x_i^{\pm}(l) \\ &+ x_i^{\pm}(l)x_j^{\pm}(m)x_i^{\pm}(k)) + (x_j^{\pm}(m)x_i^{\pm}(k)x_i^{\pm}(l) + x_j^{\pm}(m)x_i^{\pm}(l)x_i^{\pm}(k)) = 0 \\ &\qquad \text{if } (\alpha_i|\alpha_j) = -1\,. \end{aligned} \tag{2.11}$$

We denote by $\mathbf{U}'$ the subalgebra of $\mathbf{U}$ generated by the elements $x_i^{\pm}(k), a_i(l), K_i^{\pm 1}$, $\gamma^{\pm 1/2}(i = 1, 2, \dots, n, k \in \mathbb{Z}, l \in \mathbb{Z}\backslash\{0\})$.

In [Dr], it was shown that the algebra $\mathbf{U}$ (resp. $\mathbf{U}'$) is isomorphic to the quantum affine Lie algebra $U_q(D_n^{(1)})$ (resp. $U_q'(D_n^{(1)})$). We call the algebra $\mathbf{U}$ (resp. $\mathbf{U}'$) *Drinfeld's realization* of the quantum affine Lie algebra $U_q(D_n^{(1)})$ (resp. of $U_q'(D_n^{(1)})$). In order to give the precise isomorphism of $\mathbf{U}$ and $U_q(D_n^{(1)})$ (resp. $\mathbf{U}'$ and $U_q'(D_n^{(1)})$), we need the following lemma.

Lemma 2.1. *Let $I_0 = \{1, 2, \dots, n\}$ be the index set for the simple roots of a finite dimensional simple Lie algebra $\mathfrak{g}_0$ with symmetric Cartan matrix. Then for each*

$i \in I_0$, there exists a sequence of indices $i = i_1, i_2, \ldots, i_{h-1}$ such that

$$(\alpha_{i_1}|\alpha_{i_2}) = -1 ,$$
$$(\alpha_{i_1} + \alpha_{i_2}|\alpha_{i_3}) = -1 ,$$
$$\vdots$$
$$(\alpha_{i_1} + \cdots + \alpha_{i_{h-2}}|\alpha_{i_{h-1}}) = -1 , \tag{2.12}$$

where h is the Coxeter number of the Lie algebra $\mathfrak{g}_0$. □

Proposition 2.2. ([Dr]) *Let $i_1, i_2, \ldots, i_{h-1}$ be a sequence of indices in Lemma 2.1 satisfying* (2.12), *and let $\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ be the maximal root of the finite dimensional simple Lie algebra $\mathfrak{g}_0 = D_n$. Then there is a $\mathbb{C}(q^{1/2})$-algebra isomorphism $\psi : U_q(\mathfrak{g}) \to \mathbf{U}$ defined by*

$$\begin{aligned} &e_i \mapsto x_i^+(0), \quad f_i \mapsto x_i^-(0), \quad t_i \mapsto K_i \quad \text{for } i = 1, \ldots, n , \\ &e_0 \mapsto [x_{i_{h-1}}^-(0)[\cdots[x_{i_2}^-(0), x_{i_1}^-(1)]_{q^{-1}} \cdots]_{q^{-1}} K_0^{-1} , \\ &f_0 \mapsto (-q)^{h-2} K_0 [x_{i_{h-1}}^+(0)[\cdots[x_{i_2}^+(0), x_{i_1}^+(-1)]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-1}} , \\ &t_0 \mapsto \gamma K_0^{-1}, \quad q^d \mapsto q^d , \end{aligned} \tag{2.13}$$

where $K_0 = K_1 K_2^2 \cdots K_{n-2}^2 K_{n-1} K_n$, $h = 2(n-1)$, and $[x, y]_q = xy - qyx$.

Moreover, the isomorphism Ψ is independent of the choice of the sequence $i_1, i_2, \ldots, i_h$ satisfying (2.12). *The restriction of Ψ to $U_q'(D_n^{(1)})$ defines an isomorphism of $U_q'(D_n^{(1)})$ and $\mathbf{U}'$.* □

2.3. Comultiplication of the Algebra U. By Proposition 2.2, the algebra **U** is given a Hopf algebra structure. In principle, the comultiplication of the algebra **U**, which we will also denote by Δ, can be expressed using Drinfeld's isomorphism (2.13). The general formula for the comultiplication of **U** in terms of Drinfeld's generators was obtained in [Be]. However, as it was shown in [CP, JMMN, and Ko], the formula for the "main terms" of the comultiplication is sufficient for our purpose.

Theorem 2.3. *Let $k \in \mathbb{Z}_{\geqq 0}$, $l \in \mathbb{Z}_{>0}$, and let N_+^s (resp. N_-^s) be the left ideal of the algebra **U** generated by the elements $x_{i_1}^+(m_1) \cdots x_{i_s}^+(m_s)$ (resp. $x_{i_1}^-(m_1) \cdots x_{i_s}^-(m_s)$) with $m_i \in \mathbb{Z}_{\geqq 0}$. Then the comultiplication Δ of the algebra **U** satisfies the following relations:*

$$\begin{aligned} \Delta(x_i^+(k)) &= x_i^+(k) \otimes \gamma^k + \gamma^{2k} K_i \otimes x_i^+(k) \\ &\quad + \sum_{j=0}^{k-1} \gamma^{\frac{k-j}{2}} \psi_i(k-j) \otimes \gamma^{k-j} x_i^+(j) \ (\operatorname{mod} N_- \otimes N_+^2) , \\ \Delta(x_i^+(-l)) &= x_i^+(-l) \otimes \gamma^{-l} + K_i^{-1} \otimes x_i^+(-l) \\ &\quad + \sum_{j=1}^{l-1} \gamma^{\frac{l-j}{2}} \varphi_i(-l+j) \otimes \gamma^{-l+j} x_i^+(-j) \ (\operatorname{mod} N_- \otimes N_+^2) , \end{aligned}$$

$$\Delta(x_i^-(l)) = x_i^-(l) \otimes K_i + \gamma^l \otimes x_i^-(l) + \sum_{j=1}^{l-1} \gamma^{l-j} x_i^-(j) \otimes \gamma^{\frac{j-l}{2}} \psi_i(l-j) \ (\mathrm{mod}\, N_-^2 \otimes N_+)\,,$$

$$\Delta(x_i^-(-k)) = x_i^-(k) \otimes \gamma^{-2k} K_i^{-1} + \gamma^{-k} \otimes x_i^-(-k) + \sum_{j=0}^{k-1} \gamma^{j-k} x_i^-(-j) \otimes \gamma^{-\frac{k+3j}{2}} \varphi_i(j-k) \ (\mathrm{mod}\, N_-^2 \otimes N_+)\,,$$

$$\Delta(a_i(l)) = a_i(l) \otimes \gamma^{\frac{l}{2}} + \gamma^{\frac{3l}{2}} \otimes a_i(l) \ (\mathrm{mod}\, N_- \otimes N_+)\,,$$

$$\Delta(a_i(-l)) = a_i(-l) \otimes \gamma^{-\frac{3l}{2}} + \gamma^{-\frac{l}{2}} \otimes a_i(-l) \ (\mathrm{mod}\, N_- \otimes N_+)\,. \tag{2.14}$$

Clearly, our assertion is also true for the subalgebra $\mathbf{U}'$.

Proof. Fix $i \in I_0 = \{1, 2, \ldots, n\}$. By (2.6) and (2.13), we have

$$\Delta(x_i^+(0)) = x_i^+(0) \otimes 1 + K_i \otimes x_i^+(0)\,,$$

$$\Delta(x_i^-(0)) = x_i^-(0) \otimes K_i^{-1} + 1 \otimes x_i^-(0)\,.$$

Let $i = i_1, i_2, \ldots, i_{h-1}$ be a sequence of indices in I_0 satisfying (2.12). Then the inverse images of $x_i^+(-1)$ and $x_i^-(1)$ under the isomorphism Ψ are given by

$$\Psi^{-1}(x_i^+(-1)) = t_{i_2}[f_{i_2}, t_{i_3}[f_{i_3}, \ldots t_{i_{h-1}}[f_{i_{h-1}}, t_0^{-1} f_0] \ldots]]\,,$$

$$\Psi^{-1}(x_i^-(1)) = q^{h-2} t_{i_2}^{-1}[e_{i_2}, t_{i_3}^{-1}[e_{i_3}, \ldots t_{i_{h-1}}^{-1}[e_{i_{h-1}}, e_0 t_0] \ldots]]\,. \tag{2.15}$$

Since Δ is a $\mathbb{C}(q^{1/2})$-algebra homomorphism, it follows from (2.6) that

$$\Delta(x_i^+(-1)) = x_i^+(-1) \otimes \gamma^{-1} + K_i^{-1} \otimes x_i^+(-1) \ (\mathrm{mod}\, N_- \otimes N_+^2)\,,$$

$$\Delta(x_i^-(1)) = x_i^-(1) \otimes K_i + \gamma \otimes x_i^-(1) \ (\mathrm{mod}\, N_-^2 \otimes N_+)\,.$$

Using the relations

$$[x_i^+(0), x_i^-(1)] = (q - q^{-1})^{-1} \gamma^{-1/2} \psi_i(1) = \gamma^{-1/2} K_i a_i(1)\,,$$

$$[x_i^+(-1), x_i^-(0)] = -(q - q^{-1})^{-1} \gamma^{1/2} \varphi_i(-1) = \gamma^{1/2} K_i^{-1} a_i(-1)\,,$$

we obtain

$$\Delta(a_i(1)) = a_i(1) \otimes \gamma^{1/2} + \gamma^{3/2} \otimes a_i(1) \ (\mathrm{mod}\, N_- \otimes N_+)\,,$$

$$\Delta(a_i(-1)) = a_i(-1) \otimes \gamma^{-3/2} + \gamma^{-1/2} \otimes a_i(-1) \ (\mathrm{mod}\, N_- \otimes N_+)\,.$$

