Commun. Math. Phys. 173, 77 – 99 (1995)

Communications in
Mathematical
Physics

On Equivalence of Floer's and Quantum Cohomology

Vladimir Sadov

Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA and L.D. Landau Institute for Theoretical Physics, Moscow, Russia

Received: 1 February 1994/in revised form: 22 September 1994

Abstract: We show that the Floer cohomology and quantum cohomology rings of the almost Kähler manifold M, both defined over the Novikov ring of the loop space $\mathscr{L}M$, are isomorphic. We do it using a BRST trivial deformation of the topological A-model. The relevant aspect of noncompactness of the moduli of pseudoholomorphic instantons is discussed. It is shown nonperturbatively that any BRST trivial deformation of A model which does not change the dimensions of BRST cohomology does not change the topological correlation functions either.

1. Introduction

The "quantum cohomology" ring H_Q^* ($=(c,c)$ ring in terms of $N=2$ sigma models) was introduced in [1], see also [2–6]. The infinite volume limit of H_Q^* coincides with the ordinary cohomology ring $H^*(M)$ of the target space M. For any finite volume, H_Q^* is a deformation of $H^*(M)$. A natural question arises about the meaning of this deformation in classical geometry.

One way to do this in terms of the moduli space of holomorphic instantons was introduced in [2, 6, 5]. It is more or less standard by now and we refer the reader to [5], for a review of that approach. Closely related to, but not quite the same as the latter one, is the interpretation in terms of geometry of the parameterized loop space $\mathscr{L}M$ of the target space, conjectured in [1, 3]. It turns out that an appropriate object to deal with in this context is what the mathematicians call a Floer symplectic cohomology H_F^* [7–9].

H_F^* appear via the Witten–Floer [10, 11, 7] complex in $\mathscr{L}M$, whose vertices are the fixed points of some symplectomorphism ϕ of M and the edges are the "pseudoholomorphic instantons" (defined below) connecting these fixed points. It is graded by the same abelian group 2Γ as the quantum cohomology H_Q^* (and for the same reason), a phenomenon known to physicists as the anomalous conservation of fermionic number. Under some natural assumptions [7, 12] one has $\dim H_F^\iota = \sum_{\gamma\in\Gamma} b^{\iota+2\gamma}(M)$, where on the left-hand side we identify the index of Betty numbers modulo 2Γ. Moreover, there is a natural action of $H^*(M)$ on H_F^*. It is defined in terms of intersection numbers in $\mathcal{L}M$ of a finite dimensional cycle – a cell of the WF complex – with a finite codimensional one – a pullback of the cocycle on M under the natural projection $\mathcal{L}M \to M$. Having fixed the isomorphisms of vector spaces $h^i : H^i_F \equiv H^i_Q$, we may think that we have a new multiplication law (a ring structure) on $H^*(M)$. This is quite similar with how it happens for the quantum cohomology ring.

The whole ideology of the Floer theory renders it almost obvious, that there should be an isomorphism

$$H^*_Q \equiv H^*_F \ . \tag{1.1}$$

Still, there are two obstacles for just the naive identification (1.1).

The first obstacle is that, as in [7–9], H^*_F is naturally defined over integer numbers $\mathbf{Z}$. In particular, it cannot depend nontrivially on any continuous parameter. On the other hand, the quantum cohomology H^*_Q is defined over complex numbers and its ring structure depends on the Kähler structure of the target space.

The second problem is that in Floer theory by "pseudoholomorphic instantons" one understands the solutions of the equation

$$\frac{\partial X^i}{\partial \tau} + J^i_j \frac{\partial X^j}{\partial t} = \partial^i H(X,t) \ , \tag{1.2}$$

where J^i_j is an almost complex structure on M which relates the metrics G_{ij} and the Kähler form k_{ij}:

$$G_{ij} = J^n_i k_{nj} \ . \tag{1.3}$$

The function H on the right-hand side of the equation (the Hamiltonian) depends on the point on M and also periodic (with period 2π) on variable t. Fix some initial moment $t = 0$. The hamiltonian flow, generated by H, maps M to itself at each t. A map, generated when $t = 2\pi$, called a period map, gives a symplectomorphism ϕ. Thus the fixed points of ϕ are in one to one correspondence with the periodic with period 2π trajectories – the points of $\mathcal{L}M$. By difficult analytic methods [7, 8] Floer has proved that in fact H^*_F is independent of $H(X, t)$, for generic H. Unfortunately, $H = 0$ which gives the usual holomorphic instantons equation, is by no means generic for the Morse type theory.

It turns out that the first problem can be dealt with quite easily if we redefine both H^*_Q and H^*_F to be defined over some new ring, called a Novikov ring [13–15]. This trick is well known to mathematicians [16, 17][1]. It also makes possible to work with Floer theory on the Calabi–Yau manifold, which otherwise would be impossible.

Our strategy in dealing with the second difficulty will be to show how the "pseudoholomorphic instantons" appear in topological sigma model [4, 5, 18], properly deformed by adding a BRST-trivial piece to the action. Thus instead of trying to continue the Floer cohomology to the point $H = 0$ we extend the quantum cohomology to arbitrary H and show that it does not depend on H.

We obtain a family of topological theories parameterized by the Hamiltonian H. For $H = 0$ we get back a topological sigma model of Witten [4]. The local physical operators, given by BRST cohomology, are in one-to-one correspondence with elements of de Rham cohomology of M. The correlation functions of these

[1] In the last reference the Novikov ring is different from ours and used for another reason (to work with nonexact symplectomorphisms).

operators can be localized to holomorphic instantons. The operator algebra is given by the quantum cohomology H_Q^*.

For $H \neq 0$, the (off-shell) BRST operator is the same as for $H = 0$. Therefore the physical operators are the same. Moreover, as the deformation by H is BRST trivial, the topological correlation functions are the same. Hence the operator algebra is always H_Q^*. An important feature of theories with $H \neq 0$ is that it is possible to characterize also the *states* in a simple fashion. The states of topological theory are the ground (zero energy) states of the corresponding $N = 2$ supersymmetric sigma model. It turns out that perturbatively, these states are in one-to-one correspondence with the loops – critical points of the Floer functional. The non-perturbative effects of instantons "lift" some of them, leaving as the true ground states only those annihilated by the BRST operator. The whole picture is a direct generalization of one which appears in Witten's supersymmetric quantum mechanics.

Using localization, it is possible to compute the matrix elements of the BRST operator between the perturbative ground states. They turn out to coincide with matrix elements of the Floer complex. Therefore the true ground states coincide with the elements of Floer cohomology groups H_F^*. The operator algebra H_Q^* of the theory acts on the space of states $\cong H_F^*$. This action is what we are after. Using the same localization technique, one can find the matrix elements of this representation of H_Q^* on H_F^* to be the same as appear in Floer's theory under the other name. The state-operator correspondence of topological sigma model leads to identification of these matrix elements as (topological) correlation functions.

This is the outline of both the idea and the techniques used in this paper. Certainly, it does not contain a *mathematically* rigorous proof. For example, we take it for granted that the operator algebra H_Q^* of topological sigma model is commutative associative. (Recently this fact was proven [19].) There are many other fine points, part of them discussed in the last section of this paper. We explain there why the BRST trivial deformation which does not change the ranks of BRST cohomology, does not change the topological correlation functions either. In doing this, we cannot use the perturbative argument since we do not know the right vacuum of the theory *apriori*. The reason is that due to the noncompactness of the moduli spaces of pseudoholomorphic instantons, the state operator correspondence could furnish a singular map at some points. The standard example is the deformation of LG model by the relevant operator $W = x^{n+2} \to W + \varepsilon x^{n+3}$. (In this example, the BRST cohomology does change.) In the last section we show that such phenomenon does not occur in A model, the state operator correspondence is always smooth and the correlators are independent of the BRST trivial deformation.

One can understand the relations of the physical approach developed here with the mathematical approach developed by [19, 20] as follows. Mathematicians want to construct a homotopy of the Floer complex to $H = 0$. It is a technically difficult problem. It turns out that it is easier to work with generic *almost complex structure*. From the point of view of physical approach developed, it is straightforward to generalize to that situation. Since any variation of the almost complex structure corresponds to BRST trivial deformation of the theory, the topological correlation functions and operator algebra are independent of it [4, 5]. Then, one can interpret the constructions of this paper as a realization of required homotopy on the ground states of the topological sigma model given by the state-operator correspondence map. Smoothness of this map is crucial.

2. Morse Theory for Multivalued Functions: Novikov Rings

In order to organize the material better it seems convenient to begin with the explanation of ideas of the Novikov theory before having defined the Floer cohomology H_F^* in detail. The only thing we should know about H_F^* now is that it appears in a Morse theory for a multivalued function S on the loop space. That function is defined on the universal cover $\widehat{\mathscr{L}M}$ of $\mathscr{L}M$ (it is necessarily abelian) and changes by $\oint_\gamma k$ as one moves along $\widehat{\mathscr{L}M}$ by $\gamma \in \pi_1(\mathscr{L}M) = \pi_2(M)$, k is a Kähler form.

