Commun. Math. Phys. 174, 229 – 260 (1995)

Communications in
Mathematical
Physics

A Special Class of Stationary Flows for Two-Dimensional Euler Equations: A Statistical Mechanics Description. Part II

E. Caglioti[1], P.L. Lions[2], C. Marchioro[1], M. Pulvirenti[1]

[1] Dipartimento di Matematica, Università di Roma "La Sapienza," Roma, Italy
[2] Ceremade, Université Paris-Dauphine, Paris, France

Received: 14 June 1994 / in revised form: 17 November 1994

Abstract: We continue and conclude our analysis started in Part I (see [CLMP]) by discussing the microcanonical Gibbs measure associated to a N-vortex system in a bounded domain. We investigate the Mean-Field limit for such a system and study the corresponding Microcanonnical Variational Principle for the Mean-Field equation. We discuss and achieve the equivalence of the ensembles for domains in which we have the concentration at $\beta \to (-8\pi)^+$ in the canonical framework. In this case we have the uniqueness of the solutions of the Mean-Field equation. For the other kind of domains, for large values of the energy, there is no equivalence, the entropy is not a concave function of the energy, and the Mean-field equation has more than one solution. In both situations, we have concentration when the energy diverges. The Microcanonical Mean Field Limit for the N-vortex system is proven in the case of equivalence of ensembles.

Table of Contents

Research partially supported by MURST (Ministero Ricerca Scientifica e Tecnologica) and CNR-GNFM (Consiglio Nazionale delle Ricerche-Gruppo Nazionale Fisica Matematica).

1. Introduction

A systematic rigorous study of the Statistical Mechanics of point vortices as a possible approach for the understanding of the 2-D turbulence has been approached only recently, more than forty years after the first proposal due to Onsager [O]. Namely the authors of the present paper [CLMP], and M. Kiessling [Ki], studied the Statistical Mechanics of the point vortex system in the Mean Field Limit, making rigorous previous results obtained in a purely physical setup [Mo, MJ], and outlining new interesting features like possible concentration phenomena at $\beta = -8\pi$. Later on G. Eyink and H. Spohn [ES] started the study of the microcanonical ensemble, developing the theory for bounded interaction potentials (see also J. Messer and H. Spohn [MS] for a previous analysis of the Mean Field Limit of the canonical ensemble for bounded interactions).

In the present paper, which is the sequel of our previos study [CLMP], we want to go further by studying the microcanonical ensemble for the point vortex system and the stationary solutions of the incompressible Euler equation (also solutions of the Mean Field Equation), arising in the Mean Field Limit. They are solutions of a variational problem (the Maximum Entropy Principle) whose physical relevance has been recently outlined by numerical experiments due to Montgomery et al. [MMS]. Indeed in simulating the 2-D Navier–Stokes flow in a torus, solutions of the Mean Field Equation (MFE in the sequel) are closely approached in a suitable time scale. A heuristic explanation of this feature can be done by observing that the entropy production and energy dissipation rates are

$$\nu \int \frac{|\nabla \omega|^2}{\omega} dx , \tag{1.1}$$

$$\nu \int |\omega|^2 dx , \tag{1.2}$$

respectively. Here we are concerned with a positive vorticity field ω in all $\mathbb{R}^2$, for which the entropy and the energy are defined by $-\int \omega \log \omega$, and $-\frac{1}{2}\int \omega \Delta^{-1} \omega$ respectively. Here $\nu > 0$ is the viscosity coefficient. Notice now that the term expressed by (1.1) is much larger than that given by (1.2) for "most" of the ω's, so that the solution of the Navier–Stokes problem tries to make the entropy as large as possible for a practically fixed energy on a suitable time interval before the flow disappears because of the viscous dissipation.

A rigorous and detailed analysis of the dynamical problem seems, at the moment, too difficult, however a statistical analysis in terms of the vortex theory can indeed be performed and this is the main objective of this paper.

We do not wish to give further physical motivations for the present analysis and address the reader to Refs. [CLMP, Ki] and especially [ES, MP] for a more detailed presentation of the matter and additional references.

We conclude this section by outlining the contents of the paper. In the next section we establish and study the Microcanonical Variational Principle. In Sect. 3, we discuss the Thermodynamics of the solutions of the MFE and derive preliminary properties. In Sect. 4, we prove the validity of the MFE in the Mean Field Limit of a point vortex system in the Microcanonical Ensemble in the case of equivalence of ensembles. Section 5 consists of a short remark concerning the problem in the whole plane. In Sect. 6, we discuss the behavior of the solutions of the MFE for domains in which there is no concentration at inverse temperature -8π. In Sect. 7, we prove a non-uniqueness result for the solutions of the MFE. Finally in Sect. 8, we give an explicit example of concentration.

2. Microcanonical and Canonical Variational Principle

In this section we introduce the Microcanonical Variation Principle (MVP later on) and prove that for any value of the energy $E > 0$ there exists a solution. Then we show that any solution of the MVP is a solution of the MFE.

In all that follows $\Lambda \subset \mathbb{R}^2$ will be a bounded open connected set with smooth boundary. Moreover, we shall assume $|\Lambda| = \operatorname{meas} \Lambda = 1$, the general case being easily recovered by scaling.

The entropy and energy functionals are defined respectively as

$$S(\rho) = -\int_\Lambda \rho \log \rho \, dx, \quad E(\rho) = \frac{1}{2}(\rho, V_\rho), \tag{2.1}$$

where ρ is a probability density, i.e. $\rho \geqq 0$, $\int_\Lambda dx\, \rho = 1$, and

$$V_\rho(x) = \int_\Lambda V(x,y)\rho(y)dy, \text{ where } V(x,y) = -\frac{1}{2\pi}\log|x-y| + \gamma(x,y)$$

is the Green function of the Poisson problem with Dirichlet boundary condition on Λ, and $(\,,\,)$ denotes the scalar product in $L_2(\Lambda)$.

Let us consider the MVP

$$S(E) = \sup_{\rho \in P_E} S(\rho), \tag{2.2a}$$

where

$$P_E = \left\{\rho | \rho \geqq 0 \text{ a.e. in } \Lambda, \int_\Lambda \rho \, dx = 1, \; E(\rho) = E\right\}. \tag{2.2b}$$

In the next proposition we prove that the problem (2.2) has a solution for any value of $E > 0$.

Proposition 2.1. *For any $E > 0$, $S(E) < +\infty$ and there exists $\rho \in P_E$, such that $S(\rho) = S(E)$.*

Proof. $x \log x > -C$ implies $S(E) < +\infty$. Moreover, let $\rho_n \in P_E$ be a maximizing sequence for $S(E)$, with $\frac{1}{2}(\rho_n, V\rho_n) = E$ and ρ a weak limit (in the sense of the weak convergence of measure). Then we have $\rho \in L_1$ because of the bound on $\int \rho_n \log \rho_n$. Moreover $S(\rho) \geqq S(E)$ by the upper semicontinuity of the entropy. This implies $S(\rho) = S(E)$ if it can be proved that:

$$\frac{1}{2}(\rho, V\rho) = E. \tag{2.3}$$

Let us consider

$$\int \rho_n(x)V(x,y)\rho_n(y) = I(\varepsilon) + I^c(\varepsilon), \tag{2.4}$$

where

$$I(\varepsilon) = \int_{|x-y|<\varepsilon} \rho_n(x)V(x,y)\rho_n(y), \quad I^c(\varepsilon) = \int_{|x-y|>\varepsilon} \rho_n(x)V(x,y)\rho_n(y). \tag{2.5}$$

Therefore, for ε small enough,

$$\begin{aligned} 0 \leqq I(\varepsilon) &\leqq -C \int_{|x-y|<\varepsilon} \rho_n(x)\rho_n(y)\log|x-y| \\ &\leqq -C \int_{|x-y|<\varepsilon} |x-y|^{-1}\log|x-y| \\ &\quad + C \int_{|x-y|<\varepsilon} \rho_n(x)\rho_n(y)\log(\rho_n(x)\rho_n(y)), \end{aligned} \tag{2.6}$$

where we have used the properties of the function $\gamma(x,y)$ in order to obtain the first inequality. The second inequality follows by splitting the integration domain into two parts: $\rho_n(x)\rho_n(y) < |x-y|^{-1}$ and its complement. So we have

$$I(\varepsilon) \leqq -C \int_{|x-y|<\varepsilon} |x-y|^{-1}\log|x-y| + 2|S(\rho_n)| \sup_x \int_{|x-y|<\varepsilon} \rho_n(y) \leqq w(\varepsilon), \tag{2.7}$$

where $w(\varepsilon) \to 0$ as $\varepsilon \to 0$. Indeed, by the L_1 bound on $\rho_n \log \rho_n$ (entropy) $\sup_x \int_{|x-y|<\varepsilon} \rho_n(y)$ goes to 0 as ε goes to 0, uniformly in n. This completes the proof since

$$I^C(\varepsilon) \underset{n\to\infty}{\longrightarrow} \int_{|x-y|>\varepsilon} \rho(x)V(x,y)\rho(y)) \underset{\varepsilon\to\infty}{\longrightarrow} (\rho, V\rho) \text{ and (2.3) is proven.} \qquad \square$$

Now let us give some properties of the entropy function.

It is useful to define $E_0 = \frac{1}{2}(1, V1) = \frac{1}{2}\int_{\Lambda^2} V(x,y)\,dx\,dy$. Notice that, in general, $S(E) \leqq 0$ by convexity, $|\Lambda| = 1$, $\int_\Lambda \rho = 1$, and $S(E_0) \geqq -\int_\Lambda 1\log 1 = 0$. Hence $S(E_0) = 0$ and $\rho = 1$ is indeed a maximizer of $S(\rho)$ for $E = E_0$.

Proposition 2.2. *$S(E)$ is a strictly increasing negative function for $E < E_0$, and a strictly decreasing negative function for $E > E_0$.*

Proof. Let us remark that, if $E \neq E_0$ a maximizer ρ of $S(E)$ cannot be $\rho = 1$, and therefore $S(E) < 0$. Consider now the one parameter family of densities defined by

$$\rho_\lambda = (1-\lambda)\rho + \lambda; \quad \lambda \in [0,1], \tag{2.8}$$

for a given ρ solution of the MVP at energy E. Since $S(\rho_\lambda)$ is a concave function of λ, we have

$$S(\rho_\lambda) \geqq \lambda S(\rho_0) + (1-\lambda)S(\rho_1) = \lambda S(E) > S(E), \tag{2.9}$$

while $E(\rho_\lambda)$ is a convex function. In particular $E(\rho_\lambda)$ is a continuous function with respect to λ and $E(\rho_0) = E$, $E(\rho_1) = E_0$.

Now let us consider the case when $E > E_0$. Because of the continuity of $E(\rho_\lambda)$, given $E' \in (E_0, E)$ there exists $\lambda' \in (0,1)$, such that $E(\rho_{\lambda'}) = E'$. Furthermore by (2.9), we have $S(\rho_{\lambda'}) > S(E)$, and this implies $S(E') > S(E)$.

If $E < E_0$ we proceed in the same way. $\square$

Proposition 2.3. *Let ρ be a solution of the MVP at energy E, i.e. $\frac{1}{2}(\rho, V_\rho) = E$ and $S(\rho) = S(E)$. Then there exist $\beta \in \mathbb{R}$, such that ρ solves*

$$\rho = \frac{e^{-\beta V_\rho}}{\int e^{-\beta V_\rho}} \tag{2.10}$$

or, if we denote by Ψ *the solution of the Poisson equation,*

$$-\Delta\Psi = \rho, \ \Psi = 0 \quad \text{on } \partial\Lambda\,, \tag{2.11}$$

then (2.10) *is equivalent to*

$$-\Delta\Psi = \frac{e^{-\beta\Psi}}{\int e^{-\beta\Psi}}, \ \Psi = 0 \quad \text{on } \partial\Lambda\,. \tag{2.12}$$

Equation (2.12) is called Mean Field Equation, (MFE in the following).

Remark. Notice that even though (2.12) is formally the Euler–Lagrange equation associated to the MVP, a rigorous derivation of this fact is not straightforward due to a possible singularity of the functional derivative of $S(\rho)$ that is $-\log\rho$. Actually the main point of the above theorem is to prove that a solution of the MVP cannot vanish, because as we shall show later on, this is enough to apply the usual Lagrange Multipliers method.

