Commun. Math. Phys. 173, 17–42 (1995)

Communications in Mathematical Physics

The Spectral Problem for the q-Knizhnik–Zamolodchikov Equation and Continuous q-Jacobi Polynomials

Peter G.O. Freund[1], Anton V. Zabrodin[2]

Enrico Fermi Institute, Department of Physics and Mathematical Disciplines Center, University of Chicago, Chicago, IL 60637.

Received: 1 October 1994/in revised form: 10 August 1994

Abstract: The spectral problem for the q-Knizhnik–Zamolodchikov equations for $U_q(\widehat{sl_2})(0 < q < 1)$ at arbitrary non-negative level k is considered. The case of two-point functions in the fundamental representation is studied in detail. The scattering states are given explicitly in terms of continuous q-Jacobi polynomials, and the S-matrix is derived from their asymptotic behavior. The level zero S-matrix is closely connected with the kink-antikink S-matrix for the spin-$\frac{1}{2}$ XXZ antiferromagnet. An interpretation of the latter in terms of scattering on (quantum) symmetric spaces is discussed. In the limit of infinite level we observe connections with harmonic analysis on p-adic groups with the prime p given by $p = q^{-2}$.

1. Introduction

There is accumulating evidence for the idea [1] that excitation scattering in integrable models is "geometric," i.e., that the corresponding wave functions are spherical functions of certain quantum symmetric spaces. With this idea in mind, we have recently derived [2] the physical S-matrix for the scattering of kinks and antikinks for the (spin-$\frac{1}{2}$) XXX and XXZ anti-ferromagnets, starting from the level zero q-Knizhnik–Zamolodchikov (q-KZ) equations for $U_q(\widehat{sl_2})$ in the fundamental representation (the Heisenberg XXX case corresponds to $q = 1$). These q-KZ equations [3] are q-deformations of the ordinary first order differential KZ equations [4], which, in turn, are similar to the familiar Dirac and Bargmann–Wigner [5] equations. That the q-KZ equations apply in the kink-antikink problem has to do with the fact that kinks and antikinks are known to be spin $\frac{1}{2}$ excitations [6, 7]. Here we make this connection more precise and extend this work to non-negative values of the level k. This brings the continuous q-Jacobi polynomials into play and physics-wise concerns $SL_q(k+2)$-magnetics (or the corresponding generalizations of Baxter's eight vertex model). The first-order matrix q-KZ operator is cast here

[1] Work supported in part by the NSF: PHY-91-23780

[2] Permanent address: Institute of Chemical Physics, Kosygina Str. 4, SU-117334, Moscow, Russia

in the rôle of the radial part of a Dirac operator, whose "square" yields the radial part of the Laplace operator on the quantum symmetric space. A new ingredient here is the spin-orbit interaction that allows one to describe the scattering in triplet and singlet states in terms of zonal and tesseral spherical harmonics respectively. This picture leads also to useful insights concerning the p-adics-quantum-group connection [8, 9, 10, 11].

As in [2], we deal here with the *spectral* problem for the q-KZ operator and not the monodromy problem considered by others [3]. The difference between these two problems will be discussed: in a certain sense they are each other's duals.

The paper is organized as follows. We start in Sect. 2 with the classical ($q = 1$) case where the q-KZ equations are the usual differential KZ equations [4] written with trignometric rather than rational classical r-matrix. In other words, one uses a Borel type polarization instead of a parabolic polarization. The KZ equation in trigonometric form contains the highest weight of the vacuum representation which, when continued to the whole complex plane, may be viewed as a spectral parameter λ. The *spectral* problem yields a non-trivial λ-dependent S-matrix which in case of level zero is closely connected with the kink-antikink S-matrix in the spin $\frac{1}{2}$ isotropic (XXX) Heisenberg model, λ being the relative rapidity of the excitations. The precise form of this connection is discussed below in some detail.

In Sect. 3 we solve the spectral problem for the q-deformed KZ equations at arbitrary non-negative level k. The scattering states are explicitly found in terms of continuous q-Jacobi polynomials. Section 4 deals with the special case $k = 0$. The obtained S-matrix is shown to be very closely related to the kink-antikink S-matrix in the spin$-\frac{1}{2}$ XXZ antiferromagnet. The corresponding scattering states may be interpreted as spinorial harmonics on the $SL_q(2)$ *quantum* group. Another interesting special case is the limit of infinite level ($k \to \infty$) considered in Sect. 5. If $q^2 = p^{-1}$ and p is a prime number this case turns out to be closely connected with harmonic analysis on the p-adic group $PGL(2, \mathbf{Q}_p)$. We discuss some aspects of this connection and suggests an "arboreal" interpretation of spinorial harmonics on the p-adic group in terms of Bruhat–Tits trees. Section 6 contains a general discussion and conclusions. In Appendix A some technical details related to Sect. 3 and explained. Appendix B contains a brief review of the continuous q-Jacobi polynomials.

2. The Classical ($q = 1$) KZ Equation, its Spectral Problem and S-Matrix

We start from the classical ($q = 1$) KZ-equation for $\widehat{sl_2}$ in the fundamental representation for level k. The case $k = 0$ was treated in our earlier paper [2]; here we right away address the case of generic k. With normal ordering relative to a Borel polarization, consider the matrix element

$$\Psi(x_i) = \langle \Omega' | \Phi(x_2) \Phi(x_1) | \Omega \rangle \tag{2.1}$$

of the product of two vertex operators Φ between suitable vacuum states. This matrix element depends on only one variable, which in an *additive* parametrization can be chosen as

$$x = \frac{1}{2}(x_1 - x_2) . \tag{2.2}$$

The matrix element $\Psi(x_t)$ is then a $\mathbf{C}^2 \otimes \mathbf{C}^2$ valued function $\Psi(x)$ of the variable x. This $\Psi(x)$ obeys the KZ equation, which, for $\widehat{sl_2}$ at level k and in the fundamental representation, takes the form

$$(k+2)\frac{d\Psi(x)}{dx} = (r_{12}(x) + \pi_1(H))\Psi(x)\,, \tag{2.3}$$

where $r_{12}(x)$ is the familiar trigonometric solution of the *classical* Yang–Baxter equation

$$r_{12}(x) = (\coth x)\left[E \otimes F + F \otimes E + \frac{1}{2}H \otimes H - \frac{1}{2}\mathbf{1} \otimes \mathbf{1}\right] - E \otimes F + F \otimes E\,,$$

$$\pi_1(H) = i(k+2)\lambda H \otimes \mathbf{1}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

with λ the (real) parameter of our spectral problem. In the combination $k+2$ on the left-hand side of Eq. (2.3), the term 2 is the dual Coxeter number $g = 2$ of sl_2. The choice of the term proportional to the 4×4 unit matrix in the expression of $r_{12}(x)$ (the last term in the square bracket in the first equation (4)) is irrelevant as far as the Yang–Baxter equation is concerned, but considerably simplifies the argument. With respect to a basis $v_a \otimes v_b$, $a, b = \pm$ of $\mathbf{C}^2 \otimes \mathbf{C}^2$, we can expand

$$\Psi(x) = a(x)v_+ \otimes v_+ + f(x)v_+ \otimes v_- + g(x)v_- \otimes v_+ + b(x)v_- \otimes v_- \tag{2.5a}$$

In components the KZ equation (2.3) then becomes

$$\frac{d}{dx}a(x) = i\lambda a(x), \qquad \frac{d}{dx}b(x) = -i\lambda b(x)\,, \tag{2.5b}$$

$$\left(\frac{d}{dx} + \frac{1}{k+2}\coth x - \frac{1}{k+2}\sigma_1 \coth x\right)\psi(x) = i\left(\lambda\sigma_3 + \frac{1}{k+2}\sigma_2\right)\psi(x)\,, \tag{2.5c}$$

$$\psi(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \tag{2.5d}$$

where σ_i are Pauli matrices. The nonzero weight components $a(x)$ and $b(x)$ decouple and are trivial (Eqs. (2.5b)). Henceforth we ignore them. The interesting equation (2.5c) involves the zero weight sector. This equation is similar to a Dirac equation with our spectral parameter λ playing the role of mass in a "γ_5-type" mass term. It is convenient to introduce the triplet and singlet combinations

$$F^+(x) = f(x) + g(x), \qquad F^-(x) = f(x) - g(x)\,, \tag{2.6}$$

which obey the first order differential equations

$$F^{-\prime} + \frac{2}{k+2}(\coth x)F^- = \left(i\lambda + \frac{1}{k+2}\right)F^+\,, \tag{2.7a}$$

$$F^{+\prime} = \left(i\lambda = \frac{1}{k+2}\right)F^-\,. \tag{2.7b}$$

Here prime stands for derivative with respect to x. Equations (2.7) lead to the decoupled system of second order differential equations

$$\left(\frac{d^2}{dx^2}+\frac{2}{k+2}\coth x\frac{d}{dx}\right)F^+(x)=-\left(\lambda^2+\frac{1}{(k+2)^2}\right)F^+(x)\,, \tag{2.8a}$$

$$\left(\frac{d^2}{dx^2}+\frac{2}{k+2}\coth x\frac{d}{dx}-\frac{2}{(k+2)\sinh^2 x}\right)F^-(x)$$
$$=-\left(\lambda^2+\frac{1}{(k+2)^2}\right)F^-(x)\,. \tag{2.8b}$$

Equation (2.5c) is equivalent to Eqs. (2.8) supplemented by the constraint Eq. (2.7b).