Consider first a situation in general, following closely the presentation of [13]. Let X be a closed manifold. We do not specify whether it is finite dimensional or not. Of course, in the latter case, which appears in Floer's theory we have problems with compactness, but here we want to forget about it for a moment. Let $\gamma_1, \ldots, \gamma_n$ be a basis for the first homology group of X. For every closed 1-form ω on X we have n periods

$$k_j = \oint_{\gamma j} \omega \ . \tag{2.1}$$

The numbers $k_1, \ldots, k_n$ are in general irrational and their linear combinations with integral coefficients form a free abelian group. The rank k of this group is called the *irrationality* of 1-form ω. Obviously, $k \leqq n$. From now on we suppose that $k = n$ which means that ω is "generic enough."

There is a minimal free abelian covering $p : \bar{X} \to X$ such that the form $p^*\omega$ is exact:

$$p^*\omega = dS \ . \tag{2.2}$$

The monodromy group is $\mathbf{Z^n} \equiv H_1(X)$, generated by the covering transformations $T_\iota : \bar{X} \to \bar{X}$, satisfying $T_i^* S = S + k_i$. Take on X a smooth metric such that the hamiltonian flow generated by ω lifts smoothly to a ∞-continuous flow on the covering space $\bar{X}$: each trajectory ends in a critical point or intersects all the level-surfaces of the function S on $\bar{X}$. Consider now a cellular decomposition $\mathscr{C}$ (with the structure of complex) of X. For example, it can be a Morse decomposition defined as a collection of the surfaces of steepest descent starting from the critical points. This gives a collection of cells

$$\sigma_q^i, \quad q = 1, \ldots, m_i \ . \tag{2.3}$$

The complex $\mathscr{C}$ lifts to a complex $\widehat{\mathscr{C}}$ in $\bar{X}$ with a free action of the monodromy group $\mathbf{Z^n}$ on it. We can denote the cells of $\widehat{\mathscr{C}}$ by

$$t_1^{s_1} t_2^{s_2} \ldots t_n^{s_n} \sigma_q^i, \quad q = 1, \ldots, m_\iota \ , \tag{2.4}$$

then the generators $T_\iota^{\pm 1}$ of the monodromy group act by multiplication by $t_i^{\pm 1}$. We define the boundary operator on $\widehat{\mathscr{C}}$ by

$$\partial \sigma_q^i = \sum_p a_{pq}(t_1, \ldots, t_n) \sigma_p^{i-1} \ , \tag{2.5}$$

where the coefficients $a_{pq}(t_1, \ldots, t_n)$ are the formal series in t_i, t_i^{-1}. To be precise, let us give a definition

Definition. *A ring K_n (a Novikov ring) consists of all such formal power series in $t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}$ that the following two conditions are met:*

a) *There exists a number $N(a)$ such that if*

$$a = \sum u_{s_1,\dots,s_n} t_1^{s_1} \dots t_n^{s_n} , \tag{2.6}$$

then the coefficient $u_{s_1,\dots,s_n} = 0$ if $\sum_j s_j k_j < N(a)$.

b) *There is only a finite number of nonzero coefficients in any domain*

$$N_1 < \sum_j s_j k_j < N_2 . \tag{2.7}$$

Example. If $n = 1$, then

$$K_n = Z\left[t^{-1}, t\right] \equiv \left\{ a = \sum_{n \geqq N(a)} u_n t^n, \ u_n \in Z \right\} \tag{2.8}$$

– the Novikov ring coincides with all formal series with the finite negative part.

There is a natural embedding of the group ring of the monodromy group $\mathbf{Z}^\mathbf{k}$ to K_n:

$$0 \to Z[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}] \to K_n . \tag{2.9}$$

This embedding generates the local system $\mathscr{K}$ on X with coefficients in the ring K_n and the corresponding homology groups $H_*(X, \mathscr{K})$ are K_n-modules. It is the boundary operator of this local system complex that we have written in (2.5).

Having established the basic facts about Novikov rings in general situations, we are back to the loop spaces $\mathscr{L}M$. We have the canonical isomorphism[2]

$$H_1(\mathscr{L}M) = H_2(M) = \pi_2(M) . \tag{2.10}$$

To define a basic 1-form ω on the loop space $\mathscr{L}M$ let us notice that the value of vector field $\xi(z) \in \mathrm{Vect}(\mathscr{L}M)$ in the point $z = \{X(t), 0 \leqq t \leqq 2\pi\} \in \mathscr{L}M$ is a vector field $\xi(t)$ on the circle $z \subset M$. Then the value of the form ω on ξ in $z \in \mathscr{L}M$ equals

$$\omega(\xi)\ (z) = \oint_z k(\xi(t)) , \tag{2.11}$$

where k is a *Kähler* 2-form[3] on M. Now let us apply what we have developed above to the quantum cohomology ring. One notices immediately that H_Q^* is in fact already defined over K_n, if one identifies

$$t_i = e^{\oint_{\gamma_i} k} . \tag{2.12}$$

Indeed, both the fundamental two- and three-forms, defining H_Q^* take values in series in $t_1, \dots, t_n$ with integer coefficients and it is easy to see that actually they take values in K_n. Moreover, we know that there can only appear the *positive* degrees of the generators $t_1, \dots, t_n$ in all these formulas. This is because we only consider the *holomorphic* maps into M; for such maps the degree is always nonnegative. Thus in the definition of the Novikov ring given above, we can restrict ourselves to the

---

[2] We suppose that M is simply connected.

[3] That is, $[k] \in H^2(M)$ is positive on all the pseudoholomorphic curves in M.

series having no negative part. With this remark, we will use the term "Novikov ring" meaning the ring for the loop space constructed above.

From this point of view on the quantum cohomology we think of the variables $t_1, \ldots, t_n$ as indeterminates, but define H_Q^* over a bigger ring K_n instead of complex numbers C, so the Betti numbers do not change (generically).

3. A Brief Review of Floer Symplectic Cohomology

The purpose of this section is to briefly discuss the Floer theory in a way clarifying its resemblance to the quantum cohomology (modulo two obstacles, mentioned in the introduction). For example, from the very beginning we define it over the Novikov ring. Also, we don't try to give any proofs in this section, referring the reader to [7–9, 12, 16, 17, 21, 22].

Let M be a Kähler manifold with a Kähler form k. This closed 2-form defines a symplectic structure on M, providing a one to one map between vector fields v and 1-forms ω on M given by the formula $\omega = k(v, \cdot\,)$. A vector field v preserves k iff ω is closed; v is called *hamiltonian* iff $\omega = dH$ is exact. A function H is called a Hamiltonian.

The Hamiltonian equation

$$\frac{dX^i}{dt} = v^i(X, t) \tag{3.1}$$

defines a family $u_H(t)$ of diffeomorphisms of M, preserving k (called symplectomorphisms). They are characterized by the condition that $X^i(t) = u_H^i(t, X)$ solves (3.1) for all $X \in M$. The Floer theory studies fixed points of the period map Per: $u_H(t, X) \to u_H(t + 2\pi, X)$ for the Hamiltonian flows with periodic in t hamiltonians:

$$H(X, t + 2\pi) = H(X, t)\ . \tag{3.2}$$

Such points are in a one to one correspondence with the periodic trajectories of (3.1) having a period exactly 2π. It coincides with the ordinary Morse theory on M when H is independent of t, as the fixed points of $u_H(t)$ are just the critical points of $H(X)$ then. For the time dependent hamiltonians $H(X, t)$ the Floer theory is a sort of a Morse theory on the loop space $\mathscr{L}M$ of M.

Let us consider a (multivalued) function S_H on $\mathscr{L}M$,

$$S_H(z) = \int_{D^2} \phi^*(k) + \int_{S^1} H(X(t), t)dt\ . \tag{3.3}$$

Here z is a point in $\mathscr{L}M$

$$z = \{X(t) | X(0) = X(2\pi)\}\ , \tag{3.4}$$

and a smooth function ϕ furnishes a map of a disk $\phi : D^2 \to M$ with the boundary values $X(t)$, so $\partial D^2 = S^1$. Since $dk = 0$, S_H depends only on the homotopic type of ϕ with fixed boundary. This function becomes single valued on the minimal abelian cover $\widehat{\mathscr{L}M}$ of $\mathscr{L}M$ (with monodromy group $\mathbf{Z}^{\mathbf{k}}$, where $k = \operatorname{rank} H^2(M)$). When $\pi_2(M)$ has no torsion, $\widehat{\mathscr{L}M}$ coincides with the universal cover of $\mathscr{L}M$.

For a smooth vector field ξ on $\mathscr{L}M$ (in the point $z \in \mathscr{L}M$ it gives a vector field over the contour z in M) the ξ-derivative of S_H is well definite:

$$(\xi \cdot D)S_H(z) = \oint_{S^1} \{k(z'(t), \xi(t)) + dH(\xi(t))(X(t), t)\}dt \,. \tag{3.5}$$

A point z is a critical point of S_H iff (3.5) vanishes for all ξ, which happens iff $z = X(t)$ satisfies (3.1). As $z \in \mathscr{L}M$ is periodic with the period 2π by definition, it gives the fixed point of the period map we are after.