Proof. The idea of the proof is simple. If $\rho = 0$ in some non-zero measure subset of Λ, we can move some mass to this set so that the entropy greatly increases, (in particular the entropy increases more than linearly in the added mass) and in such a way that the energy does not decrease when $E > E_0$ or does not increase when $E < E_0$. Thus let us suppose (by absurdum) that $\Lambda = A \cup A^C$, where

$$A = \{x|\rho(x) > 0\}, \qquad A^C = \{x|\rho(x) = 0\}, \quad \text{meas}(A^C) > 0\,. \tag{2.13}$$

Then we shall prove that

$$\Psi = 2E \text{ a.e.} \quad \text{in } A. \tag{2.14}$$

Let us consider first the case $E > E_0$. Suppose that (2.14) is violated. Then we can show that the sets:

$$B_\varepsilon^+ = \{x \in A|\Psi(x) > 2E + \varepsilon\}, \qquad B_\varepsilon^- = \{x \in A|\Psi(x) < 2E - \varepsilon\} \tag{2.15}$$

have both a positive measure, for ε sufficientely small. Indeed we have only to prove that meas$(A - B_0^+) > 0$ and meas$(A - B_0^-) > 0$. For instance if meas$(A - B_0^-) = 0$, then $\Psi(x) > 2E$ a.e. in A and

$$2E = \int_\Lambda \rho\Psi = \int_A \rho\Psi > 2E\int_\Lambda \rho = 2E\int_\Lambda \rho = 2E \tag{2.16}$$

so the contradiction proves our claim.

Next, let us consider the two-parameters family of densities defined as

$$\rho\alpha_1, \alpha_2 = (1 - \alpha_1 - \alpha_2)\rho + \alpha_1\frac{1_{B^+}}{\text{meas}(B^+)} + \alpha_2\frac{1_{A^C}}{\text{meas}(A^C)}\,, \tag{2.17}$$

$$\text{for } \alpha_1 > 0, \ \alpha_2 > 0, \ \alpha_1 + \alpha_2 = \alpha < 1\,;$$

where 1_D is the characteristic function of the set D, and where we have replaced B_ε^+ by B^+ for simplicity. Let us now compute the energy and the entropy of ρ_{α_1,α_2}.

The energy is:

$$\begin{aligned} E(\rho_{\alpha_1,\alpha_2}) &= \frac{1}{2}(\rho_{\alpha_1,\alpha_2}, V\rho_{\alpha_1,\alpha_2}) \\ &= E + \alpha_1(\langle\Psi\rangle_{B^+} - 2E) + \alpha_2(\langle\Psi\rangle_A - 2E) + o(\alpha)\,, \\ &> E + \alpha_1\varepsilon - 2E\alpha_2\,, \end{aligned} \tag{2.18}$$

where $\langle\Psi\rangle_D = \frac{\int_D \Psi}{\text{meas}(D)}$. The last step follows from tha fact that $\Psi > 2E + \varepsilon$ in B^+, and that Ψ is non-negative in A. Now let us fix $\lambda \in (0,1)$ such that

$$\frac{\lambda}{1-\lambda} > \frac{2E}{\varepsilon}\,, \tag{2.19}$$

and set $\alpha_1 = \lambda\alpha$, $\alpha_2 = (1-\lambda)\alpha$. With this choice, if $\alpha > 0$ is sufficiently small, we find

$$E(\rho_{\alpha_1,\alpha_2}) > E\,. \tag{2.20}$$

On the other hand, the entropy of ρ_{α_1,α_2} is given by

$$\begin{aligned} S(\rho_{\alpha_1,\alpha_2}) &= -\int \left[(1-\alpha_1-\alpha_2)\rho + \alpha_1\frac{1_{B^+}}{\text{meas}(B^+)} + \alpha_2\frac{1_{A^C}}{\text{meas}(A^C)}\right] \\ &\quad\cdot \log\left[(1-\alpha_1-\alpha_2)\rho + \alpha_1\frac{1_{B^+}}{\text{meas}(B^+)} + \alpha_2\frac{1_{A^C}}{\text{meas}(A^C)}\right] \\ &= -\int_A \left[(1-\alpha_1-\alpha_2)\rho + \frac{\alpha_1 1_{B^+}}{\text{meas}(B^+)}\right] \cdot \log\left[(1-\alpha_1-\alpha_2)\rho + \frac{\alpha_1 1_{B^+}}{\text{meas}(B^+)}\right] \\ &\quad - \int_{A^C} \frac{\alpha_2}{\text{meas}(A^C)} \log\left[\frac{\alpha_2}{\text{meas}(A^C)}\right]. \end{aligned} \tag{2.21}$$

Therefore:

$$S(\rho_{\alpha_1,\alpha_2}) = S + c_1\alpha_1 + c_2\alpha_2 - \alpha_2\log\left[\frac{\alpha_2}{\text{meas}(A^C)}\right] + o(\alpha)\,, \tag{2.22}$$

for some constants c_1 and c_2. By (2.22),

$$S(\rho_{\alpha_1,\alpha_2}) = S + c(\lambda)\alpha - (1-\lambda)\alpha\log\alpha\,, \tag{2.23}$$

where $c(\lambda)$ is a constant depending only on λ. Therefore we deduce, for α sufficiently small:

$$S(\rho_{\alpha_1,\alpha_2}) > S\,. \tag{2.24}$$

Thus we proved (2.14) when $E > E_0$. If $E < E_0$ we proceed in the same way, replacing B^+ by B^-, with the only difference that we want to exihibit a density ρ' such that $E(\rho') < E$ and $S(\rho') < S(\rho)$.

To conclude the proof we must consider again separately the cases $E > E_0$ and $E < E_0$.

If $E > E_0$ we introduce

$$\rho_A = \frac{1_A}{\text{meas}(A)} \,. \tag{2.25}$$

If $E > E_0$ we consider:

$$\rho_\lambda = (1-\lambda)\rho + \lambda\rho_A, \quad \lambda \in [0,1] \,, \tag{2.26}$$

a one-parameter family of densities. The entropy $S(\rho_\lambda)$ has its maximum value for $\lambda = 1$. By concavity, when $\lambda > 0$, we have

$$S(\rho_\lambda) \geqq (1-\lambda)S + \lambda\, S(\rho_1) = S + \lambda(S(\rho_1) - S) > S \,, \tag{2.27}$$

(since $S < S(\rho_1)$). The energy $E(\rho_\lambda)$ is given by

$$E(\rho_\lambda) = \frac{1}{2}\left((1-\lambda)\rho + \lambda\frac{1_A}{\text{meas}(A)},\ V\left[(1-\lambda)\rho + \lambda\frac{1_A}{\text{meas}(A)}\right]\right) \,. \tag{2.28}$$

Since $\Psi = 2E$ a.e. in A, we have:

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \int_A \Psi\left(\frac{1_A}{\text{meas}(A)} - \rho\right) = \int_A \Psi\left(\frac{1_A}{\text{meas}(A)} - \rho\right) = 0 \,. \tag{2.29}$$

Therefore, because of the convexity of the energy functional

$$E(\rho_\lambda) = E + c\lambda^2 \,, \tag{2.30}$$

where $c \geqq 0$ and $c = 0$ if and only if $\rho = \frac{1_A}{\text{meas}(A)}$. Hence, for any $\lambda \in (0,1)$ we have $S(\rho_\lambda) > S$ and $E(\rho_\lambda) > E$ and this contradicts Proposition 2.2. Therefore we have $\Psi = 2E$ in A, and $\rho = \frac{1_A}{\text{meas}(A)}$. But this is impossible. In fact $\rho = \frac{1_A}{\text{meas}(A)}$ implies $\rho \in L_\infty(\Lambda)$, therefore, by elliptic regularity, $\Psi \in W^{2,p}(\Lambda)$ for any p. This implies that Ψ is two times differentiable a.e. in Λ, implying $\rho(x) = -\Delta\Psi(x) = 0$ a.e. in A. Since $\rho = 0$ in A^C by definition, this is impossible. Therefore we have proved that when $E > E_0$, $\rho > 0$ a.e. in Λ.

Now, let us consider the case $E < E_0$. Let us recall that $\Psi = 2E$ a.e. in A. Notice also that there must exist some non-zero measure set $B, A^C \supset B$, such that $\Psi < 2E - \varepsilon$ in B for some $\varepsilon > 0$. If it is not so we would have $\Psi > 2E$ a.e. in Λ and this is impossible because $\Psi \in H_0^1(\Lambda)$.

Let us consider the one parameter family of densities defined as

$$\rho_\lambda = (1-\lambda)\rho + \lambda\frac{1_B}{\text{meas}(B)} \,. \tag{2.31}$$

If we compute the first derivative of $E(\rho_\lambda)$ with respect to λ we find

$$\frac{\partial E}{\partial \lambda} = \int_\Lambda \Psi\left(\frac{1_B}{\text{meas}(B)} - \rho\right) = \langle\Psi\rangle_B - 2E < 2E - \varepsilon - 2E = -\varepsilon < 0 \,. \tag{2.32}$$

Therefore, if λ is small enough, $E(\rho_\lambda) < E$. For the entropy $S(\rho_\lambda)$ we find

$$\begin{aligned} S(\rho_\lambda) &= -\int_\Lambda \rho_\lambda \log\rho_\lambda = -\int_A (1-\lambda)\rho\ \log[(1-\lambda)\rho] - \lambda\int_B \frac{1}{\text{meas}(B)} \log\frac{\lambda}{\text{meas}(B)} \\ &= S - c\lambda - \lambda\log\lambda \,, \end{aligned} \tag{2.33}$$

where c is a constant that does not depend on λ. Therefore, when λ is sufficiently small, $S(\rho_\lambda) > S$, and this combined with the fact that $E(\rho_\lambda) < E$ leads to a contradiction.

We now complete the proof. Let ρ^* be a maximum for $S(E)$. Then taking variations $\rho^* \to \rho^*(1+\varphi) = \rho$ we find:

$$E(\rho) = E(\rho^*) + (\rho^*\varphi, V\rho^*) + \frac{1}{2}(\rho^*\varphi, V\rho^*\varphi)\,, \tag{2.34}$$

$$m(\rho) = m(\rho^*) + \int \rho^*\varphi \quad \left(\text{where } m(\rho) = \int \rho\right)\,. \tag{2.35}$$

The conditions $\rho^* = 0$ a.e. and $V\rho^* = \Psi^* = \text{const}$ a.e. in Λ are both impossible (indeed $\Psi \in H_0^1(\Lambda)$) so that the two functionals E and m are linearly independent (as follows by computing the derivatives of m and E in ρ^*). This allows us to compute the variations of $S(\rho)$ in ρ^* which are compatible with the energy and mass constraints, and find:

$$\rho^* \log \rho^* = \alpha\Psi^*\rho^* + \gamma\rho^*\,. \tag{2.36}$$

Again by $\rho^* > 0$ a.e. we conclude the proof. □

As a consequence of the MFE we can prove:

Proposition 2.4. *$S = S(E)$ is a continuous function.*

Proof. The monotonicity of $S(E)$ for $E > E_0$ implies that there exist left and right limits of $S(E')$ given respectively by

$$\lim_{E' \to E^-} S(E') = S^-(E), \qquad \lim_{E' \to E^+} S(E') = S^+(E)\,; \tag{2.37}$$

where $S^-(E) \geqq S(E) \geqq S^+(E)$. We shall prove that $S^+(E) = S(E)$, and next that $S^-(E) = S(E)$.

Let ρ be a solution of the MVP at energy, E, then, by Proposition 2.2, there exists f, with $\int f = 0$, such that

$$E(\rho + \varepsilon f) = E + \varepsilon E'_f + o(\varepsilon), \qquad S(\rho + \varepsilon f) = S + \varepsilon S'_f + o(\varepsilon)\,, \tag{2.38}$$

where E'_f, S'_f are the derivative of the energy and entropy functional w.r.t. ε at $\varepsilon = 0$, and where $S'_f = \alpha E'_f$. This means that for $E' > E$ (with $E' - E$ sufficiently small) we have

$$S(E') \geqq S(E) + \beta(E' - E) + o(E' - E)\,, \tag{2.39}$$

therefore by taking the limit of both members of (2.39) we find $S^+(E) \geqq S(E)$. Since we know that $S^+(E) < S(E)$ we have that $S^+(E) = S(E)$.

