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Modular Invariance and Characteristic Numbers

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Abstract: We prove that a general miraculous cancellation formula, the divisibility of certain characteristic numbers, and some other topological results related to the generalized Rochlin invariant, the η-invariant and the holonomies of certain determinant line bundles, are consequences of the modular invariance of elliptic operators on loop space.

1. Motivations

In [AW], a gravitational anomaly cancellation formula, which they called the miraculous cancellation formula, was derived from very non-trivial computations. See also [GS] and [GSW], pp. 347–361. This is essentially a formula relating the L-class to the $\hat{A}$-class and a twisted $\hat{A}$-class of a 12-dimensional manifold. More precisely, let M be a smooth manifold of dimension 12, then this miraculous cancellation formula is

$$L(M) = 8\hat{A}(M,T) - 32\hat{A}(M),$$

where $T = TM$ denotes the tangent bundle of M and the equality holds at the top degree of each differential form. Here recall that, if we use $\{\pm x_j\}$ to denote the formal Chern roots of $TM \otimes \mathbb{C}$, then

$$L(M) = \prod_j \frac{x_j}{\tanh x_j/2}, \qquad \hat{A}(M) = \prod_j \frac{x_j/2}{\sinh x_j/2},$$

and

$$\hat{A}(M,T) = \hat{A}(M)\,\mathrm{ch}\,T \text{ with } \mathrm{ch}\,T = \sum_j e^{x_j} + e^{-x_j}.$$

Using a computer, Ochanine obtained the expressions of the top degree terms of $\hat{A}(M)$ and $L(M)$ in terms of the Pontryagin classes $\{p_j\}$ of M, also see [AW] or [GSW]:

$$\hat{A}(M) = -\frac{31}{967680}p_1^3 + \frac{11}{241920}p_1p_2 - \frac{1}{60480}p_3,$$

and

$$L(M) = \frac{2}{945} p_1^3 - \frac{13}{945} p_1 p_2 + \frac{62}{945} p_3 \, .$$

By using this cancellation formula, Zhang [Z] derived an analytical version of Ochanine's Rochlin congruence formula for 12-dimensional manifolds. He achieved this by considering the adiabatic limit of the η-invariant of a circle bundle over a characteristic submanifold of M. In [LM], it was shown that a general formula of such type implies that the generalized Rochlin invariant is a spectral invariant.

On the other hand, Hirzebruch [H] and Landweber [L] used elliptic genus to prove Ochanine's result: the signature of an $8k+4$-dimensional compact smooth spin manifold is divisible by 16. Actually Hirzebruch derived a general formula in [H] relating the signature to the indices of certain twisted Dirac operators of a compact smooth spin manifold of any dimension.

In this paper, using the modular invariance of certain elliptic operators on loop space, which we would like to call the general elliptic genera, we derive a more general miraculous cancellation formula about the characteristic classes of a smooth manifold and a real vector bundle on it. This formula includes all of the above results and is much more explicit. What is more important is that it is actually a formula about differential forms. Combining with the index formula of [APS] for manifold with boundary and the holonomy formulas of determinant line bundles [BF], we are able to show that several seemingly unrelated topological results (in Corollaries 1–6) are connected together by this general miraculous cancellation formula.

In the following we will state our results in Sect. 2 and prove them in Sect. 3. In Sect. 4 we discuss some other generalizations of the cancellation formula. Also given in this section is a formula relating the generalized Rochlin invariants to the holonomies of certain determinant line bundles. Section 5 contains general discussions about other applications of modular invariance in topology and some speculations on the relationship between loop space, double loop space and cohomology theory.

Our main results in this paper grew out of discussions with W. Zhang. The main idea is essentially due to Hirzebruch [H] and Landweber [L]. My sole contribution is to emphasize the role of modular invariance which reflects many beautiful properties of the topology of a manifold and its loop space. Note that we have shown in [Liu1] that modular invariance also implies the rigidity and vanishing of many naturally derived elliptic operators on loop space. We have further proved, also using modular invariance, a loop space analogue of the Atiyah–Hirzebruch $\hat{A}$-vanishing theorem for a spin manifold with compact Lie group symmetry. We refer the reader to [Liu1] and [Liu2] for the details. In a joint paper with W. Zhang [LZ], combining the results of [Z] and the cancellation formula in this paper, we have derived the analytical expressions of the Ochanine–Finashin's invariant [F] and some other topological invariants. I would like to thank the referee for many helpful comments which made this paper much more readable than its original version. I am also very grateful to A. Dessai, M. Hopkins, T. Li, and S. Ochanine for many very helpful discussions.