The trajectories of the gradient flow of S_H are the solutions[4] $X^i(t,\tau) : S^1 \times R \to M$ of the partial differential equation (1.2). Two terms

$$G^i(X(t,\tau)) = J^i_j \frac{\partial X^j}{\partial t} - \partial^i H(X,t) \tag{3.6}$$

may be considered as a vector field on $\mathscr{L}M$ evaluated at $z(\tau)$ so (1.2) is a gradient flow equation on $\mathscr{L}M$:

$$\frac{\partial z(\tau)}{\partial \tau} = -G(z) \,. \tag{3.7}$$

On the other hand, when $H(X,t) = 0$ (1.2) is just the Cauchy–Riemann equation for the holomorphic instantons.

The function $S_H(z)$ decreases along the trajectories of (3.7),

$$\frac{\partial S_H(z)}{\partial \tau} = -\oint_{S^1} \left| \frac{\partial X(t,\tau)}{\partial \tau} \right|^2 dt \,. \tag{3.8}$$

Thus the set of trajectories for which $\int_R \oint_{S^1} \left| \frac{\partial X(t,\tau)}{\partial \tau} \right|^2 dt d\tau$ is finite coincides with one of those for which S_H is bounded. Such trajectories connect the critical points of S_H. We define the Morse complex $\mathscr{C}_H$ as the set of bounded trajectories

$$\mathscr{C}_H = \{X(t,\tau) - \text{a solution of } (1.2) | \int_R \int_{S^1} \left| \frac{\partial X(t,\tau)}{\partial \tau} \right|^2 dt d\tau < \infty\} \,. \tag{3.9}$$

Let us define $\mathscr{C}_H(z_+, z_-)$ as a set of trajectories in $\mathscr{C}_H$ such that $z(\tau) \to z_\pm$ when $\tau \to \pm\infty$ and a set of k-trajectories $\mathscr{C}^k_H(x,y)$ going from x to y as the set of all k-tuples $z_1(\tau), \ldots, z_k(\tau)$ such that $z_i(\tau) \in \mathscr{C}_H(x_{i-1}, x_i)$, $x_0 = x$, $x_k = y$. A shift of the variable τ preserves $\mathscr{C}_H(z_+, z_-)$, so it makes sense to consider the quotient by the translational symmetry

$$\widehat{\mathscr{C}}_H(z_+, z_-) = C_H(z_+, z_-)/\mathbf{R} \,. \tag{3.10}$$

The set $\mathscr{X}$ of the critical points is graded by the analog of the Morse index μ. But unless $c_1(M) = 0$, i.e. unless M is a Calabi–Yau manifold, this is not a $\mathbf{Z}$

[4] They are also called the pseudoholomorphic instantons.

grading. Let $\Gamma \subset Z$ be a lattice generated by the set of periods of $c_1(M)$ on $\pi_2(M)$, then there is a function $\mu : \mathscr{X} \to Z/2\Gamma$ such that

$$\dim \mathscr{C}_H(x, y) = [\mu(x) - \mu(y)] \pmod{2\Gamma} . \tag{3.11}$$

There is the same situation for the quantum cohomology, where μ is called a fermionic number. The ambiguity in (3.11) occurs because [23, 7] a sequence of paths in $\mathscr{L}M$ can diverge by splitting off a (pseudo)holomorphic sphere (instanton) w, and for a joint of a path $z(\tau) \in \mathscr{C}_H(x, y)$ with w we have

$$\mu(z\#w) = \mu(x) - \mu(y) + 2c_1(w) . \tag{3.12}$$

This "splitting off a sphere" phenomenon also results in that the compactness properties of the cells $\widehat{\mathscr{C}}_H(z_+,z_-)$ are not so good as they are in the finite dimensional Morse theory. But it can happen only for those components with dimensions bigger than 2 (basically, because an S^1 action on the 2-sphere gives an additional degree of freedom). Thus, as it was in the finite dimensional case, the 0-dimensional component of $\widehat{\mathscr{C}}_H(z_+,z_-)$ is finite and the 1-dimensional component is compact up to the boundaries from $\widehat{\mathscr{C}}^2_H(z_+,z_-)$.

Now let Z^* be a free module generated by the critical points $\mathscr{X}$ over the Novikov ring K. This module is graded by μ. For every isolated trajectory $z(\tau)$ belonging to the 0-dimensional component of $\widehat{\mathscr{C}}_H(z_+,z_-)$, let $\sigma(z)$ denote its orientation Wi, F and $\rho(z) = t_1^{s_1(z)} \dots t_n^{s_n(z)}$ denote the homomorphism which the local system $\mathscr{K}$. (defined in Sect. 2) associates with the path $z(\tau)$. We define the matrix element of the coboundary operator by the formula

$$\langle \delta y, x \rangle = \sum_z \sigma(z)\rho(z) , \tag{3.13}$$

where the sum is taken over all isolated trajectories in $\mathscr{C}_H(z_+,z_-)$. Then the action of the coboundary operator on Z^* is given by

$$\delta y = \sum_{x \in \mathscr{X}} \langle \delta y, x \rangle x . \tag{3.14}$$

These formulas are to be compared with that obtained in [11] for the finite dimensional situation:

$$\langle \delta y, x \rangle = \sum_z \sigma(z) e^{h(y)-h(x)} . \tag{3.15}$$

When the Morse function is single valued, as in [11], the factors $e^h(x)$ can be absorbed by redefinition of the vertices x. When this is not the case, as in the Floer theory, the set of "phase factors" $\rho(z) = e^{h(y)-h(x)}$ forms the nontrivial 1-cocycle on $\mathscr{X}$ and cannot be canceled out by any renormalization. We see that the formula (3.15), which naturally appears in the context of supersymmetric quantum mechanics, knows already about the Novikov ring. The formula (3.13) should be considered as its counterpart for the topological sigma model where it computes the matrix element of the BRST operator between two wavefunctionals localized on the loops x and y respectively.

Computing the intersection numbers (weighed by ρ) of cycles in $\mathscr{L}M$ with cells $\mathscr{C}_H(z_+, z_-)$ of the Floer complex, we could define the cup product $\cup : H^*(\mathscr{L}M) \times H^*_F \to H^*_F$. But because of noncompactness the intersection numbers are only defined for the particular dual classes of $H^*(\mathscr{L}M)$, pulled back from M by the zero time evaluation map $\pi : z = \{X(t)|X(0) = X(2\pi)\} \to X(0)$. This is similar to what we have for the quantum cohomology where we should compute the intersections of only those cycles on the moduli space pulled back from M.

In order to define a restricted cup operation $\alpha\cup : H_F^* \to H_F^*$ we represent the cohomology class $\alpha \in H^p(M)$ by the dual singular simplex $\alpha : \bigcup_i \Delta_i^{|\alpha|} \to M$, $|\alpha| = \dim M - p$ and set

$$\alpha \cap \mathscr{C}_H(x, y) = \{(\lambda, z) \in \cup\Delta^{|\alpha|} \times \mathscr{C}_H(x, y) | \alpha(\lambda) = X(0)\} . \tag{3.16}$$

We have $\dim(\alpha \cap \mathscr{C}_H(x, y)) = p - \dim M$. Then we define the weighed intersection number as

$$\langle x, \alpha \cup y\rangle = \sum_p \sigma(p)\rho(p) , \tag{3.17}$$

where p runs over 0-dimensional part of $\dim(\alpha \cap \mathscr{C}_H(x, y))$ and $\sigma(p)$ is the usual relative orientation factor ± 1. The point p lies on one particular path $z_p(\tau)$ and

$$\rho(p) = \rho(z_p) = t_1^{S_1} \cdots t_n^{S_n} \tag{3.18}$$

is the homomorphism which the local system $\mathscr{K}$ associates to z_p. Finally, the cup operation $\alpha \cup : Z^* \to Z^*$ is defined as

$$\alpha \cup y = \sum_{x \in \mathscr{Z}} x\langle x, \alpha \cup y\rangle . \tag{3.19}$$

So defined the cup operation commutes with the coboundary operator δ and therefore descends to H_F^*. The subtlety here is that for two arbitrary $x, y \in Z^*$ the matrix element (3.17) depends on α itself, not only on its cohomology class[5]. It becomes independent of the choice of any particular representative for $[\alpha] \in H^*(M)$ only after we descend from Z^* to H_F^*. We will understand this better in the next section, in terms of decoupling of BRST-trivial states from the correlation functions of topological sigma model.

From the point of view of the topological sigma model the bracket $\langle x, \alpha \cup y\rangle \equiv \langle x|\alpha|y\rangle$ should be interpreted as a matrix element of the operators corresponding to α, taken between vacua x and y. In the next section we give such interpretation and relate these matrix elements with three point correlation functions of the A-model.

In the Floer theory, we consider the elements of de Rham cohomology of M as linear operators, acting on H_F^*, so there is a homomorphism

$$v : H^*_{\mathrm{de\,Rham}}(M) \to \mathrm{End}(H_F^*) . \tag{3.20}$$

It is not obvious that the image of v is a *ring*, i.e. that it is preserved by the operator multiplication[6]. We shall see it is true only when we identify this image as the operator algebra of topological A-model, which is closed (and associative [19]). It would be very interesting to be able to prove this fact directly from the definitions (3.17), (3.19).

---

[5] I thank D. Kazhdan who pointed out this fact.

[6] I thank I. Singer and C. Taubes for the discussions that helped to realize importance of this.