Now let $\rho_{E'}$, with $E' < E$, be solutions of the MVP at energy E'. Let us consider the limit $E' \to E^-$ of $\rho_{E'}$. By repeating here the argument used in Proposition 2.1, in order to prove the existence of a solution of the MVP, we find that $\rho_{E'} \to \rho^*$, up to the extraction of a subsequence, weakly in the sense of measure. Moreover ρ^* and ρ^* log ρ^* are in $L_1(\Lambda)$ and $E(\rho^*) = E$. Finally by the upper semicontinuity of the entropy we have

$$S(\rho^*) \geqq \limsup_{E' \to E^-} S(\rho_{E'}) = S^-(E)\,, \tag{2.40}$$

and since $E(\rho^*) = E$ then $S(E) \geqq S^-(E)$ which achieves the proof. □

3. Free Energy and Entropy Functionals

In this paragraph we recall some facts about the Canonical Variational Principle (CVP later on) introduced in [CLMP] and [Ki]. Moreover we establish a correspondence between the solutions of the MVP and CVP (whenever possible).

Consider the free energy functional

$$f_\beta(\rho) = -\frac{S(\rho)}{\beta} + E(\rho) = \frac{1}{\beta}\int_\Lambda \rho \log \rho + \frac{1}{2}(\rho, V_\rho) \tag{3.1}$$

and the associated CVP that is the study of

$$f(\beta) = f_\Lambda(\beta) = \begin{cases} \inf f_\beta(\rho); & \beta > 0 \\ \sup f_\beta(\rho); & \beta < 0 \end{cases}, \tag{3.2}$$

where the extrema are taken on

$$P = \left\{ \rho \geqq 0, \int_\Lambda \rho = 1, \int_\Lambda \rho \log \rho < +\infty \right\}. \tag{3.3}$$

In the sequel we shall use the notation f_Λ instead of f (see (3.2)), when the dependence on the domain Λ has to be stressed. It is useful to analyse also the functional

$$g_\beta(\Psi) = -\frac{1}{2}\int_\Lambda |\nabla \Psi|^2 - \frac{1}{\beta}\log \int_\Lambda e^{-\beta\Psi}, \tag{3.4}$$

with the associated variational principle

$$g(\beta) = g_{\bar\Lambda}(\beta) = \sup\{g_\beta(\Psi) | \Psi \in H_0^1(\Lambda)\}. \tag{3.5}$$

Notice that the MFE (2.10) is also the Euler equation associated to the CVP. Therefore we have two different variational principles, CVP and MVP, which correspond to the same Euler equation. As we shall see later on this fact is not sufficient for the equivalence of the two sets of solutions.

We now summarize some known results [CLMP] which will be useful in the sequel. Before to do this let us recall that a domain Λ is strictly star-shaped if (modulo translations of the origin) there exists $\alpha > 0$, such that

$$x \cdot \nu(x) \geqq \alpha > 0 \quad \text{when} \quad x \in \partial\Lambda,$$

where $\nu(x)$ is the outer normal to Λ in x.

Theorem 3.1.

i) *The variational principles* (3.2) *and* (3.5) *have solutions in the range of inverse temperatures* $\beta \in (-8\pi, +\infty)$, *satisfies the MFE.*

ii) $g(\beta) = f(\beta) < +\infty$, *for any* $\beta \in [-8\pi, +\infty)$.

iii) *If* Λ *is strictly star-shaped then there exists* $\beta_P < 0$ *such that, for* $\beta \leqq \beta_P$, *there are no smooth solutions of the MFE.*

iv) *If* $\Lambda = \Lambda_0$ *is the disk* (*with measure one*), *then there are no solutions of the MFE when* $\beta \leqq -8\pi$. *In particular when* $\beta = -8\pi$, *if* ρ_n *is a maximizing sequence* *for* $I_{\Lambda_0} \equiv f_{\Lambda_0}(-8\pi)$, *then*

$$\rho_n \longrightarrow \delta_{x_0}, \quad \textit{weakly in the sense of measures}, \tag{3.6}$$

x_0 *being the center of the disk.*

v) *For* $x \in \Lambda$, *let us define*

$$I_\Lambda(x) = \sup\left\{\limsup_n f_{-8\pi}(\rho_n)|\rho_n \geqq 0, \int \rho_n = 1, \rho \longrightarrow \delta_x \right.$$
$$\left. \text{weakly in the sense of the measures}\right\}, \tag{3.7}$$

where δ_x *is the delta measure supported at the point* x. *Then*:

$$I_\Lambda(x) = \frac{1}{2}\gamma(x,x) + I_{\Lambda_0}\,. \tag{3.8}$$

As consequence, if we define $I_\Lambda \equiv \sup_x I_\Lambda(x)$, *we have*

$$I_\Lambda = \frac{1}{2}\gamma(x_0,x_0) + I_{\Lambda_0}\,, \tag{3.9}$$

where x_0 *is a maximum point for* $\gamma(x,x)$ (*we can say, formally, that* I_Λ *is the free energy of a delta function at inverse temperature* β). *Clearly it is* $f_{-8\pi}(\Lambda) \geqq I_\Lambda$.

The following alternative holds:

a) *If* $f_\Lambda(-8\pi) > I_\Lambda$, *then for every maximizing sequence* ρ_n *for* $f_\Lambda(-8\pi)$

$$\rho_n \longrightarrow \rho \tag{3.10}$$

(*up to the extraction of a subsequence, and weakly in the sense of measure*) *and* ρ *is a smooth solution of the MFE.*

b) *If* $f_\Lambda(-8\pi) = I_\Lambda$, *then it is possible to find* ρ_n, *a maximizing sequence for* $f_\Lambda(-8\pi)$, *such that*

$$\rho_n \longrightarrow \delta_{x_0}, \quad \textit{where } x_0 \textit{ is a maximum point for } \gamma(x,x)\,. \tag{3.11}$$

Remark. Both possibilities in the alternative expressed in the above theorem can occur. For instance if Λ is a disk, the solutions of the MFE can be studied explicitly [CLMP] and the concentration ($\rho_n \longrightarrow \delta_{x_0}$) does occur at $\beta = -8\pi$. Furthermore, as we shall see in Sect. 8, the concentration also occurs in a simply connected domain sufficiently close to a disk. On the contrary there are regions (for instance rectangles when the ratio between the sides is large enough) for which the concentration does not occur [CLMP]. We are not able to characterize the behavior of the solutions at -8π fully.

In the sequel we shall call domains of the first kind those for which the concentration at -8π occurs, and domains of second kind the others.

Theorem 3.2.

i) *The solution of the MFE for* $\beta \geqq 0$ *is unique.*

Let Λ *be a simply connected domain, then*

ii) *The solution of the MFE for* $\beta \in (-8\pi, 0]$ *is unique.*

iii) *The set of solutions* $\{\Psi_\beta, \beta \in (-8\pi, 0]\}$ *is a regular branch* (*see* [*Su*]).

The uniqueness for positive β follows by D. Gogny and P.L. Lions [GL]. For negative β the situation is much more intricate. Point ii) and iii) follows by T. Suzuki

[Su]. In [Su] it is considered the equation

$$-\Delta u = \lambda e^u \quad \text{in } \Lambda\,,$$
$$u = 0 \qquad \text{on } \partial\Lambda\,. \tag{3.12}$$

In particular, if $\Sigma = \lambda\int_\Lambda e^u$, then by [Su] Lemma 1 and Lemma 2 it follows that there exists a unique solution of (3.12) for $\Sigma \in [0, 8\pi)$, and the set of these solutions form a regular branch. This implies immediately ii) and iii). In fact, if (β, Ψ) solves the MFE if we set

$$\lambda = \frac{-\beta}{\int_\Lambda e^{-\beta\Psi}}, \qquad u = -\beta\Psi\,. \tag{3.13}$$

Therefore, there exists a unique solution of MFE when $\beta \in (-8\pi, 0]$, and the set (β, Ψ), $\beta \in (-8\pi, 0]$ forms a regular branch.

Let $E = E(\beta)$ be the energy of the solution of the MFE at inverse temperature $\beta > -8\pi$. Then if Λ is a domain of the second kind, we define $E_c = E[(-8\pi)^+] < +\infty$, while if Λ is a domain of the first kind we define $E_c = +\infty$. For these domains we are now in position to prove the complete equivalence between the MVP and CVP, while for the others, as we shall in the sequel, we have the equivalence of the ensembles of solutions for the set of energies $E \in (0, E_c)$ (in both cases Λ is assumed simply connected).

Proposition 3.3. *Let Λ be a simply connected domain, then*

i) *$F(\beta) = -\beta f(\beta)$ is a strictly convex, decreasing function, defined for $\beta \geqq -8\pi$.*

ii) *F is differentiable for $\beta > -8\pi$ and $E(\beta) = -F(\beta) = \frac{1}{2}(\rho_\beta, V\rho_\beta)$, where ρ_β solves the MFE at inverse temperature β. In particular $E(\beta)$ is a continuous function.*

iii) $S(E) = \inf_\beta\{F(\beta) + \beta E\}\,,$ (3.14)

and hence S is a smooth concave function of E.

iv) *If ρ_E is solution of the MVP then $\rho_{E(\beta)} = \rho_\beta$. In particular ρ_E solves the MFE (equivalence of the ensembles for $E < E_c$) and the solution is unique.*

Proof.

i) The monotonicity of F was proven in [CLMP] (actually it is an easy consequence of the Moser and Jensen inequalities). The convexity follows by a direct inspection. For $\beta = \frac{\beta_1+\beta_2}{2}$,

$$F(\beta) = -\frac{1}{2}S(\rho) + \frac{\beta_1}{2}E(\rho) - \frac{1}{2}S(\rho) + \frac{\beta_2}{2}E(\rho) < \frac{1}{2}F(\beta_1) + \frac{1}{2}F(\beta_2)\,. \tag{3.15}$$

ii) We have:

$$F(\beta_2) \geqq S(\rho_1) - \beta_2 E(\rho_1) = F(\beta_1) - (\beta_2 - \beta_1)E(\rho_1)\,,$$
$$F(\beta_1) \geqq S(\rho_2) - \beta_1 E(\rho_2) = F(\beta_2) - (\beta_1 - \beta_2)E(\rho_2)\,, \tag{3.16}$$

where ρ_i maximizes F at β_i, $i = 1, 2$.

From (3.16):

$$-E(\rho_2) \leqq \frac{F(\beta_2) - F(\beta_1)}{\beta_2 - \beta_1} \leqq -E(\rho_1) \quad \text{if } \beta_1 > \beta_2 \tag{3.17}$$

and

$$-E(\rho_1) \leqq \frac{F(\beta_1) - F(\beta_2)}{\beta_1 - \beta_2} \leqq -E(\rho_2) \quad \text{if } \beta_1 > \beta_2 \,. \tag{3.18}$$

On the other hand, if $\beta_1 \to \beta_2$, $E(\rho_1) \to E(\rho_2)$. Indeed $\Psi_1 = V\rho_1$ is a maximizing sequence for $F(\beta_2)$ and converges in H_0^1 for subsequences. The uniqueness of the solution in β_2 gives the assertion.

The continuity of $E(\beta)$ is a consequence of the regularity of the branch of solutions $\{\Psi_\beta,\, \beta \in (-8\pi, 0]\}$, see Point ii) of Proposition 2.1.

iii) The function $E(\beta) : (-8\pi, +\infty) \to \mathbb{R}^+$ is a decreasing continuous function whose inverse will be denoted by $\beta = \beta(E)$. Therefore we have

$$\inf_\beta \{F(\beta) + \beta E\} \leqq F(\beta(E)) + \beta(E)E = S(\beta(E)) \leqq S(E)\,, \tag{3.19}$$

and if $S(E) = S(\rho)$ then:

$$S(E) = S(\rho) + \beta E - \beta E \leqq F(\beta) - \beta E; \quad \beta \in (-8\pi, +\infty)\,, \tag{3.20}$$

implying iii) by general arguments on the Legendre transform.

iv) By the previous step we have, for all $E \in \mathbb{R}^+$:

$$F(\beta(E)) + \beta(E)E = S(E)\,, \tag{3.21}$$

so that there is a one-to-one correspondence between the MVP and the CVP solutions due to the fact that $E \to \beta(E)$ is bijective in from $(0, E_c)$ to $(-8\pi, +\infty)$.