2. Results

Let M be a dimension $8k+4$ smooth manifold and V be a rank $2l$ real vector bundle on M. We introduce two elements in $K(M)[[q^{\frac{1}{2}}]]$ which consists of formal power series in q with coefficients in the K-group of M,

$$\Theta_1(M,V) = \bigotimes_{n=1}^{\infty} S_{q^n}(TM - \dim M) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(V - \dim V)\,,$$

$$\Theta_2(M,V) = \bigotimes_{n=1}^{\infty} S_{q^n}(TM - \dim M) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-\frac{1}{2}}}(V - \dim V)\,,$$

where $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the upper half plane, is a parameter. Recall that for an indeterminate t,

$$\Lambda_t(V) = 1 + tV + t^2\Lambda^2 V + \cdots, \quad S_t(V) = 1 + tV + t^2 S^2 V + \cdots$$

are two operations in $K(M)[[t]]$. They satisfy

$$S_t(V) = \frac{1}{\Lambda_{-t}(V)}, \qquad \Lambda_t(V - W) = \Lambda_t(V) S_{-t}(W)\,.$$

We can formally expand $\Theta_1(M,V)$ and $\Theta_2(M,V)$ into Fourier series in q

$$\Theta_1(M,V) = A_0 + A_1 q + \cdots\,,$$

$$\Theta_2(M,V) = B_0 + B_1 q^{\frac{1}{2}} + \cdots\,,$$

where the A_j's and B_j's are elements in $K(M)$. Let $\{\pm 2\pi i y_j\}$ be the formal Chern roots of $V \otimes \mathbb{C}$. If V is spin and $\Delta(V)$ is the spinor bundle of V, one knows that the Chern character of $\Delta(V)$ is given by

$$\mathrm{ch}\,\Delta(V) = \prod_{j=1}^{l} (e^{\pi i y_j} + e^{-\pi i y_j})\,.$$

In the following, we do not assume that V is spin, but still formally use $\mathrm{ch}\,\Delta(V)$ for the short-hand notation of $\prod_{j=1}^{l}(e^{\pi i y_j} + e^{-\pi i y_j})$ which is a well-defined cohomology class on M. Let p_1 denote the first Pontryagin class. Our main results include the following theorem:

Theorem 1. *If $p_1(V) = p_1(M)$, then*

$$\hat{A}(M)\,\mathrm{ch}\,\Delta(V) = 2^{l+2k+1} \sum_{j=0}^{k} 2^{-6j} b_j\,,$$

where the b_j's are integral linear combinations of the $\hat{A}(M)\,\mathrm{ch}\,B_j$'s.

Here still we only take the top degree terms of both sides. More generally we have that for any A_j as in the Fourier expansion of $\Theta_1(M,V)$, the top degree term of

$$\hat{A}(M)\,\mathrm{ch}\,\Delta(V)\,\mathrm{ch}\,A_j$$

is an integral linear combination of the top degree terms of the $\hat{A}(M)\operatorname{ch} B_j$'s. All of the b_j's can be computed explicitly, for example

$$b_0 = -\hat{A}(M), \quad b_1 = \hat{A}(M)\operatorname{ch} V + (24(2k+1) - 2l)\hat{A}(M).$$

Take $V = TM$, then $\hat{A}(M)\operatorname{ch} \Delta(V) = L(M)$. We get, at the top degree,

$$L(M) = 2^3 \sum_{j=0}^{k} 2^{6k-6j} b_j .$$

When $\dim M = 12$, one recovers the miraculous cancellation formula of [AW] and [GS].

The complicated expressions of the $\hat{A}(M)$ and $L(M)$ in dimension 12, as shown in the last section, indicate the usefulness of modular invariance. We can certainly expect more delicate applications of deeper results about modular forms in topology.

For a dimension $8k$ manifold M we have a similar cancellation formula,

$$L(M) = \sum_{j=0}^{k} 2^{6k-6j} b_j .$$

One can also express the top degree of $\hat{A}(M)$ in terms of those of the twisted L-classes.

Now assume M is compact. As an easy corollary of Theorem 1, we have the following

Corollary 1. *If M and V are spin and $p_1(V) = p_1(M)$, then*

$$\operatorname{Ind} D \otimes \Delta(V) = 2^{l+2k+1} \cdot \sum_{j=0}^{k} 2^{-6j} b_j ,$$

where D is the Dirac operator on M.

The b_j's are integral linear combinations of the $\operatorname{Ind} D \otimes B_j$'s. This gives

Corollary 2. *If M and V are spin with* $\dim V \geqq \dim M$ *and* $p_1(V) = p_1(M)$, *then* $\operatorname{Ind} D \otimes \Delta(V) \equiv 0 \pmod{16}$, *especially* $\operatorname{sign}(M) \equiv 0 \pmod{16}$.