4. Floer Theory as a Topological Quantum Field Theory

4.1. Pseudoholomorphic Instantons in the Topological A-Model. In this section we give a physical interpretation of the Floer theory in terms of the topological sigma model [4, 18, 5]. It leads to identification of H_F^* as a quantum cohomology ring.

As usual, we start from the $N = 2$ supersymmetric sigma model and perform a topological twist so that one of the SUSY generators becomes a 1-form. Then the corresponding charge is the BRST operator Q. The $N = 2$ chiral multiplet of fields of the model contains

$$\begin{aligned} \text{Bosons: } & \text{world sheet scalar } X^i - \text{target space coordinates}\,, \\ & \text{world sheet one form } F^i_\alpha - \text{target space vector}\,, \\ \text{Fermions: } & \text{world sheet scalar } \chi^i - \text{target space vector}\,, \\ & \text{world sheet one form } \rho^i_\alpha - \text{target space vector}\,. \end{aligned} \tag{4.1}$$

The field F^i_α is what is called the auxiliary field. Both ρ^i_α and F^i_α satisfy a self duality constraint

$$\begin{aligned} \rho^i_\alpha &= i\varepsilon^\beta_\alpha J^i_j \rho^j_\beta\,, \\ F^i_\alpha &= \varepsilon^\beta_\alpha J^i_j F^j_\beta\,. \end{aligned} \tag{4.2}$$

The name "auxiliary" stresses that F^i_α serves to close the $N = 2$ algebra off shell and that it can be set to zero on shell. Here we want to show how it can be used to localize the path integral of the topological theory to the pseudoholomorphic instantons satisfying (1.2) with a nontrivial right-hand side.

The BRST action on the fields of the multiplet is given by

$$\begin{aligned} &[Q, X^i] = i\chi^i\,, \\ &\{Q, \chi^i\} = 0\,, \\ &\{Q, \rho^i_\alpha\} = F^i_\alpha + \partial_\alpha X^i + \varepsilon^\beta_\alpha J^i_j \partial_\beta X^j - i\Gamma^i{}_{jk}\chi^j\rho^k_\alpha + \frac{i}{2}\varepsilon^\beta_\alpha D_k J^i_j \chi^k \rho^j_\beta\,. \end{aligned} \tag{4.3}$$

We don't need an awkward explicit formula for the commutator $[Q, F^i_\alpha]$; it is enough to know that it is fermionic and equals to zero on the subvariety $\chi^i = 0$, $\rho^i_\alpha = 0$ in the field space.

The local physical operators (observables) of the model are the BRST cohomology, isomorphic to de Rham cohomology of M [4, 18]. To any p-form $\omega = A_{i_1} \dots {}_{i_p} dx^{i_1} \dots dx^{i_p}$ there corresponds an operator[7] $\mathcal{O}_\omega = A_{i_1} \dots {}_{i_p} \chi^{i_1} \dots \chi^{i_p}$ and

$$[Q, \mathcal{O}_\omega] = \mathcal{O}_{d\omega}\,. \tag{4.4}$$

The isomorphism $H^*_{\mathrm{BRST}} = H^*_{\mathrm{de\,Rham}}(M)$ follows from (4.4). It is important to note that we computed the *off-shell* BRST cohomology. We see that the auxiliary field F^i_α does not show up in the formula for $\mathcal{O}_\omega$. (It does appear in the nonlocal physical

[7] This local operator is a scalar on the world sheet. Besides, there are the nonlocal physical operators which are integrals of 1- and 2-forms, the whole hierarchy related by the "descent equation" [5].

operators though. But in this paper we are only concerned with the local operators, which by the state operator correspondence are related to the ground states.)

The next step is the computation of the matrix elements of the physical operators (the correlators). First of all, we should specify our two dimensional action. The standard choice, coming from $N = 2$ sigma model, is

$$A_0 = \int X^*(k) + \left\{Q, \int \tfrac{1}{2} g^{\alpha\beta} G_{ij} \rho^i_\alpha \left(\partial_\alpha X^j - \tfrac{1}{2} F^j_\beta\right)\right\} . \tag{4.5}$$

Then using the BRST fermionic symmetry, the path integral

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \int \mathcal{D}X^i \mathcal{D}\chi^i \mathcal{D}\rho^i_\alpha \mathcal{D}F^i_\alpha\, \mathcal{O}_1 \cdots \mathcal{O}_m e^{A_0[X,\chi,\rho,F]} \tag{4.6}$$

can be localized [5] on the variety $\mathscr{E}_0$ of fixed points of Q (classically, we should treat Q as a vector field in the field space).

But first we want to get rid of the auxiliary field F^i_α. It can be done either by taking the Gaussian in F^i_α integral (4.6) or by using the equations of motion for F^i_α. If we choose the standard action (4.5), the (algebraic) equations of motion give $F^i_\alpha = 0$ so in (4.6) we can drop simultaneously the integration in F^i_α and the F^i_α-dependent piece of the action in the exponent.

From (4.3) we see that on $\mathscr{E}_0$ the fermions vanish: $\chi^i = 0$, $\rho^i_\alpha = 0$ and the fields $X^i(z, \bar{z})$ satisfy the Cauchy–Riemann equation for the holomorphic maps from the world sheet into M.

Up to now this was a well known story [5] about A-model, leading to the notion of the quantum cohomology ring, characterized by the two- and three-point correlation functions on sphere. At this moment we can note that the auxiliary field F^i_α is bosonic. Hence there is nothing wrong if it has a nontrivial expectation $\Phi^i_\alpha(X, z, \bar{z})$ on shell. The expectation $\Phi^i_\alpha(X, z, \bar{z})$ should satisfy the same self duality condition (4.2) as F^i_α. We can *actually* give to F^i_α the expectation $\Phi^i_\alpha(X)$ on shell, if we add to the action A_0 a BRST trivial piece

$$A = A_0 + \{Q, \int g^{\alpha\beta} \rho^i_\alpha \Phi^i_\beta\} . \tag{4.7}$$

In the sense of topological theory, the deformation (4.7) is trivial. We explicitly see that the *local* physical operators are independent of Φ^i_α, since *off-shell* they do not contain F^i_α. Also, all the correlation functions (4.6) remain the same. This last point is not quite trivial because of the possible problems with noncompactness. In Sect. 6 we argue that these problems do not really appear in our setup.

The conditions (4.3) give now for the X^i fields on the fixed points locus $\mathscr{E}_H$ the equation

$$\partial_\alpha X^i + \varepsilon^\beta_\alpha J^i_j \partial_\beta X^j = -\Phi^i_\alpha \tag{4.8}$$

(and the fermions vanish on $\mathscr{E}_H$ as before).

To study the Floer theory, we only need the case when the worldsheet is a cylinder $S^1 \times R^1$. Then we can introduce the global coordinates ($t \in S^1, \tau \in R^1$), the same as in Sect. 3. The field Φ^i_α has two components, related to each other by (4.2):

$$\Phi^i_t = J^i_j \Phi^i_\tau . \tag{4.9}$$

On the cylinder we can consistently take $\Phi^i_\alpha(X, t)$ to be a (periodic) function of the space coordinate t independent of the time coordinate τ. Then the energy is conserved.

Also, we need to consider only the *hamiltonian* vector fields Φ^j_τ, that is

$$\Phi^j_\tau = -\partial^j H \ . \tag{4.10}$$

The function $H(X,t)$ is a hamiltonian. Then (4.8) is just the pseudoholomorphic instantons equation (1.2) we know from the Floer theory.

The only geometry of the world sheet we need to consider to compute the matrix elements (3.13), (3.17) in the Floer theory is that of the cylinder. Suppose now that our A-model lives over an arbitrary Riemann surface. Let us briefly discuss what are the restrictions on $\Phi^i_\alpha(X,z,\bar z)$. First, there is no good way to divide globally the coordinates into space and time. If we want, as we usually do, that the energy be conserved we have to consider only the coordinate independent fields $\Phi^i_\alpha(X)$. Second, the consistency of the fermionic sector now requires that *both* components of $\Phi^i_\alpha(X)$ be the hamiltonian fields. This condition, together with selfduality (4.2), forms an overdetermined system of equations for two hamiltonians $H_\alpha(X)$. The compatibility condition for this system is equivalent to the requirement for J^i_j to be *a complex structure* on M.

4.2. The Operators and the States . The family $\mathscr{E}_H$ of deformations of the standard variety $\mathscr{E}_0$ gives a family of localizations of the same topological field theory. Hence the set of the observables for $\mathscr{E}_H$ is the same as for $\mathscr{E}_0$ and coincides with the de Rham cohomology $H^*_{\mathrm{de\,Rham}}(M)$. What is new is that for generic $H \neq 0$ it is possible to localize the *states* in the theory to the critical loops set $\mathscr{X}$ of the Floer complex. Indeed, the action is (we work on the cylinder):

$$A = \int \{(\partial_\tau X^i + J^i_j \partial_t X^j + \Phi^i_\tau)^2 + i g^{\alpha\beta} \rho^i_\alpha D_\beta \chi^i + \cdots\} dt d\tau \ , \tag{4.11}$$

so the bosonic piece of the potential energy of the string configuration z is given by

$$E[z] = \oint \left(J^i_j \frac{dX^i}{dt} + \Phi^i_\tau \right)^2 dt \ . \tag{4.12}$$

The set of the minima of $E[z]$ coincides with $\mathscr{X}$. Note that the potential energy functional comes entirely from the BRST trivial piece of the action. Making the coefficient before it arbitrarily large we make the walls around minima arbitrarily steep, so classically the string never flies away from the minima. Thus the wave-functionals Ψ of the physical states should be the linear combinations of those $|x_i\rangle$ localized to $x_i \in \mathscr{X}$,

$$\Psi = \sum_{x_i \in \mathscr{X}} \lambda_i |x_i\rangle \ , \tag{4.13}$$

additionally satisfying the BRST condition $Q\Psi = 0$, modulo Q-trivial vectors.