Equation (2.8a) is a special case of a general theorem on the connection between solutions to the KZ equations and zonal spherical functions proved by Matsuo [12]. Both Eqs. (2.8) can also be extracted from Cherednik's papers [13], as has been pointed out to us by A. Veselov. It is convenient to introduce the new variable z, the new functions $G^{\pm}$, and the new parameters n, l defined by

$$z=\cosh x,\quad G^{\pm}(z)=F^{\pm}(x)\,,$$
$$l=\tfrac{1}{k+2},\qquad n=-l+i\lambda\,,$$

in terms of which Eqs. (2.8) become

$$\left[\frac{d^2}{dz^2}+\frac{2l+1}{z^2-1}z\frac{d}{dz}\right]G^+(z)=\frac{n(n+2l)}{z^2-1}G^+(z)\,, \tag{2.9a}$$

$$\left[\frac{d^2}{dz^2}+\frac{2l+1}{z^2-1}z\frac{d}{dz}\right]G^-(z)=\left(n(n+2l)+\frac{2l}{z^2-1}\right)\frac{1}{z^2-1}G^-(x)\,, \tag{2.9b}$$

The constraint is

$$\frac{dG^+(z)}{dz}=(i\lambda-l)(z^2-1)^{-1/2}G^-(z)\,. \tag{2.9c}$$

Equation (2.9a) is the familiar differential equation obeyed by the Gegenbauer functions $C_n^{\nu}(z)$ for $\nu=l$. We normalize them by the condition $C_n^{\nu}(1)=1$. The following useful representation in terms of the hypergeometric function holds:

$$C_n^{\nu}(z)={}_2F_1(n+2\nu,-n;\nu+1/2;(1-z)/2)\,.$$

Equation (2.9b) can be reduced to the same form by a simple transformation so that the solution regular at $x=0$ ($z=1$) has the form

$$G^+(z)=AC^l_{-l+i\lambda}(z)\,, \tag{2.10a}$$

$$G^-(z)=A\frac{l+i\lambda}{2l+1}(z^2-1)^{1/2}C^{l+1}_{-l-1+i\lambda}(z)\,, \tag{2.10b}$$

where A is an arbitrary constant. So we see that the condition of regularity at $x=0$ makes the space of solutions to the matrix Eq. (2.5c) one-dimensional. Note that this boundary condition is exactly the same as that for spherical functions on symmetric spaces. In particular for $k=0$, so that $l=\frac{1}{2}$, the Gegenbauer functions reduce to Legendre functions in agreement with [2] and $G^-(z)$ in this particular case turns into an associated Legendre function.

In this paper we consider the "radial" scattering problem for the system (2.5c) on the half-line of positive x with the boundary condition discussed above. From the known aysmptotics of Gegenbauer functions [14], we get

$$G^{+}(\cosh x)|_{x\to\infty} = Ac(\lambda)e^{i\lambda x - lx} + Ac(-\lambda)e^{-i\lambda x - lx}\,, \tag{2.11a}$$

$$G^{-}(\cosh x)|_{x\to\infty} = Ac(\lambda)e^{i\lambda x - lx} + A\frac{l+i\lambda}{l-i\lambda}c(-\lambda)e^{-i\lambda x - lx}\,, \tag{2.11b}$$

where the c-function is

$$c(\lambda) = 2^{2l-1}\pi^{-1/2}\Gamma(l+1/2)\frac{\Gamma(i\lambda)}{\Gamma(i\lambda+l)}\,. \tag{2.12}$$

The solution (2.11) of the system (2.5c) involves a superposition of incoming and outgoing waves, and corresponds therefore to a scattering state. One can rewrite it in terms of the initial components $f(x), g(x)$:

$$f(x)|_{x\to\infty} = Ac(\lambda)e^{i\lambda x - lx} + \frac{Al}{l-i\lambda}c(-\lambda)e^{-i\lambda x - lx}\,, \tag{2.13a}$$

$$g(x)|_{x\to\infty} = -\frac{Ai\lambda}{l-i\lambda}c(-\lambda)e^{-i\lambda x - lx}\,, \tag{2.13b}$$

The scattering matrix should transform incoming "particles" (waves) to outgoing ones. In the case at hand it is more convenient to define the *inverse* S^{-1} of the S-matrix as an operator that transforms outgoing waves to incoming ones. So, if the outgoing wave at infinity has the form $\binom{1}{0}e^{i\lambda x - lx}$ (as in (2.13)) then the corresponding incoming wave is generally of the form $\begin{pmatrix}(S^{-1})_{11}\\(S^{-1})_{21}\end{pmatrix}e^{-i\lambda x - lx}$, where $(S^{-1})_{11}$ and $(S^{-1})_{21}$ are, by defnintion, matrix elements of the inverse S-matrix. We obtain from (2.13):

$$(S^{-1}(\lambda))_{11} = \frac{c(-\lambda)}{c(\lambda)}\frac{l}{l-i\lambda}\,, \tag{2.14a}$$

$$(S^{-1}(\lambda))_{21} = -\frac{c(-\lambda)}{c(\lambda)}\frac{i\lambda}{l-i\lambda}\,. \tag{2.14b}$$

These scattering amplitudes satisfy the unitarity condition

$$|(S^{-1}(\lambda))_{11}|^2 + |(S^{-1}(\lambda))_{21}|^2 = 1\,. \tag{2.14c}$$

So instead of a full 2×2 S-matrix we have obtained the two scattering amplitudes (2.14). Processes with other types of polarization cannot be realized in our system because, as was already noted, the space of physical solutions to (2.5c) (or to (2.9)) is one-dimensional.

However, the obtained amplitudes look like matrix elements of a full-fledged S-matrix. Furthermore, one can formally reconstruct this S-matrix (let us call it the *extended* scattering matrix $S^{(ext)}$) from the known data using unitarity. Under the natural assumption $S_{11}^{(ext)} = S_{22}^{(ext)}$ (isotropy property) it then follows that

$$S_{12}^{(ext)}\bar{S}_{11}^{(ext)} = -\bar{S}_{21}^{(ext)}S_{11}^{(ext)}\,,$$

hence

$$S^{(ext)}(\lambda) = \frac{c(\lambda)}{c(-\lambda)(l+i\lambda)} \begin{pmatrix} l & i\lambda \\ i\lambda & l \end{pmatrix} . \tag{2.15}$$

This S-matrix is diagonal in the triplet/singlet $((+)/(-))$ basis and the corresponding eigenvalues are

$$S_+(\lambda) = \frac{c(\lambda)}{c(-\lambda)}, \qquad S_-(\lambda) = \left(\frac{l-i\lambda}{l+i\lambda}\right) S_+(\lambda) . \tag{2.16}$$

By construction we have

$$((S^{(ext)}(\lambda))^{-1})_{11} = (S^{-1}(\lambda))_{11}, \qquad ((S^{(ext)}(\lambda))^{-1})_{21} = (S^{-1}(\lambda))_{21} .$$

Of special importance is the case $k=0$, which we treated in [2]. Here we shall indicate the S-matrix elements and c-function for this $k=0$ case by using the superscript (0). The eigenvalues are

$$S_+^{(0)}(\lambda) = \frac{c^{(0)}(\lambda)}{c^{(0)}(-\lambda)}, \qquad S_-^{(0)}(\lambda) = \left(\frac{1-2i\lambda}{1+2i\lambda}\right) S_+^{(0)}(\lambda) , \tag{2.17}$$

with

$$c^{(0)}(\lambda) = \pi^{-1/2} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda + \frac{1}{2})} . \tag{2.18}$$

In this case it is possible to give a physical interpretation to the full S-matrix $S^{(ext)}(\lambda)$ which so far was a formal object. Consider the Laplace–Beltrami operator on the real hyperbolic plane H:

$$\Delta = \frac{\partial^2}{\partial x^2} + \coth x \frac{\partial}{\partial x} + \frac{1}{\sinh^2 x} \frac{\partial^2}{\partial \varphi^2} . \tag{2.19}$$

Its eigenfunctions $\Phi_m(x,\varphi)$ with appropriate boundary conditions (finiteness at the origin) are the spherical functions.

$$\Delta \Phi_m(x,\varphi) = -\left(\lambda^2 + \frac{1}{4}\right) \Phi_m(x,\phi) , \tag{2.20}$$

where m is the angular momentum:

$$-i\frac{\partial}{\partial\varphi} \Phi_m(x,\varphi) = m\Phi_m(x,\varphi) . \tag{2.21}$$

Zonal spherical functions (S-waves in physical language) correspond to $m=0$, tesseral ones to integer $m \neq 0$. It is readily seen that the second order equations (2.8) (for $k=0$), derived from the KZ equation, are precisely the radial parts of Eq. (2.19) with $m=0$ for

$$\Phi_0(x,\varphi) = F^+(x) , \tag{2.22a}$$

and $m=1$ for

$$\Phi_1(x,\varphi) = e^{i\varphi} F^-(x) . \tag{2.22b}$$

Thus $F^-(x)$ is the radial part of the P-wave.