When $V = TM$ Corollaries 1 and 2 were derived in [H]. See also [L].

Recall the definition of the Rochlin invariant. Suppose given a compact smooth dimension $8k+3$ manifold N with spin structure w, which is the spin boundary of a spin manifold M with spin structure W. The Rochlin invariant $R(N, w)$ is defined to be

$$R(N, w) = \operatorname{sign}(M) (\operatorname{mod} 16) .$$

Ochanine's theorem implies that $R(N, w)$ is well-defined. Let us formally write

$$b_j = \hat{A}(M)\operatorname{ch} \beta_j$$

with β_j integral linear combinations of the B_j's. Another corollary of Theorem 1 is the following.

Corollary 3. *The Rochlin invariant of* (N,w) *is a spectral invariant of* N *and is given by the explicit formula*

$$R(N,w) \equiv -\eta(\Delta) + \sum_{j=0}^{k} 2^{6k-6j+2}(\eta(\beta_j) + h_{\beta_j})(\mathrm{mod}\,16)\,.$$

Here $\eta(\Delta)$ and $\eta(\beta_j)$ are the η-invariants associated to the signature operator $d_s = D \otimes \Delta(M)$ and $D \otimes \beta_j$ respectively; h_{β_j} is the complex dimension of the kernel of the Dirac operator on N twisted by the restriction of β_j to N. They depend on the geometry of (N,w). See Sect. 3 for their precise definitions.

For an oriented smooth compact manifold M of dimension $8k+4$, let F be its submanifold which is the Poincaré dual of the second Stiefel–Whitney class $w_2(M)$. Denote by $F \cdot F$ the self-intersection of F. Let D_F denote the Dirac operator on F, β_j^F be the restriction of β_j to F. In [LZ], combining the general miraculous cancellation formula in Theorem 1 and the results in [Z], we have proved the following analytic version of the generalized Rochlin congruence formula of Finashin [F]:

Corollary 4.

$$\frac{\mathrm{sign}\,(M) - \mathrm{sign}\,(F \cdot F)}{8} \equiv \sum_{j=0}^{k} 2^{6k-6j-1}(\eta(\beta_j) + h_{\beta_j})(\mathrm{mod}\,2)\,.$$

Here $\eta(\beta_j)$ and h_{β_j} are the corresponding η-invariants and the dimension of the kernel of a certain twisted Dirac operator on F by the restriction of β_j to F. The proof of Corollary 4 and some other more general congruence results in this direction can be found in [LZ].

For the divisibility of the twisted signature and the relation between the generalized Rochlin invariant and the holonomies of determinant line bundles, see Sect. 4.

3. Proofs

The proof of Theorem 1 requires the Jacobi theta-functions [Ch] which are

$$\theta(v,\tau) = 2q^{\frac{1}{8}} \sin \pi v \prod_{j=1}^{\infty} (1-q^j)(1-e^{2\pi i v}q^j)(1-e^{-2\pi i v}q^j)\,,$$

$$\theta_1(v,\tau) = 2q^{\frac{1}{8}} \cos \pi v \prod_{j=1}^{\infty} (1-q^j)(1+e^{2\pi i v}q^j)(1+e^{-2\pi i v}q^j)\,,$$

$$\theta_2(v,\tau) = \prod_{j=1}^{\infty}(1-q^j)(1-e^{2\pi i v}q^{j-\frac{1}{2}})(1-e^{-2\pi i v}q^{j-\frac{1}{2}}),$$

and

$$\theta_3(v,\tau) = \prod_{j=1}^{\infty}(1-q^j)(1+e^{2\pi i v}q^{j-\frac{1}{2}})(1+e^{-2\pi i v}q^{j-\frac{1}{2}})\,.$$

These are holomorphic functions for $(v,\tau) \in \mathbb{C} \times \mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane. Up to some complex constants they have the following remarkable transformation formulas:

$$\theta\left(\frac{v}{\tau},-\frac{1}{\tau}\right)=i\tau^{\frac{1}{2}}e^{-\frac{v^2}{\tau}}\theta(v,\tau),\qquad \theta(v,\tau+1)=\theta(v,\tau)\,;$$

$$\theta_1\left(\frac{v}{\tau},-\frac{1}{\tau}\right)=\tau^{\frac{1}{2}}e^{-\frac{v^2}{\tau}}\theta_2(v,\tau),\qquad \theta_1(v,\tau+1)=\theta_1(v,\tau)\,;$$

$$\theta_2\left(\frac{v}{\tau},-\frac{1}{\tau}\right)=\tau^{\frac{1}{2}}e^{-\frac{v^2}{\tau}}\theta_1(v,\tau),\qquad \theta_2(v,\tau+1)=\theta_3(v,\tau)\,;$$

$$\theta_3\left(\frac{v}{\tau},-\frac{1}{\tau}\right)=\tau^{\frac{1}{2}}e^{-\frac{v^2}{\tau}}\theta_3(v,\tau),\qquad \theta_3(v,\tau+1)=\theta_2(v,\tau)\,.$$