The quadratic in fermions piece of the action is (we have used the equations of motion for X^i to simplify it)

$$A_F = \int \{-iG_{ij} \rho^i_\tau \partial_\tau \chi^j + \rho^i_\tau \hat M_{ij} \chi^j)\} dt d\tau \ . \tag{4.14}$$

The mass operator $\hat M_{ij}$ here is given by

$$\hat M_{ij} = J_{ij} \partial_t + (D_k J_{ij}) J^{km} \Phi_{m\tau} + D_i \Phi_{j\tau} \ . \tag{4.15}$$

Note that the mass operator (4.15) is just a Hessian of the Floer functional $S[z]$:

$$\hat{M}_{ij} = \frac{\delta^2 S[z]}{\delta X^i \delta X^j} \, . \tag{4.19}$$

From the usual interpretation in the quantum field theory, we conclude that the modes having positive masses (eigenvalues of M_{ij}), correspond to the particles and those having negative masses correspond to antiparticles[8]. In order to have the stable *fermionic* vacuum at each minimum $z \in \mathscr{Z}$ of E, the Dirac sea of antiparticles should be completely filled. The different minima $x, y \in \mathscr{Z}$ have the different mass matrices M_{ij} and hence the different fermionic vacua. Their fermionic numbers differ by $\mu(x) - \mu(y)$ modulo the anomaly lattice 2Γ (generated by evaluation of $2c_1(M)$ on the group $\pi_2(M)$).

It should also be possible to say this another way [24, 25], using semi-infinite differential forms on the loop space $\mathscr{L}M$. The Hessian (4.19) defines a polarization of the tangent bundle $T\mathscr{L}M$ at the critical point

$$T\mathscr{L}M = T\mathscr{L}M_- \oplus T\mathscr{L}M_+ \, . \tag{4.20}$$

In turn, it gives a polarization in the Clifford algebra *Cliff* of $T\mathscr{L}M$ (*Cliff* is generated by taking the canonical anticommutators for the fields χ^i and ρ^i_τ). We may consider a Verma module of *Cliff* associated with this polarization and formally present its vacuum vector as $\det T\mathscr{L}M_-$ – a "semi-infinite form". For the different critical points in $\mathscr{Z}$, the corresponding polarizations in *Cliff* are hopefully compatible with each other and it is possible to define (modulo 2Γ) a relative degree $\mu(x) - \mu(y)$ of two different vacua x and y.

To find the physical states (4.13) we should compute the BRST cohomology on the space of the wavefunctionals localized to $\mathscr{Z}$. The action of the BRST operator Q on their space is encoded in the matrix elements $\langle x|Q|y\rangle$, where $x, y \in Z$. We use the path integral representation

$$\langle x|Q|y\rangle = \int \mathscr{D}X^i \mathscr{D}\chi^i \mathscr{D}\rho^i_\chi Q e^{A[X,\chi,\rho]} \, , \tag{4.21}$$

where the path integral is computed with the boundary conditions

$$\begin{aligned} X(\tau = -\infty, t) &= x(t) \, , \\ X(\tau = +\infty, t) &= y(t) \, , \end{aligned} \tag{4.22}$$

and localize it to the pseudoholomorphic instantons. The term multiplying the exponent is the BRST charge

$$Q = \oint \left(J_{ij}\{\partial_\tau X^i + J^i_k \partial_t X^k + \Phi^i_\tau\}\chi^j + \tfrac{1}{2} D_k J_{ij} \rho^i_\tau \chi^k \chi^j \right) dt \tag{4.23}$$

with fermionic number 1, so (4.21) is zero unless the space of the instantons connecting x to y is one dimensional modulo 2Γ. It means that $\mu(x) - \mu(y) \equiv 1$ and that (4.21) is localized to the sum over the same instantons as appear in the expression (3.13) for the coboundary δ of the Floer complex. For each such instanton the Q-nontrivial piece of the action A gives a factor $\rho = t_1^{s_1} \cdots t_n^{s_n}$ exactly the same

[8] In the language of the Morse theory, the particles correspond to the stable and antiparticles to the unstable cells of the minimum z.

as the multiplier of the local system $\mathscr{K}$. The integration over fermions brings the factor $\sigma = \pm 1$ the same as in (3.13). We see that the matrix elements (4.21) of the BRST operator Q coincide with that (3.13) of the coboundary operator δ of the Floer complex. Thus the Floer cohomology H_F^* computes just the physical states of our topological sigma model.

Actually, we can find the matrix elements of any observable $\mathcal{O}_\omega$ in the same fashion:

$$\langle x|\mathcal{O}_\omega|y\rangle = \int \mathscr{D}X^i \mathscr{D}\chi^i \mathscr{D}\rho_\alpha^i \mathcal{O}_\omega e^{A[X,\chi,\rho]} , \tag{4.24}$$

computing the path integral with the same boundary conditions (4.22) and localizing it to the instanton configurations. If ω has a fermion number p, then (4.24) is zero unless the space of instantons connecting x to y is p-dimensional, so $\mu(x) - \mu(y) = p$. Repeating the computation for the matrix elements of Q we see that (4.24) coincides with the matrix element (3.17) of the operator $\omega \cup$ in the Floer theory.

Now we can understand better the remark following the formula (3.17). Unless $Q|x\rangle = Q|y\rangle = 0$, i.e. unless $|x\rangle$ and $|y\rangle$ are the physical states, the matrix element $\langle x|\mathcal{O}_{d\Omega}|y\rangle \neq 0$ in general, so (4.24) depends on the choice of the representative $\mathcal{O}_\omega$ for the BRST cohomology class or equivalently, on the choice of the representative ω for de Rham cohomology $H^*_{\text{de Rham}}(M)$. Only after we restrict to the physical states (4.13) that the matrix elements of the observables become the functions on $H^*_{\text{de Rham}}(M)$.

So far we dealt with operators and states independently. But it is a general fact that in the topological theories there is a one to one correspondence between the operators and states. Now we want to work out this correspondence explicitly. It will enable us to identify the *matrix elements* (4.24) with the *3-point correlation functions* in the A-model and thereby to establish the isomorphism of the Floer's and quantum cohomology.

To do that, we specify to the hamiltonians $H(X)$ independent of t and such that the corresponding Hamiltonian flows on M have no periodic trajectories at all[9]. For such $H(X)$, the critical loops of the Floer functional $S[z]$ are just the points and coincide with the critical points of $H(X)$ on M and the critical set $\mathscr{X}$ is described by the usual Morse theory. In [11], which we try to generalize in this paper, the Morse theory on M is related to Supersymmetric Quantum Mechanics (SQM). This SQM is nothing but the zero-modes approximation of the string theory, described by our topological sigma model.

Let us show that the SQM approximation for the matrix elements of the BRST operator Q is exact. To see it is true it suffices to show that the instantons, which appear in (3.13), (4.21) are point-like, i.e. correspond to the propagation of the string as a point, not as a loop. It would mean that only the zero modes are important. But this follows from the fact that the relevant in (3.13) instantons are *isolated*, i.e. belong to the one dimensional cell of the Floer complex. If an instanton could be represented as a joint of a point-like trajectory with a 2-sphere, there would be 2-parametric freedom to move this sphere around, so such an instanton would belong to at least a 2-dimensional cell. This proves that the matrix elements of the coboundary operator of the Floer complex coincide with that of the coboundary operator of the Witten complex on M, so their cohomology, as abelian groups, are canonically isomorphic. This is the statement of

[9] It is always possible to choose a pair J_t^i, H such that this condition is met, see [7].

the Theorem 5 of [7], and our dimension-counting argument is borrowed from its proof.

Of course, the matrix elements (4.24) of the observables $\mathcal{O}_\omega$ are localized to the 2- and higher dimensional cells of the Floer complex and cannot be computed just by SQM.