Remark. The equivalence of the ensembles when $E \in [0, E_c]$ holds for Λ bounded regular. In fact, in this case, i.e. when $\beta > 0$, the MFE admits a unique solution (see point i) of Theorem 3.2.

In Figs. 1, 2, 3 is plotted the qualitative behaviours of $F(\beta)$, $E(\beta)$, $S(E)$ respectively.

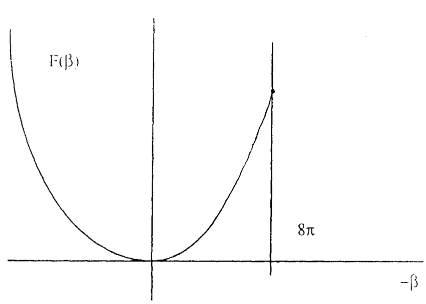

Fig. 1.

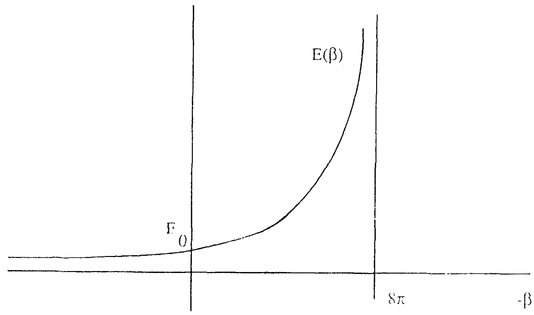

Fig. 2.

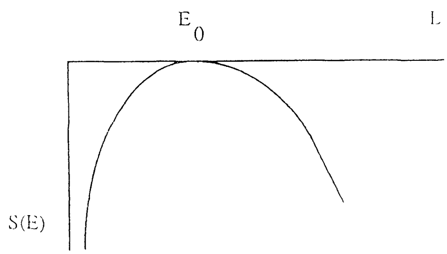

Fig. 3.

The above analysis allows us to perform the limit for the vortex system in the microanonical ensemble when Λ is a simply connected domain. Notice that, by the concavity of the entropy, we have a single value E_0 for which $S'(E_0) = 0$. E_0 is (in general $S'(E) = \beta$) the energy which corresponds to $\beta = 0$ (infinite temperature). Therefore if $E > E_0$, $\beta < 0$ and

$$S(E) = \sup\{S(\rho)|E(\rho) \geqq E\}\,, \tag{3.22}$$

while if $E < E_0$, $\beta > 0$ and

$$S(E) = \sup\{S(\rho)|E(\rho) \leqq E\}\,. \tag{3.23}$$

4. The Microcanonical Mean Field Limit

The arguments we present in this section work for those values of the energy E satisfying $0 < E < E_c$ that is whenever the equivalence of CVP and MVP is ensured. We remark that for domains of the first kind any value of the energy is allowed. Λ is always assumed to be simply connected.

In Λ we consider a vortex system given by the Hamiltonian

$$H_N = \frac{1}{N}\sum_{i<j} V(x_i,x_j) + \frac{1}{N}\sum_{i}\gamma(x_i,x_i), \quad x_i \in \Lambda,\ i = 1, 2, \ldots, N\,, \tag{4.1}$$

where $V(x,y) = -\frac{1}{2\pi}\log|x-y| + \gamma(x,y)$ is the Green function of the Poisson equation in Λ with Dirichlet boundary conditions.

Given $E > E_0$, $(E < E_c)$ let us consider on the phase space Λ^N, the microcanonical measure $\mu_{N,E}(dx_1, \ldots, dx_N)$ defined as

$$\mu_{N,E}(dX_N) = \frac{\chi_E(X_N)}{\Omega_+(E)} dX_N, \tag{4.2}$$

where

$$X_N = \{x_1, \ldots, x_N\},$$

$$\chi_E(X_N) = \chi(\{H_N > EN\}),$$

$$\Omega_N^+(E) = \int dX_N \chi_E(X_N). \tag{4.3}$$

Now we shall characterize the microcanonical measure in the mean field limit, i.e. in the limit $N \to \infty$.

Before doing it, it is useful to recall some known results [CLMP, Ki] about the mean field limit for the canonical measure.

The Gibbs measure at an inverse temperature β for the Hamiltonian system (4.1) is given by

$$m_\beta^N(dX_N) = \frac{e^{-\beta H_N(X_N)}}{Z_N(\beta)} dX_N, \tag{4.4}$$

where $Z_N(\beta) = \int_{\Lambda^N} e^{-\beta H_N(X_N)}$, is called partition function. The following theorem holds.

Theorem 4.1.

i) $0 < Z_N(\beta) < \infty$ *if and only if* $\beta \in (-8\pi, +\infty)$.

ii) *If* $\beta \in (-8\pi, +\infty)$ *then there exists* $\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta) = f(\beta)$; *where* $f(\beta)$ *is defined in* (3.1) *and* (3.2).

iii) *Let us define the n-particle correlation functions for* $n < N$, *as*

$$\rho_N^n(X_j) = \int_{\Lambda^{N-n}} dx_{n+1} \ldots dx_N \, m_\beta^N(X_N). \tag{4.5}$$

Let $\{d\rho^n\}_{n=1}^\infty$ *be a weak cluster point, in the sense of the weak convergence of measures, of the sequence* ρ_N^n, *i.e. there exists a subsequence* N_k *for which*

$$\int d\rho^n(X_n)\varphi(X_n) = \lim_{k\to\infty} \int dX_n \rho_{N_k}^n(X_n)\varphi(X_n) \tag{4.6}$$

for all bounded and continuous φ *and all* $n \geqq 1$. *Then, the measures* $d\rho^n$ *are absolutely continuous, i.e.*

$$d\rho^n(X_n) = \rho^n(X_n)dX_n, \tag{4.7}$$

and the following representation holds:

$$\rho^j(x_1, \ldots, x_N) = \int \nu(d\rho) \prod_{k=1}^{j} \rho(x_k), \tag{4.8}$$

where ν *is a Borel probability measure on* $L_1(\Lambda)$ *endowed with the weak topology. Furthermore,* ν *is supported on the solutions* $\rho \in L_\infty(\Lambda)$ *of the CVP* (2.11).

Now we can characterize the Mean Field limit for the microcanonical ensemble. We first prove:

Theorem 4.2. *Let $E > E_0$ (i.e. $\beta < 0$). Then*:

$$\lim_{N\to\infty} \frac{1}{N} \log \Omega_N^+(E) = S(E)\,. \tag{4.9}$$

Proof.

$$\frac{1}{N} \log \Omega_N^+(E) = \frac{1}{N} \log \frac{\int dx_N \chi_E(X_N) e^{-\beta H_N} e^{\beta H_N}}{Z_N(\beta)} + \frac{1}{N} \log Z_N(\beta)\,, \tag{4.10}$$

where $Z_N(\beta) = \int e^{-\beta H_N}$, and we choose β such that $-8\pi < \beta < \beta(E)$ and $\beta(E)$ is the inverse function of $E(\beta)$ given by ii) of Proposition 3.3. Then

$$\frac{1}{N} \log \Omega_N^+(E) = \frac{1}{N} \log Z_N(\beta) + \frac{1}{N} \log \frac{E_\beta^N(e^{\beta H_N} \chi_E)}{E_\beta^N(\chi_E)} + \frac{1}{N} \log E_\beta^N(\chi_E)\,, \tag{4.11}$$

(where E_β^N indicates the expectation with respect to the probability measure m_β^N),

$$\geqq \frac{1}{N} \log Z_N(\beta) + \frac{\beta E_\beta^N\left(\frac{H_N}{N}\chi_E\right)}{E_\beta^N(\chi_E)} + \frac{1}{N} \log E_\beta^N(\chi_E)\,, \tag{4.12}$$

(by Jensen inequality).

Notice that by the uniqueness of the MFE and Theorem 4.1 we know that $\frac{H_N}{N} \to E(\beta)$ a.e. with respect to the Gibbs measure m_β^N, and hence $E_\beta^N(\chi_E) \to 1$ if $E < E(\beta)$. Moreover by Theorem 4.1,

$$\frac{1}{N} \log Z_N(\beta) \to S(\beta) - \beta E(\beta)\,, \tag{4.13}$$

and therefore:

$$\liminf \frac{1}{N} \log \Omega_N^+(E) \geqq S(\beta)\,, \tag{4.14}$$

for any $\beta \in (-8\pi, \beta(E))$. This implies

$$\liminf \frac{1}{N} \log \Omega_N^+(E) \geqq \sup_{-8\pi \leqq \beta < \beta(E)} S(\beta) = S(E)\,. \tag{4.15}$$

In addition, by (4.11), since $\beta < 0$, we have:

$$\frac{1}{N} \log \Omega_N^+(E) \leqq \frac{1}{N} \log Z_N(\beta) + \beta E + \frac{1}{N} \log E_\beta^N(\chi_E) \to S(\beta) - \beta(E(\beta) - E)\,. \tag{4.16}$$

If we take the infimum over all values of β we obtain (4.9).

Remark. When $E < E_0$ (that is for positive temperatures) we can obtain the same result by defining

$$\Omega_N^-(E) = \int dx_N \chi_E(X_N), \qquad \chi_E(X_N) = \chi(\{H_N < EN\})\,. \tag{4.17}$$

Theorem 4.3. *Let us define* (*for $E < E_c$*), *the microcanonical correlation functions as* (*see* (4.2))

$$\phi_N^j(X_j) = \int \mu_E^N(X_N) dx_{j+1} \ldots . dx_N\,, \tag{4.18}$$

and let $\rho^E(dX_j)$ *be a weak accumulation point. Then*:

i) $\rho^E(dX_j) = \rho_E^j dX_j$ *that is the correlation functions are absolutely continuous with respect to the Lebesgue measure,*

ii) $\rho_E^j(X_j) = \prod_{k=1}^j \rho(x_k)$, *where* ρ *is the solutions of the CVP* (4.1) *at inverse temperature* $\beta = \beta(E)$.

Proof. The argument is the same as in [MS] (see also [CLMP, Ki and ES]). The subadditivity of the entropy implies

$$\frac{1}{j}\int \rho_E^j \log \rho_E^j \leqq \frac{1}{N}\int \mu_E^N \log \mu_E^N = -\frac{1}{N}\log \Omega(E,N)\,. \tag{4.19}$$

we know that the right-hand side converges. This implies that the left-hand side is uniformly bounded and we deduce i).

Moreover, for all j,

$$-\frac{1}{j}\int \rho_E^j \log \rho_E^j \geqq \lim_N -\frac{1}{N}\log \Omega(E,N) = S(E)\,, \tag{4.20}$$

which implies:

$$\int \nu^E(d\rho)\int \rho \log \rho < +\infty\,, \tag{4.21}$$

and therefore

$$-\int \nu^E(d\rho)\int \rho\log\rho \geqq S(E)\,. \tag{4.22}$$

Finally, by the fact that ν^E is supported on those ρ for which $E(\rho) \geqq E$, we obtain ii). □

It is obvious that the same analysis can be performed in the case when $H_N < NE$, that is when $\beta > 0$.

We would be really interested in the microcanonical measures defined by

$$\mu_E^N(X_N) = \frac{\delta(H-E)}{\Omega(E,N)}dX_N\,, \tag{4.23}$$

$$\Omega(E) = \int dx_N \delta(H-E)\,. \tag{4.24}$$

Clearly the definition we have used here allow us to avoid the problems due to the singularity of the δ-function. It is possible to extend the proof we gave above to the case

$$\mu_I^N(X_N) = \frac{\chi_I(X_N)}{\Omega(I;N)}dX_N\,, \tag{4.25}$$

$$\Omega(I;N) = \int dx_N \chi_I(X_N)\,; \tag{4.26}$$

where $I = (E_1, E_2)$, $0 < E_1 < E_2 < E_c$, is an energy interval, and $\chi_I(X_N) = \chi\left(\frac{H_N}{N} \in I\right)$. By the previous arguments we obtain

$$\lim_{N\to\infty}\frac{1}{N}\log\Omega(I;N) = \sup_{E\in I} S(E)\,. \tag{4.27}$$

Now using the continuity of the function $S(E)$ we find

$$\lim_{I\to\{E\}}\lim_{N\to\infty}\frac{1}{N}\log\Omega(I;N) = S(E)\,. \tag{4.28}$$

5. Microcanonical Ensemble in all R^2

When $\Lambda = R^2$ and the additional invariance of the momentum of inertia is explicitly taken into account, the MFE reads as:

$$-\Delta\Psi = -\frac{e^{-\beta\Psi+\lambda x^2}}{\int e^{-\beta\Psi+\lambda x^2}}\,, \tag{5.1}$$

being $\beta \in (-8\pi, +\infty)$, $\lambda < 0$, $\nabla\Psi \to 0$ as $|x| \to \infty$.