These transformation formulas, which are simple consequences of the Poisson summation formula, will be the key of our argument. Let

$$\Gamma_0(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})|c\equiv 0\ (\mathrm{mod}\,2)\right\}\,,$$

$$\Gamma^0(2)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})|b\equiv 0\ (\mathrm{mod}\,2)\right\}$$

be two modular subgroups of $SL_2(\mathbb{Z})$. Let

$$S=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\qquad T=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}$$

be the two generators of $SL_2(\mathbb{Z})$. Their actions on $\mathbb{H}$ are given by

$$S:\tau\rightarrow-\frac{1}{\tau},\qquad T:\tau\rightarrow\tau+1\,.$$

Recall that a modular form over a modular subgroup Γ is a holomorphic function $f(\tau)$ on $\mathbb{H}$ such that for any

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma\,,$$

it satisfies the transformation formula

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=\chi(g)(c\tau+d)^k f(\tau)\,,$$

where $\chi:\Gamma\rightarrow\mathbb{C}^*$ is a character of Γ and k is called the weight of f. We also assume f is holomorphic at all cusps of $\mathbb{H}/\Gamma$.

Obviously, at $v=0$, $\theta_j(0,\tau)$ for $j=1,2$ are modular forms of weight $\frac{1}{2}$ over $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively; and

$$\theta'(0,\tau)=\frac{\partial}{\partial v}\theta(v,\tau)|_{v=0}$$

is a modular form of weight $\frac{3}{2}$ over $SL_2(\mathbb{Z})$. We will not need $\theta_3(v,\tau)$ in this paper.

With the notations as above we then have the following

Lemma 1. *Assume* $p_1(V) = p_1(M)$, *then* $P_1(\tau) = \hat{A}(M)\operatorname{ch}\Delta(V)\operatorname{ch}\Theta_1(M,V)$ *is a modular form of weight* $4k+2$ *over* $\Gamma_0(2)$; $P_2(\tau) = \hat{A}(M)\operatorname{ch}\Theta_2(M,V)$ *is a modular form of weight* $4k+2$ *over* $\Gamma^0(2)$.

Proof. Let $\{\pm 2\pi i y_\nu\}$ and $\{\pm 2\pi i x_j\}$ be the corresponding formal Chern roots of $V \otimes \mathbb{C}$ and $TM \otimes \mathbb{C}$. In terms of the theta-functions, we get

$$P_1(\tau) = 2^l \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \right) \prod_{\nu=1}^{l} \frac{\theta_1(y_\nu,\tau)}{\theta_1(0,\tau)},$$

$$P_2(\tau) = \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0,\tau)}{\theta(x_j,\tau)} \right) \prod_{\nu=1}^{l} \frac{\theta_2(y_\nu,\tau)}{\theta_2(0,\tau)}.$$

Applying the transformation formulas of the theta-functions, we have

$$P_1\left(-\frac{1}{\tau}\right) = 2^l \tau^{4k+2} P_2(\tau), \;\; P_1(\tau+1) = P_1(\tau),$$

where for the first equality, we need the condition $p_1(V) = p_1(M)$.

It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST, while the generators of $\Gamma^0(2)$ are STS, T^2STS from which the lemma easily follows. ♠

Write $\theta_j = \theta_j(0,\tau)$. We introduce four explicit modular forms [Liu2],

$$\delta_1(\tau) = \frac{1}{8}(\theta_2^4 + \theta_3^4), \qquad \varepsilon_1(\tau) = \frac{1}{16}\theta_2^4\theta_3^4,$$

$$\delta_2(\tau) = -\frac{1}{8}(\theta_1^4 + \theta_3^4), \qquad \varepsilon_2(\tau) = \frac{1}{16}\theta_1^4\theta_3^4.$$

They have the following Fourier expansions in q:

$$\delta_1(\tau) = \frac{1}{4} + 6q + \cdots, \qquad \varepsilon_1(\tau) = \frac{1}{16} - q + \cdots,$$

$$\delta_2(\tau) = -\frac{1}{8} - 3q^{\frac{1}{2}} + \cdots, \qquad \varepsilon_2(\tau) = q^{\frac{1}{2}} + \cdots,$$

where $\cdots$ are the higher degree terms all of which have integral coefficients. Let $M(\Gamma)$ denote the ring of modular forms over Γ with integral Fourier coefficients. We have

Lemma 2. δ_1, δ_2 *are modular forms of weight 2 and* $\varepsilon_1, \varepsilon_2$ *are modular forms of weight 4, and furthermore* $M(\Gamma^0(2)) = \mathbb{Z}[8\delta_2(\tau), \varepsilon_2(\tau)]$.