We are ready now to construct a canonical isomorphism between the states H_F^* and the operators $H^*(M)$ of the topological sigma model. Let us define two nondegenerate pairings of K-modules (K is a Novikov ring)

$$(.,.) : H_F^p \otimes_K H^p(M) \to K \; ,$$
$$\{.,.\} : H_F^{n-p} \otimes_K H^p(M) \to K \tag{4.25}$$

in the following way. The lowest- and highest degree cohomology $H_W^0 \subset H_F^0$ and $H_W^n \subset H_F^n$ of the Witten complex on M are always generated by one element each. Let us call them bot and top respectively. They are the in and out vacua of the theory dual to each other:

$$\langle \text{bot}|\mathcal{O}_\Omega|\text{top}\rangle = 1 \; , \tag{4.26}$$

where Ω is the top class in $H^n(M)$ and the correlation functions (4.6) of the sigma model are

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \langle \text{bot}|\mathcal{O}_1 \cdots \mathcal{O}_m|\text{top}\rangle \; . \tag{4.27}$$

The formula (4.27) is the fundamental relation in the quantum field theory. Then the pairings we want to define are:

$$(\omega, x) = \langle \text{bot}|\mathcal{O}_\omega|x\rangle \; , \tag{4.28}$$

where $\omega \in H^p(M)$, $x \in H_F^p$ and

$$\{\omega, y\} = \langle y|\mathcal{O}_\omega| \text{ top}\rangle \; , \tag{4.29}$$

where $\omega \in H^p(M), y \in H_F^{n-p}$. Both these pairings are nondegenerate, because in the SQM approximation, when $t_i = 0$ for all i, the pairing $(.,)$ is the Poincaré duality and $\{.,\}$ is the Poincaré isomorphism (using the canonical duality $(H_p(M))^* = H^p(M)$). Hence their determinants, as functions of $t_1, \dots, t_m$, are not equal to zero.

These pairings are related to the two-point correlation functions of observables (the quantum intersection numbers)

$$\langle \mathcal{O}_{\omega_1} \mathcal{O}_{\omega_2} \rangle = \sum_{x \in H_F^*} \langle \text{bot}|\mathcal{O}_{\omega_1}|x\rangle\langle x|\mathcal{O}_{\omega_2}| \text{ top}\rangle = \sum_{x \in H_F^*} (\mathcal{O}_{\omega_1}, x)\{\mathcal{O}_{\omega_2}, x\} \; , \tag{4.30}$$

which is just a statement of completeness of the physical states of the quantum theory. Thus the pairings above give both the isomorphism between the operators and the states: $h^p : H_F^p = H^p(M)$ *and* the quantum intersection matrix. The 3-point functions are related to the matrix elements (4.24) by the formula

$$\langle \mathcal{O}_{\omega_1} \mathcal{O}_{\omega_2} \mathcal{O}_{\omega_3} \rangle = \sum_{x, y \in H_F^*} (\mathcal{O}_{\omega_1}, x)\langle x|\mathcal{O}_{\omega_2}|y\rangle\{\mathcal{O}_{\omega_3}, y\} \; . \tag{4.31}$$

Note that the (super)commutativity and associativity of the algebra of observables $\mathcal{O}_\omega$ give the relations for the matrix elements (4.24), equivalent to the super-commutativity and associativity of the cup operation $\omega\cup$ in the Floer theory.

In other words, the cohomology $H^*(M)$ with the cup product multiplication from the Floer theory is isomorphic to the quantum cohomology ring.

5. Some Examples

Now we would like to present some computations, partly described in [7], showing how our "matrix" approach really works.

5.1. Projective spaces. This is the simplest possible example. The homology H_F^* and the action of $H^*_{\text{de Rham}}(\mathbf{CP}^1)$ on it were computed by Floer himself in [7]. It is very instructive to repeat his argument.

Let us take the hamiltonian vector field

$$v = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\,. \tag{5.1}$$

It does not have periodic trajectories and only fixed points are the north (N) and the south (S) poles of the sphere $S^2 = \mathbf{CP}^1$. Thus in the loop space $\mathscr{L}\mathbf{CP}^1$ the critical set of the Floer functional (3.3) consists just of these two points. By the universal cover map $\widehat{\mathscr{L}\mathbf{CP}^1} \to \mathscr{L}\mathbf{CP}^1$ the set (N,S) is covered by the points $(\dots,N^{(-1)},S^{(-1)},N^{(0)},S^{(0)},N^{(1)},S^{(1)},\dots)$ so that $N^{(k)} \to N$ and $S^{(k)} \to S$. In fact, this picture represents [25] the affine Weyl diagram for $\widehat{sl(2)}$.

The trajectories going from $N^{(k)}$ to $S^{(k)}$ are the "classical" ones. Downstairs they cover a 2-parametric set of trajectories of the vector field v. As a set of points, this set coincides with the base projective line itself.

On the other hand, the trajectories from $S^{(k)}$ to $N^{(k+1)}$ are essentially stringy – they have homotopic type of $\mathbf{CP}^1$. Again, there is a 2-parametric family of them due to the action of $\mathbf{C}^*$ on the world sheet (a sphere with two marked points). Each such trajectory covers the base $\mathbf{CP}^1$.

Again, this can be represented by the affine Weyl diagram, if we denote the 2-dimensional cells of the Floer complex by the arrows connecting the vertices $(\dots,N^{(-1)},S^{(-1)},N^{(0)},S^{(0)},N^{(1)},S^{(1)},\dots)$.

As in the usual Morse theory for $\mathbf{CP}^1$, there are no 1-dimensional cells, hence the coboundary operator is trivial and the Floer cohomology is represented by N (of degree 0) and S (of degree 2). Integrated over $\mathbf{CP}^1$, the first Chern class of the tangent bundle gives 2, so the fermionic number anomaly is 4.

Now let us find the matrix elements of the generator x of $H^*_{\text{de Rham}}(\mathbf{CP}^1)$ between S and N. The element $\langle N|x|S\rangle$ comes from integration of (the pullback of) x on $\mathscr{C}^\varepsilon(N,S) = \mathbf{CP}^1$, each trajectory is homotopicaly trivial in the loop space. Therefore, the "classical" answer $\langle N|x|S\rangle = 1$ holds true.

Unlike the usual Morse theory, there is also a nontrivial matrix element $\langle S|x|N\rangle$. Formally it is possible because $\mu(N) - \mu(S) = 2 - 4 = -2 \equiv 2(\text{mod}\,4)$. There is indeed a 2-parametric family of stringy trajectories from S to N as we saw above. The integral of the pullback of x over it is again the integral of x over $\mathbf{CP}^1$. But now the homotopic type of each trajectory is not trivial and to obtain the matrix element (3.17) we need to multiply the integral of x by (3.18) equal to $t = \exp - \int k$. Thus $\langle S|x|N\rangle = t$. Therefore, the matrix representation of the operator $\mathcal{O}_x$ is

$$\mathcal{O}_x = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} . \tag{5.2}$$

It satisfies the relation

$$\mathcal{O}_x \mathcal{O}_x = t , \tag{5.3}$$

well known for $\mathbf{CP^1}$ sigma model.

A straightforward generalization of the example above is $\mathbf{CP^n}$. It was also done in [7]. Again, there are no 1-dimensional cells in the Floer complex and H_F^* is spanned by the critical points of (S^1 equivariant) Morse function on $\mathbf{CP^n}$ of degrees $0, 2, \ldots, 2n$. The fermionic number anomaly is $2(n+1)$.

The de Rham cohomology $H^*_{\text{de Rham}}(\mathbf{CP^n})$ is generated as a ring by one element x. We can again find the matrix representation for the operator $\mathcal{O}_x$, which generates the *quantum* ring. In this example it is also true that $\mathcal{O}_{x^k}\mathcal{O}_{x^l} = \mathcal{O}_{x^{k+1}}$, for $k + l < n + 1$; nothing like this should be true in general.

There are two simple remarks we want to make before we write the formula for $\mathcal{O}_x$. Let M be an almost Kähler manifold with $c_1 > 1$ and $x \in H^2(M)$. Then the matrix elements of x on the main diagonal and above, like $\langle z_m|x|z_{m'}\rangle$, $m' \geqq m$, do not depend on $t_1, \ldots, t_n$ and can be computed classically. Indeed, to get the matrix element the pullback of x should be integrated over some two-cycle of the Floer complex. But if this cycle consisted of stringy trajectories, (which would lead to t_i dependence), then by the index theorem it would belong to the cell of the dimension at least 4. The other simplification for the matrix elements of $x \in H^2(M)$ is that the 2-cells of the Floer complex, for the purposes of intersection theory, are representable by the 2-cells on M itself, just like it was for $\mathbf{CP^1}$. This is because these 2-cells consist either of point-like classical trajectories or of a single 2-sphere in M, reparametrized by $\mathbf{C}^*$.

All the matrix elements below the main diagonal are always due to the stringy paths. The formula for $\mathcal{O}_x$ is

$$\mathcal{O}_x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ t & 0 & \cdots & 0 & 0 \end{pmatrix} . \tag{5.4}$$

To obtain (5.4) we note that as $c_1 = (n+1)x$, only $\langle z_{2n}|x|z_0\rangle$ can be a nontrivial matrix element below diagonal (as $0 - 2n + 2(n+1) \cdot 1 = 2$). The 2-cell $\mathscr{C}^2(z_{2n}, z_0)$ consists of paths from z_{2n} to z_0 of degree 1 and coincides with a straight line (z_{2n}, z_0) for the appropriate choice of hamiltonian. It explains the degree 1 of t as well as a numerical coefficient 1 before it in (5.4). The operator $\mathcal{O}_x$ satisfies

$$\mathcal{O}_x^n = t . \tag{5.5}$$

5.2. Flag Spaces. A less trivial generalization of the example with $\mathbf{CP^1}$ is the flag space Fl_n, which can be realized as a coset

$$Fl_n = U(n)/(U(1))^n . \tag{5.6}$$

Its second cohomology group is $\mathbf{Z^{n-1}}$ generated by $x_1, \ldots, x_{n-1}$. Cohomology of Fl_n, as a ring [26], is generated by x_i with the relations which are homogeneous components of the single relation

$$\prod_{i=1,\dots,n} (1+x_i) = 1\,. \tag{5.7}$$

We will compute the matrix elements of x_1, x_2 in the Floer theory for the simplest nontrivial example of 3-dimensional flag space Fl_3. A way we do this, using an affine Weyl group for $\widehat{sl(3)}$, can be generalized for all other flag spaces.