This is probably the most interesting situation from a physical point of view. In this case, we have unique radial solutions, concentration at $(-8\pi)^+$ and no radial solutions for $\beta < -8\pi$ (see [CLMP] and [CK]). So everything goes on as for domains of the first kind. Indeed all the considerations we have developed so far extends with some care to the present case. For instance the variational principle (3.4), (3.5) is not well posed due to the obvious divergence of $\|\nabla\Psi\|_{L_2}$. However the variational principle (3.1), (3.2) makes sense and can be solved (see [CLMP]).

6. The MVP in Absence of Concentration

The theory of the MVP and its connection with the vortex theory has been achieved in the previous sections for domains of the first kind and for domains of the second kind when $E > E_c$ (in both cases for Λ simply connected). For domains of the second kind when $E > E_c$, the situation is more involved. In particular the Microcanonical solutions at energies larger than E_c do not correspond to any solution of the CVP so that the equivalence of the ensembles does not holds. In this section we analyse this range of energies. We start by considering the behavior of the entropy function $S(E)$ and of the solutions when $E \to \infty$ and prove that, in this limit, the corresponding solutions of the MVP concentrate to a delta function. The non-equivalence of the ensembles and the concentration for the MVP will allow us the prove a non-uniqueness theorem for the MFE in a star-shaped domain of the second kind when $\beta \leqq -8\pi$ and this will be the argument of Sect. 7.

Proposition 6.1. *Let $E \geqq E_c$, then there exits C_1, C_2 such that*

$$-8\pi E + C_1 \leqq S(E) \leqq -8\pi E + C_2\,; \tag{6.1}$$

where $C_2 = S(E_c) + 8\pi E_c = f(-8\pi)$.

Proof. Let ρ be such that $E(\rho) = E$, $E > E_c$, and $S(E) > S(E_c) - 8\pi(E - E_c)$ (by contradiction). Then

$$f_{-8\pi}(\rho) = \frac{1}{8\pi}[S(E) + 8\pi E] \geqq \frac{1}{8\pi}[S(E_c) + 8\pi E_c] = 8\pi f(-8\pi)\,. \tag{6.2}$$

This is clearly false by the definition of CVP so that the inequality from above in (6.1) is proven.

The estimate from below is obtained by estimating from below the Entropy functional by using a suitable ρ. Let us consider a circle $B_{x_0,\varepsilon}$ of center x_0 a maximum point for $\gamma(x,x)$, and radius ε, ε such that the $B_{x_0,\varepsilon}$ is contained in Λ. Let $\rho_{B,\beta}$

be the solution of the MFE in the circle $B_{x_0,\varepsilon}$, at inverse temperature β. Then, let us construct a density function in all Λ by extending $\rho_{B,\beta}$ in all Λ, i.e.

$$\rho_{\beta,\Lambda} = \begin{cases} \rho_{B,\beta} & \text{in } B \\ 0 & \text{in } \Lambda - B \end{cases} . \tag{6.3}$$

The entropy of $\rho_{\beta,\Lambda}$ in Λ is the same as $\rho_{B,\beta}$ in B, that is

$$S_\Lambda(\rho_{\beta,\Lambda}) = S_B(\rho_{\beta,B}) , \tag{6.4}$$

while the energy is different because of the contribution due to the regular part of the γ function, i.e.:

$$E_\Lambda(\rho_{\beta,\Lambda}) = E_B(\rho_{\beta,B}) + \int_B \rho_{B,\beta}(x)\gamma_\Lambda(x,y)\rho_{B,\beta}(y) - \gamma_B . \tag{6.5}$$

Here the last term is the values of the γ function on the circle, that is constant for radial solutions. But in the circle, thanks to the equivalence of the ensemble, we have, for any $\beta > 0$,

$$S_B(E) = S_B(E_0) + \int_{E_0}^{E} dE\, \beta(E) \geqq S_B(E_0) - 8\pi(E - E_0) , \tag{6.6}$$

where we have used $\beta(E) > -8\pi$ if $E < \infty$. Hence,

$$E_\Lambda(\rho_{\beta,\Lambda}) > E_B(\rho_{\beta,B}) + \min_{x,y\in B} \gamma_\Lambda(x,y) - \gamma_B > E_B(\rho_{\beta,B}) + C , \tag{6.7}$$

and (6.1) is proven. □

The main result of this section is the following:

Theorem 6.1. *Let ρ_E be the solution of the MVP at energy E. Then, up to the extraction of a subsequence,*

$$\rho_E \underset{E\to\infty}{\longrightarrow} \delta_{x_0}(\textit{weakly in the sense of the measures}) , \tag{6.8}$$

where x_0 is a maximum point for $\gamma(x,x)$.

We shall prove Theorem 6.1 later on. For the moment, notice that as a corollary of the above two propositions we have the

Proposition 6.2. *In a domain of the second kind, i.e. in a domain in which $I_\Lambda < f_\Lambda(-8\pi)$, $S(E)$ is not a concave function.*

Proof. This corollary is an immediate consequence of the two previous propositions and of Theorem 3.1.
In fact, since $\rho_E \underset{E\to\infty}{\longrightarrow} \delta_{x_0}$, we have, by point v) of Theorem 3.1 that $f_{-8\pi}(\rho_E) = S(\rho_E) + 8\pi\rho_E$ satisfies

$$\limsup_{E\to\infty} f_{-8\pi}(\rho_E) = I_\Lambda , \tag{6.9}$$

and

$$f_{-8\pi}(\rho_{Ec}) = f_{-8\pi} , \tag{6.10}$$

while we have by assumption $f_{-8\pi} > I_\Lambda$. This means that $f_{-8\pi}(\rho_E)$ cannot be concave. The same is true for $S(E)$ that is obtained by adding a linear function of E to $f_{-8\pi}(\rho_E)$. □

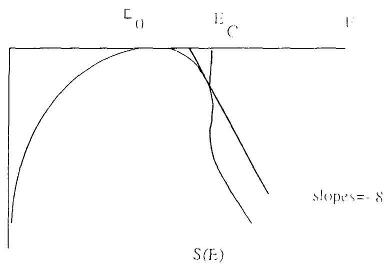

Fig. 4.

A possible qualitative behaviour of S is plotted in Fig. 4.

We now prove Theorem 6.1. Before giving a rigorous proof of this fact we want to give an idea of why there is a concentration phenomenon when the energy diverges. When the energy is large then the vorticity tends to concentrate. It is also clear that this concentration may be obtained in many ways. For example, the whole vorticity may concentrate in a unique or in many clusters. In the following, we want to show (heuristically) how the concentration of the vorticity in a single cluster is selected for a thermodynamical reason, (for analogous ideas for the gravitational case, see [Ki]$_2$).

Let us consider the two cases in which the vorticity clusters into:

i) A unique cluster
ii) Two identical separated clusters.

For the sake of simplicity we think that the clusters are circular and that the vorticity is constant in the circles. In the case of a unique cluster we have all the vorticity is in a circle of radius r_1. Thus the energy will be approximately (modulo an additive constant) given by:

$$E_1 \approx -\frac{1}{4\pi}\log(r_1)\,. \tag{6.11}$$

In the second case the energy will be given by the sum of the energy of the two clusters plus the interaction energy between the two clusters. The interaction term will be negligible in the limit $E \to \infty$, because this term is bounded by a constant; we have in fact assumed that the two clusters are well separated. Therefore in this case we find

$$E_2 \approx -\frac{1}{8\pi}\log(r_2)\,. \tag{6.12}$$

Hence if we want $E_1 = E_2 = E$ we find $r_1 \approx \exp(-4\pi E)$, $r_2 \approx \exp(-8\pi E)$, which implies for the entropies,

$$\begin{aligned} S_1 &\approx \log(r_1{}^2) = 2\log(r_1) \approx -8\pi E\,, \\ S_2 &\approx \log(r_2{}^2) = 2\log(r_2) \approx -16\pi E\,. \end{aligned} \tag{6.13}$$

So it is clear that when $E \to \infty$ the vorticity should not separate into two clusters. Therefore, we expect that when $E \to \infty$, the solution concentrates in a single point (becoming a Dirac mass at the point). We can finally note that this heuristic argument gives the right behaviour of $S(E)$ at large values of the energy, say $S \approx -8\pi E$.

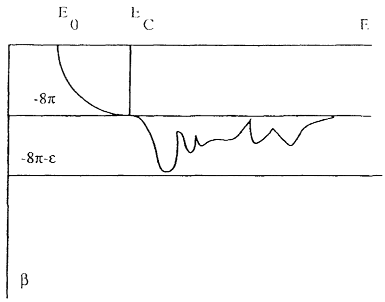

Fig. 5.

The rigorous proof follows this idea. We shall use the following lemma.

Lemma 6.1.
i) *Let Λ be a bounded domain, and $f \geqq 0$ such that $\int_\Lambda f = m \leqq 1$. If we define*

$$E(f) = -\frac{1}{4\pi} \int_{\Lambda^2} f(x)V(x,y)f(y), \qquad S(f) = -\int_\Lambda f \log f\,, \tag{6.14}$$

we have

$$S(f) < -\frac{8\pi}{m} E(f) + mC_2 - m \log m\,; \tag{6.15}$$

where C_2 is a constant defined in (6.1).
ii) *Given $f_1 \geqq 0$, $f_2 \geqq 0$, we set*:

$$E_{ij} = \frac{1}{2} \int_{\Lambda^2} f_i(x)V(x,y)f_{j(y)}, \quad (1 \leqq i,j \leqq 2)\,, \tag{6.16}$$

we have

$$E_{12} < \sqrt{E_{11}E_{22}}. \tag{6.17}$$

Proof. The proof of inequality (6.15) is consequence of Proposition 6.1 and an obvious scaling. The inequality (6.17) is nothing but the Schwarz' inequality.

Lemma 6.2. *Let A,B be two disjoint subsets of Λ, such that $d(A,B) > 0$. ($d(A,B) = \inf_{x\in A, y\in B}|x-y|$). Then, let ρ_E be a solution of the MVP at energy E, and let us define $m_D(\rho_E) = \int_D \rho_E$. Then we have*:

$$\lim_{E\to\infty} \min(m_A(\rho_E), m_B(\rho_E)) = 0\,. \tag{6.18}$$

Proof. Let us consider a solution ρ of the MVP in a domain Λ at energy E, and let us suppose that there exist two domains A,B in Λ, such that $\int_A \rho_E = \varepsilon_A > 0$, $\int_B \rho_E = \varepsilon_B > 0$, with $d(A,B) = d > 0$. Let us define $C = \Lambda - A \cup B, \rho_X = \chi_X \rho$; we obtain $\rho = \rho_A + \rho_B + \rho_C$.

With this definition the energy may be written as

$$E = \frac{1}{2} \int_{\Lambda^2} \rho(x)V(x,y)\rho(y) = E_{AA} + E_{BB} + E_{CC} + 2E_{AB} + 2E_{AC} + 2E_{BC}\,, \tag{6.19}$$

where

$$E_{XY} = \frac{1}{2} \int_{XY} \rho_X(x)\, V(x,y)\, \rho_Y(y)\,. \tag{6.20}$$

We can note (and this is the key point of the proof) that E_{AB} is bounded; i.e.

$$E_{AB} = \frac{1}{2}\int_{XY} \rho_A(x)V(x,y)\rho_B(y)$$
$$= \frac{-1}{4\pi}\int_{XY} \rho_A(x) \log|x-y|\rho_B(y) + \frac{1}{2}\int_{XY} \rho_A(x)\gamma(x,y)\rho_B(y) < c_2 . \quad (6.21)$$

This allows us to write:

$$E \leqq E_{AA} + E_{BB} + E_{CC} + 2E_{AB} + 2E_{AC} + 2E_{BC} + c_2 . \quad (6.22)$$

For the entropy we have simply

$$S = S_A + S_B + S_C; \qquad S_X = -\int_X \rho \log \rho . \quad (6.23)$$

Moreover by Lemma 6.2 we can write the following inequalities:

$$S_X \leqq \frac{-8\pi E_X + C_A}{m_X}; \quad X = A, B, C, \quad (6.24)$$

and

$$E_{XY} \leqq \sqrt{E_X E_Y}; \quad X, Y = A, B, C . \quad (6.25)$$

We shall check that (6.22–25) imply that

$$S(E) \leqq -(8\pi + c_3)E + c_4 ; \quad (6.26)$$

where $c_3 > 0; c_3, c_4$ do not depend on E, then, as we will see, the assertion follows rather easily.