The rest of the formulas can be proved inductively using the relations (2.11). □

2.4. Evaluation Modules. Let $V = (\bigoplus_{i=1}^n \mathbb{C}(q^{1/2}) v_i) \oplus (\bigoplus_{i=1}^n \mathbb{C}(q^{1/2}) v_{\bar{i}})$ be the $2n$-dimensional natural representation of the quantum group $U_q(D_n)$. Thus the $U_q(D_n)$-

module structure on V is given as follows:

$$
\begin{aligned}
&e_i v_{i+1} = v_i, \quad e_i v_{\bar{i}} = v_{\overline{i+1}}, \quad e_i v_j = 0 \quad \text{if } j \neq i+1, \bar{i}\,, \\
&f_i v_i = v_{i+1}, \quad f_i v_{\overline{i+1}} = v_{\bar{i}}, \quad f_i v_j = 0 \quad \text{if } j \neq i, \overline{i+1}\,, \\
&t_i v_i = q v_i, \quad t_i v_{i+1} = q^{-1} v_{i+1}, \quad t_i v_{\bar{i}} = q^{-1} v_{\bar{i}}, \quad t_i v_{\overline{i+1}} = q v_{\overline{i+1}}\,, \\
&t_i v_j = v_j \quad \text{if } j \neq i, i+1, \bar{i}, \overline{i+1}\,, \\
&e_n v_{\bar{n}} = v_{n-1}, \quad e_n v_{\overline{n-1}} = v_n, \quad e_n v_j = 0 \quad \text{if } j \neq \overline{n-1}, \bar{n}\,, \\
&f_n v_{n-1} = v_{\bar{n}}, \quad f_n v_n = v_{\overline{n-1}}, \quad f_n v_j = 0 \quad \text{if } j \neq n-1, n\,, \\
&t_n v_{n-1} = q v_{n-1}, \quad t_n v_n = q v_n, \quad t_n v_{\overline{n-1}} = q^{-1} v_{\overline{n-1}}, \quad t_n v_{\bar{n}} = q^{-1} v_{\bar{n}}\,, \\
&t_n v_j = v_j \quad \text{if } j \neq n-1, n, \overline{n-1}, \bar{n}\,,
\end{aligned} \tag{2.16}
$$

where $i = 1, 2, \ldots, n-1, j = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$.

We define the $U_q'(D_n^{(1)})$-module structure on V by

$$
\begin{aligned}
&e_0 v_1 = v_{\bar{2}}, \quad e_0 v_2 = v_{\bar{1}}, \quad e_0 v_j = 0 \quad \text{for } j \neq 1, 2\,, \\
&f_0 v_{\bar{1}} = v_2, \quad f_0 v_{\bar{2}} = v_1, \quad f_0 v_j = 0 \quad \text{for } j \neq \bar{1}, \bar{2}\,, \\
&t_0 v_1 = q^{-1} v_1, \quad t_0 v_2 = q^{-1} v_2, \quad t_0 v_{\bar{1}} = q v_{\bar{1}}, \quad t_0 v_{\bar{2}} = q v_{\bar{2}}\,, \\
&t_0 v_j = v_j \quad \text{for } j \neq 1, 2, \bar{1}, \bar{2}\,.
\end{aligned} \tag{2.17}
$$

Since V is a finite dimensional vector space over $\mathbb{C}(q^{1/2})$, it does not admit a $U_q(D_n^{(1)})$-module structure. But we can define a $U_q(D_n^{(1)})$-module structure on the affinization of V (cf. [KMN]). The *affinization* of V is the $U_q(D_n^{(1)})$-module $V_z = V \otimes \mathbb{C}(q^{1/2})[z, z^{-1}]$ with the $U_q(D_n^{(1)})$-modules structure defined by

$$
\begin{aligned}
&e_i(v \otimes z^m) = e_i v \otimes z^{m+\delta_{i,0}}, \qquad f_i(v \otimes z^m) = f_i v \otimes z^{m-\delta_{i,0}}\,, \\
&t_i(v \otimes z^m) = t_i v \otimes z^m, \qquad q^d(v \otimes z^m) = q^m v \otimes z^m\,,
\end{aligned} \tag{2.18}
$$

for $i = 0, 1, \ldots, n$, $m \in \mathbb{Z}$, $v \in V$. The affinization V_z of V is also called the *evaluation module of V at z*. Let us denote by E_{ij} the matrix unit of $\mathrm{End}_{\mathbb{C}(q^{1/2})} V$ such that $E_{ij} v_k = \delta_{jk} v_i$ for $i, j, k = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$. Then the $U_q(D_n^{(1)})$-module structure on V_z can be expressed as follows:

$$
\begin{aligned}
&e_i = E_{i,i+1} + E_{\overline{i+1},\bar{i}}, \quad f_i = E_{i+1,i} + E_{\bar{i},\overline{i+1}}\,, \\
&t_i = q(E_{ii} + E_{\overline{i+1},\overline{i+1}}) + q^{-1}(E_{\bar{i},\bar{i}} + E_{i+1,i+1}) + \sum_{j \neq i, i+1, \bar{i}, \overline{i+1}} E_{jj}\,, \\
&e_n = E_{n-1,\bar{n}} + E_{n,\overline{n-1}}, \quad f_n = E_{\bar{n},n-1} + E_{\overline{n-1},n}\,, \\
&t_n = q(E_{n-1,n-1} + E_{nn}) + q^{-1}(E_{\overline{n-1},\overline{n-1}} + E_{\bar{n},\bar{n}}) + \sum_{j \neq n-1, n, \overline{n-1}, \bar{n}} E_{jj}\,,
\end{aligned}
$$

$$e_0 = z(E_{\bar{2},1} + E_{\bar{1},2}), \quad f_0 = z^{-1}(E_{2,\bar{1}} + E_{1,\bar{2}}),$$

$$t_0 = q(E_{\bar{1},\bar{1}} + E_{\bar{2},\bar{2}}) + q^{-1}(E_{11} + E_{22}) + \sum_{j \neq 1,2,\bar{1},\bar{2}} E_{jj},$$

$$q^d(v \otimes z^m) = q^m v \otimes z^m \tag{2.19}$$

for $i = 1,2,\ldots,n-1$, $v \in V$, $m \in \mathbb{Z}$.

We now consider the $\mathbf{U}$-module structure on V_z induced by Proposition 2.2. Note that, as a $U_q(D_n^{(1)})$-module, V_z has level 0. Thus γ acts on V_z as the identity. We have already seen that q^d acts on V_z by (2.19). The action of the rest of Drinfeld's generators of the algebra $\mathbf{U}$ on V_z is given in the following theorem.

Theorem 2.4. *The* $\mathbf{U}$*-module structure on the evaluation module* V_z *is defined as follows*:

$$x_i^+(k) = (q^i z)^k E_{i,i+1} + (q^{2n-i-2}z)^k E_{\overline{i+1},\bar{i}},$$

$$x_i^-(k) = (q^i z)^k E_{i+1,i} + (q^{2n-i-2}z)^k E_{\bar{i},\overline{i+1}},$$

$$x_n^+(k) = (q^{n-1}z)^k (E_{n-1,\bar{n}} + E_{n,\overline{n-1}}),$$

$$x_n^-(k) = (q^{n-1}z)^k (E_{\bar{n},n-1} + E_{\overline{n-1},n}),$$

$$a_i(l) = \frac{[l]}{l}((q^i z)^l (q^{-1}E_{ii} - q^l E_{i+1,i+1}) + (q^{2n-i-2}z)^l (q^{-l} E_{\overline{i+1},\overline{i+1}} - q^l E_{\bar{i},\bar{i}})),$$

$$a_n(l) = \frac{[l]}{l}(q^{n-1}z)^l ((q^{-l}E_{n-1,n-1} - q^l E_{\bar{n},\bar{n}}) + (q^{-l}E_{n,n} - q^l E_{\overline{n-1},\overline{n-1}})) \tag{2.20}$$

for $i = 1,2,\ldots,n-1$, $k \in \mathbb{Z}$, *and* $l \in \mathbb{Z}\backslash\{0\}$.

Proof. The idea of proof is the same as that of Theorem 2.3. Fix $i \in I_0\backslash\{n\} = \{1,2,\ldots,n-1\}$. Since $x_i^+(0) = \Psi(e_i)$ and $x_i^-(0) = \Psi(f_i)$, we have from (2.19) that

$$x_i^+(0) = E_{i,i+1} + E_{\overline{i+1},\bar{i}},$$

$$x_i^-(0) = E_{i+1,i} + E_{\bar{i},\overline{i+1}}$$

on V_z. Recall that the inverse images of $x_i^+(-1)$ and $x_i^-(1)$ under the isomorphism Ψ are given by (2.15). Using the formulas (2.19), we obtain

$$x_i^+(-1) = (q^i z)^{-1} E_{i,i+1} + (q^{2n-i-2}z)^{-1} E_{\overline{i+1},\bar{i}},$$

$$x_i^-(1) = (q^i z) E_{i+1,i} + (q^{2n-i-2}z) E_{\bar{i},\overline{i+1}}$$

on V_z. The relations

$$[x_i^+(0), x_i^-(1)] = \gamma^{-1/2} K_i a_i(1), \quad [x_i^+(-1), x_i^-(0)] = \gamma^{1/2} K_i^{-1} a_i(-1)$$

yield

$$a_i(1) = (q^i z)(q^{-1}E_{ii} - qE_{i+1,i+1}) + (q^{2n-i-2}z)(q^{-1}E_{\overline{i+1},\overline{i+1}} - qE_{\bar{i},\bar{i}}),$$

$$a_i(-1) = (q^i z)^{-1}(qE_{ii} - q^{-1}E_{i+1,i+1}) + (q^{2n-i-2}z)^{-1}(qE_{\overline{i+1},\overline{i+1}} - q^{-1}E_{\bar{i},\bar{i}})$$

on V_z. The rest of the formulas can be proved inductively by using the relations (2.11) and (2.19).