Note that in this *two-dimensional* setting any superposition of the form $A\Phi_0(x,\varphi) + B\Phi_1(x,\varphi)$ is an eigenfunction of Δ. This allows one to define in this scalar problem a 2×2 scattering matrix for incoming and outgoing waves at infinity $x \longrightarrow +\infty$. Using the above formulas for the asymptotics one finds that this S-matrix is diagonal in the basis of S- and P-waves with eigenvalues (2.17). Being transformed to the (f, g)-basis according to (2.6), it acquires the form

$$|S^{(0)}(\lambda) = \frac{c(\lambda)}{c(-\lambda)(1+2i\lambda)} \begin{pmatrix} 1 & 2i\lambda \\ 2i\lambda & 1 \end{pmatrix}, \tag{2.23}$$

which coincides with $S^{(ext)}(\lambda)$ at $k = 0$.

The physical meaning of this formal prescription can be understood in terms of a spin-orbit type interaction for a fictitious particle moving on the two-dimesional hyperboloid. Remember that the S-wave motion of a particle on the hyperboloid (hyperbolic plane) H is equivalent to a *relative* motion of two scalar particles on the line interacting via $\sinh^{-2}$-potential. Similarly, the relative motion of two spin-1/2 particles on the line can be described in terms of a fictitious particle moving on H in a superposition of S- and P-waves and carrying an additional quantum number. In the scalar case, to extract the dynamics of relative motion, one has to restrict the whole space of states to the subspace with zero total momentum, thus passing to the center mass frame. A spin analog of this is the restriction to states with *zero total z-projection of the spin* (i.e. what we called zero weight sector with basic vectors $v_+ \otimes v_-$ and $v_- \otimes v_+$). The spin counterpart of the relative momentum is then precisely the just mentioned additional quantum number, which turns out to be the *total* spin of the system s: $s = 0$ (the singlet state $v_{sing} = v_+ \otimes v_- - v_- \otimes v_+$) or $s = 1$ (the triplet state with zero z-projection of its spin $v_{tripl} = v_+ \otimes v_- + v_- \otimes v_+$). So, in the sense of this analogy it may be appropriate to refer to the quantum number s as "relative spin." If the *total* angular momentum (spin plus orbital momentum) on H is conserved, the general form of the fictitious particle wave function is any linear superposition of the form

$$\psi(x, \varphi) = A v_{sing} \Phi_0(x, \varphi) + B v_{tripl} \Phi_1(x, \varphi). \tag{2.24}$$

(It is implied that the axis of the hyperboloid lies in the xy-plane so that the orbital momentum has zero z-projection.) This argument justifies the prescription above and makes it clear why the S-matrix (2.23) is diagonal in the single/triplet basis. So, we see that the KZ equation (2.5c) is a special reduction of this more general problem.

The physical interpretation of the extended KZ scattering matrix described above looks quite natural due to the following observation. As shown in [2], $S_+^{(0)}(\lambda)$ and $-S_-^{(0)}(\lambda)$ (2.17) (the minus sign to be discussed below) are the eigenvlaues of the Heisenberg XXX model scattering matrix for elementary excitations. These were classified in [6, 7] where their scattering was studied by means of the algebraic Bethe ansatz technique. It was shown that the elementary excitations in the XXX (and XXZ) model have spin 1/2 and can be created only in pairs. In the physics literature they are called kinks (if the z-component of spin is $+1/2$) and antikinks (if the z-component of spin is $-1/2$). One can see that the S-matrix (2.23) differs from the kink-antikink scattering matrix of the XXX-model (i.e. the scattering matrix in the subspace with zero z-component of spin) only by the constant factor σ_1. Note that the multiplication by the Pauli matrix σ_1 changes the sign of one of the eigenvalues.

This remarkable coincidence of S-matrices for systems of very different nature undoubtedly holds for profound reasons. We show in Sect. 4 that it can be extended to the q-deformed case.

We wish to draw attention here to two points.

First, as far as sl_2 representations are concerned, $\Psi(x)$, valued in the tensor product of two two-dimensional representations, decomposes into one singlet and three triplet components. Yet, as we saw, the two non-vanishing weight triplet components decouple, leaving a two-dimensional "spin-$\frac{1}{2}$"-like system obeying the Dirac-like equation (2.5). A further study of this metamorphosis would be of interest.

Second, the steps leading from the coupled first-order system (2.5) to the decoupled second order system (2.8), completely parallel those which lead from the coupled first order Dirac equations to the eigenfunctions of the decoupled second order Laplace operator. It will be worth keeping this in mind, when we repeat these steps at the quantum level.

3. q-Deformed Case

Let us start by recalling some facts about q-KZ equations for 2-point functions in the $U_q(\widehat{sl_2})$-case ($0 < q < 1$). We use the notations of [15], in a slightly modified form. Consider a correlation function of two q-vertex operators

$$\Psi(z_1,z_2) = \langle\Omega'|\Phi(z_2)\Phi(z_1)|\Omega\rangle \in V \otimes V\,, \tag{3.1}$$

where $V \cong \mathbf{C}^2$ is a linear space on which now 2-dimensional representations of $U_q(\widehat{sl_2})$ act. We fix a basis $\{v_+, v_-\}$ in V. With a proper definition of q-vertex operators, $\Psi(z_1,z_2)$ depends only on z_1/z_2 (we have now switched from the additive parametrization of Sect. 2, to a *multiplicative* parametrization), so we consider the function

$$\Psi(z) = \langle\Omega'|\Phi(z^{-1})\Phi(z)|\Omega\rangle\,. \tag{3.2}$$

For level k vertex operators, $\Psi(z)$ satisfies the first order q-KZ difference equation:

$$\Psi(q^{k+2}z) = \rho(z)(q^{-\phi} \otimes 1)R(z)\Psi(z)\,, \tag{3.3}$$

where the R-matrix $R(z)$ is defined by explicit action in $V \otimes V$ as follows:

$$R(z)v_\pm \otimes v_\pm = v_\pm \otimes v_\pm\,, \tag{3.4a}$$

$$R(z)v_+ \otimes v_- = \frac{q(1-z^2)}{1-q^2z^2}v_+ \otimes v_- + \frac{(1-q^2)z^2}{1-q^2z^2}v_- \otimes v_+\,,$$

$$R(z)v_- \otimes v_+ = \frac{1-q^2}{1-q^2z^2}v_+ \otimes v_- + \frac{q(1-z^2)}{1-q^2z^2}v_- \otimes v_+\,. \tag{3.4b}$$

Let us introduce a special notation $R_0(z)$ for the zero-weight part (3.4b) of this R-matrix:

$$R_0(z) = \begin{pmatrix} \frac{q(1-z^2)}{1-q^2z^2} & \frac{1-q^2}{1-q^2z^2} \\ \frac{(1-q^2)z^2}{1-q^2z^2} & \frac{q(1-z^2)}{1-q^2z^2} \end{pmatrix}\,. \tag{3.4c}$$

The operator $q^{-\phi}$ acts on the basis vectors by multiplication:

$$q^{-\phi}v_{\pm} = q^{\mp 2i\lambda}v_{\pm}\,, \tag{3.5}$$

where λ is a spectral parameter. In the q-KZ equations considered in [15, 3] λ takes a particular value depending on the choice of vacuum states Ω, Ω' in (3.2). When we are interested in the spectral problem for the difference operator in (3.3) (rather than monodromy properties of the solutions) λ plays the role of spectral parameter (this becomes evident from Eq. (3.10) below). We have introduced i in (3.5) (just like in Eq. (2.5c) in the classical case) so that the continuous spectrum will correspond to real values of λ.