Let us prove (6.26). Let a, b, c are such that $E_{AA} = a^2,\ E_{BB} = b^2,\ E_{CC} = c^2$. Then we may write

$$E \leqq a^2 + b^2 + c^2 + 2ac + 2bc + c_2 , \quad (6.27)$$

while for S, using (6.15) and (6.17), we may write

$$S \leqq \frac{-8\pi a^2 + C_A}{m_A} + \frac{-8\pi b^2 + C_A}{m_B} + \frac{-8\pi c^2 + C_A}{m_C}$$
$$= -8\pi\left(\frac{a^2}{m_A} + \frac{b^2}{m_B} + \frac{c^2}{m_C}\right) + c_3 . \quad (6.28)$$

Hence, the proof of (6.26) is reduced to an eigenvalue problem between two quadratic forms.

In fact let us define the matrices

$$S = \begin{pmatrix} \frac{1}{m_A} & 0 & 0 \\ 0 & \frac{1}{m_B} & 0 \\ 0 & 0 & \frac{1}{m_C} \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (6.29)$$

then (6.26) is implied by

$$\sup_{v \in R^3} \frac{(v, Ev)}{(v, Sv)} < \lambda < 1 . \quad (6.30)$$

This fact is easily proved by noting that S and E are positive matrices and that the eigenvalue equation $D(\lambda) = |E - \lambda S| = 0$, with some straightforward algebraic manipulations yields

$$-\lambda^3 + \lambda^2 - (m_A + m_B)\lambda - m_A m_B m_C = 0\,. \tag{6.31}$$

This equation has no solution for $\lambda \geqq 1$. We then conclude easily, since we know from Proposition 6.2 that $S(E) \geqq -8\pi E + C_2$, and therefore we find a contradiction with (6.19). □

Remark. In Lemma 6.2 we have proved that it is not possible to find two distinct regions in Λ, that are at a distance larger than 0, in both of which the mass is different from 0 definitively for $E \to \infty$. As an almost immediate consequence of Lemma 6.2 we can prove Theorem 6.1.

Proof of Theorem 6.1. Let ρ_E be a solution of the MVP at energy E, and given a domain D, let us define $m_E(D) = \int_{D\cap\Lambda} \rho_E$. Let $B_r(x)$ be the ball of center $x \in \Lambda$, and radius r, and let us consider the function $m_E(B_r(x))$. Let $x_E(r)$ be a maximum point for $m_E(B_r(x))$. By Lemma 6.2, we see that $m_E(B_{2r}(x_r(E)) \to 1$ when E goes to infinity; in fact $m_E(B_r(x)) > c_r$, where c_r is a constant that does not depend on E, implies that the mass out of a ball $B_{2r}(x_r(E))$ vanishes as $E \to \infty$. Furthermore, by the compactness of Λ, we may assume, extracting subsequences if necessary, that $x_E(r) \underset{E\to\infty}{\longrightarrow} x(r)$. It is easy to prove that $m_E(B_{3r}(x(r))) \underset{E\to\infty}{\longrightarrow} 1$; in fact $x_r(E) \to x_r$, implies $|x_r(E) - x_r| < r$ for r large, so we have that $B_{3r}(x_r) \supset B_{2r}(x_r(E))$ for r large.

Now, let us consider the sequence x_r, as r goes to 0. We have obviously $|x_r - x_{r'}| \leqq 6\min(r, r')$. Hence $x_r \underset{r\to 0}{\longrightarrow} x \in \Lambda$. It is clear by construction that for any $r > 0$ we have $m_E(B_r(x)) \underset{E\to\infty}{\longrightarrow} 1$, and thus $\rho_E \to \delta_x$ as $E \to \infty$.

Next there remains to prove that x is a maximum point of $\gamma(x,x)$. This last point may be shown by contradiction. Let x^* be a maximum point of $\gamma(x,x)$ and let ρ_E converge weakly to $\delta_{x'}$, where x' is not a maximum point of $\gamma(\cdots)$. Then, see point v) of Theorem 2.1, we have that

$$\limsup_{E\to\infty} f_{-8\pi}(\rho_E) \leqq I_\Lambda(x') < I_\Lambda(x^*)\,; \tag{6.32}$$

while we have shown in Theorem 3.1, (see point v)) that it is possible to find a sequence ρ'_E, such that $E(\rho'_E) \to \infty$ and $f_{-8\pi}(\rho'_E) \to I_\Lambda(x^*)$. So, for E sufficently large we find $f_{-8\pi}(\rho_E) < f_{-8\pi}(\rho'_E)$, and this leads to a contradiction with $f_{-8\pi} = E + 8\pi S$. □

7. Non-uniqueness of the Solutions of the MFE

Let Λ be a domain of the second king (that is a domain for which there is no concentration at $(-8\pi)^+$) and strictly star-shaped. Thus, (see Theorem 3.1) there exists β_P such that the MFE on Λ does not admit solutions if $\beta \leqq \beta_P$. For these domains we shall prove the following alternative.

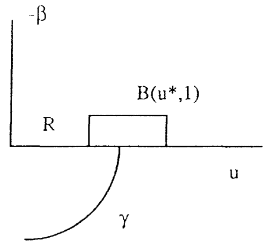

Fig. 6.

Theorem 7.1. *Either*

i) *the MFE has at least two solutions at* -8π, *or*

ii) *the MFE has a unique solution at* -8π, *and there exists a sequence of inverse temperature* $\{\beta_n\}_{n=1}^{\infty} : \beta_n < -8\pi,\ \beta_n \to -8\pi$ *as* $n \to \infty$, *for which the MFE has at least two solutions for each* β_n.

The strategy of the proof is the following. We know that we have a unique solution of the MFE up to -8π (not included). Let us consider the set of limit points of these solutions when $\beta \to (-8\pi)^+$. In the case of non-uniqueness of the limit point, we can easily show that there exists a continuum of solutions of the MFE at $\beta = -8\pi$. Otherwise, if we have a unique solution at -8π, say u^*, we can show, by using the Leray–Schauder toplogocial argument about the existence of continua of solutions, that it is possible to continue this solution beyond -8π. Namely we can prove that there exist an $\varepsilon > 0$ and a connected set of solutions up to $-8\pi-\ \varepsilon$. Then, we consider the solutions of the MVP as $E \to \infty$. By using a result by Nagasaki and Suzuki [NaSu] and the fact that the solutions of the MFE concentrate to a Dirac delta as $E \to \infty$, (see Theorem 6.1) we obtain that β_E (the inverse temperature of these solutions) accumulates at -8π. Hence we have solutions of MFE, close to u^*, for any $\beta \in (-8\pi-\varepsilon, -8\pi]$ and there are solutions of the MFE with β_E that accumulates at -8π (observe that $\beta_E < -8\pi$ because of the uniqueness up to -8π). So, for any of those β_E, we have at least two solutions of the mean field equation, see Fig. 6.

Proof. In order to prove Theorem 7.1 it is useful to recall some facts.

Fact 1. If $\beta > -8\pi$, there exists a unique solution u_β of the MFE.
This result is a consequence of [SU] and we have already described this result in Theorem 3.1.

Fact 2. Application of Leray–Schauder topological argument.

The MFE can be written in the form $u = T(-\beta, u)$, where

$$T(-\beta, u) = (-\Delta)^{-1} \left(\frac{e^{-\beta(u+u_0)}}{\int e^{-\beta(u+u_0)}} \right) - u_0 \,, \tag{7.1}$$

where $u_0 = (-\Delta)^{-1}\ 1$, is the solution of the MFE at 0 inverse temperature; in fact

$$T(-\beta, u) = u \quad \text{if and only if } \ -\Delta(u + u_0) = \frac{e^{-\beta(u+u_0)}}{\int e^{-\beta(u+u_0)}} \,. \tag{7.2}$$

Because of compactness of $T : \mathbb{R} \times \mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ may be chosen, for our purposes, as $C^1(\Lambda)$ equipped with the usual norm denoted by $\| \ \|$, and because of the existence of the solution $0 = T(0,0)$, we know, by Leray–Schauder topological argument [LS] that there exists a connected unbounded component C of solutions $(-\beta, u)$ in $[0, +\infty) \times \mathbb{B}$, containing (0, 0).

Now because of Fact 1, $C \cap ([0, 8\pi) \times \mathbb{B})$ is the curve (β, u_β) for $\beta \in (-8\pi, 0]$. Then, two cases are possible:

Case 1: As $\beta \to -8\pi$ $(\beta > -8\pi)$, u_β has, up to the extraction of subsequences, several limit points in $\mathbb{B}$. In this case, let $C_{-8\pi}$ be the set of these limits:

$$C_{-8\pi} = \{u | \exists \beta_n > -8\pi, \ \beta_n \to -8\pi \text{ as } n \to \infty, \ u_{\beta n} \to u\} . \tag{7.3}$$

Obviously, if $u \in C_{-8\pi}$, then u is a smooth solution of the MFE at -8π. This means that in Case 1 we have non-uniqueness at -8π. Furthermore, when $\beta > -8\pi$, u_β induces a continuous function $(-8\pi, 0] \to \mathbb{B}$. If u_β does not have a unique limit point it must have a continuum of limit points. Therefore, in Case 1 we have a continuum of solution at -8π.

Case 2: u_β converges to some u^* as $\beta \to (-8\pi)^+$. (7.4)

In this case, we shall prove, using the fact that C is connected and unbounded, that the following statement holds. Let us also mention that we make a systematic use in all that follows of the elementary observation: if (β_n, u_n) is a bounded sequence of solutions in $\mathbb{R} \times \mathbb{B}$ of solutions of MFE then u_n is relatively compact in $\mathbb{B}$ by elliptic regularity.

Statement 1. i) There exists $\varepsilon > 0$ and a family of solutions $u_\beta \in C$ for any $\beta \in (-8\pi - \varepsilon, -8\pi)$.

ii) There exists a family of solutions such that $\beta < -8\pi$, $u_\beta \to u^*$ as $\beta \to -8\pi$.

First of all we prove i). Let $\gamma = C \cap ([0, 8\pi] \times \mathbb{B})$, and let us suppose that i) is false, that is for all n there exists $\beta_n \in (-8\pi - \frac{1}{n}, -8\pi)$ such that there are not solution $u_{\beta n}$ of the MFE at inverse temperature β_n in C. Then

$$\Gamma_n^1 = C \cap ([\beta_P, \beta_n] \times \mathbb{B}) \text{ is a closed subset of } C , \tag{7.5}$$

and

$$\Gamma_n^2 = C \cap ([\beta_n, -8\pi] \times E) \text{ is a closed subset of } C . \tag{7.6}$$

Therefore, $\Gamma_n^1 \cap \Gamma_n^2 = \emptyset$, and $\Gamma_n^1 \cup \Gamma_n^2 = C$. So, since $\Gamma_n^2 \neq \emptyset$, and C is connected, we find $\Gamma_n^1 = \emptyset$, therefore there are no solutions of MFE in C_1, at inverse temperature $\beta \leqq \beta_n$.

Now, taking the limit $n \to \infty$, we find $C = \gamma$. This is a contradiction with the fact that C must be unbounded and i) is proven.

Now, given β, let u_β be a minimizer of Min $\{\|u - u^*\| : u | (\beta, u) \in C\}$.
We shall now prove that $u_\beta \to u^*$ as $\beta \to -8\pi$. If this were not true, then we could find $\beta_n \to -8\pi$, $\beta_n < -8\pi$, β_n increases, $\|u_{\beta n} - u^*\| > \alpha > 0$. This implies that $\|u_{\beta n}\| \to \infty$; (otherwise we could take a limit, up to subsequences, of $u_{\beta n} \to \hat{u}$ which would be a solution at -8π).