The Floer theory for the loop spaces of flags was considered in [25]. The critical point set in $\mathscr{L}Fl_n$ is parameterized by the elements of the *finite* Weyl group. The cover map $\widehat{\mathscr{L}Fl_n} \to \mathscr{L}Fl_n$ covers it by the *affine* Weyl group. A simplest example to look at is $Fl_2 = \mathbf{CP}^1$, considered in the previous section. There are no 1-cells in the Floer complex; to describe all the 2-cells we need an auxiliary geometric construction.

Let us think of the elements x_1, x_2 as of two simple roots of $sl(3)$. There are two standard embeddings of $sl(2)$ to $sl(3)$ sending a simple root of $sl(2)$ either to x_1 or to x_2. These maps can be continued to the maps of groups $SL(2) \to SL(3)$ which, in turn, induce two maps $\mathbf{CP}^1 \to Fl_3$ which send the generator x of $H_2(\mathbf{CP}^1)$ to either x_1 or x_2.

This construction extends to give the maps of the (universal covers of) *loop* spaces. Looking at the diagram for the affine Weyl group for $\widehat{sl(3)}$[10] we see that it can be covered by the straight lines parallel to any simple root. Each such straight line corresponds to some particular embedding $\widehat{\mathscr{L}\mathbf{CP}^1} \to \widehat{\mathscr{L}Fl_3}$. A pattern of critical points along the straight line coincides with that one for $\widehat{\mathscr{L}\mathbf{CP}^1}$: when projected downstairs to the base flag space Fl_3 it covers *two* (which ones, depends on the particular straight line) critical points we denote by N, S. These two points we can identify with the critical points on the preimage $\mathscr{L}\mathbf{CP}^1$. The arrangement of critical points upstairs is $(\dots, N^{(-1)}, S^{(-1)}, N^{(0)}, S^{(0)}, N^{(1)}, S^{(1)}, \dots)$.

Now let us choose on Fl_3 such a hamiltonian vector field that it respects both embeddings $\mathbf{CP}^1 \to Fl_3$. In the finite-dimensional Morse theory it means that the *whole flow*, connecting N to S, belongs to the preimage $\mathbf{CP}^1$. Then this property is promoted to the loop spaces. It means that the matrix elements of x_1, x_2 between N and S can effectively be computed within a Floer theory for a projective line $\mathbf{CP}^1$. Not every pair of critical points whose indices differ by 2(mod4) lies on a straight line parallel to a simple root. For them, the degree count shows that only the matrix elements $\langle \mathrm{top}|x_i|\mathrm{bot}\rangle \propto t_1 t_2$ can be non-zero, and the commutativity of two matrix operators $\mathcal{O}_{x_1}$, $\mathcal{O}_{x_2}$ fixes the integral coefficients in front of $t_1 t_2$.

Ultimately, the operators $\mathcal{O}_{x_1}$ and $\mathcal{O}_{x_2}$ are given by

$$\mathcal{O}_{x_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ t_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ t_1 t_2 & 0 & 0 & 0 & t_1 & 0 \end{pmatrix}, \tag{5.8}$$

[10] It is a 2-lattice with hexagonal point group.

$$\mathcal{O}_{x_2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ t_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -t_2 & 0 & 0 & 0 & 0 \\ -t_1 t_2 & 0 & 0 & -t_2 & 0 & 0 \end{pmatrix} . \tag{5.9}$$

They generate an algebra with relations

$$\begin{aligned} &\mathcal{O}_{x_1}^2 + \mathcal{O}_{x_1}\mathcal{O}_{x_2} + \mathcal{O}_{x_2}^2 = (t_1 + t_2), \\ &\mathcal{O}_{x_1}^3 = t_1\mathcal{O}_{x_1} - t_1\mathcal{O}_{x_1}, \\ &\mathcal{O}_{x_2}^3 = t_2\mathcal{O}_{x_2} - t_2\mathcal{O}_{x_1}, \\ &\mathcal{O}_{x_1}^2\mathcal{O}_{x_2} + \mathcal{O}_{x_1}\mathcal{O}_{x_2}^2 = t_2\mathcal{O}_{x_1} + t_1\mathcal{O}_{x_2} , \end{aligned} \tag{5.10}$$

which can be considered as a two-parametric deformation of (5.7).

5.3. Calabi–Yau Manifolds. Almost all the known nontrivial examples of quantum cohomology of CY manifolds are obtained using mirror symmetry. Mirror symmetry reduces the problem of computation of quantum cohomology $H_Q^*(M)$ to some computation in the Variation of Mixed Hodge Structure Theory of the mirror pair W. The latter problem is usually relatively easier.

As a first application of the general theory we may just transfer all these results to the Floer theory, which give the first examples of the latter for the CY manifolds. Note, that unless we consider the Floer complex over the Novikov ring, the boundary operator is not defined – its matrix elements are represented by the divergent series due to summation over all maps of the cylinder to itself. The same is true for the matrix elements of de Rham cohomology.

On the other hand, in the string theory computations for the 3-dimensional CY, we are mostly interested in the matrix elements of the second de Rham cohomology ($H^{1,1}(M)$, to be precise), which generate the marginal deformations of the corresponding superconformal theory. A remark in the section about projective spaces tells that these can be computed just looking at the 2-cycles of the CY manifold itself, without going up to the loop space. We hope it may make it possible to do a direct computation in some examples.

6. Conclusion: Noncompactness and Non-Generic Moduli of Instantons

Roughly speaking, there are two ingredients in the Floer theory. One of them, the algebraic one, is very neat; we tried to stress it in Sect. 3. The second – analytic – ingredient is what makes the Floer theory so difficult. In this paper we tried to substitute the language of the topological quantum field theory for the analytic language of Floer *et. al.* From the mathematician's point of view, it may seem as trading a bad thing for the worse one.

A real advantage of our approach is that it makes the independence of H_F^* of the hamiltonian H and the almost complex structure J an immediate consequence of the BRST invariance, *without any assumptions about H.* In particular, the choice $H = 0$ is admissible. On the other hand, for $H = 0$ and J being a true complex structure the path integral defining the sigma model has a combinatorial definition [5] in terms of the moduli spaces of holomorphic instantons. It is an object of study of algebraic geometry which does not require infinite-dimensional analysis to deal with.

We hope that it may be helpful to think about the same object using two languages. For the topological theory, it may also be useful to have a picture where all the physical states are localized to the particular loops.

There are two fine points which we want to touch upon in conclusion. The first is a problem of non-compactness of moduli space of (almost) holomorphic instantons. The effects on the "infinity" can be crucial when considering the effects of "BRST trivial" deformations. The Landau–Ginzburg A_n model with superpotential $W = x^{n+2}$ is an example. The deformation $W \to W + \varepsilon x^{n+3}$ is "BRST trivial" but changes the theory completely: $A_n \to A_{n+1}$. It is important to realize that the problem in this example appears already at the level of BRST cohomology: the deformation brings "from infinity" one extra ground state or equivalently, makes physical an extra operator x^{n+1}. This changes completely the structure of the operator state correspondence. In particular, the dependence of the vacuum $|\mathrm{top}\rangle$ corresponding to the unity operator is *not* continuous. This makes unapplicable the standard perturbative argument used to prove the BRST invariance of the correlation functions. And indeed, the correlation functions are not even continuous at $\varepsilon = 0$.

Let us examine in light of that example the situation occurring in our theory with $H \neq 0$. We have seen in Sect. 4 that the set of local physical operators (BRST cohomology) is *independent* of H (even at the level of representatives). We need to prove that the correlation functions of these operators are also independent of H. The correlation function can be considered as the matrix element $\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle = \langle \mathrm{bot} | \mathcal{O}_1 \cdots \mathcal{O}_m | \mathrm{top} \rangle$ of the product $\mathcal{O}_1 \cdots \mathcal{O}_m$ between two vacua. By the state operator correspondence the vacuum $|\mathrm{bot}\rangle$ is mapped to 1 and the vacuum $|\mathrm{top}\rangle$ is mapped to $\mathcal{O}_\Omega$, where Ω is a top form on M. They are normalized by

$$\langle \mathrm{bot} | \mathcal{O}_\Omega | \mathrm{top} \rangle = 1 \ . \tag{6.1}$$

If the H deformation changes the vacua $|\mathrm{top}\rangle$ and/or $|\mathrm{bot}\rangle$ discontinuously the perturbation theory argument cannot be applied and the correlation functions may change. Let us show it does not happen. This is obvious when M is Calabi–Yau so that the sigma model is conformal. Then the fermionic number is strictly conserved. Since $|\mathrm{bot}\rangle$ is the *unique* state with fermionic number 0 and $|\mathrm{top}\rangle$ is the *unique* state with fermionic number $n = \dim M$, any perturbation which preserves BRST cohomology can only *rescale* them. But the relevant scaling factor is fixed by (6.1). This proves the statement.