Finally, $\rho(z)$ in (3.3) is the scalar multiplier defined in [15]:

$$\rho(z) = q^{-1/2}\cdot\frac{(q^2z^2;q^4)^2_\infty}{(z^2;q^4)_\infty(q^4z^2;q^4)_\infty}\,, \tag{3.6}$$

where the standard notation [18]

$$(z;q)_n = \prod_{j=0}^{n-1}(1-zq^j); \qquad (z;q)_\infty = \lim_{n\to\infty}(z;q)_n \tag{3.7}$$

is used. It is shown in [3] that this multiplier comes from restriction of the universal R-matrix for $U_q(\widehat{sl_2})$ to the tensor product of two 2-dimensional representations. Though crucial in the monodromy problem, $\rho(z)$ is irrelevant for our purposes here, because one can gauge it away without altering the spectral properties.

Due to the specific form (3.4) of the R-matrix, the $v_+\otimes v_+$ and $v_-\otimes v_-$ components of $\Psi(z)$ decouple, and each of them obeys a scalar first order difference equation as in the classical case (2.5b). Again, the non-trivial equations come from the zero-weight sector of the R-matrix (3.4). After reduction to the zero-weight subspace of $V\otimes V$ we obtain for the two components of

$$\psi(z) \equiv f(z)v_+\otimes v_- + g(z)v_-\otimes v_+\,, \tag{3.8}$$

the following system of difference equations

$$f(q^{k+2}z) = q^{-2i\lambda}\frac{q(1-z^2)}{1-q^2z^2}f(z) + q^{-2i\lambda}\frac{1-q^2}{1-q^2z^2}g(z)\,,$$

$$g(q^{k+2}z) = q^{2i\lambda}\frac{(1-q^2)z^2}{1-q^2z^2}f(z) + q^{2i\lambda}\frac{q(1-z^2)}{1-q^2z^2}g(z)\,. \tag{3.9}$$

There is a more suggestive form of (3.9) which resembles the classical spectral problem (2.7). Using the notation $R_0(z)$ (3.4c) for the zero-weight part of the R-matrix and introducing the diagonal 2×2 matrix $\Lambda = diag\{q^{2i\lambda}, q^{-2i\lambda}\}$, one can rewrite (3.9) in the form

$$T^{-1}R_0(z)\psi(z) = \Lambda\psi(z)\,, \tag{3.10}$$

where T is the shift operator: $T\psi(z) = \psi(q^{k+2}z)$. This equation does look like a finite-difference analog of (2.7).

Guided by the classical limit we interpret (3.10) as the radial part of a discrete "Dirac-like" equation for a particle on a curved quantum space. Let us rewrite

(3.9) in terms of discrete "radial coordinate" n which we assume to be a non-negative integer. To do this, it is convenient to redefine the parameters as

$$p = q^{k+2}\,, \tag{3.11}$$

$$l = \frac{1}{k+2}\,, \tag{3.12}$$

$$u = 2l\lambda\,, \tag{3.13}$$

so that $q = p^l$. From now on we consider the case $k + 2 \geqq 0$. Setting $z = p^{n+l}$ and calling $f(p^{n+l}) = f_n$, $g(p^{n+l}) = g_n$, we obtain the following system of recurrence relations:

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = (1 - p^{2n+4l})^{-1} \begin{pmatrix} p^{-iu+l}(1 - p^{2n+2l}) & p^{-iu}(1 - p^{2l}) \\ p^{iu}(1 - p^{2l})p^{2n+2l} & p^{iu+l}(1 - p^{2n+2l}) \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix}. \tag{3.14}$$

The natural boundary condition is finiteness of f_n and g_n at $n = 0$.

A comment on the choice of the integer values of n ($n = 0, 1, 2, \ldots$) in (3.14) is in order. This choice means a specific truncation of the initial system (3.9) (defined on the whole complex plane) to a *discrete* recurrence relation on the one-dimensional half-infinite lattice. Of course, different truncations of this kind may not be equivalent. We note, however, that our choice is a distinguished one and looks quite natural because it ensures the closed similarity to the continuous Eq. (2.5c) in the following sense. The space of regular solutions to (2.5c) is one-dimensional and the same is true for the discrete Eq. (3.14) provided f_0 and g_0 are finite. Indeed, the matrix $R_0(p^{n+l})$ on the r.h.s. of (3.14) becomes degenerate at $n = 0$ and projects any two-dimensional vector $\begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$ into the 1-dimensional subspace spanned by the vector $\begin{pmatrix} 1 \\ p^{2iu+l} \end{pmatrix}$. If the initial point is moved away from $n = 0$, the matrix R_0 is non-dengenerate in any point of the lattice, hence the space of solutions is 2-dimensional. The analysis of this general case is beyond the scope of this paper.

By a straightforward but somewhat lengthy calculation it can be shown that the linear combinations

$$F_n^{\pm} = f_n \pm p^{-iu} g_n \tag{3.15}$$

obey second-order recurrence relations of the form

$$\frac{1 - p^{2n+4l}}{1 - p^{2n+2l}} F_{n+1}^{\pm} + p^{2l} \frac{1 - p^{2n-2}}{1 - p^{2n+2l-2}} F_{n-1}^{\pm} = \mp \frac{(1 - p^2)(1 - p^{2l})p^{2n+2l-2}}{(1 - p^{2n+2l})(1 - p^{2n+2l-2})} F_n^{\pm} + p^l(p^{iu} + p^{-iu}) F_n^{\pm}\,, \tag{3.16a}$$

and the system (3.14) is equivalent to the decoupled pair of Eqs. (3.16a) with the constraint

$$2\frac{1 - p^{2n+4l}}{1 - p^{2n+2l}} F_{n+1}^{+} = \left(p^l(p^{iu} + p^{-iu}) + \frac{(1 - p^{2l})(1 + p^{2n+2l})}{1 - p^{2n+2l}} \right) F_n^{+} - (1 - p^{2l} + p^l(p^{iu} - p^{-iu}) F_n^{-}\,. \tag{3.16b}$$

Notice that (3.16a) is a set of two equations, one for F_n^+ and one for F_n^-. The equivalence of Eqs. (3.16) and (3.14) then means that, given a solution of Eq. (3.16a) for F_n^+, one can find a corresponding F_n^- from the linear relation (3.16b). This F_n^- then satisfies the other Eq. (3.16a) and f_n, g_n calculated according to (3.15) satisfy Eq. (3.14). Equations (3.15) and (3.16) are the quantum counterparts of the "classical" equations (2.6) and (2.8). In Appendix A we show how one can come to (3.16) from (3.14) without knowing the combinations (3.15) *a priori*.

A usual form of the boundary condition for second-order recurrence relations is fixing values of the unknown function at two initial points (say, $n = 0$ and $n = 1$). Note, however, that in (3.16) the values of F_0^+ and F_0^- are irrelevant (provided they are finite) because the coefficient in front of $F_{n-1}^{\pm}$ becomes zero for $n = 1$. We see once again that in the case at hand the regularity at $n = 0$ already determines the solution up to an arbitrary constant.

With these boundary conditions, Eqs. (3.16) are the recurrence relations for certain q-Jacobi polynomials. (See, for example, [18] and Appendix B.) This observation allows us to solve (3.16). Specifically,

$$F_n^+ = F_1^+ R_{n-1}^{(l-1/2,l+1/2)}\left(\frac{1}{2}(p^{iu} + p^{-iu}); p\right), \tag{3.17a}$$

$$F_n^- = F_1^+ \frac{1 - p^{iu+l}}{1 + p^{iu+l}} \cdot \frac{p^{-n+1} - p^{n+2l}}{1 - p^{2l+1}} R_{n-1}^{(l+1/2,l-1/2)}\left(\frac{1}{2}(p^{iu} + p^{-iu}); p\right), \tag{3.17b}$$

where $R_n^{(\alpha,\beta)}(x; p)$ are the *continuous q-Jacobi polynomials*. For their definition and brief review of their properties, see Appendix B. The initial value F_1^+ is a free parameter.

For real values of u in the Brillouin zone $-\frac{\pi}{\log p} < u \leqq \frac{\pi}{\log p}$ the wave functions (3.17) describe scattering states for the q-KZ operator (l.h.s. of Eq. (3.10), i.e. they are solutions to (3.10) and their asymptotics at infinity is a superposition of incoming and outgoing waves. To see this, it is necessary to find the asymptotics of $F_n^{\pm}$, or in other words, of the continuous q-Jacobi polynomials at large n. Fortunately, this is known [18], so that we have

$$F_n^+|_{n\to\infty} = Ap^{nl-l}(p^{inu}c_+(-u) + p^{-inu}c_+(u)), \tag{3.18a}$$

$$F_n^-|_{n\to\infty} = A\frac{1 - p^{iu+l}}{1 + p^{iu+l}} p^{nl-l}(p^{inu}c_-(-u) + p^{-inu}c_-(u)), \tag{3.18b}$$

where A is a non-essential constant,

$$c_\pm(u) = \frac{p^{iu}}{1 \pm p^{iu+l}} \cdot \frac{\Gamma_{p^2}(iu)}{\Gamma_{p^2}(iu + l)}, \tag{3.19}$$

and the q-gamma function is defined as [18]

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty}(1 - q)^{1-x}. \tag{3.20}$$

For details see (B9)–(B11) in Appendix B. The expression (3.18) does indeed look like a superposition of incoming and outgoing waves.