The proof is quite easy. In fact δ_2 and ε_2 generate a graded polynomial ring which has dimension $1 + [\frac{k}{2}]$ in degree $2k$. On the other hand one has a well-known upper bound for this dimension: $1 + \frac{k}{6}[SL_2(\mathbb{Z}) : \Gamma^0(2)]$ which is $1 + \frac{k}{2}$. Also note that the leading terms of $8\delta_2, \varepsilon_2$ in the lemma have coefficients 1 which immediately gives the integrality. The modularity follows from the transformation formulas of the theta-functions and the fact that $\Gamma^0(2)$ is generated by STS and T^2STS.

Now we can *prove Theorem 1*. By Lemmas 1 and 2 we can write

$$P_2(\tau) = b_0(8\delta_2)^{2k+1} + b_1(8\delta_2)^{2k-1}\varepsilon_2 + \cdots + b_k(8\delta_2)\varepsilon_2^k\,,$$

where the b_j's are integral linear combinations of the $\hat{A}(M)$ ch B_j's.

Applying the modular transformation $S : \tau \to -\frac{1}{\tau}$, we have

$$\delta_2\left(-\frac{1}{\tau}\right) = \tau^2\delta_1(\tau), \qquad \varepsilon_2\left(-\frac{1}{\tau}\right) = \tau^4\varepsilon_1(\tau)\,,$$

$$P_2\left(-\frac{1}{\tau}\right) = 2^{-l}\tau^{4k+2}P_1(\tau)\,.$$

Therefore

$$P_1(\tau) = 2^l[b_0(8\delta_1)^{2k+1} + b_1(8\delta_1)^{2k-1}\varepsilon_1 + \cdots + b_k(8\delta_1)\varepsilon_1^k]\,.$$

At $q = 0, 8\delta_1 = 2$ and $\varepsilon_1 = 2^{-4}$. One gets the result by a simple manipulation. the cancellation formula for dimension $8k$ case can be proved in the same way. ♠

One can certainly get more divisibility results on characteristic numbers from the above formulas.

Corollary 1 is an easy consequence of Theorem 1. Note that in the case that M and V are spin, all of the b_j's are integral linear combinations of the $\operatorname{Ind} D \otimes B_j$'s. *Corollary 2* follows from Corollary 1, since in dimension $8k+4$, each b_j is an even integer.

For Corollary 3, we first recall the definition of η-invariant. Let $(N, w) = \partial(M, W)$ be as in Sect. 2. Let E be a real vector bundle on M. Consider a twisted Dirac operator $D \otimes E$ on M. With a suitable choice of metrics on M, N and E, in a neighborhood of N one can write

$$D \otimes E = \sigma\left(\frac{\partial}{\partial u} + D_N \otimes E|_N\right)\,,$$

where D_N is the Dirac operator on N, $E|_N$ is the restriction of E to N, σ is the bundle isomorphism induced by the symbol of $D \otimes E$ and u is the parameter in the normal direction to N. Let $\{\lambda_j\}$ be the nonzero eigenvalues of $D_N \otimes E|_N$ and h_E be the complex dimension of its zero eigenvectors. Then the η-invariant associated to $D \otimes E$, which we denote by $\eta(E)$, is given by evaluating at $s = 0$ of the function

$$\eta(s) = \sum_j \operatorname{sign}\lambda_j \cdot \lambda_j^{-s}\,.$$

One has the following index formula from [APS]:

$$\operatorname{Ind} D \otimes E = \int_M \hat{A}(M)\operatorname{ch} E - \frac{\eta(E) + h_E}{2}\,.$$

For convenience, we will call $\eta(E)$ the η-invariant associated to $D \otimes E$. One should note that $\eta(E)$ and h_E are actually geometrical invariants of N.