We still have to show, that $\|u_{\beta n}\|$ cannot go to $+\infty$. Let us consider in $\mathbb{R} \times \mathbb{B}$ the rectangle

$$R_{u^*} = [-\beta_n, -8\pi] \times B(u^*, 1) , \tag{7.7}$$

where $B(u^*, 1)$ is the closed unit ball in $\mathbb{B}$ centered at u^*, see Fig. 6.

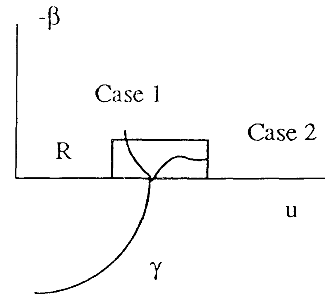

Fig. 7.

Then $\partial R_{u^*} \cap C \neq \emptyset$; because C is connected and unbounded. Hence, there exists $(\gamma_n, u_n) \in \partial R_{u^*} \cap C$. There are two possibilities:

a) $\gamma_n = \beta_n$, see Fig. 7. This is impossible, since for n large

$$1 = \|u_n - u^*\| < \|u_{\beta n} - u^*\| \to +\infty\,, \tag{7.8}$$

and this contradicts the definition of $u_{\beta n}$.

b) There exists (γ_n, u_n) in C, where $\gamma_n \in [-\beta_n, -8\pi]$, and $\|u_n - u^*\| = 1$, (see Fig. 7). This is also impossible. Indeed, we can consider the sequence u_n as $n \to \infty$, which is bounded because $\|u_n - u^*\| = 1$. Then, $u_n \to \hat{u}$ in $\mathbb{B}$ (up to subsequences), and $\hat{u}$ is a solution at -8π and $\|\hat{u} - u^*\| = 1$, and this is impossible in view of the uniqueness of the solutions at -8π. This completes the proof of Statement 1.

There remains to prove that the microcanoical solutions have inverse temperature β and accumulates to -8π when $E \to \infty$ (see Statement 2).

Let us recall the Nagasaki–Suzuki result [NaSu]:

Fact 3. Let us consider the following problem

$$\begin{aligned} -\Delta u &= \lambda e^u \quad \text{in } \Lambda, \\ u &= 0 \quad \text{on } \partial\Lambda; \end{aligned} \tag{7.9}$$

and let us denote by

$$\Sigma = \lambda \int_\Lambda e^u\,. \tag{7.10}$$

By [NaSu], see also Thm. 1 in [Su], we have that if (u_n, λ_n) is a sequence of solutions of (7.9) such that $\lambda_n \to 0$ as $n \to \infty$, then, extracting subsequences if necessary, Σ_n converges to $8\pi m; m \in 0 \cup \mathbb{N} \cup \{\infty\}$; where

$$m = 0 \quad \text{if and only if } \|u_n\|_{L_\infty} \to 0 \text{ as } n \to \infty,$$

$$0 < m < +\infty \quad \text{if and only } u|_S \to \infty,$$

and

$$\|u_n\|^{\text{loc}}_{L\,\infty(\Lambda - S)} \in O(1) \text{ for some finite set } S \text{ of}$$

m-points, (m-points blow up),

$m = +\infty$ if and only if $u(x) \to \infty$ for any x in Λ.

Now, because of the concentration phenomenon, we are able to prove the following:

Statement 2. Let $\rho_E, \Psi_E = (-\Delta)^{-1}\rho_E$ be the solutions of th MVP at energy E, and let β_E be the corresponding inverse temperatures. Then, as $E \to \infty, \beta_E$ converges to -8π.

Proof. As we have seen in Sect. 3 if (β, Ψ) is a solution of MFE then

$$\lambda = \frac{-\beta}{\int\limits_{\Lambda} e^{-\beta\Psi}}, \qquad u = -\beta\Psi, \tag{7.11}$$

solves (7.9), and $\Sigma = -\beta$. Furthermore, let us note that we know that $\rho_E \to \delta_{x_0}$ as $E \to \infty$ and;

$$\Psi_E \underset{\text{a.e.}}{\to} V(x,x_0) = (-\Delta)^{-1}\delta_{x_0} \quad \text{as } E \to \infty\,. \tag{7.12}$$

Next, given a solution of the MVP (β_E, Ψ_E), we set $\lambda_E = \frac{-\beta_E}{\int_\Lambda e^{-\beta_E\Psi_E}}$, $u_E = -\beta_E\Psi_E$, $\Sigma_E = -\beta_E$. We shall now prove that λ_E goes to 0 when $E \to \infty$. Because β_E is bounded (indeed $\beta_P < \beta_E < -8\pi$) this is equivalent to prove that $\int_\Lambda e^{-\beta_E\Psi_E} \to \infty$. If it is not the case there exists a constant c such that

$$\int\limits_{\Lambda} e^{-\beta_E\Psi_E} \leqq c\,. \tag{7.13}$$

By Fatou's Lemma,

$$c \geqq \liminf_{E\to\infty} \int\limits_{\Lambda} e^{-\beta_E\Psi_E} \geqq \liminf_{E\to\infty} \int\limits_{\Lambda} e^{8\pi\Psi_E} \geqq \int\limits_{\Lambda} e^{8\pi V(x,x_0)} = +\infty\,, \tag{7.14}$$

Therefore, as $E \to \infty$, $\lambda_E \to 0$ and by Fact 3, $\beta_E = \Sigma_E$ accumulates at $-8\pi m$. Finally, $m = 1$ because $\beta_E \in [\beta_P, -8\pi)$ and we know that we have a one-point blow up. This completes the proof of Statement 2 and therefore the one of Theorem 7.1.

Remark.

1. In the case when there exists a unique solution at -8π, we have shown that there is a set β_n, $\beta_n \in (-8\pi-\varepsilon, -8\pi)$ for which we have constructed two solutions of the MFE. One of these solutions is close to u^*, the unique solution at -8π, while the other is close to the "singular solution" describing the concentration, see Fig. 6.
2. In the case when the set of limit points of $\{u_\beta\colon \beta \in (-8\pi, 0]\}$ is not a single point, we have shown along the proof, that we have a continuum of solutions of the MFE at -8π. In fact, working a bit more we may replace i) in Theorem 7.1 by i$'$) the set of solutions at -8π of the MFE is unbounded and contains a continuum of solutions.

8. Concentration for Solutions of the CVP at $\beta = -8\pi$ for Simply Connected Domains Close to a Circle

In this section we shall consider the problem of the concentration in the case in which Λ is a simply connected domain sufficiently close to a disk. In this case it is possible to prove that the concentration does occur just by exploiting an argument due to Suzuki and Nagasaki [SuNa]. We first present this result and then we give an alternative direct proof of the concentration, for a particular class of domains close to a disk, based on variational argument. This second proof is a consequence of some result upon the concentration in a circle in the presence of an external field.

Let us recall some known facts.

Fact 1. As we have shown (see Theorem 3.2), Lemma 1 and Lemma 2 of [Su] implies that, if Λ is a simply connected domain, the MFE admits a unique solution for $\beta \in (-8\pi, 0]$.

Fact 2. T. Suzuki and K. Nagasaki proved (see [SuNa] Thm. 3) that if Λ is a simply connected domain sufficiently close to a disk, then there exists a branch of solutions (β, Ψ), blowing up, (that is it concentrates at -8π) from above (i.e. $\beta > -8\pi$), at $\beta = -8\pi$, containing the solutions $\beta = 0$, $\Psi = (-\Delta^{-1})1$.

As in the case of Theorem 3.1 the result due to Suzuki and Nagasaki is formulated for Eq. (3.12). In order to obtain the above mentioned result it is sufficient to exploit the correspondence with MFE, see (3.12) and (3.13).

The concentration for simply connected domains sufficiently close to a disk is an immediate consequence of these facts. Indeed, we have:

Theorem 8.1. *If Λ is a simply connected domain sufficiently close to a disk, then the solution ρ_β of the CVP at inverse temperature β in Λ concentrates to a Dirac mass as $\beta \to -8\pi$.*

Proof. Fact 2 says that there exist a branch of solutions (β, Ψ) that concentrates to a point when $\beta \to -8\pi$. Furthermore, this blow up happens form above, that is for $\beta > -8\pi$. Therefore Theorem 8.1 follows by the uniqueness of the solutions for $\beta > -8\pi$ (Fact 1). $\square$

A more direct analysis shows this is result for a special class of small deformations of the unit disk. Let us consider first the problem for a disk in the presence of an external field. Consider the disk $\Lambda_0 = \{x||x| < 1\}$ and two functionals

$$f^{\lambda}_{\alpha,\beta}(\rho) = \frac{1}{2}(\rho, V_\rho) - \frac{1}{\beta}\int dx\, \rho \ln\rho - \frac{\lambda}{\beta}\int \rho |x|^\alpha \tag{8.1}$$

with the constraints $\rho \geqq 0$ a.e.; $\int dx\, \rho = 1$ and

$$g^{\lambda}_{\alpha,\beta}(\Psi) = -\frac{1}{2}\int dx |\nabla\Psi|^2 - \frac{1}{\beta}\log \int dx\, e^{-\beta\Psi} e^{\lambda|x|^\alpha}\,. \tag{8.2}$$

For $\beta < 0$ we maximize each functional and we obtain the same mean field equation

$$-\Delta\Psi = \frac{e^{-\beta\Psi} e^{\lambda|x|^\alpha}}{\int dx\, e^{-\beta\Psi} e^{\lambda|x|^\alpha}}\,. \tag{8.3}$$

It is important to notice that the CVP (8.1), (8.2) and the corresponding Mean Field Equation (8.3) can be naturally obtained by considering the Mean Field Limit for the Canonical ensemble for point vortices in a circle under the action of a radial external field $U(x) = \frac{\lambda}{\beta}|x|^\alpha$.

Here we want to study the behavior when β approaches -8π. We note that, when $\lambda > 0$, the external field is repulsive. Therefore we have a competition between the free energy functional without the external field, that induces a concentration, and the external field which has the opposite action. In the following theorem we show under which conditions there is concentration. As usual in this paper, by concentration we mean that there exists a maximizing sequence ρ_n of $f^{\lambda}_{\alpha,-8\pi}$, converging weakly (in the sense of the measures) to δ_{x_0} up to the extraction of subsequences. (see Theorem 7.1 in [CLMP]).

Lemma 8.2.
i) *If* $\lambda < 0$ *and* $\alpha > 0$ *there is concentration at* 0.
ii) *If* $\lambda > 0$ *and* $\alpha \leqq 2$ *there is no concentration.*
iii) *If* $\lambda > 0$ *(*λ *sufficiently small, for a given* α*), and* $\alpha > 2$, *there is concentration at* 0.

Proof. The first point is obvious; in fact both the terms appearing in the free energy, the free energy functional without the external field, and the external field itself, are maximized when the solution concentration to δ_0.

ii) Let us consider an inverse temperature η close to -8π and the radial functions

$$\rho_\eta = \frac{A}{(1+Br^2)^2}; \qquad A = \frac{1+B}{\pi}, \qquad B = -\frac{\eta}{8\pi+\eta}. \tag{8.4}$$

If we evaluate explicitly the free energy $f^{\lambda}_{2,-8\pi}(\rho_\eta)$ for ρ_η at the temperature -8π when $\alpha = 2$ we find

$$f^{\lambda}_{2,-8\pi}(\Psi) = \frac{1}{8\pi B^2}\log(1+B) - \frac{(B+1)}{8\pi B} + \frac{1}{8\pi}\log\pi + \frac{1}{4\pi} + \frac{\lambda}{4\pi}\frac{1+B}{B^2}\log(1+B) - \frac{\lambda}{4\pi B}. \tag{8.5}$$

It is easy to note that when $B \gg 1$ (that is $\eta + 8\pi \ll 1$)

$$f^{\lambda}_{2,-8\pi}(\rho) > \frac{1}{8\pi}(\log\pi + 1) = I_{\Lambda_0}, \tag{8.6}$$

where I_{Λ_0} is, by construction the free energy of the Dirac delta in the center of Λ_0.