Now we are in a position to derive the S-matrix. Let us rewrite the asymptotics (3.18) in terms of the original components f_n and g_n, $f_n = (1/2)(F_n^+ - F_n^-)$, $g_n = (1/2)p^{iu}(F_n^+ - F_n^-)$ (as follows from Eq. (3.15)):

$$f_n|_{n\to\infty} = \frac{Ap^{nl-l}}{1+p^{iu+l}}\left(\frac{(1-p^{2l})p^{-iu}}{1-p^{-2iu+2l}}c_p(-u)p^{inu} + p^{iu}c_p(u)p^{-inu}\right), \quad (3.21a)$$

$$g_n|_{n\to\infty} = \frac{Ap^{nl-l}}{1+p^{iu+l}} \cdot \frac{p^l(p^{iu}-p^{-iu})}{1-p^{-2iu+2l}}c_p(-u)p^{inu}, \quad (3.21b)$$

where

$$c_p(u) = \frac{\Gamma_{p^2}(iu)}{\Gamma_{p^2}(iu+l)}. \quad (3.22)$$

The S-matrix can be readily determined from (3.21). The picture is qualitatively the same as in Sect. 2. Again, it is more convenient to define the inverse S^{-1} of the S-matrix first. The operator S^{-1} transforms outgoing particles (waves) to incoming ones. So, if the outgoing wave at infinity has the form $\binom{1}{0}p^{-inu+nl}$, (as in (3.21)) then the corresponding incoming wave is generally of the form $\begin{pmatrix}(S^{-1})_{11}\\(S^{-1})_{21}\end{pmatrix}p^{inu+nl}$, where $(S^{-1})_{11}$ and $(S^{-1})_{21}$ are, by definition, matrix elements of the inverse S-matrix. We obtain from (3.21):

$$(S^{-1}(u))_{11} = \frac{c_p(-u)}{c_p(u)}\frac{1-p^{2l}}{p^{2iu}-p^{2l}}, \quad (3.23a)$$

$$(S^{-1}(-u))_{21} = \frac{c_p(-u)}{c_p(u)}\frac{p^l(p^{2iu}-1)}{p^{2iu}-p^{2l}}, \quad (3.23b)$$

These scattering amplitudes satisfy the unitarity condition of the same form as (2.14c). Using unitarity and isotropy one can formally reconstruct from these data the full (extended) scattering matrix $S_{q-KZ}^{(ext)}$:

$$S_{q-KZ}^{(ext)}(u) = \frac{c_p(u)}{c_p(-u)}\begin{pmatrix}\frac{p^{2iu}(1-p^{2l})}{1-p^{2iu+2l}} & \frac{p^l(1-p^{2iu})}{1-p^{2iu+2l}}\\ \frac{p^{2iu+l}(1-p^{2iu})}{1-p^{2iu+2l}} & \frac{p^{2iu}(1-p^{2l})}{1-p^{2iu+2l}}\end{pmatrix}. \quad (3.24)$$

Again, as in Sect. 2, we call it *extended* because this S-matrix can be physically realized in this system only for one particular type of polarization of waves at infinity. We have

$$((S_{q-KZ}^{(ext)}(u))^{-1})_{11} = (S^{-1}(u))_{11},$$

$$((S_{q-KZ}^{(ext)}(u))^{-1})_{21} = (S^{-1}(u))_{21}.$$

Recalling the definition of the R-matrix (3.4) and its zero-weight part $R_0(z)$ (3.4c), we see that (3.24) can be represented in the form

$$S^{(ext)}_{q-KZ}(u) = \begin{pmatrix} 0 & p^{-iu} \\ p^{iu} & 0 \end{pmatrix} R_0(p^{iu})U(p^{iu})$$

$$= \Lambda^{-1}\sigma_1 R_0(p^{iu})U(p^{iu}), \tag{3.25}$$

where $\Lambda = diag\{p^{iu}, p^{-iu}\}$ as before (see (3.10) and (3.13)) and

$$U(p^{iu}) = p^{iu}\frac{c_p(u)}{c_p(-u)} \tag{3.26}$$

is the scalar unitarizing factor. Note that $|R_0(p^{iu}), \Lambda^{-1}\sigma_1] = [S^{(ext)}_{q-KZ}(u), \Lambda^{-1}\sigma_1] = 0$. So the local (bare) "$S$-matrix" $R_0(z)$ in (3.10) (connecting two neighboring sites) partially reproduces itself in the global scattering, getting "dressed" by the scalar infinite product factor $U(z)$.

So we have solved the scattering problem for the difference operator in the l.h.s. of the q-KZ Eq. (3.10). A crucial part of the argument is the transition from the first-order matrix difference equation to the pair of second-order recurrence relations (3.16) that can be actually solved in terms of the continuous q-Jacobi polynomials. This procedure is completely parallel to that in the "classical" ($q \to 1$) limit though much more involved technically.

It is interesting to note that in the q-deformed case, the choice (3.15) of linear combinations leading to a reasonable pair of decoupled second-order equations is *not unique*. Here by a reasonable equation we mean one which can be actually solved, with the asymptotics of the solutions being known explicitly. The other possibility is to take (see (A26))

$$\tilde{F}^{\pm}_n = f_n \pm p^{-2iu\pm l} g_n \tag{3.27}$$

instead of (3.15) (the classical limit is the same). In this case, Eq. (3.14) is equivalent to the following second order recurrence relations:

$$\frac{1-p^{2n+4l-2}}{1-p^{2n+2l-2}}\tilde{F}^{+}_{n+1} + p^{2l}\frac{1-p^{2n-2}}{1-p^{2n+2l-2}}\tilde{F}^{+}_{n-1} = p^l(p^{iu}+p^{-iu})\tilde{F}^{+}_n\,, \tag{3.28a}$$

$$\frac{1-p^{2n+4l}}{1-p^{2n+2l-2}}\tilde{F}^{-}_{n+1} + p^{2l}\frac{1-p^{2n}}{1-p^{2n+2l-2}}\tilde{F}^{-}_{n-1} = p^l(p^{iu}+p^{-iu})\frac{1-p^{2n}}{1-p^{2n-2}}\tilde{F}^{-}_2\,, \tag{3.28b}$$

with the constraint

$$(p^l + p^{-l})\tilde{F}^{+}_{n+1} = (p^{iu}+p^{-iu})\tilde{F}^{+}_n + (p^{2l-iu} - p^{iu})\tilde{F}^{-}_n \tag{3.28c}$$

(for details see Appendix A). Now finiteness of f_0 and g_0 leads to the vanishing of $\tilde{F}^{-}_1$. The same condition $\tilde{F}^{-}_1 = 0$ is forced by (3.28b). One can see that the solutions of (3.28a) and (3.28b) can also be expressed thought continuous q-Jacobi polynomials (of a different type):

$$\tilde{F}_n^+ = \tilde{F}_1^+ R_{n-1}^{(l-1/2,l-1/2)}(1/2(p^{iu} + p^{-iu}); p), \tag{3.29a}$$

$$\tilde{F}_n^- = \tilde{F}_2^- \cdot \frac{p^{-(n-1)} - p^{n-1}}{p^{-1} - p} R_{n-2}^{(l+1/2,l+1/2)}(1/2(p^{iu} + p^{-iu}); p), \tag{3.29b}$$

where

$$\tilde{F}_2^- = \frac{p^l(1 - p^2)(p^{-iu} - p^{iu+2l})}{(1 + p^{2l})(1 - p^{2+4l})} \tilde{F}_1^+ . \tag{3.29c}$$

From the known asymptotics of the r.h.s. of (3.29) (see Appendix B) one obtains the same S-matrix elements (3.23).

The polynomials $R_n^{(\alpha,\alpha)}(x; p)$ in (3.29) are known also as Rogers–Askey–Ismail polynomials [16, 17, 18] or Macdonald polynomials for root system A_1 [8]. The appearance of Macdonald polynomials in the context of q-KZ equations was already discussed by Cherednik [13, 19].