The formulas for $x_n^+(k), a_n(l)$ $(k \in \mathbb{Z}, l \in \mathbb{Z}\backslash\{0\})$ are proved in a similar way. □

Now let $V^* = (\bigoplus_{i=1}^n \mathbb{C}(q^{1/2})v_i^*) \oplus (\bigoplus_{i=1}^n \mathbb{C}(q^{1/2})v_{\bar{i}}^*)$ be the dual space of V, and recall that the $U_q'(\mathfrak{g})$-module structure on V^* is given by

$$(x \cdot v^*)(u) = v^*(S(x) \cdot u) \tag{2.21}$$

for $x \in U_q'(D_n^{(1)})$, $u \in V$, $v^* \in V^*$. Let V_z^* be the affinization of V^*, and denote by E_{ij}^* the matrix unit of $\mathrm{End}_{\mathbb{C}(q^{1/2})}V^*$ such that $E_{ij}^*v_k^* = \delta_{jk}v_i^*$ for $i,j,k = 1,\dots,n,\bar{1},\dots,\bar{n}$. Then by (2.18) and (2.21), the $U_q(D_n^{(1)})$-module structure on V_z^* is given by

$$\begin{aligned}
e_i &= (-q^{-1})(E_{i+1,i}^* + E_{\bar{i},\overline{i+1}}^*), \quad f_i = (-q)(E_{i,i+1}^* + E_{\overline{i+1},\bar{i}}^*),\\
t_i &= q(E_{i+1,i+1}^* + E_{\bar{i},\bar{i}}^*) + q^{-1}(E_{ii}^* + E_{\overline{i+1},\overline{i+1}}^*) + \sum_{j \neq i,i+1,\bar{i},\overline{i+1}} E_{jj}^*,\\
e_n &= (-q^{-1})(E_{\overline{n-1},n}^* + E_{\bar{n},n-1}^*), \quad f_n = (-q)(E_{n-1,\bar{n}}^* + E_{n,\overline{n-1}}^*),\\
t_n &= q^{-1}(E_{n-1,n-1}^* + E_{nn}^*) + q(E_{\overline{n-1},\overline{n-1}}^* + E_{\bar{n},\bar{n}}^*) + \sum_{j \neq n-1,n,\overline{n-1},\bar{n}} E_{jj}^*,\\
e_0 &= (-q^{-1}z)(E_{1,\bar{2}}^* + E_{2,\bar{1}}^*), \quad f_0 = (-qz^{-1})(E_{\bar{1},2}^* + E_{\bar{2},1}^*),\\
t_0 &= q(E_{11}^* + E_{22}^*) + q^{-1}(E_{\bar{1},\bar{1}}^* + E_{\bar{2},\bar{2}}^*) + \sum_{j \neq 1,2,\bar{1},\bar{2}} E_{jj}^*,
\end{aligned}$$

$$q^d(v^* \otimes z^m) = q^m v^* \otimes z^m \tag{2.22}$$

for $i = 1,2,\dots,n-1$, $v^* \in V^*$, $m \in \mathbb{Z}$.

As in the case with V_z, the evaluation module V_z^* is given a $\mathbf{U}$-module structure induced by Proposition 2.2. In particular, γ acts on V_z^* as the identity, and q^d acts on V_z^* by (2.22). The action of the rest of Drinfeld's generators on V_z^* is given in the following theorem, which can be proved by the same argument for Theorem 2.4.

Theorem 2.5. *The* $\mathbf{U}$*-module structure on the evaluation module* V_z^* *is defined as follows*:

$$\begin{aligned}
x_i^+(k) &= (-q^{-1})((q^{-i}z)^k E_{i+1,i}^* + (q^{-(2n-i-2)}z)^k E_{\bar{i},\overline{i+1}}^*),\\
x_i^-(k) &= (-q)((q^{-i}z)^k E_{i,i+1}^* + (q^{-(2n-i-2)}z)^k E_{\overline{i+1},\bar{i}}^*),\\
x_n^+(k) &= (-q^{-1})(q^{-(n-1)}z)^k (E_{\overline{n-1},n}^* + E_{\bar{n},\overline{n-1}}^*),\\
x_n^-(k) &= (-q)(q^{-(n-1)}z)^k (E_{n-1,\bar{n}}^* + E_{n,\overline{n-1}}^*),
\end{aligned}$$

$$a_i(l) = \frac{[l]}{l}((q^{-i}z)^l(q^{-l}E^*_{i+1,i+1} - q^lE^*_{ii}) + (q^{-(2n-i-2)}z)^l(q^{-l}E^*_{\bar{i},\bar{i}} - q^lE^*_{\overline{i+1},\overline{i+1}})),$$

$$a_n(l) = \frac{[l]}{l}(q^{-(n-1)}z)^l((q^{-l}E^*_{\overline{n-1},\overline{n-1}} - q^lE^*_{nn}) + (q^{-l}E^*_{\bar{n},\bar{n}} - q^lE^*_{n-1,n-1})) \quad (2.23)$$

for $i = 1,\ldots,n-1$. $k \in \mathbb{Z}$, *and* $l \in \mathbb{Z}\backslash\{0\}$. □

3. Level One Representations of $U_q(D_n^{(1)})$

3.1. Frenkel–Jing Construction. The level one irreducible representations of the quantum affine Lie algebra $U_q(D_n^{(1)})$ are realized on the Fock space of the tensor product of the group algebra $\mathbb{C}[\mathring{Q}]$ of the root lattice of the Lie algebra D_n and the symmetric algebra generated by the elements $a_j(-k)$, $k \in \mathbb{N}$, $j = 1,\ldots,n$. To construct vertex operators between the irreducible representations, we need to work on the group algebra $\mathbb{C}[\mathring{P}]$ of weight lattice. However, the latter poses some inconvenience to deal with (see the remark below). We instead consider the following group algebra of the lattice $\mathring{P}'$:

$$\mathring{P}' = \mathbb{Z}\lambda_1 + \mathbb{Z}\alpha_1 + \cdots \mathbb{Z}\alpha_{n-1}, \quad (3.1)$$

where the element λ_1 together with other λ_i's are the fundamental weights of the Lie algebra D_n. Note that the λ_i's are the finite dimensional analogues of the fundamental weights Λ_i's:

$$\Lambda_i = \lambda_i + \Lambda_0 \quad (i = 1,\ldots,n).$$

We identify the algebra $\mathbb{C}[\mathring{Q}]$ with a subalgebra of $\mathbb{C}[\mathring{P}']$ via:

$$\alpha_n = 2\lambda_1 - 2\alpha_1 - \cdots - 2\alpha_{n-2} - \alpha_{n-1}. \quad (3.2)$$

We also need the following weights:

$$\lambda_0 = 0,$$

$$\lambda_i = \alpha_1 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_{n-2}) + \frac{1}{2}i(\alpha_{n-1} + \alpha_n)$$

$$\text{for } i = 1,2,\ldots,n-2,$$

$$\lambda_{n-1} = \frac{1}{2}\left(\alpha_1 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n\right),$$

$$\lambda_n = \frac{1}{2}\left(\alpha_1 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n\right). \quad (3.3)$$

The inner product on $\mathring{P}$ induces an inner product on $\mathring{P}'$. There exists a central extension of the group algebra $\mathbb{C}[\mathring{P}']$:

$$1 \to \mathbb{Z}_2 \to \mathbb{C}\{\mathring{P}'\} \to \mathbb{C}[\mathring{P}'] \to 1$$

such that

$$e^{\alpha}e^{\beta} = (-1)^{(\alpha|\beta)}e^{\beta}e^{\alpha} \tag{3.4}$$

for $\alpha, \beta \in \mathring{Q}$, the root lattice, and here we also use the same symbol to denote corresponding elements in the central extension.

In fact we can construct the central extension as the associative algebra $\mathbb{C}\{\mathring{P}'\}$ generated by $e^{\alpha_1}, \dots, e^{\alpha_{n-1}}$ and e^{λ_1} subject to the relations:

$$\begin{aligned} e^{\alpha_i}e^{\alpha_j} &= (-1)^{(\alpha_i|\alpha_j)}e^{\alpha_j}e^{\alpha_i}, \\ & \qquad\qquad (1 \leqq i,j \leqq n-1) \\ e^{\lambda_1}e^{\alpha_i} &= (-1)^{\delta_{1i}}e^{\alpha_i}e^{\lambda_1} \end{aligned} \tag{3.5}$$

For any element $\alpha = m'\lambda_1 + \sum_{j=1}^{n-1} m_j\alpha_j \in \mathring{P}'$, we define

$$e^{\alpha} = e^{m'\lambda_1}e^{m_1\alpha_1} \cdots e^{m_{n-1}\alpha_{n-1}} . \tag{3.6}$$

In particular, we have,

$$e^{\alpha_n} = e^{2\lambda_1}e^{-2\alpha_1} \cdots e^{-2\alpha_{n-2}}e^{-\alpha_{n-1}} . \tag{3.7}$$

Note that in general $e^{\alpha}e^{-\alpha} = \varepsilon \in \{\pm 1\}$.

Proposition 3.1. *The algebra $\mathbb{C}\{\mathring{P}'\}$ is a central extension of $\mathbb{C}[\mathring{P}']$ with the property that*

$$e^{\alpha}e^{\beta} = (-1)^{(\alpha|\beta)}e^{\beta}e^{\alpha}$$

for $\alpha, \beta \in \mathring{Q} = 2\mathbb{Z}\lambda_1 + \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_{n-1}$, the root lattice of D_n, and moreover,

$$\begin{aligned} e^{\lambda_1}e^{\alpha_i} &= (-1)^{\delta_{i1}}e^{\alpha_i}e^{\lambda_1} \\ & \qquad\qquad (1 \leqq i \leqq n), \\ e^{\lambda_n - \lambda_{n-1}} &= (-1)^{n-1}(e^{\lambda_{n-1}-\lambda_n})^{-1} . \end{aligned} \tag{3.8}$$

Proof. Note that

$$\begin{aligned} e^{\lambda_n - \lambda_{n-1}} &= e^{\lambda_1}e^{-\alpha_1} \cdots e^{-\alpha_{n-2}}e^{-\alpha_{n-1}} , \\ e^{\lambda_{n-1} - \lambda_n} &= e^{-\lambda_1}e^{\alpha_1} \cdots e^{\alpha_{n-2}}e^{\alpha_{n-1}} . \end{aligned} \tag{3.9}$$

The relations are directly verified from the defining relations (3.5). □

We remark that we could also consider the following method to construct central extensions of the group algebra $\mathbb{C}[\mathring{P}]$ (cf. [FLM, DL]). Let ω_p be a primitive p^{th} root of unity, and we assume that p is an even integer. For any skew-symmetric $\mathbb{Z}$-bilinear map c_0:

$$c_0 : \quad \mathring{P} \times \mathring{P} \to \mathbb{Z}_p ,$$

there associates a central extension $\hat{P}$ of P:

$$1 \to \langle \omega_p \rangle \to \hat{P} \xrightarrow{-} P \to 1$$

such that

$$aba^{-1}b^{-1} = \omega_p^{c_0(\bar{a}|\bar{b})}$$

for $a, b \in \hat{P}$. Moreover, we can have a c_0 such that

$$c_0(\alpha, \beta) = (\alpha|\beta) \mod 2\mathbb{Z}$$

for $\alpha, \beta \in \mathring{Q}$, an even sublattice of P. Thus the central extension $\hat{P}$ factors through the root lattice $\mathring{Q}$ as a central extension over $\mathbb{Z}_2$ and

$$aba^{-1}b^{-1} = (-1)^{(\bar{a}|\bar{b})}$$

for $\bar{a}, \bar{b} \in \mathring{Q}$. Then the group algebra $\mathbb{C}[\hat{P}]$ will serve our purpose. However, the disadvantage of this construction is that our vertex operators will contain some clumsy constants involving the commutator map c_0.