If M is not Calabi–Yau, the fermionic number is conserved only modulo fermionic anomaly. Still we can define $|\mathrm{top}\rangle$ and $|\mathrm{bot}\rangle$ using the operator state correspondence as above (one can also use the SQM approximation). Suppose that the deformation could lead to

$$|\mathrm{top}\rangle \to |\mathrm{top}'\rangle = |\mathrm{top}\rangle + \sum r_p q_1^{-p_1} \dots q_k^{-p_k} |p\rangle \;,$$

$$|\mathrm{bot}\rangle \to |\mathrm{bot}'\rangle = |\mathrm{bot}\rangle + \sum s_{p'} q_1^{N_1 - p_1'} \dots q_k^{N_k - p_k'} |p'\rangle \;. \tag{6.2}$$

In (6.2), $|p\rangle$ and $|p'\rangle$ are the ground states of (multi) degrees $(p_1, \dots, p_k)$ and $(p_1', \dots, p_k')$. $(N_1, \dots, N_k)$ is a multidegree of $|\mathrm{top}\rangle$. The coefficients r_p and $s_{p'}$ are q-independent. The formula (6.2) gives the most general form of the deformation compatible with anomalous conservation of fermionic number. Having the q-dependence completely fixed, let us consider the quasiclassical limit $q_i \to 0$. Since the fermionic number is conserved in this limit, we expect that the deformation should tend to zero with q_i. We see this is true for $|\mathrm{top}\rangle$ for any $s_{p'}$. On the other hand, it can only be true for $|\mathrm{bot}\rangle$ if all the coefficients r_p are equal to zero.

Now let us compute the matrix element $\langle \mathrm{bot}' | \mathcal{O}_\omega | \mathrm{top}' \rangle$ of any operator $\mathcal{O}_\omega$, $\deg \omega < n$. On one hand, since $|\mathrm{bot}'\rangle = |\mathrm{bot}\rangle$, this matrix element is given by

$$\langle \mathrm{bot}' | \mathcal{O}_\omega | \mathrm{top}' \rangle = \langle \mathrm{bot} | \mathcal{O}_\omega | \mathrm{top} \rangle + \sum s_{p'} q_1^{N_1 - p_1'} \dots q_k^{N_k - p_k'} \langle \mathrm{bot} | \mathcal{O}_\omega | p' \rangle \;.$$

On the other hand, one can compute it directly. Since $\deg \mathcal{O}_\omega < \deg|\mathrm{top}\rangle - \deg|\mathrm{bot}\rangle = n$ and $c_1(M)$ is positive, this matrix element does not have instanton corrections, can be computed in SQM and equals zero. Of course, the same is true about $\langle \mathrm{bot} | \mathcal{O}_\omega | \mathrm{top} \rangle$, so we end up with the equation $(\omega, \sum [s_{p'} q_1^{N_1 - p_1'} \dots q_k^{N_k - p_k'}] p') = 0$ for any $\omega \in H^*(M)$, where we denoted $(\omega, p') = \langle \mathrm{bot} | \mathcal{O}_\omega | p' \rangle$. It was explained in Sect. 4 that the pairing $(.,.)$ is nondegenerate over $\mathbf{C}[q_i, q_i^{-1}]$. Therefore, the only solution is $s_{p'} = 0$.

We just have derived that the vacua $|\mathrm{top}\rangle$ and $|\mathrm{bot}\rangle$ cannot be deformed by any perturbation not changing the BRST cohomology. The H independence of the correlation functions follows. Indeed, since the vacua remain the same we can treat the H deformation within the perturbation theory:

$$\begin{aligned}\langle \mathcal{O}_1 \dots \mathcal{O}_k \rangle_{\varepsilon H} &= \langle \mathrm{bot} | \mathcal{O}_1 \dots \mathcal{O}_k e^{\varepsilon \{Q,A\}} | \mathrm{top} \rangle = \langle \mathrm{bot} | \mathcal{O}_1 \dots \mathcal{O}_k \left(1 + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \{Q,A\}^n \right) | \mathrm{top} \rangle \\ &= \langle \mathrm{bot} | \mathcal{O}_1 \dots \mathcal{O}_k 1 | \mathrm{top} \rangle = \langle \mathcal{O}_1 \dots \mathcal{O}_k \rangle_0 \;. \end{aligned} \tag{6.3}$$

The second point we want to discuss is the problem of "non-generic" moduli spaces of instantons. It might happen (and almost always happens in the (almost) holomorphic case) that the dimension of some connected component of the moduli space of instantons is strictly greater than its lower bound given by the Riemann–Roch. (Then in order to obtain the correlation function one has to compute the Euler characteristic of a certain vector bundle on this (properly compactified) component of the moduli space [5, 6].) From the point of view of the real differential geometry, this situation is not generic. The key point here is that there are no "generic deformations" in the algebraic geometry in a sense of differential geometry, so we cannot resolve the "degeneration" within (almost) complex-analytic setup.

The way out is to break the holomorphicity. This is exactly what we did with (1.2). Deformed by H, the theory is no longer complex-analytic. (For example, the real dimension of cells of the Floer complex can be odd and the intersection numbers can have arbitrary signs.) Thus the H deformations enable one to reduce any degenerate situation to the generic one. The topological correlation functions being H independent, the results of computation are not affected. (That is why we do not have to discuss the multiplication in Floer's theory when the situation is not generic. In principle, it could be done along the lines of [5, 6].)

Acknowledgements. I thank A. Astashkevich, M. Bershadsky, S.P. Novikov, S. Piunikhin and C. Vafa for the useful discussions. The remarks of D. Kazhdan and I. Singer were important at the final stage of the work

Research supported in part by the Packard Foundation and by NSF grant PHY-87-14654

References

1. Lerche, W., Vafa, C., Warner, N.: Chiral Rings in $N = 2$ Superconformal Theory. Nucl. Phys. **B324**, 427 (1989)
2. Candelas, P., Green, P., Parkes, L., de la Ossa, X.: A pair of Calabi–Yau manifolds as an exactly soluble superconformal field theory. Nucl. Phys. **B359**, 21 (1991)
3. Vafa, C.: Topological Mirrors and Quantum Rings. In: Essays in Mirror Symmetry, ed. Yau, S.-T. 1992
4. Witten, E.: Topological sigma model. Commun. Math. Phys. **118**, 411 (1988)
5. Witten, E.: Mirror manifolds and Topological Field Theory. In Essays in Mirror Symmetry, ed. Yau, S.-T. 1992
6. Aspinwall, P., Morrison, P.D.: Topological Field Theory and Rational Curves. Oxford preprint, 1991
7. Floer, A.: Symplectic Fixed Points and Holomorphic Spheres. Commun. Math. Phys. **120**, 575–611 (1989)
8. Floer, A.: The unregularised gradient flow of the symplectic action. Comm. Pure Appl. Math. **41**, 775–813 (1988)
9. Floer, A.: Witten's complex and infinite dimensional Morse theory. J. Diff. Geom. **30**, 207–221 (1989)
10. Milnor, J.: Lectures on the h-cobordism theorem. Math. Notes, Princeton, NJ: Princeton Univ. Press, 1965
11. Witten, E.: Supersymmetry and Morse theory. J. Diff. Geom. **17**, 661–692 (1982)
12. Dostoglou, S., Salamon, D.: Instanton homology and symplectic fixed points. Preprint 1992
13. Novikov, S.: Quasiperiodic structures in topology. In: Topological methods in Mathematical Physics, ed. by L. Goldberg and A. Phillips, Berkeley KA: Publish or Perish, 1993
14. Novikov, S.: Russ. Math. Surv. **37**, 5, 3–49 (1982)
15. Le Tu Quok: Russ. Math. Surv. **44**, 3, (1991)
16. Hofer, H., Salamon, D.: Floer homology and Novikov rings. Preprint 1992
17. Le Hong Van, K.: Ono Symplectic fixed points, Calabi invariants and Novikov homology. Preprint MPI/93–27
18. Bailieu, L., Singer, I.: The topological sigma model. Commun. Math. Phys. **125**, 227 (1989)
19. Ruan, Y., Tian, G.: Mathematical theory of quantum cohomology. Preprint, January, 1994
20. Piunikhin, S.: Quantum and Floer cohomology have the same ring structure. Preprint hep-th/9401130
21. Salamon, D.: Morse theory, the Conley index and the Floer homology. Bull. L.M.S. **22**, 113–140 (1990)
22. Salamon, D., Zehnder, E.: Morse theory for periodic solutions of Hamiltonian equations and the Maslov index. Comm. Pure Appl. Math. **XLV**, 1303–1360 (1992)

23. Gromov, M.: Pseudoholomorphic curves in Symplectic Manifolds. Invent. Math. **82**, 307–347 (1985)
24. Cohen, R.L., Jones, J.D.S., Segal, G.B.: Floer's Infinite Dimensional Morse Theory and Homotopy Theory. Preprint 1993
25. Feigin, B., Frenkel, E.: Affine Kac–Moody algebras and semi-infinite flag manifolds. Commun. Math. Phys. **128**, 161–185 (1990)
26. Bott, R., Tu, F.: Differential Forms in Algebraic Topology

Communicated by R.H. Dijkgraaf