4. The Spectral Problem at Level Zero

In this section we discuss the general result of Sect. 3 in the special case of level zero ($k = 0$ in (3.9)). This deserves particular attention because, as we shall see, the spectral problem (3.14) for this simplest case yields the physical S-matrix for the XXZ spin$-1/2$ anti-ferromagnet suggesting at the same time a nice geometrical interpretation in terms of scattering on the quantum group $SL_q(2, \mathbf{R})$.

For $k = 0, p = q^2, l = 1/2, u = \lambda$ (see (3.11)–(3.13)) it is more convenient to use the original notation q and λ. Specializing (3.15) and (3.17) to this case we obtain

$$F_n^+ = F_1^+ R_{n-1}^{(0,1)} \left(\frac{1}{2}(q^{2i\lambda} + q^{-2i\lambda}); q^2 \right) , \tag{4.1a}$$

$$F_n^- = F_1^+ \cdot \frac{1 - q^{2i\lambda+1}}{1 + q^{2i\lambda+1}} \cdot \frac{q^{-2n} - q^{2n}}{q^{-2} - q^2} R_{n-1}^{(1,0)} \left(\frac{1}{2}(q^{2i\lambda} + q^{-2i\lambda}); q^2 \right) , \tag{4.1b}$$

with the asymptotics

$$F_n^+|_{n\to\infty} = Aq^{n-1} \left(\frac{q^{-2i\lambda} \Gamma_{q^4}(-i\lambda)}{(1 + q^{-2i\lambda+1})\Gamma_{q^4}(-i\lambda + 1/2)} q^{2in\lambda} + \frac{q^{2i\lambda} \Gamma_{q^4}(i\lambda)}{(1 + q^{2i\lambda+1})\Gamma_{q^4}(i\lambda + 1/2)} q^{-2in\lambda} \right) , \tag{4.2a}$$

$$F_n^-|_{n\to\infty} = Aq^{n-1} \frac{1 - q^{2i\lambda+1}}{1 + q^{2i\lambda+1}} \cdot \left(\frac{q^{-2i\lambda} \Gamma_{q^4}(-i\lambda)}{(1 - q^{-2i\lambda+1})\Gamma_{q^4}(-i\lambda + 1/2)} q^{2in\lambda} + \frac{q^{2i\lambda} \Gamma_{q^4}(i\lambda)}{(1 - q^{2i\lambda+1})\Gamma_{q^4}(i\lambda + 1/2)} q^{-2in\lambda} \right) , \tag{4.2b}$$

where

$$F_n^{\pm} = f_n \pm q^{-2i\lambda} g_n \tag{4.3}$$

and A is a constant. Had we included the factor $\rho(z)$ (3.6) in the R-matrix, its effect would have been to force $F_n^{\pm}$ to vanish at negative n because the zeros of $\rho(z)$ lie just at the points $q^{2n+1}, n < 0$.

The S-matrix (3.25) is

$$S_{q-KZ}^{(ext)}(\lambda)|_{k=0} = \Lambda^{-1}\sigma_1 R_0(q^{2i\lambda}) \cdot q^{2i\lambda} \frac{\Gamma_{q^4}(i\lambda)\Gamma_{q^4}(-i\lambda + 1/2)}{\Gamma_{q^4}(-i\lambda)\Gamma_{q^4}(i\lambda + 1/2)}, \tag{4.4}$$

where

$$\Lambda^{-1}\sigma_1 = \begin{pmatrix} 0 & q^{-2i\lambda} \\ q^{2i\lambda} & 0 \end{pmatrix} \tag{4.5}$$

commutes with $R_0(q^{2i\lambda})$. Note that $F_n^{\pm}$ (4.3) are just the eigenvectors of $R_0(q^{2i\lambda})$. The eigenvalues of $S_{q-KZ}^{(ext)}(\lambda)$ for $k = 0$ are given by

$$S_{\pm}(\lambda) = q^{4i\lambda} \frac{1 \pm q^{-2i\lambda+1}}{1 \pm q^{2i\lambda+1}} \cdot \frac{\Gamma_{q^4}(i\lambda)\Gamma_{q^4}(-i\lambda + 1/2)}{\Gamma_{q^4}(-i\lambda)\Gamma_{q^4}(i\lambda + 1/2)}. \tag{4.6}$$

This S-matrix actually coincides (up to the trivial matrix factor $\Lambda^{-1}\sigma_1$) with the kink-antikink scattering matrix $S_{k-a}(\lambda)$ for the spin$-1/2$ XXZ model in the anti-ferromagnetic regime (with anisotropy parameter $-\log q$):

$$S_{q-KZ}^{(ext)}(\lambda)|_{k=0} = \Lambda^{-1}\sigma_1 S_{k-a}(\lambda). \tag{4.7}$$

For the definition of $S_{k-a}(\lambda)$ see, for example, Eq. (6.18) in [15]. The eigenvalues of $S_{k-a}(\lambda)$ (corresponding to scattering with a given parity) are $\pm S_{\pm}(\lambda)$. The role of the extra matrix factor $\Lambda^{-1}\sigma_1$ is to change the sign of S_-. In the classical limit, this factor reduces to the constant matrix σ_1 and produces the minus sign noted there.

Unfortunately, at present we are unable to suggest a physical interpretaion of the S-matrix (4.4) similar to the one given for (2.23) in Sect. 2. The difficulty stems from the lack of a clear geometrical meaning of (non-commutative) angular coordinates on the quantum group. Here is a very preliminary discussion of this point.

The q-Jacobi polynomials $R_n^{(\alpha,\beta)}$ are known to provide (for some values of α, β) the full set of spherical harmonics on the quantum group $SL_q(2)$ [20]. Recalling that in this case $\alpha = |m - n|, \beta = |m + n|$, where $m(n)$ is the number of the left (right) $SO(2)$-harmonics, we see that the values of α, β in (4.1) correspond to spinorial harmonics $(1/2, 1/2)$ and $(1/2, -1/2)$. This is in line with the Dirac–Bargmann–Wigner analogy.

Thus, Eq. (4.7) suggests an interpretation of the scattering of physical excitations in the XXZ model in terms of purely geometrical scattering of a spinning particle on the quantum group. To be more precise, we need to consider the scattering on the dual object to the compact real form of $SL_q(2)$. Indeed, our coordinate variable n is just the spectral index of the q-Jacobi polynomials. In the case of the S-wave scattering (involving only zonal spherical harmonics) this dual object can be in some sense identified with a *non-compact* real form of $SL_q(2)$. Analytically, this shows up in the nice self-duality symmetry of Macdonald polynomials (which play the role of zonal spherical functions on $SL_q(2)$) with respect to the exchange of the argument and spectral index [11, 12]. However, for spinorial harmonics this duality comes into play in a less straightforward way which is not completely clear to us at the moment. To illustrate this, let us consider the classical ($q \to 1$) limit of the q-Jacobi polynomials.

If we take the limit $q \to 1$ for fixed $\varphi = 2\lambda \log q$ (so that $q^{2i\lambda} \to e^{i\varphi}$) the q-Jacobi polynomials $R_n^{(\alpha,\beta)}(\frac{1}{2}(q^{2i\lambda} + q^{-2i\lambda}); q^2)$ go to the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos \varphi)$ up to normalization. They yield the restriction of spherical harmonics on the compact real form of $SL(2)$ to the maximal torus parametrized by the coordinate φ. The scattering in the index n *after* taking this limit gives a trivial S-matrix having nothing to do with the $S_{k-a}(\lambda)$ at $q = 1$ (2.23). This means that the limits $n \to \infty$ and $q \to 1$ are not interchangeable. To achieve agreement with the S-matrix (2.12) we need another classical limit! This is $q \to 1$ and $n \to \infty$ for fixed $x \sim n \log q$ (now x is a continuous variable) and fixed λ. Then, by considering the asymptotics at *large* x we recover the correct XXX S-matrix (2.23). In this case the q-Jacobi polynomials go to Legendre (or Gegenbauer) functions $P_{-\alpha-1/2+i\lambda}(\cosh x)$ and *not* to Jacobi functions (as one would except). This can be easily seen from (B6) and was already pointed out by Koornwinder [22].

Finally, in the case $k = 0$ the spectral problem for $\tilde{F}_n^+$ (3.28a) gives yet another eigenvalue of full XXZ S-matrix corresponding to *kink-kink* (or antikink-antikink) scattering. This is just the result obtained in our earlier papers. [11, 12] by considering the scalar spectral problem on a quantum hyperbolic plane. This eigenvalue turns out to be equal to $U(q^{2i\lambda})$ (3.26).

5. The Limit of Infinite Level

The limit of infinite level $k \to \infty$ is obscure in the original version of the q-KZ eqs. (3.9) because $q^{k+2} = p$ goes to zero. However, the form (3.14) is more appropriate for taking this limit since p does not appear in the shift operator explicitly and $p^l = p^{1/(k+2)} = q$ is fixed.