This fact ensures us that there is no concentration for $\alpha = 2$; we have in fact exhibited a test function whose free energy is greater than the free energy of δ_0 (recall that if there is concentration, concentration must take place as 0, i.e. the maximum point in Λ of $\gamma(x,x)$, by the same argument used in the case without the external field). Moreover, when $\alpha < 2$, we note that

$$f^{\lambda}_{\alpha,-8\pi}(\rho) \geqq f^{\lambda}_{2,-8\pi}(\rho); \tag{8.7}$$

and ii) is shown.

iii) Suppose that the solution is radial (see later). We multiply both sides of (8.3) by $r^2\frac{\partial\Psi}{\partial r}$, and integrate on r. Then we integrate by parts and use the Stokes Theorem to evaluate $\frac{\partial\Psi}{\partial r}|_{r=1}$. We have:

$$\frac{1}{8\pi}\frac{e^{8\pi\Gamma}}{Z} = \frac{\alpha\Gamma}{Z}\int_0^1 dr\, r^{\alpha+1}e^{8\pi}(\Psi + \Gamma r^\alpha),$$

$$Z = \int dx\, e^{8\pi}(\Psi + \Gamma r^\alpha); \tag{8.8}$$

where $\Gamma = -\frac{\lambda}{8\pi}$, $r = |x|$. ((8.8) is indeed the Pohozaev identity [Po] written for a radial Ψ).

Let us now assume, arguing by contradiction, that there exists a smooth solution $\Psi(Z < \infty)$. Using the fact that Ψ is radial, by the Stokes theorem we find that $|\partial_r \Psi| \leqq (2\pi r)^{-1}$ and since $\Psi(r = 1) = 0$ we have

$$\Psi \leqq -\frac{1}{2\pi}\log(r). \tag{8.9}$$

Therefore, from (8.7) we find

$$\frac{1}{8\pi}\frac{e^{8\pi\Gamma}}{Z} \leqq \frac{\alpha\Gamma}{Z}\int_0^1 \mathrm{d}r\, r^{\alpha-3}e^{8\pi\Gamma r^\alpha} \leqq \frac{\alpha\Gamma}{Z}\frac{e^{8\pi\Gamma}}{\alpha-2}. \tag{8.10}$$

Therefore, if Γ is small we find a contradiction: in fact (8.10) cannot be satisfied unless $Z = \infty$. In conclusion, if λ is sufficiently small, the solution must be a Dirac delta. Actually we have only proved that if the solution is radial then Z must diverge. But this is sufficient by Theorem 7.1 of [CLMP] (which also applies here with the same proof) that ensures that if $\beta = -8\pi$, $Z = \infty$, then the solution concentrates.

Finally, we must prove that the solution is radial. We do it by using a theorem due to B. Gidas, W.M. Ni and L. Nirenberg [GNN] which states that if u solves the equation

$$\begin{aligned} -\Delta u &= g(r,u) \quad \text{on } \Lambda_0\,, \\ u &= 0 \quad \text{on } \partial\Lambda_0\,, \end{aligned} \tag{8.11}$$

where Λ_0 is the unit ball in R^2, $u > 0$ in Λ_0, and g is locally Lipschitz in u and strictly decreasing in r, then u is radial. To apply this result we define

$$\Phi = \Psi - \frac{\lambda}{\beta}(r^\alpha - 1). \tag{8.12}$$

Hence, (8.2) yields the following equation for Φ:

$$\begin{aligned} -\Delta\Phi &= \frac{e^{-\beta\Phi}}{\int_\Lambda e^{-\beta\Phi}} + \frac{\lambda}{\beta}\alpha^2 - r^{\alpha-2} \quad \text{on } \Lambda_0\,, \\ \Phi &= 0 \quad \text{on } \partial\Lambda_0\,. \end{aligned} \tag{8.13}$$

Then, we observe that, since $\alpha > 2$ and $\beta < 0$, $\frac{\lambda}{\beta}\alpha^2 r^{\alpha-2}$ is a strictly decreasing function of r. Therefore, in order to apply the theorem by B. Gidas, W.M. Ni and L. Nirenberg there only remains to prove that Φ is strictly positive in Λ. Suppose the contrary. Let us assume also that there is no concentration. Then, taking the limit as λ goes to 0 we find that Ψ, extracting subsequences, if necessary, converges to some Ψ^* in C^2, while $\frac{\partial\Psi}{\partial v} \to \frac{\partial\Psi^*}{\partial v} < 0$ ($\frac{\partial\Psi^*}{\partial v} < 0$ is due to the Hopf Lemma [Ho] applied to a solution of (8.13) when $\lambda = 0$). Then, since $\Phi = 0$ and $\partial\Lambda_0$, and $\frac{\partial\Phi}{\partial v} = \frac{\partial\Psi}{\partial v} + O(\lambda)$, we find $\Phi > 0$ near $\partial\Lambda_0$ for λ small. Furthermore, we can note that on any compact subset of $\Lambda_0\Phi \to \Psi^* > 0$, therefore, for λ small, $\Phi > 0$ in Λ_0; and this concludes the proof of Lemma 8.2. □

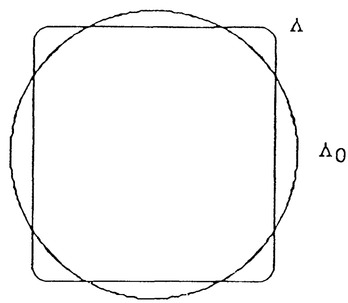

Fig. 8.

Remark. Lemma 8.2 with $\alpha = 2$, any λ, applies to a physically meaningful system: point vortices in a circle, studied in the mean field limit when we take into account the moment of inertia as in Sect. 5, see [SON].

We now use the results obtained above in order to prove the existence of a class of domains, close to the circle in a sense we shall specify later on, for which there is concentration.

Let Λ be a simply connected domain such that $|\Lambda| = |\Lambda_0|$, where Λ_0 is the disk of unit radius centered at the origin. Let $f : \Lambda \to \Lambda_0$ be a conformal transform of Λ in Λ_0, and let z_0 be the pole of this transformation. The Jacobian of this transformation is (when $z \in \Lambda$)

$$J(z) = \exp[-4\pi\gamma(z,z_0)]\{1 + (2\pi)^2(\nabla\gamma(z,z_0))^2(z - z_0)^2 + 2(2\pi)^2\nabla\gamma(z,z_0)(z - z_0)|z - z_0|^2\}\,. \tag{8.14}$$

Let us choose the zero of the transformation to be the maximum point of $\gamma(z) = \frac{1}{2}\gamma(z,z)$. We obtain the following inequalities:

$$J^{-1}(\xi) \leqq \exp[8\pi\gamma(z_0)]\exp[\lambda|\xi|^\alpha]\,. \tag{8.15}$$

Generically $\alpha = 2$, but there exists a class of domains for which, because of particular symmetries, $\alpha > 2$. For example, see Fig. 8. if the domain is invariant by rotations of an angle $\frac{\pi}{2}$ then $\alpha = 4$.

The free energy functional may be written, through the conformal map, on $H_0^1(\Lambda_0)$ as

$$g_\beta(\Psi) = -\frac{1}{2}\int_\Lambda |\nabla\Psi|^2 - \frac{1}{\beta}\log\int_\Lambda e^{-\beta\Psi} = -\frac{1}{2}\int_{\Lambda_0}|\nabla\Psi|^2 - \frac{1}{\beta}\log\int_{\Lambda_0} J^{-1}\cdot e^{-\beta\Psi}\,. \tag{8.16}$$

Therefore, we can estimate from above the free energy functional Λ by the free energy functional in Λ_0 in the presence of a suitable external field; that is

$$g^{\lambda}_{\alpha,\beta}(\Psi) = -\frac{1}{2}\int_{\Lambda_0} |\nabla\Psi|^2 - \frac{1}{\beta}\log\int_{\Lambda_0} J^{-1}\cdot e^{-\beta\Psi} \leqq -\frac{1}{2}\int_{\Lambda_0} |\nabla\Psi|^2$$
$$-\frac{1}{\beta}\log\int_{\Lambda_0} e^{-\beta\Psi+\lambda|z|^{\alpha}}, \tag{8.17}$$

with a suitable choice of λ and $\alpha < 2$.

Let $g^{\lambda}_{\alpha,\beta}(\Psi)$ be the last term of inequality (8.17). Because of Lemma 8.2 (see point iii)

$$g^{\lambda}_{\alpha,\beta}(\Psi) = I_{\Lambda_0} \quad \text{(that is the free energy of the Dirac delta for the circle)}, \tag{8.18}$$

and therefore

$$f_{-8\pi}(\Lambda) \leqq I_{\Lambda_0} + \frac{1}{2}\gamma(z_0,z_0) = I_{\Lambda}\,. \tag{8.19}$$

This implies the concentrations by point v) of Theorem 3.1.

Note that in Theorem 3.1 we deal, for the sake of simplicity, with domains of area 1, so in Theorem 3.1 Λ_0 is a disk of area 1 and not, as it is here, the disk of radius 1 whose area is π. Nevertheless it is easy to convince oneself, by simple scaling arguments, that (3.9), i.e. $I_{\Lambda_0} + \frac{1}{2}\gamma(z_0,z_0) = I_{\Lambda}$, holds also for domains of general area, where Λ_0 is the disk whose area is the same as Λ. $\square$

Acknowledgements. Three of us (E.C., C.M. and M.P.) are indebted to G. Tarantello for fruitful and illuminating discussions. We thank M.K.H. Kiessling for having pointed out additional references and improvements to a previous version of the paper. Finally we also thank the referees for deep and accurate remarks.

References

[CK] Chanillo, S., Kiessling, M.H.K.: Commun. Math. Phys. **160**, 217–238 (1994)

[CLMP] Caglioti, E., Lions, P.L., Marchioro, C., Pulvirenti, M.: Commun. Math. Phys. **143**, 501–525 (1992)

[ES] Eyink, G.L., Spohn, H.: J. Stat. Phys. **70**, Nov. 3/4 (1993)

[GL] Gogny, D., Lions, P.L.: RAIRO Modél. Math. et Anal. Num. **23**, 137–153 (1989)

[GNN] Gidas, B., Ni, W.M., Nirenberg, L.: Commun. Math. Phys. **68**, 203–243 (1979)

[H] Hopf, E.: J. Rat. Mech. Anal. **1**, 87–123 (1952)

[Ki] Kiessling, M.K.H.: Comm. Pure Appl. Math. **46**, 27–56 (1993)

[Ki]$_2$ Kiessling, M.K.H.: J. Stat. Phys. **55**, 203–257 (1989)

[LS] Leray, J., Schauder.: Topology et equations functionnels, Ann. Sci. Ecole Norm. Sup. **3**, 45–78 (1934)

[MS] Messer, J., Spohn, H.: J. Stat. Phys. **29**, 561–578 (1982)

[M] Moser, J.: Indiana Univ. Math. J. **20**, 1077–1092 (1971)

[MMS] Montgomery, D., Matthaeus, W.H., Stribling, W.T., Martinez, D., Oughton, S.: Phys. Fluids **A4**, 3–6 (1992);
Matthaeus, W.H., Stribling, W.T., Martinez, D., Oughton, S., Montgomery, D.: Phys. Rev. Lett. **66**, 2731 (1991)

[Mo] Montgomery, D.: Phys. Lett. **39A**, 7–8 (1972)

[MoJ] Montgomery, D., Joyce, G.: Phys. Fluids **17**, 1139–1145 (1974)

[MP] Marchioro, C., Pulvirenti, M.: Mathematical Theory of Incompressible Nonviscous Fluids, Appl. Math. Sci. **96**, Berlin, Heidelberg, New York: Springer 1994
[NaSu] Nagasaki, K., Suzuki, T.: Asymptotic Analysis **3**, 173–188 (1990)
[O] Onsager, L.: Suppl. Nuovo Cimento **279** (1949)
[Po] Pohozaev, S.I.: Soviet Math. Dokl. **6**, 1408–1411 (1965)
[SON] Smith, R.A., O'Neil, T.: Phys. Fluids. **B2**, 2961–2975 (1990)
[Su] Suzuki, T.: Ann. Inst. H. Poincaré **9**, 4, 367–398 (1992)
[SuNa] Suzuki, T., Nagasaki, K.: Trans. Am. Math. Soc. **309** (1988)

Communicated by J.L. Lebowitz