The subalgebra of $U_q(D_n^{(1)})$ generated by elements $\gamma^{\pm 1}$ and $a_j(k)$ ($k \in \mathbb{Z}\backslash\{0\}$, $j = 1, \ldots, n$) is an infinite dimensional Heisenberg algebra, denoted by $U_q(\hat{\mathfrak{h}})$. Let $Sym(\hat{\mathfrak{h}}^-)$ be the symmetric algebra over $\mathbb{C}(q^{1/2})$ generated by the elements 1 and $a_j(-k)$, $k \in \mathbb{N}$, $j = 1, \ldots, n$ of the Heisenberg subalgebra $U_q(\hat{\mathfrak{h}})$. Then the space $Sym(\hat{\mathfrak{h}}^-)$ provides a natural representation of $U_q(\hat{\mathfrak{h}})$ with $\gamma = q$ (or $c = 1$), where the action is induced from the left multiplication modulo the relations

$$a_j(n) \cdot 1 = 0 \quad (n \in \mathbb{N}),$$

$$[a_i(k), a_j(l)] = \delta_{k+l,0}\frac{[(\alpha_i|\alpha_j)k]}{k}[k].$$

For $i = 0, 1, n-1, n$, let

$$W_i = Sym(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}\{\mathring{Q}\}e^{\lambda_i}, \tag{3.10}$$

where we formally enlist the element $e^{\lambda_{n-1}}$, and define

$$e^{\lambda_n} = e^{\lambda_n - \lambda_{n-1}}e^{\lambda_{n-1}} = e^{\lambda_1}e^{-\alpha_1}\cdots e^{-\alpha_{n-1}}e^{\lambda_{n-1}} \tag{3.11}$$

as an element in the space $\mathbb{C}\{\mathring{P}'\}e^{\lambda_{n-1}}$. Note that, as vector spaces,

$$\mathbb{C}\{\mathring{P}\} = \mathbb{C}\{\mathring{Q}\} \oplus \mathbb{C}\{\mathring{Q}\}e^{\lambda_1},$$

$$\mathbb{C}\{\mathring{P}'\}e^{\lambda_{n-1}} = \mathbb{C}\{\mathring{Q}\}e^{\lambda_{n-1}} \oplus \mathbb{C}\{\mathring{Q}\}e^{\lambda_n}. \tag{3.12}$$

We extend the action of the Heisenberg algebra $U_q(\hat{\mathfrak{h}})$ to the space W_i by letting its elements acting freely on the twisted group algebra $\mathbb{C}\{\mathring{P}'\}$. Define the operators $e^{\alpha}, \partial_\alpha$ and d on the space W_i by

$$e^{\alpha} \cdot f \otimes e^{\beta} = f \otimes e^{\alpha}e^{\beta},$$

$$\partial_\alpha \cdot f \otimes e^{\beta} = (\alpha|\beta) f \otimes e^{\beta},$$

$$d \cdot f \otimes e^{\beta} = \left(-\sum_{j=1}^{l} n_j - \frac{(\beta|\beta)}{2} + \frac{(\lambda_i|\lambda_i)}{2}\right) f \otimes e^{\beta}, \tag{3.13}$$

where $f \otimes e^{\beta} = a_{j_1}(-n_1)\cdots a_{j_l}(-n_l) \otimes e^{\beta} \in W_i$.

Proposition 3.2. [FJ] *The space W_i becomes the irreducible representation $V(\Lambda_i)$ $(i = 0, 1, n-1, n)$ of the quantum affine Lie algebra $U_q(D_n^{(1)})$ under the action*:

$$\gamma \mapsto q, \quad K_j \mapsto q^{\partial_{\alpha_j}}, \quad a_j(k) \mapsto a_j(k) \quad (1 \leqq j \leqq n),$$

$$x_j^{\pm}(z) \mapsto X_j^{\pm}(z) = \exp\left(\pm\sum_{k=1}^{\infty} \frac{a_j(-k)}{[k]} q^{\mp k/2} z^k\right) \exp\left(\mp\sum_{k=1}^{\infty} \frac{a_j(k)}{[k]} q^{\mp k/2} z^{-k}\right)$$
$$\times e^{\pm\alpha_j} z^{\pm\partial_j+1},$$

and the degree operator d acts by (3.13). *The highest weight vectors are respectively*:

$$|\Lambda_0\rangle = 1 \otimes 1, \quad |\Lambda_i\rangle = 1 \otimes e^{\lambda_i}, \; i = 1, n-1, n. \qquad \square$$

3.2. Vertex Operators for Level one Representations of Quantum Affine Lie Algebra $U_q(D_n^{(1)})$. We recall the notion of vertex operators and some of the properties [FR, DJO]. Let V be a finite dimensional representation of the derived quantum affine Lie algebra $U_q'(D_n^{(1)})$ with the associated affinization space V_z (recall Sect. 2). The *vertex operators* are $U_q(D_n^{(1)})$-intertwining operators between an irreducible module and another one tensored by the affinization V_z. A vertex operator belongs to type I if the affinization V_z lies in the right factor of the tensor product, and it is of type II if the affinization V_z lies in the left factor of the tensor product.

The existence of vertex operators is described in the following theorem.

Proposition 3.3. [FR, DJO] *Let $V(\lambda)$ and $V(\mu)$ be two irreducible representations of $U_q(D_n^{(1)})$. Then we have*

$$\mathrm{Hom}_{U_q(D_n^{(1)})}(V(\lambda), V(\mu)\hat{\otimes} V_z) \simeq \{v \in V | wt(v) = \lambda - \mu \bmod \delta \text{ and }$$
$$e_i^{\langle \mu, h_i \rangle + 1} v = 0 \text{ for } i = 0, \ldots, n\},$$

where the isomorphism is defined by sending an element $\Phi \in \mathrm{Hom}_{U_q}(V(\lambda), V(\mu))$ to an element $v \in V$ such that

$$\Phi|\lambda\rangle = |\mu\rangle \otimes v + (\textit{higher terms in the powers of } z),$$

and $\hat{\otimes}$ denotes a suitable completion of the tensor product. (*We will omit $\hat{}$ from now on.*)

Similar statements are also true for the vertex operators of type II.

We will consider only level one representations for the quantum affine Lie algebra $U_q(D_n^{(1)})$. There are only four irreducible level one modules for $U_q(D_n^{(1)})$:

$$V(\Lambda_0), \quad V(\Lambda_1), \quad V(\Lambda_{n-1}), \quad V(\Lambda_n).$$

The vertex operators can be equivalently formulated as intertwining operators between modules of derived quantum affine Lie algebra $U_q'(D_n^{(1)})$ of the form:

$$\tilde{\Phi}_\lambda^{\mu V} : V(\lambda) \to \hat{V}(\mu) \otimes V,$$

$$\tilde{\Phi}_\lambda^{V\mu} : V(\lambda) \to V \otimes \hat{V}(\mu),$$

where the space $\hat{V}(\mu) = \prod_\nu V(\mu)_\nu$ is a completion of $V(\mu)$. Equivalently, we consider the vertex operators:

$$\tilde{\Phi}_\lambda^{\mu V}(z) : V(\lambda) \to \hat{V}(\mu) \otimes V_z ,$$

$$\tilde{\Phi}_\lambda^{V\mu}(z) : V(\lambda) \to V_z \otimes \hat{V}(\mu) ,$$

viewed as $U_q'(D_n^{(1)})$-modules. Then the operators

$$\Phi_\lambda^{\mu V}(z) = \tilde{\Phi}_\lambda^{\mu V}(z) z^{\Delta_\mu - \Delta_\lambda} ,$$

$$\Phi_\lambda^{V\mu}(z) = \tilde{\Phi}_\lambda^{V\mu}(z) z^{\Delta_\mu - \Delta_\lambda} \tag{3.14}$$

satisfy the relation:

$$(d \otimes 1)\Phi_\lambda^{\mu V}(z) - \Phi_\lambda^{\mu V}(z) d = -\left(z\frac{d}{dz} + \Delta_\lambda - \Delta_\mu \right) \Phi_\lambda^{\mu V}(z) . \tag{3.15}$$

Here we set $\Delta_\lambda = (\lambda|\lambda + 2\rho)/2(k + \check{h})$, where $k = 1$ is the level and $\check{h} = 2n - 2$ is the dual Coxeter number for the Lie algebra $\mathfrak{g} = D_n$. We have explicitly

$$\Delta_{\Lambda_0} = 0, \qquad \Delta_{\Lambda_1} = \frac{1}{2} ,$$

$$\Delta_{\Lambda_{n-1}} = \frac{n}{8}, \qquad \Delta_{\Lambda_n} = \frac{n}{8} .$$

Let V be the natural representation of $U_q(D_n)$ with the basis (see Sect. 2):

$$\{v_1, \ldots, v_n, v_{\bar{1}}, \ldots, v_{\bar{n}}\} .$$

We define the components of vertex operator in the following manner. For $\tilde{\Phi}_\lambda^{\mu V} : V(\lambda) \to V(\mu) \otimes V_z$, we write

$$\tilde{\Phi}_\lambda^{\mu V}(z)|u\rangle = \sum_{j=1}^{n} \tilde{\Phi}_{\lambda j}^{\mu V}(z)|u\rangle \otimes v_j + \sum_{j=1}^{n} \tilde{\Phi}_{\lambda \bar{j}}^{\mu V}(z)|u\rangle \otimes v_{\bar{j}} , \tag{3.16}$$

for $|u\rangle \in V(\lambda)$. The components of type II vertex operators are defined similarly.

We also consider the intertwining operators of modules of the following form:

$$\tilde{\Phi}_{\lambda V}^{\mu}(z) : V(\lambda) \otimes V_z \to V(\mu) \otimes \mathbb{C}[z, z^{-1}]$$

by means of the vertex operators with respect to the dual space V_z^*:

$$\tilde{\Phi}_{\lambda V}^{\mu}(z)(|v\rangle \otimes v_i) = \tilde{\Phi}_{\lambda i}^{\mu V^*}(z)|v\rangle \tag{3.17}$$

for $|v\rangle \in V(\lambda)$ and $i = 1, \ldots, n, \bar{1}, \ldots, \bar{n}$.