Taking the limit $k \to \infty$ in (3.14) we get

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \begin{pmatrix} q^{-2i\lambda} & 0 \\ 0 & q^{2i\lambda} \end{pmatrix} \begin{pmatrix} q & 1-q^2 \\ 0 & q \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix}, \quad n \geqq 1\,. \tag{5.1}$$

The boundary condition is $g_1/f_1 = q^{4i\lambda+1}$ and follows from the finiteness of f_0 and g_0 (it is implied that $p^n \to \delta_{n,0}$ as $p \to 0$). The corresponding scattering matrix can be determined as above. Note that the scalar dressing factor (3.26) in this limit becomes an elementary function. Equation (5.1) can be viewed also as a large n limit of the general equation (3.14).

The second matrix on the r.h.s. of (5.1) is just the zero-weight piece of the inverse of the familiar constant R-matrix for $U_q(sl_2)$ in the fundamental representation. The appearance of $U_q(sl_2)$ in this context looks quite natural since the limit of infinite level usually means "forgetting the affinization" of affine (quantum) algebras and thus leads to finite-dimensional Lie algebras.

On the other hand, we will presently show that for the particular values of $q^2 = P^{-1}$, where P is a prime number, Eq. (5.1) has an nice interpretation in terms of harmonic analysis of the P-adic group $SL(2, \mathbf{Q}_P)$. [In view of the extensive

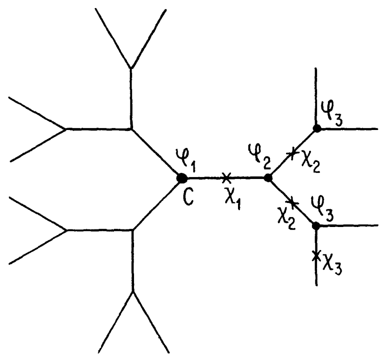

Fig. 1. Bruhat–Tits tree ($P = 2$) with central vertex C

use above of the symbol p as defined in Eq. (3.11), we are straying here from number theoretic custom and denote a prime by P]. To see this, let us derive from (5.1) the second-order recurrence relations for $F_n^{\pm} = f_n \pm q^{-2i\lambda} g_n$ and for $\tilde{F}_n^{\pm} = f_n \pm q^{-4i\lambda \pm 1} g_n$. From (3.16) and (3.28) we obtain:

$$PF_{n+1}^{\pm} + F_{n-1}^{\pm} = 2xP^{1/2}F_n^{\pm} + \delta_{n,1}(F_0^{\pm} \mp F_1^{\pm}); \quad n \geqq 1 \tag{5.2a}$$

with the constraint

$$2(F_{n+1}^{+} - P^{i\lambda - 1/2}F_n^{+}) = (1 + P^{-i\lambda - 1/2})(1 - P^{i\lambda - 1/2})(F_n^{+} - F_n^{-}), \tag{5.2b}$$

and

$$\begin{cases} P\tilde{F}_{n+1}^{+} + \tilde{F}_{n-1}^{+} = 2xP^{1/2}\tilde{F}_n^{+} + \delta_{n,1}(\tilde{F}_0^{+} - \tilde{F}_2^{+}); & n \geqq 1 \quad (5.3a) \\ P\tilde{F}_{n+1}^{-} + \tilde{F}_{n-1}^{-} = 2xP^{1/2}\tilde{F}_n^{-}; & n \geqq 2,\ \tilde{F}_1^{-} = 0, \quad (5.3b) \end{cases}$$

with the constraint

$$(P^{1/2} + P^{-1/2})\tilde{F}_n^{+} = (P^{i\lambda} + P^{-i\lambda})\tilde{F}_n^{+} + (P^{i\lambda - 1} - P^{-i\lambda})\tilde{F}_n^{-}, \tag{5.3c}$$

where $x = 1/2(P^{i\lambda} + P^{-i\lambda})$.

Consider Eq. (5.3a) first. One easily recognizes it as the recurrence relation for the Mautner–Cartier polynomials [23] which are zonal spherical functions on the P-adic symmetric space $H_P = PGL(2, \mathbf{Q}_P)/PGL(2, \mathbf{Z}_P)$ of the group $PGL(2, \mathbf{Q}_P) = GL(2, \mathbf{Q}_P)/\mathbf{Q}_P^*$, where $PGL(2, \mathbf{Z}_P)$ is the maximal compact subgroup. The δ-symbol term in (5.3a) provides the proper boundary condition. The space H_P is known as a Bruhat–Tits tree [24] and it can be represented as the homogeneous tree with each vertex being joined to $P + 1$ "neighboring" vertices by edges (Fig. 1). The $PGL(2, \mathbf{Q}_P)$-invariant Laplacian acting on scalar-valued functions of the vertices can be naturally defined as a sum of the values at all nearest neighbors of a given vertex minus the value of the function at this vertex times the number of the nearest neighbors ($P + 1$ in this case). This definition naturally generalizes the mean value theorem for harmonic functions. One can arbitrarily choose a "central" vertex of the tree and consider zonal spherical functions defined by the two conditions: 1) they are eigenfunctions of the Laplacian; 2) they are "spherically symmetric," i.e., take one and the same value at all points at the same distance from the center. Calling φ_n the value of the function of the vertices on distance $n + 1$ from the center, one can easily write down the eigenvalue equation (Fig. 1):

$$P\varphi_{n+1} + \varphi_{n-1} = (E + P + 1)\varphi_n, \; n \geqq 2\,, \tag{5.4a}$$

$$(P + 1)\varphi_2 = (E + P + 1)\varphi_1\,, \tag{5.4b}$$

where E is the eigenvalue and n stands not for the distance of the vertex from the center, but for this distance plus one. Introducing the new spectral variable $x = 1/2(P^{i\lambda} + P^{-i\lambda})$ by

$$E + P + 1 = 2xP^{1/2}\,, \tag{5.5}$$

we see that (5.3a) and (5.4) coincide.

The Mautner–Cartier polynomials are P-adic analogs of the Legendre (or Gegenbauer) functions. The polynomials $F_n^{\pm}(x)$ in (5.2) should thus provide a P-adic analog of the Jacobi-functions with particular values of α and β. Acually, (5.3a) and (5.2) differ only by the boundary condition at the origin.

We now suggest an "arboreal" interpretation of Eq. (5.2) for F_n^+ similar to that for $\tilde{F}_n^+$ described above. Suppose we now consider functions χ_n of the edge rather than of the vertex. We can again consider a spherically symmetric function: its values depend only on the distance n of the edge to the center C (Fig. 1). The definition of the Laplacian remains the same as before with the change from vertices to edges, the nearest neighbors of an edge being all the edges having a common end with it (there are $2P$ of them). Now the eigenvalue equation looks like

$$P\chi_{n+1} + \chi_{n-1} + (P-1)\chi_n = (E + 2P)\chi_n, \quad n \geqq 2\,,$$

$$P\chi_2 + P\chi_1 = (E + 2P)\chi_1\,. \tag{5.6}$$

Recalling (5.5) this can be rewritten as

$$P\chi_{n+1} + \chi_{n-1} = 2xP^{1/2}\chi_n, \quad n \geqq 2\,,$$

$$P\chi_2 + \chi_1 = 2xP^{1/2}\chi_1\,, \tag{5.7}$$

which *is* exactly Eq. (5.2a) for F_n^+.

Keeping in mind the analogy with the Dirac equation, we can say that the R-matrix for $U_q(sl_2)$ in (5.1) provides a "square root" of the Laplace operator on the P-adic tree ($q^2 = P^{-1}$). This is one more face of the P-adics-quantum group connection [8, 9, 10, 11]. This opens a way to introduce spinorial harmonics on P-adic groups that would be very interesting for a number of reasons.

6. Discussion

In considering the spectral problem for the q-KZ equation, we were motivated by the scalar spectral problem for the Laplace operator on a curved (quantum) space. The latter was known to yield the scattering phase for the kink-kink scattering in the spin$-1/2$ XXZ antiferromagnet [10, 11, 12]. Since the physical excitations in this model are spin $-1/2$ kinks it is natural to expect that their scattering matrix would come from the spectral problem for a matrix first-order operator on the same quantum space whose "square" gives the Laplace operator in the spirit of Dirac's trick. The difference operator appearing in q-KZ equations is a natural candidate for this. We have shown that this is indeed the case and the physical S-matrix of the XXZ-model can be derived this way (for the q-KZ equation at zero level).