For the *proof of Corollary 3*, we take $V = TM$ in Theorem 1 and apply this formula to $L(M)$ and each $b_j = \hat{A}(M)\operatorname{ch}\beta_j$. For β_j the [APS] formula gives us

$$\operatorname{Ind} D \otimes \beta_j = \int_M \hat{A}(M)\operatorname{ch}\beta_j - \frac{\eta(\beta_j) + h_{\beta_j}}{2}\,.$$

For $L(M)$ we have

$$\operatorname{sign}(M) = \int_M L(M) - \eta(\Delta),$$

where $\eta(\Delta)$ is the η-invariant associated to the signature operator d_s. Put these two formulas into the equality of Theorem 1 with $V = TM$, we get

$$R(N, w) \equiv -\eta(\Delta) + \sum_{j=0}^{k} 2^{6k-6j+2}(\eta(\beta_j) + h_{\beta_j}) \pmod{16}.$$

Here we have used the fact that $\operatorname{Ind} D \otimes \beta_j$ is even in dimension $8k+4$. ♠

4. Generalizations

Let M and V be as in Sect. 2. We introduce two more elements in $K(M)[[q^{\frac{1}{2}}]]$:

$$\Phi_1(M, V) = \Theta_1(M, TM) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V - \dim V),$$

$$\Phi_2(M, V) = \Theta_2(M, TM) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-\frac{1}{2}}}(V - \dim V),$$

where $\Theta_j(M, TM)$ is defined as in Sect. 2 with $V = TM$.

Similarly introduce two cohomology classes on M:

$$Q_1(\tau) = L(M) \operatorname{ch} \Delta(V) \operatorname{ch} \Phi_1(M, V),$$

$$Q_2(\tau) = \hat{A}(M) \operatorname{ch} \Phi_2(M, V).$$

Expressing them in terms of the theta-functions, we have

$$Q_1(\tau) = 2^{l+4k+2} \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0,\tau)\theta_1(x_j,\tau)}{\theta(x_j,\tau)\theta_1(0,\tau)} \right) \prod_{v=1}^{l} \frac{\theta_1(y_v,\tau)}{\theta_1(0,\tau)},$$

$$Q_2(\tau) = \left(\prod_{j=1}^{4k+2} x_j \frac{\theta'(0,\tau)\theta_2(x_j,\tau)}{\theta(x_j,\tau)\theta_2(0,\tau)} \right) \prod_{v=1}^{l} \frac{\theta_2(y_v,\tau)}{\theta_2(0,\tau)}.$$

If $p_1(V) = 0$, then it is easy to see that the top degree terms of these two classes are modular forms over $\Gamma_0(2)$ and $\Gamma^0(2)$ respectively. Similar method to the proof of Theorem 1 can be used to derive an expression of $L(M) \operatorname{ch} \Delta(V)$ in terms of the integral linear combination of the $\hat{A}(M) \operatorname{ch} D_j$'s, where the D_j's are the Fourier coefficients of $\Phi_2(M, V)$. More explicitly we have

$$L(M) \operatorname{ch} \Delta(V) = 2^{l+3} \sum_{j=0}^{k} 2^{6k-6j} d_j,$$

where each d_j is an integral linear combination of the $\hat{A}(M) \operatorname{ch} D_j$'s. As a corollary, one gets

Corollary 5. *Let V be a rank $2l$ spin vector bundle on a spin manifold M of dimension $8k+4$; if $p_1(V) = 0$, then* $\operatorname{Ind} d_s \otimes \Delta(V) \equiv 0 \pmod{2^{l+4}}$.

We remark that this corollary can also be derived from Corollary 2 by taking $V \oplus TM$ as the V there. One may also introduce other similar elements and play the same game to get new cancellation formulas.

Using Corollary 3, we can relate the Rochlin invariant to the holonomy of certain determinant line bundles. This generalizes the formula in [LMW] to higher dimension. We first recall some basic notations.

Given a smooth family of $8k+2$-dimensional compact smooth spin manifolds $\pi : Z \to N$ and a spin vector bundle E on Z, let D_x be the Dirac operator on the fiber $M_x = \pi^{-1}(x)$. To each x we associate the one dimensional complex vector space

$$(\Lambda^T \ker D_x \otimes E|_{M_x})^* \otimes \Lambda^T \operatorname{coker} D_x \otimes E|_{M_x},$$

where Λ^T denotes the top degree wedge product, which patch together to give a well-defined smooth complex line bundle on N, called the determinant line bundle and denoted by $\det D \otimes E$. When E, N and Z are equipped with smooth metrics, then one has the following equality of differential forms [BF]:

$$c_1(\det D \otimes E)_Q = \int_{M_x} \hat{A}(M_x) \operatorname{ch} E,$$

where the left-hand side is the first Chern form with respect to the Quillen metric [BF].

Let M be a dimension $8k+2$ compact smooth spin manifold, and f be a diffeomorphism of M. Denote by $(M \times S^1)_f$ the mapping torus of f which is defined to be

$$M \times [0,1]/(x,0) \sim (f(x),1).$$

Each spin structure w on M naturally induces a spin structure which we still denote by w, on the mapping torus [LMW].