Using Proposition 3.3, we know there exist vertex operators only between modules $V(\Lambda_i)$ and $V(\Lambda_{i+1})$ for $i = 0$ or $n - 1$. The normalization takes the following form:

$$\tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1}V}(z)|\Lambda_i\rangle = |\Lambda_{i+1}\rangle \otimes v_{\overline{i+1}} + \text{higher terms in } z\,,$$

$$\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V}(z)|\Lambda_{i+1}\rangle = |\Lambda_i\rangle \otimes v_{i+1} + \text{higher terms in } z\,,$$

$$\tilde{\Phi}_{\Lambda_i}^{\Lambda_{i+1}V^*}(z)|\Lambda_i\rangle = |\Lambda_{i+1}\rangle \otimes v_{i+1}^* + \text{higher terms in } z\,,$$

$$\tilde{\Phi}_{\Lambda_{i+1}}^{\Lambda_i V^*}(z)|\Lambda_{i+1}\rangle = |\Lambda_i\rangle \otimes v_{\overline{i+1}}^* + \text{higher terms in } z\,, \tag{3.18}$$

where $i = 0, n-1$. For type II vertex operators we take a similar normalization. For example,

$$\tilde{\Phi}_{\Lambda_i}^{V\Lambda_{i+1}}(z)|\Lambda_i\rangle = |\Lambda_{i+1}\rangle \otimes v_{\overline{i+1}} + \text{higher terms in } z\,. \tag{3.19}$$

Proposition 3.4. *The vertex operator $\tilde{\Phi}$ of type* I *is determined by its component $\tilde{\Phi}_{\bar{1}}(z)$. More explicitly, with respect to V_z, we have*:

$$\tilde{\Phi}_i(z) = [\tilde{\Phi}_{i+1}(z), f_i]_q \quad \textit{for } i = 1,\ldots,n-1\,,$$

$$\tilde{\Phi}_{\overline{i+1}}(z) = [\tilde{\Phi}_{\bar{i}}(z), f_i]_q \quad \textit{for } i = 1,\ldots,n-1\,,$$

$$\tilde{\Phi}_n(z) = [\tilde{\Phi}_{\overline{n-1}}(z), f_n]_q\,,$$

$$\tilde{\Phi}_{n-1}(z) = [\tilde{\Phi}_n(z), f_{n-1}]_q = [\tilde{\Phi}_{\bar{n}}(z), f_n]_q\,, \tag{3.20}$$

and with respect to V_z^*, we have:

$$\tilde{\Phi}_{i+1}^*(z) = [f_i, \tilde{\Phi}_i^*(z)]_{q^{-1}} \quad \textit{for } i = 1,\ldots,n-1\,,$$

$$\tilde{\Phi}_{\bar{i}}^*(z) = [f_i, \tilde{\Phi}_{\overline{i+1}}^*(z)]_{q^{-1}} \quad \textit{for } i = 1,\ldots,n-1\,,$$

$$\tilde{\Phi}_{\bar{n}}^*(z) = [f_n, \tilde{\Phi}_{n-1}^*(z)]_{q^{-1}}\,,$$

$$\tilde{\Phi}_{\overline{n-1}(z)}^* = [f_n, \tilde{\Phi}_n^*(z)]_{q^{-1}} = [f_{n-1}, \tilde{\Phi}_{\bar{n}}^*(z)]_{q^{-1}}\,. \tag{3.21}$$

Proof. The natural representation V of $U_q(D_n)$ is described by (2.16), which implies that for $1 \leqq i \leqq n-2$,

$$\begin{aligned}\Phi(z)(f_i u) &= \Phi_1(f_i u) \otimes v_1 + \cdots + \Phi_n(z)(f_i u) \otimes v_n \\ &\quad + \Phi_{\bar{n}}(f_i u) \otimes v_{\bar{n}} + \cdots + \Phi_{\bar{1}}(z)(f_i u) \otimes v_{\bar{1}} \\ &= (\Delta f_i)\Phi(z)u \\ &= \Phi_i(z)u \otimes v_{i+1} + \Phi_{\overline{i+1}} + \Phi_{\overline{i+1}}(z)u \otimes v_{\bar{i}} + f_i\Phi_1(z)u \otimes t_i^{-1}v_1 \\ &\quad + \cdots + f_i\Phi_{\bar{1}}(z) \otimes t_i^{-1}v_{\bar{1}}\,.\end{aligned}$$

Thus we deduce

$$[\Phi_j(z), f_i] = 0 \quad \text{if } j \neq i+1, \bar{i}\,,$$

$$\Phi_i(z) = [\Phi_{i+1}(z), f_i]_q\,,$$

$$\Phi_{\overline{i+1}}(z) = [\Phi_{\bar{i}}, f_i]_q\,.$$

Using the intertwining property

$$\Phi(x)(f_i u) = (\Delta f_i)\Phi(z)u$$

for $i = n-1, n$ we obtain the remaining relations of (3.19). The case of V_z^* can be proved similarly. □

For the type II vertex operators, we have the following similar result.

Proposition 3.5. *Let $\tilde{\Phi}(z)$ be a type* II *vertex operator with respect to $V_z : V(\lambda) \longrightarrow V_z \otimes V(\mu)$. Then $\tilde{\Phi}(z)$ is determined by the component $\tilde{\Phi}_1(z)$. More precisely, with respect to V_z, we have*:

$$\tilde{\Phi}_{i+1}(z) = [\tilde{\Phi}_i(z), e_i]_q \quad i = 1, \ldots, n-1 ,$$

$$\tilde{\Phi}_{\bar{i}}(z) = [\tilde{\Phi}_{\overline{i+1}}(z), e_i]_q \quad i = 1, \ldots, n-1 ,$$

$$\tilde{\Phi}_{\bar{n}}(z) = [\tilde{\Phi}_{n-1}(z), e_n]_q ,$$

$$\tilde{\Phi}_{\overline{n-1}}(z) = [\tilde{\Phi}_{\bar{n}}(z), e_{n-1}]_q = [\tilde{\Phi}_n(z), e_n]_q , \tag{3.22}$$

and with respect to V_z^*, we have:

$$\tilde{\Phi}_i^*(z) = q^2[e_i, \tilde{\Phi}_{i+1}^*(z)]_{q^{-1}} \quad \textit{for } i = 1, \ldots, n-1 ,$$

$$\tilde{\Phi}_{\overline{i+1}}^*(z) = q^2[e_i, \tilde{\Phi}_{\bar{i}}^*(z)]_{q^{-1}} \quad \textit{for } i = 1, \ldots, n-1 ,$$

$$\tilde{\Phi}_n^*(z) = q^2[e_n, \tilde{\Phi}_{\overline{n-1}}^*(z)]_{q^{-1}} ,$$

$$\tilde{\Phi}_{n-1}^*(z) = q^2[e_n, \tilde{\Phi}_{\bar{n}}^*(z)]_{q^{-1}} = q^2[e_{n-1}, \tilde{\Phi}_n^*(z)]_{q^{-1}} . \quad \square \tag{3.23}$$

3.3. Bosonization. To further determine vertex operators, we express them in terms of Heisenberg generators and group algebra of the weight lattice. To this end, we find the relations between Drinfeld generators and vertex operators.

Theorem 3.6. *Let $\tilde{\Phi}(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z$ be a vertex operator of type* I, *where $(\lambda, \mu) = (\Lambda_0, \Lambda_1), (\Lambda_1, \Lambda_0), (\Lambda_{n-1}, \Lambda_n), (\Lambda_n, \Lambda_{n-1})$. Then we have for each $j = 1, \ldots, n$,*

$$[\tilde{\Phi}_{\bar{1}}(z), X_j^+(w)] = 0 ,$$

$$t_j \tilde{\Phi}_{\bar{1}}(z) t_j^{-1} = q^{\delta_{j1}} \tilde{\Phi}_{\bar{1}}(z) ,$$

$$[a_j(k), \tilde{\Phi}_{\bar{1}}(z)] = \delta_{j1} \frac{[k]}{k} q^{\frac{4n-1}{2}k} z^k \tilde{\Phi}_{\bar{1}}(z) ,$$

$$[a_j(-k), \tilde{\Phi}_{\bar{1}}(z)] = \delta_{j1} \frac{[k]}{k} q^{-\frac{4n-3}{2}k} z^{-k} \tilde{\Phi}_{\bar{1}}(z) . \tag{3.24}$$

Proof. From the partial comultiplication formulas in Theorem 2.3, it follows that,

$$\begin{aligned}\tilde{\Phi}(z)X_j^+(k)u &= \sum \tilde{\Phi}_i(z)X_j^+(k)u \otimes v_i \\ &= \Delta(X_j^+(k))\sum \tilde{\Phi}_i(z)u \otimes v_i \\ &= (X_j^+(k) \otimes \gamma^k + \gamma^{2k}K_j \otimes X_j^+(k) \\ &\quad + \sum_l \gamma^{(k-l)/2}\psi_j(k-l) \otimes \gamma^{k-l}X_j^+(l) + \cdots)\sum \tilde{\Phi}_i(z)u \otimes v_i\,.\end{aligned}$$

Then from the action of $X_j^+(l)$ on the evaluation module V_z, we see that no terms containing the vector $v_{\bar{1}}$ will survive the action. Hence $\tilde{\Phi}_{\bar{1}}(z)$ commutes with $X_j^+(k)$.

As for the relations between $a_j(k)\,(k > 0)$ and $\tilde{\Phi}_{\bar{1}}(z)$, we consider

$$\begin{aligned}\tilde{\Phi}(z)a_j(k)u &= (a_j(k) \otimes \gamma^{k/2} + \gamma^{3k/2} \otimes a_j(k) + \cdots)\sum \tilde{\Phi}_i(z)u \otimes v_i \\ &= a_j(k)\tilde{\Phi}_{\bar{1}}(z)u \otimes v_{\bar{1}} + \delta_{j1}q^{3k/2}\Phi_{\bar{1}}(z)u \otimes \left(-\frac{[k]}{k}(q^{2n-3}z)^k\right)q^k v_{\bar{1}} + \cdots\,,\end{aligned}$$

which yields

$$[a_j(k), \tilde{\Phi}_{\bar{1}}(z)] = \delta_{j1}\frac{[k]}{k}q^{\frac{4n-1}{2}k}z^k\tilde{\Phi}_{\bar{1}}(z)\,.$$

The remaining relations can be proved similarly. □

The same argument will lead to the following result.

Theorem 3.7. (a) *If $\tilde{\Phi}(z)$ is a vertex operator of type* II *associated with the evaluation module V_z, then*

$$\begin{aligned}&[\tilde{\Phi}_1(z), X_j^-(w)] = 0\,,\\ &t_j\tilde{\Phi}_1(z)t_j^{-1} = q^{-\delta_{j1}}\tilde{\Phi}_1(z)\,,\\ &[a_j(k), \tilde{\Phi}_1(z)] = -\delta_{j1}\frac{[k]}{k}q^{k/2}z^k\tilde{\Phi}_1(z)\,,\\ &[a_j(-k), \tilde{\Phi}_1(z)] = -\delta_{j1}\frac{[k]}{k}q^{-3k/2}z^{-k}\tilde{\Phi}_1(z)\,.\end{aligned} \tag{3.25}$$

(b) *If $\tilde{\Phi}(z)$ is a vertex operator of type* I *associated with V_z^*, then*

$$\begin{aligned}&[\tilde{\Phi}_1(z), X_j^+(w)] = 0\,,\\ &t_j\tilde{\Phi}_1(z)t_j^{-1} = q^{\delta_{j1}}\tilde{\Phi}_1(z)\,,\\ &[a_j(k), \tilde{\Phi}_1(z)] = \delta_{j1}\frac{[k]}{k}q^{\frac{3}{2}k}z^k\tilde{\Phi}_1(z)\,,\\ &[a_j(-k), \tilde{\Phi}_1(z)] = \delta_{j1}\frac{[k]}{k}q^{-\frac{1}{2}k}z^{-k}\tilde{\Phi}_1(z)\,.\end{aligned} \tag{3.26}$$

(c) *If $\tilde{\Phi}(z)$ is a vertex operator of type* II *associated with V_z^*, then*

$$
\begin{aligned}
[\tilde{\Phi}_{\bar{1}}(z), X_j^-(w)] &= 0\,, \\
t_j \tilde{\Phi}_{\bar{1}}(z) t_j^{-1} &= q^{-\delta_{j1}} \tilde{\Phi}_{\bar{1}}(z)\,, \\
[a_j(k), \tilde{\Phi}_{\bar{1}}(z)] &= -\delta_{j1} \frac{[k]}{k} q^{\frac{-4n+5}{2}k} z^k \tilde{\Phi}_{\bar{1}}(z)\,, \\
[a_j(-k), \tilde{\Phi}_{\bar{1}}(z)] &= -\delta_{j1} \frac{[k]}{k} q^{\frac{4n-7}{2}k} z^{-k} \tilde{\Phi}_{\bar{1}}(z)\,. \quad \square
\end{aligned}
\tag{3.27}
$$

In order to construct an operator satisfying the commutation relations, we introduce some auxiliary Heisenberg operators.

Lemma 3.8. *Let*

$$
a_{\bar{1}}(k) = -\frac{k[(n-1)k]}{[k]^2[2(n-1)k]} \left(\sum_{i=1}^{n-2} \frac{[2(n-i-1)k]}{[(n-i-1)k]} a_i(k) + a_{n-1}(k) + a_n(k) \right)\,. \tag{3.28}
$$

Then on the Heisenberg algebra $U_q(\hat{\mathfrak{h}})$, we have

$$
[a_j(k), a_{\bar{1}}(l)] = \delta_{j1}\delta_{k,-l}\,. \tag{3.29}
$$

Proof. Write $v = q^k$, then $\frac{[2(n-i-1)k]}{[(n-i-1)k]} = v^{n-i-1} + v^{-n+i+1}$. It follows from the Dynkin diagram of the Lie algebra $D_n^{(1)}$ that

$$
\begin{aligned}
[a_1(k), a_{\bar{1}}(-k)] &= \frac{k[(n-1)k]}{[k]^2[2(n-1)k]} \left(\sum_{i=1}^{n-2} \frac{[2(n-i-1)k]}{[(n-i-1)k]} [a_1(k), a_1(-k)] + \cdots \right) \\
&= \frac{k[(n-1)k]}{[k]^2[2(n-1)k]} \left(\frac{[2(n-2)k][2k][k]}{[(n-2)k]k} - \frac{[2(n-3)k][k]^2}{[(n-3)k]k} \right) \\
&= \frac{1}{v^{n-1} + v^{-n+1}} ((v^{n-2} + v^{-n+2})(v + v^{-1}) - (v^{n-3} + v^{-n+3})) \\
&= 1\,.
\end{aligned}
$$

The rest of the relations are shown similarly. $\square$

Proposition 3.4 asserts that the vertex operators of type I are determined by their $\bar{1}$-components, and the type II ones are given by their 1-components. The following result thus completely determines the vertex operators.

Theorem 3.9. *The $\bar{1}$-components of the vertex operator $\tilde{\Phi}(z)_{\Lambda_{i\pm1}}^{\Lambda_i V}$ of type* I *with respect to $V_z : V(\Lambda_i) \longrightarrow V(\Lambda_{i\pm1}) \otimes V_z$ can be realized explicitly as follows*:

$$
\begin{aligned}
\tilde{\Phi}_{\bar{1}}(z) &= \exp\left(\sum \frac{[k]}{k} q^{\frac{4n-1}{2}k} a_{\bar{1}}(-k) z^k \right) \exp\left(\sum \frac{[k]}{k} q^{-\frac{4n-3}{2}k} a_{\bar{1}}(k) z^{-k} \right) \\
&\quad \times e^{\lambda_1} (q^{2n-1}z)^{\partial_{\lambda_1} + a} b\,,
\end{aligned}
\tag{3.30}
$$

where $a = 0, 1, 1/2, 1/2, b = 1, 1, 1, (-1)^{n-1}$ *for the case of* $V(\Lambda_0) \to V(\Lambda_1), V(\Lambda_1) \to V(\Lambda_0), V(\Lambda_{n-1}) \to V(\Lambda_n)$ *and* $V(\Lambda_n) \to V(\Lambda_{n-1})$. *For the vertex operators of type* I *associated with the dual evaluation module* V_z^*, *we have correspondingly*

$$\tilde{\Phi}_{\bar{1}}^*(z) = \exp\left(\sum \frac{[k]}{k} q^{3k/2} a_{\bar{1}}(-k) z^k\right) \exp\left(\sum \frac{[k]}{k} q^{-k/2} a_{\bar{1}}(k) z^{-k}\right) \times e^{\lambda_1} (qz)^{\partial_{\lambda_1} + a} b\,, \tag{3.31}$$

where a is the same as above and $b = 1,\ q^{2n-2},\ (-q)^{n-1},\ q^{n-1}$, *respectively.*

Proof. The idea of the proof is to verify that the given operator satisfies all the commutation relations of Theorem 3.6 and the normalization.

Consider the situation associated with V_z first. Since the proof of the four cases are similar, we look at the case of $V(\Lambda_0) \longrightarrow V(\Lambda_1) \otimes V_z$. The commutation relations with Heisenberg generator $a_j(n)$ are clearly satisfied due to the two exponential factors in $\tilde{\Phi}_{\bar{1}}(z)$ and Lemma 3.8, as in the usual situation of the theory of vertex operators for affine Lie algebras [FLM]. The factor e^{λ_1} guarantees the commutation relation with t_j for $j = 1, \ldots, n$. To see the commutation with $X_j^+(z)$, we need to use the notion of bosonic normal operators of vertex operators, which rearrange the monomials in Heisenberg generators $a_j(n)$ and $e^\alpha, \partial_\alpha$ so that the $a_j(n)$ $(n \in \mathbb{N})$ and ∂_α appear first. Thus we have

$$\tilde{\Phi}_{\bar{1}}(z) X_j^+(w) =: \tilde{\Phi}_{\bar{1}}(z) X_j^+(w) : \left(1 - q^{-(2n-1)} \frac{w}{z}\right)^{\delta_{1j}} (q^{2n-1} z)^{\delta_{1j}} \varepsilon(\lambda_1, \alpha_j)\,,$$

$$X_j^+(w) \tilde{\Phi}_{\bar{1}}(z) =: X_j^+(w) \tilde{\Phi}_{\bar{1}}(z) : \left(1 - q^{2n-1} \frac{z}{w}\right)^{\delta_{ji}} w^{\delta_{j1}} \varepsilon(\alpha_j, \lambda_1)\,,$$

where ε is the cocycle associated with the central extension (3.2). Moreover, we have

$$[\tilde{\Phi}_{\bar{1}}(z), X_j^+(w)] =: \tilde{\Phi}_{\bar{1}}(z) X_j^+(w) : \{(q^{2n-1}z - w)^{\delta_{j1}} \varepsilon(\lambda_1, \alpha_j) - (w - q^{2n-1}z)^{\delta_{j1}} \varepsilon(\alpha_j, \lambda_1)\} = 0\,,$$

since $\varepsilon(\lambda_1, \alpha_j)\varepsilon(\alpha_j, \lambda_1) = (-1)^{(\lambda_1|\alpha_j)}$.

Finally, we note that

$$\tilde{\Phi}_{\Lambda_0 \bar{1}}^{\Lambda_1 V}(0)|\Lambda_0\rangle = 1 \otimes e^{\lambda_1} = |\Lambda_1\rangle\,,$$

which gives the exact normalization required.