Let $\gamma : S^1 \to N$ be an immersed circle in N. Pulling back from the family $\pi : Z \to N$, we get an $8k+3$ dimensional manifold which is isomorphic to a mapping torus $(M \times S^1)_f$. Let $H(E,w)$ denote the holonomy of the line bundle $\gamma^* \det D \otimes E$ with spin structure w on M around the circle. Scaling the (induced) metric on S^1 by ε^{-2} and letting ε go to zero, we have the following holonomy formula [BF]:

$$H(E,w) = \lim_{\varepsilon \to 0} \exp(-\pi i(\eta(E) + h_E)).$$

Given two spin structures w_1, w_2 on M, from Corollary 3 we have

$$\begin{aligned} &R((M \times S^1)_f, w_1) - R((M \times S^1)_f, w_2) \\ &= \sum_{j=0}^{k} 2^{6k-6j+2}\{(\eta^{w_1}(\beta_j) + h_{\beta_j}^{w_1}) - (\eta^{w_2}(\beta_j) + h_{\beta_j}^{w_2})\}, \end{aligned}$$

where the superscript w_j denotes the invariant of the corresponding spin structure. Therefore the above holonomy formula gives us the following

Corollary 6.

$$\begin{aligned} &\exp\left\{-\frac{\pi i}{4}(R((M \times S^1)_f, w_1) - R((M \times S^1)_f, w_2))\right\} \\ &= \prod_{j=0}^{k} (H(\beta_j, w_1) H(\beta_j, w_2)^{-1})^{2^{6k-6j}}. \end{aligned}$$

Note that in this equality, the β_j's are considered as elements in $K((M \times S^1)_f)$ by restriction. More precisely they are the restriction to $(M \times S^1)_f$ of the corresponding β_j's on K, where $\partial K = (M \times S^1)_f$.

We remark that the generalized Rochlin invariants are closely related to the Arf invariants, or the generalized Kervaire invariants, as well as the theta multipliers. Therefore our formula in Corollary 6 also gives expressions of these topological invariants in terms of the holonomies of determinant line bundles.

For any oriented compact smooth manifold M with $p_1(M) = 0$, the top degree term of

$$\hat{A}(M)\,\mathrm{ch}\bigotimes_{n=1}^{\infty} S_{q^n}(TM - \dim M)$$

is a modular form over $SL_2(\mathbb{Z})$ of weight $k = \frac{1}{2}\dim M$. When M is spin with $p_1(M) = 0$, this cohomology class is the index density of the Dirac operator on the loop space LM. See [W], [Liu1] for the other aspects of this operator. From elementary theory of modular forms, we can easily get the following

Corollary 7. *If* $\hat{A}(M) = 0, p_1(M) = 0$ *and* $\dim M < 24$, *then the top degree term of*

$$\hat{A}(M)\,\mathrm{ch}\bigotimes_{n=1}^{\infty} S_{q^n} TM$$

vanishes.

This corollary applies to the case that M is spin with positive scaler curvature or with S^1-action. It is interesting to find out its geometric meaning. As pointed out by Li, using [De], we can easily derive the following corollary from our loop space $\hat{A}$-vanishing theorem in [Liu1]:

Corollary 8. *For any compact homogeneous spin manifold with* $p_1 = 0$,

$$\mathrm{Ind}\, D \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM$$

vanishes.

Similarly one can consider

$$\hat{A}(M)\,\mathrm{ch}\,(\Delta^+(M) - \Delta^-(M)) \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(TM - \dim M)\,,$$

and

$$\mathrm{ch}\,(\Delta^+(V) - \Delta^-(V)) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^m}(V - \dim V)\,,$$

which is the index density of the Euler characteristic operator on LM. Here

$$\mathrm{ch}\,(\Delta^+(V) - \Delta^-(V)) = \prod_{j=1}^{l} (e^{\pi i y_j} - e^{-\pi i y_j})$$

and $\mathrm{ch}(\Delta^+(M) - \Delta^-(M))$ denotes the corresponding class for TM.

If $p_1(V) = p_1(M)$, the top degree term of the above Euler index density on LM is a modular form of weight $k - l$ over $SL_2(\mathbb{Z})$. Here $2l = \mathrm{rank}\, V$ and $2k = \dim M$. When $l > k$ or $k = l$ and the Euler characteristic of V is zero, the top degree term of this class vanishes.