Since the verification of commutation relations for the other three cases are quite similar to the above, we only check the given operators satisfy the normalization. Again we look at the case of $\tilde{\Phi}_{\bar{1}} = \tilde{\Phi}_{\Lambda_1}^{\Lambda_0 V}$ to illustrate the idea. To this end, we observe that in this case,

$$\tilde{\Phi}_1 = [\cdots[\tilde{\Phi}_{\bar{1}}, f_1]_q, \ldots, f_{n-2}]_q, f_n]_q, f_{n-1}]_q, \ldots, f_1]_q\,.$$

Note also that

$$\begin{aligned}
&\tilde{\Phi}_{\bar{1}}(0)X_1^-(0)\cdots X_{n-2}^-(0)X_n^-(0)\cdots X_1^-(0)\cdot 1\otimes e^{\lambda_1}\\
&= \tilde{\Phi}_{\bar{1}}(0)X_1^-(0)\cdots X_{n-2}^-(0)X_n^-(0)\cdots X_2^-(0)\cdot 1\otimes e^{-\alpha_1}e^{\lambda_1}\\
&= \cdots\cdots\\
&= \tilde{\Phi}_{\bar{1}}(0)\cdot 1\otimes e^{-\alpha_1,-\cdots-\alpha_{n-2}}e^{-\alpha_n}(e^{-\lambda_1+\alpha_1+\cdots+\alpha_{n-1}})^{-1}\\
&= 1\otimes 1\,,
\end{aligned}$$

due to $(\lambda_1|\lambda_1)=1$ and the appearance of the factor $(q^{2n-1}z)^{\partial_{\lambda_1}+1}$ in $\tilde{\Phi}^{\Lambda_0 V}_{\Lambda_1\bar{1}}$. Now we notice that any nontrivial permutation of the product of

$$\tilde{\Phi}_{\bar{1}}(0)X_1^-(0)\cdots X_{n-2}^-(0)X_n^-(0)\cdots X_1^-(0)$$

will annihilate the vector $|\Lambda_1\rangle$. Thus we have

$$\tilde{\Phi}_1(0)|\Lambda_1\rangle = |\Lambda_0\rangle\,.$$

The normalization of $\tilde{\Phi}^{\Lambda_{n-1}V}_{\Lambda_n}$ is given by Proposition 3.1:

$$e^{\lambda_{n-1}-\lambda_n}e^{\lambda_n-\lambda_{n-1}} = (-1)^{n-1}\,. \qquad \square$$

By the same argument, we get the realization for the vertex operators of type II.

Theorem 3.10. *The* 1*-components of the vertex operators* $\tilde{\Phi}(z)^{V\Lambda_i}_{\Lambda_{i\pm1}}$ *of type* II *with respect to* $V_z : V(\Lambda_{i\pm1}) \longrightarrow V_z\otimes V(\Lambda_i)$ *can be realized explicitly as follows*:

$$\tilde{\Phi}_1(z) = \exp\left(-\sum\frac{[k]}{k}q^{k/2}a_{\bar{1}}(-k)z^k\right)\exp\left(-\sum\frac{[k]}{k}q^{-\frac{3}{2}k}a_{\bar{1}}(k)z^{-k}\right)$$
$$\times e^{-\lambda_1}(qz)^{-\partial_{\lambda_1}+a}b\,, \tag{3.32}$$

where $a=1,2,3/2,3/2$, $b=1,1,1,(-1)^{n-1}$ *for the case of* $V(\Lambda_1)\to V(\Lambda_0)$, $V(\Lambda_0)\to V(\Lambda_1)$, $V(\Lambda_{n-1})\to V(\Lambda_n)$ *and* $V(\Lambda_n)\to V(\Lambda_{n-1})$. *For the vertex operators of type* II *associated with the dual evaluation module* V_z^*, *we have correspondingly*

$$\tilde{\Phi}_{\bar{1}}(z) = \exp\left(-\sum\frac{[k]}{k}q^{-\frac{4n-5}{2}k}a_{\bar{1}}(-k)z^k\right)$$
$$\times\exp\left(-\sum\frac{[k]}{k}q^{\frac{4n-7}{2}k}a_{\bar{1}}(k)z^{-k}\right)e^{-\lambda_1}(q^{-2n+3}z)^{-\partial_{\lambda_1}+a}b\,, \tag{3.33}$$

where $a=1,3,3/2,3/2$ *and* $b=1,q^{-2n+2},(-q)^{-n+1},q^{-n+1}$ *for the four cases.*

4. Integral Representations for Correlation Functions

In this section, we derive an integral formula for the correlation functions of the vertex models associated to the vector representation of $U_q(D_n^{(1)})$ as an application of the bosonization of the vertex operators.

4.1. Vertex Models. We give the mathematical definition of the vertex models following [DFJMN] and [IIJMNT]. As explained in the introduction, we take

$$\mathrm{End}_{\mathbb{C}}\Big(\bigoplus_{\lambda\in\Omega} V(\lambda)\Big) \cong \bigoplus_{\lambda,\mu\in\Omega} V(\lambda)\widehat{\otimes}V(\mu)^*$$

as the space of states $\mathscr{F}$, where $\Omega = \{\Lambda_0, \Lambda_1, \Lambda_{n-1}, \Lambda_n\}$ and $\widehat{}$ means a suitable completion. In the following, we use λ and μ as an element of Ω. We give the left and right action of $U_q(D_n^{(1)})$ on $\mathscr{F}$ as follows:

$$x\cdot f = \sum x_{(1)}\circ f\circ S(x_{(2)}), \qquad f\cdot x = \sum S^{-1}(x_{(2)})\circ f\circ x_{(1)}\,,$$

where $f\in\mathscr{F}$, $x\in U$, $\Delta(x) = \sum x_{(1)}\otimes x_{(2)}$. The space $\mathscr{F}$ regarded as the right module is denoted by $\mathscr{F}^r$. Let

$$\mathscr{F}_{\lambda\mu} := \mathrm{Hom}(V(\mu), V(\lambda)) \cong V(\lambda)\otimes V(\mu)^*\,.$$

There is a natural inner product between $\mathscr{F}^r_{\lambda\mu}$ and $\mathscr{F}_{\mu\lambda}$ as follows:

$$\langle f|g\rangle = \frac{\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho}fg)}{\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho})} \quad \text{for } f\in\mathscr{F}^r_{\lambda\mu},\ g\in\mathscr{F}_{\mu\lambda}\,,$$

where $\rho = \sum_{i=0}^n \Lambda_i$. It is invariant under the action of $U_q(D_n^{(1)})$, i.e. $\langle fx|g\rangle = \langle f|xg\rangle$ for all $x\in U_q(D_n^{(1)})$. We use the vertex operator

$$\tilde{\Phi}_\lambda^{\mu V}(z) : V(\lambda)\to V(\mu)\otimes V_z$$

to incorporate the local structure into $\mathscr{F}$. We need the following proposition.

Proposition 4.1.

1) $$\tilde{\Phi}^\lambda_{\mu V}(z)\tilde{\Phi}_\lambda^{\mu V}(z) = \frac{(q^2\xi;\xi^2)_\infty(\xi^2;\xi^2)_\infty}{(q^2;\xi^2)_\infty(\xi;\xi^2)_\infty} id_{V(\lambda)}\,,$$

2) $$\tilde{\Phi}_\mu^{\lambda V}(z)\tilde{\Phi}^\mu_{\lambda V}(z) = \frac{(q^2\xi;\xi^2)_\infty(\xi^2;\xi^2)_\infty}{(q^2;\xi^2)_\infty(\xi;\xi^2)_\infty} id_{V(\lambda)\otimes V}\,,$$

for $(\lambda,\mu) = (\Lambda_0,\Lambda_1),(\Lambda_1,\Lambda_0),(\Lambda_{n-1},\Lambda_n),(\Lambda_n,\Lambda_{n-1})$, *where* $\xi = q^{2n-2}$,$(a;p)_\infty = \prod_{l=0}^\infty(1-ap^l)$.

Proof. Using our bosonization formulas, the proposition can be proved by a direct calculation. However, it is very cumbersome. In [DO, Appendix], the explicit forms of the 2-point functions of the vertex operators for level 1 modules for $U_q(D_n^{(1)})$ case are calculated by solving the quantum Knizhnik–Zamolodchikov equations in [FR]. The proof is done in the same way as in [DFJMN, IIJMNT]. □

Setting $z = 1$, we obtain the $U'_q(D_n^{(1)})$-module homomorphism

$$\tilde{\Phi}_\lambda^{\mu V} : V(\lambda) \to \widehat{V}(\mu) \otimes V .$$

Let

$$\tilde{\Phi}_\lambda^{(N)} := \tilde{\Phi}_{\lambda^{(N-1)}}^{\lambda^{(N)}V} \cdots \tilde{\Phi}_{\lambda'}^{\lambda'' V} \tilde{\Phi}_\lambda^{\lambda' V} ,$$

where the sequence $(\lambda^{(m)})$ is given by $(\lambda^{(m)}) = (\Lambda_0, \Lambda_1, \Lambda_0, \Lambda_1, \ldots)$ or $(\lambda^{(m)}) = (\Lambda_1, \Lambda_0, \Lambda_1, \Lambda_0, \ldots)$. Note that the sequence $(\lambda^{(m)}) = (\lambda, \lambda', \ldots)$ is determined by the first λ ([cf. [KMN]). Then $\tilde{\Phi}_\lambda^{(N)}$ converges and gives the following isomorphism by Proposition 4.1:

$$\mathscr{F}_{\lambda\mu} = V(\lambda) \otimes V(\mu)^* \cong V(\lambda^{(m)}) \otimes \underbrace{V \otimes \cdots \otimes V}_{\text{N−times}} \otimes V(\mu)^* .$$

Using this isomorphism, the space $\mathscr{F}$ is equipped with the local structure. Now, we define the local operators. For $L \in \operatorname{End} V^{\otimes N}$, let

$$\mathscr{L}_{(\lambda)} := (\tilde{\Phi}_\lambda^{(N)})^{-1} (id_{V(\lambda^{(N)})} \otimes L)(\tilde{\Phi}_\lambda^{(N)}) .$$

By Proposition 4.1, we know

$$(\tilde{\Phi}_\lambda^{(N)})^{-1} = \left(\frac{(q^2; \xi^2)_\infty (\xi; \xi^2)_\infty}{(q^2\xi; \xi^2)_\infty (\xi^2; \xi^2)_\infty} \right)^N \tilde{\Phi}_{\lambda' V}^{\lambda} \tilde{\Phi}_{\lambda'' V}^{\lambda'} \cdots \tilde{\Phi}_{\lambda^{(N)} V}^{\lambda^{(N-1)}} ,$$

where $\tilde{\Phi}_{\lambda V}^{\mu} = \tilde{\Phi}_{\lambda V}^{\mu}(1)$. The action of L on $\mathscr{F}_{\lambda\mu}$ is defined as follows:

$$L \cdot f := \mathscr{L}_{(\lambda)} \circ f .$$

By the above considerations, $\mathscr{F}$ is understood naively as the subspace of the infinite tensor product $V^{\otimes\infty}$. Using the dual vertex operators (essentially the same as the type I vertex operators, e.g. see [DFJMN])

$$\tilde{\Phi}_{V\lambda'}^{*\mu'} : V \otimes V(\lambda')^* \to V(\mu')^* ,$$

we define the shift operator $T : \mathscr{F} \to \mathscr{F}$ by

$$T : \mathscr{F}_{\lambda\lambda'} = V(\lambda) \otimes V(\lambda')^* \cong V(\mu) \otimes V \otimes V(\lambda')^* \cong V(\mu) \otimes V(\mu')^* = \mathscr{F}_{\mu\mu'} .$$

The Hamiltonian $\mathscr{H}$ is defined by

$$\mathscr{H} = \text{negative const.} \times (T^2 d T^{-2} - d) .$$

The space $\mathscr{F}_{\lambda\lambda}$ has the unique canonical element $id_{V(\lambda)}$. We call it the *vacuum* and denote it by $|\text{vac}\rangle_\lambda \in \mathscr{F}_{\lambda\lambda}$, ${}_\lambda\langle\text{vac}| \in \mathscr{F}_{\lambda\lambda}^r$. In fact, the vacuum vector is the eigenvector of $\mathscr{H}$ which has the lowest eigenvalue 0 (cf. [DFJMN]). We denote the correlation function ${}_{\lambda}\langle \mathrm{vac}|L|\mathrm{vac}\rangle_{\lambda}$ by $\langle L\rangle^{(\lambda)}$.

4.2. Integral Formulas. In [JMMN], an integral representation of correlation functions of the XXZ-model was given using the bosonization of the vertex operators for the level 1 modules over $U_q(\widehat{sl}_2)$. We can apply the same method to our case. Let

$$P^{m_1m_2\cdots m_N}_{m'_1m'_2\cdots m'_N}(z_1,\dots,z_N|x,y|\lambda) := \left(\frac{(q^2;\xi^2)_\infty(\xi;\xi^2)_\infty}{(q^2\xi;\xi^2)_\infty(\xi^2;\xi^2)_\infty}\right)^N$$

$$\times\frac{\mathrm{tr}_{V(\lambda)}(x^{-d}y^{2\overline{\rho}}\tilde{\Phi}^{\lambda}_{\lambda' V m'_1}(z_1)\cdots\tilde{\Phi}^{\lambda^{(N-1)}}_{\lambda^{(N)}Vm'_N}(z_N)\tilde{\Phi}^{\lambda^{(N)}V}_{\lambda^{(N-1)}m_N}(z_N)\cdots\tilde{\Phi}^{\lambda' V}_{\lambda m_1}(z_1))}{\mathrm{tr}_{V(\lambda)}(x^{-d}y^{2\overline{\rho}})},$$

where $\overline{\rho}$ is the classical analogue of ρ. Then we have

$$\langle L\rangle^{(\lambda)} = P^{m_1m_2\cdots m_N}_{m'_1m'_2\cdots m'_N}(z,\dots,z|\xi^2,q^{-1}|\lambda)$$

for $L = E_{m'_Nm_N}\otimes\cdots\otimes E_{m'_1m_1}$.

In the following, we only concentrate on one-point functions ($N=1$ case) $P^m_{m'}(z|x,y|\Lambda_i)$ for $i=0,1,n-1,n$. Let

$$h(z) = (z;x)_\infty(q^2z^{-1};x)_\infty\,,$$

$$k(z) = (z;x)_\infty(q^2z;x)_\infty(z^{-1}x;x)_\infty(q^2z^{-1}x;x)_\infty\,,$$

$$\Theta_i(z_1,\dots,z_n) = y^{(2\overline{\rho}|\lambda_i)}\sum_{\alpha\in\overline{Q}} x^{\frac{(\alpha|\alpha)}{2}+(\alpha|\lambda_i)}z_1^{(\lambda_1|\alpha)}\cdots z_n^{(\lambda_n|\alpha)}\,.$$

Then, using the same technique developed in [JMMN], we obtain the following:

$$P^m_{m'}(z|x,y|\Lambda_i) = \frac{g}{(2\pi\sqrt{-1})^{2n-2}}\oint\frac{d\eta_1\cdots d\eta_{n-1}}{\eta_1\cdots\eta_{n-1}}\frac{d\eta'_1\cdots d\eta'_{n-1}}{\eta'_1\cdots\eta'_{n-1}}$$

$$\times\frac{a_iK_{(m,m')}\Theta_i\prod_{l=1}^{n-2}k\left(\frac{\eta'_l}{\eta_l}\right)}{(x;x)^n_\infty\mathrm{tr}_{V(\Lambda_i)}(q^{-2\rho})\prod_{s=1}^4\prod_{l=0}^{n-2}h(w_l^{(s)})},$$

where

$$g = (q^2;\xi))_\infty(\xi;\xi)_\infty((q^2;\xi^2))_\infty(\xi^2;\xi^2)_\infty)^{2n-2}\,,$$

$$a_i = \frac{1}{q^{2n}z^2},\ \frac{1}{\eta_1\eta'_1},\ \frac{(-1)^{n-1}}{q^nz\eta_{n-1}},\ \frac{(-1)^{n-1}}{q^nz\eta'_{n-1}},\quad \text{for } i=0,1,n-1,n\,,$$

$$w_l^{(1)} = \frac{q\eta_{l+1}}{\eta_l},\quad w_l^{(2)} = \frac{q\eta'_{l+1}}{\eta_l},\quad w_l^{(3)} = \frac{q\eta_{l+1}}{\eta'_l},\quad w_l^{(4)} = \frac{q\eta'_{l+1}}{\eta'_l}\,,$$

for $l = 0, \ldots, n-2$, and $\eta_0 = qz$, $\eta'_0 = q^{2n-1}z$,

$$\Theta_l = \Theta_l\left(\frac{\eta_0\eta'_0\eta_2\eta'_2}{(\eta_1\eta'_1)^2}y^2, \ldots, \frac{\eta_{n-3}\eta'_{n-3}\eta_{n-1}\eta'_{n-1}}{(\eta_{n-2}\eta'_{n-2})^2}y^2, \frac{\eta_{n-2}\eta'_{n-2}}{(\eta_{n-1})^2}y^2, \frac{\eta_{n-2}\eta'_{n-2}}{(\eta'_{n-1})^2}y^2\right),$$

$$K_{(j,j)} = \frac{q^{2n-j}z\eta_{n-1}\eta'_{n-1}}{\eta'_{j-1}}(1 - q^2 w_{j-1}^{(1)^{-1}})\prod_{l=0}^{j-1}(1 - q^2 w_l^{(2)^{-1}})\prod_{l=0}^{j-2}(1 - w_l^{(3)})$$

$$\times \prod_{l=j}^{n-2}\frac{R(\eta'_{l-1}, \eta'_l, \eta'_{l+1}, \eta_l, \eta_{l+1})}{(1-q^2)(q^2\frac{\eta'_l}{\eta_l} - 1)} \quad \text{for } j = 1, \ldots, n-1,$$

$$K_{(\bar{j},\bar{j})} = \frac{q^j z\eta'_{j-1}\eta_{n-1}\eta'_{n-1}}{\eta_j\eta'_j}(1 - w_{j-1}^{(4)})\prod_{l=0}^{j-2}(1 - q^2 w_l^{(2)^{-1}})\prod_{l=0}^{j-1}(1 - w_l^{(3)})$$

$$\times \prod_{l=j}^{n-2}\frac{R(\eta_{l-1}, \eta_l, \eta_{l+1}, \eta'_l, \eta'_{l+1})}{(1-q^2)(1 - q^2\eta'_l/\eta_l)(\frac{\eta'_l}{\eta_l})} \quad \text{for } j = 1, \ldots, n-1,$$

$$R(\eta_{l-1}, \eta_l, \eta_{l+1}, \eta'_l, \eta'_{l+1}) = \{(\eta'_l - q\eta_{l-1})(\eta'_l - q\eta_{l+1})(\eta'_l - q\eta'_{l+1})(\eta_l - q\eta'_l)$$
$$-q(\eta_{l-1} - q\eta'_l)(\eta_{l+1} - q\eta'_l)(\eta'_{l+1} - q\eta'_l) \times (\eta'_l - q\eta_l)\}/(\eta_l\eta_{l+1}\eta'_l\eta'_{l+1}),$$

$$K_{(n,n)} = q^n z\eta'_{n-1}\prod_{l=0}^{n-2}(1 - q^2 w_l^{(2)^{-1}})\prod_{l=0}^{n-2}(1 - w_l^{(3)}),$$

$$K_{(\bar{n},\bar{n})} = q^n z\eta_{n-1}(1 - q^2 w_{n-2}^{(1)^{-1}})(1 - w_{n-2}^{(4)})\prod_{l=0}^{n-3}(1 - q^2 w_l^{(2)^{-1}})\prod_{l=0}^{n-3}(1 - w_l^{(3)}),$$

$$K_{(m,m')} = 0 \quad \text{otherwise}.$$

All the contours of the variables $\eta_1, \ldots, \eta_{n-1}, \eta'_1, \ldots, \eta'_{n-1}$ are counterclockwise and are in the following region:

(i) for $(m, m') = (j, j)$,

$$q^2 < w_0^{(1)} < 1, \ldots, q^2 < w_{j-2}^{(1)} < 1,\ q^2 < w_j^{(1)} < 1, \ldots, q^2 < w_{n-2}^{(1)} < 1,$$

$$q^2 < w_j^{(2)} < 1, \ldots, q^2 < w_{n-2}^{(2)} < 1,\ q^2 < w_{j-1}^{(3)} < 1, \ldots, q^2 < w_{n-2}^{(3)} < 1,$$

$$q^2 < w_0^{(4)} < 1, \ldots, q^2 < w_{n-2}^{(4)} < 1,$$

(ii) for $(m, m') = (\bar{j}, \bar{j})$,

$$q^2 < w_0^{(1)} < 1, \ldots, q^2 < w_{n-2}^{(1)} < 1,\ q^2 < w_{j-1}^{(2)} < 1, \ldots, q^2 < w_{n-2}^{(2)} < 1,$$

$$q^2 < w_j^{(3)} < 1, \ldots, q^2 < w_{n-2}^{(3)} < 1,$$

$$q^2 < w_0^{(4)} < 1, \ldots, q^2 < w_{j-2}^{(4)} < 1,\ q^2 < w_j^{(4)} < 1, \ldots, q^2 < w_{n-2}^{(4)} < 1.$$

Remark. When $N = 1$, the integral does not depend on the spectral parameter z. In fact, z disappears after rescaling the integral variables. It remains to calculate the explicit form of one-point functions as in [JMMN] and [Ko]. At present, it seems to be difficult. The trace $\operatorname{tr} q^{-2\rho}$ in the integral can be expressed by using the above theta function. If we can calculate the integral, the constant related to the Θ_i will be cancelled with the trace $\operatorname{tr} q^{-2\rho}$.

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