5. Discussions

Modular invariance is one of the most fundamental principles in modern mathematical physics. In [Liu1], we proved that many elliptic operators derived from loop spaces have modular invariance which in turn implies their rigidity. We note that formal application of the Lefschetz fixed point formula on loop spaces together with modular invariance gives us a lot of information about index theory on loop spaces. Many topological results, such as rigidity and divisibility, are much easier to understand when we look at them while standing on loop spaces, therefore they are infinite dimensional phenomena. On the other hand many seemingly unrelated topological results discussed above are also intrinsically connected together by the modular invariance of elliptic operators derived from loop spaces.

Another interesting point is not directly related to modular invariance. It is only a kind of wild speculation. As observed by Witten and discussed in detail by Atiyah [A1], the inverse of the $\hat{A}$-genus is actually the equivariant Euler class of the normal bundle of M in its loop space LM with respect to the natural S^1-action on LM. Therefore up to certain normalization, the Atiyah–Singer index formula can be formally derived from the Duistermaat–Heckman localization formula on LM. We note that, up to certain normalization, the inverse of the loop space $\hat{A}$-genus

$$q^{-\frac{k}{12}}\hat{A}(M)\,\mathrm{ch}\bigotimes_{n=1}^{\infty}S_{q^n}TM\,,$$

where $2k=\dim M$, is formally equal to the equivariant Euler class of the normal bundle of M in the double loop space LLM with respect to the natural $T=S^1\times S^1$-action. Up to the normalization constant, we have the following formal product formula which is a consequence of the Eisenstein product formula [We]:

$$\prod_{m,n}(x+m\tau+n)=q^{\frac{1}{12}}(e^{\pi ix}-e^{-\pi ix})\prod_{n=1}^{\infty}(1-e^{2\pi ix}q^n)(1-e^{-2\pi ix}q^n)\,.$$

This means that the index formula on loop space, especially elliptic genera, can also be derived from the formal application of the Duistermaat–Heckman formula to the double loop space LLM!

In fact, one knows that the K-theory Euler class of TM is $\Delta^+(M)-\Delta^-(M)$. On the other hand, from the formal product formula

$$\prod_{n=-\infty}^{\infty}(x+n)=e^{\pi ix}-e^{-\pi ix}$$

which holds up to a normalization constant [A1], we have the following equality:

$$\mathrm{ch}(\Delta^+(M)-\Delta^-(M))=\text{the equivariant Euler class of } TLM|_M\,,$$

where $TLM|_M$ denotes the restriction of the tangent bundle of LM to M. Similarly the equivariant K-theory Euler class of $TLM|_M$ is

$$q^{\frac{k}{12}}(\Delta^+(M)-\Delta^-(M))\otimes\bigotimes_{n=1}^{\infty}\Lambda_{-q^n}TM$$

and its Chern character is

$$q^{\frac{k}{12}}\prod_{j=1}^{k}(e^{\pi ix_j}-e^{-\pi ix_j})\prod_{n=1}^{\infty}(1-e^{2\pi ix_j}q^n)(1-e^{-2\pi ix_j}q^n),$$

which is the equivariant cohomology Euler class of $TLLM|_M$ with respect to the natural T-action. Here $\{\pm 2\pi ix_j\}$ denote the formal Chern roots of $TM\otimes\mathbb{C}$.

Let us use $\Leftrightarrow$ to mean correspondence, then modulo certain normalization, we can put the above discussions in the following way:

$$\begin{gathered}\text{The equivariant cohomology Euler class of } TLM|_M\\ \Leftrightarrow \text{the } K\text{-theory Euler class of } TM\,;\\ \text{The equivariant cohomology Euler class of } TLLM|_M\\ \Leftrightarrow \text{the equivariant } K\text{-theory Euler class of } TLM|_M\,.\end{gathered}$$

Another interesting point is that, under the natural transformation of cohomology theories

$$\mathrm{Ell}^*(M)\to K(M)[[q]],$$

the Euler class of elliptic cohomology is transformed to the equivariant K-theory Euler class of $TLM|_M$. We remark that the elliptic cohomology here may be slightly different from the one in [L]. Actually the elliptic cohomology in [L] is associated to the signature operator on loop space, while the one here should be associated to the Dirac operator on loop space, which conjecturally should exist.

These discussions make it reasonable to speculate that the equivariant cohomology of LM should correspond to the K-theory of M, while the equivariant cohomology of the double loop space LLM should correspond to the equivariant K-theory of LM which in turn corresponds to the elliptic cohomology of M. We can put these into the following diagram:

$$\begin{aligned}&H_{S^1}(LM)\Leftrightarrow K(M),\\ &H_T^*(LLM)\Leftrightarrow K_{S^1}(LM),\\ &K_{S^1}(LM)\Leftrightarrow \mathrm{Ell}^*(M)\Leftrightarrow H_T^*(LLM)\,.\end{aligned}$$

This also means that looping M once lift the equivariant cohomology theory one order higher.

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