However, this correspondence is perfect only for the 2-dimensional zero-weight (zero z-projection of spin) subspace of the linear space $\mathbf{C}^2 \otimes \mathbf{C}^2$ on which the solutions of the q-KZ equation take values. In terms of the XXZ-model, this corresponds to kink-antikink scattering. As for the kink-kink or antikink-antikink channels, they seem to have nothing to do with the $(++)$ and $(--)$ components of the q-KZ solutions, since the latter have trivial scattering. The conceptual understanding of this situation is obscure.

Anyway, we succeeded in describing at least the kink-antikink scattering in the spin$-1/2$ XXZ antiferromagnet in terms of particular continuous q-Jacobi polynomials which yield the spinorial harmonics on the quantum group $SL_q(2)$. This picture is in good agreement with the conjecture made in our earlier paper [1] that the scattering processes in integrable systems are of a *purely geometric* nature.

The above connection with the XXZ model comes already at the level k equal to zero. In Sect. 3 we solved the scattering problem for the $U_q(\widehat{sl}_2)$-q-KZ equation in the fundamental representation for arbitrary level. Does this have any interpretation in terms of excitation scattering in integrable models? Based on the results for the corresponding scalar spectral problem [11] we can conjecture that this case is related to generalized $SL_q(k+2)$-magnetics, or equivalently, to the $\mathbf{Z}_{k+2}$ Baxter statistical model on the square lattice.

A generalization to q-KZ equations for multipoint functions would also be of interest. It should correspond to the multiparticle scattering in integrable models which is known to factorize.

Matsuo [25] has found a Jackson-integral representation of the Jordan–Pochhammer type for solutions of the q-KZ equation. Combining this with our results, one could obtain a new Jackson integral representation for the q-Jacobi polynomials.

Finally, let us remark on the relationship between the monodromy (or connection) problem on the one hand and the scattering problem on the other hand. Our results suggest that they are in some sense dual to each other. The usual setting of the monodromy (connection) problems for q-KZ equations is quite the opposite to what we have done: the spectral parameter λ is fixed once and for all, and one compares the solutions regular at $z = 0$ with those regular at $z = \infty$ (in the variable n this corresponds to $n = -\infty$ and $n = \infty$). Then the physical S-matrices appear as the ratios of two special solutions and they are now functions of z, not λ. This indicates a remarkable duality between the coordinate variable and the spectral parameter.

Acknowledgements. We thank A. Gorsky, S. Shatashvili and P. Wiegmann for discussions and I. Cherednik and T. Koornwinder for sending us their papers. We are grateful to the referee for pointing out a mistake in the original version of this paper. One of us (A.Z.) wishes to thank Prof. J. Peter May for the hospitality of the Mathematical Disciplines Center at the University of Chicago.

Appendix A

In this appendix we show how one can find the specific linear combinations of f_n, g_n satisfying the equation for q-Jacobi polynomials.

Let us write the q-KZ equation (3.14) in the form

$$\begin{pmatrix} f_{n+1} \\ g_{n+1} \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix} . \tag{A1}$$

It is clear that f_n and g_n satisfy decoupled second-order recurrence relations, but generally these are not very helpful. Our task is to extract an equation which can be solved! In he limit $q = 1$ the situation is quite similar though much simpler: $f(x)$ and $g(x)$ themselves obey two decoupled second order differential equations of rather complicated form but, fortunately, the "right" linear combinations $f \pm g$ leading to the familiar hypergeometric equations are more or less obvious from the very beginning.

The idea is to apply to (A1) a similarity transformation in such a way that the resulting second order equations would be as simple as possible. So let us take a non-degenerate constant matrix $U = \{u_{ij}\}; i,j = 1,2$ and apply it to (A1). We get

$$\begin{pmatrix} \tilde{f}_{n+1} \\ \tilde{g}_{n+1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_n & \tilde{b}_n \\ \tilde{c}_n & \tilde{d}_n \end{pmatrix} \begin{pmatrix} \tilde{f}_n \\ \tilde{g}_n \end{pmatrix} , \tag{A2}$$

where

$$\begin{pmatrix} \tilde{f}_n \\ \tilde{g}_n \end{pmatrix} = U^{-1} \begin{pmatrix} f_n \\ g_n \end{pmatrix} , \tag{A3}$$

$$\begin{pmatrix} \tilde{a}_n & \tilde{b}_n \\ \tilde{c}_n & \tilde{d}_n \end{pmatrix} = U^{-1} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} U . \tag{A4}$$

The matrix equation (A2) is equivalent to the pair of decoupled second order equations for $\tilde{f}_n$ and $\tilde{g}_n$:

$$\tilde{b}_{n-1}\tilde{f}_{n+1} + \tilde{b}_n \Delta_{n-1}\tilde{f}_{n-1} = (\tilde{a}_n\tilde{b}_{n-1} + \tilde{d}_{n-1}\tilde{b}_n)\tilde{f}_n , \tag{A5a}$$

$$\tilde{c}_{n-1}\tilde{g}_{n+1} + \tilde{c}_n \Delta_{n-1}\tilde{g}_{n-1} = (\tilde{d}_n\tilde{c}_{n-1} + \tilde{a}_{n-1}\tilde{c}_n)\tilde{g}_n \tag{A5b}$$

together with the constraint

$$\tilde{g}_n = \tilde{b}_n^{-1}(\tilde{f}_{n+1} - \tilde{a}_n\tilde{f}_n) , \tag{A5c}$$

where

$$\Delta_n = \det R_0(p^{n+l}) = \frac{p^{2l}(1 - p^{2n})}{1 - p^{2n+4l}} . \tag{A6}$$

The matrix elements in (A4) have the following general form:

$$\tilde{a}_n = \frac{\alpha_0 - \alpha_1 p^{2n}}{1 - p^{2n+4l}}, \qquad \tilde{b}_n = \frac{\beta_0 - \beta_1 p^{2n}}{1 - p^{2n+4l}} ,$$

$$\tilde{c}_n = \frac{\gamma_0 - \gamma_1 p^{2n}}{1 - p^{2n+4l}}, \qquad \tilde{d}_n = \frac{\delta_0 - \delta_1 p^{2n}}{1 - p^{2n+4l}} , \tag{A7}$$

where the full n-dependence is indicated explicitly. Calling $y = p^{iu}$ for brevity, we have from (A4):

$$\begin{aligned}
\alpha_0 &= (\det U)^{-1}[y^{-1}p^l u_{11}u_{22} + y^{-1}(1-p^{2l})u_{21}u_{22} - yp^l u_{12}u_{21}]\,,\\
\alpha_1 &= (\det U)^{-1}[y^{-1}p^{3l} u_{11}u_{22} + y(1-p^{2l})p^{2l}u_{11}u_{12} - yp^{3l} u_{12}u_{21}]\,,\\
\beta_0 &= (\det U)^{-1}[y^{-1}p^l u_{12}u_{22} + y^{-1}(1-p^{2l})u_{22}^2 - yp^l u_{12}u_{22}]\,,\\
\beta_1 &= (\det U)^{-1}[y^{-1}p^{3l} u_{12}u_{22} + y(1-p^{2l})p^{2l}u_{12}^2 - yp^{3l} u_{12}u_{22}]\,,\\
\gamma_0 &= (\det U)^{-1}[-y^{-1}p^l u_{11}u_{21} - y^{-1}(1-p^{2l})u_{21}^2 + yp^l u_{11}u_{21}]\,,\\
\gamma_1 &= (\det U)^{-1}[-y^{-1}p^{3l} u_{11}u_{21} - y(1-p^{2l})p^{2l}u_{11}^2 + yp^{3l} u_{11}u_{21}]\,,\\
\delta_0 &= (\det U)^{-1}[-y^{-1}p^l u_{12}u_{21} - y^{-1}(1-p^{2l})u_{21}u_{22} + yp^l u_{11}u_{22}]\,,\\
\delta_1 &= (\det U)^{-1}[-y^{-1}p^{3l} u_{12}u_{21} - y(1-p^{2l})p^{2l}u_{11}u_{12} + yp^{3l} u_{11}u_{22}]\,. \qquad \text{(A8)}
\end{aligned}$$

Equations (A5) are now written as:

$$\begin{aligned}
&(1-p^{2n+4l})(\beta_0 - \beta_1 p^{2n-2})\tilde{f}_{n+1} + p^{2l}(1-p^{2n-2})(\beta_0 - \beta_1 p^{2n})\tilde{f}_{n-1}\\
&\quad = ((\alpha_0 - \alpha_1 p^{2n})(\beta_0 - \beta_1 p^{2n-2}) + (\delta_0 - \delta_1 p^{2n-2})(\beta_0 - \beta_1 p^{2n}))\tilde{f}_n \qquad \text{(A9a)}
\end{aligned}$$

and

$$\begin{aligned}
&(1-p^{2n+4l})(\gamma_0 - \gamma_1 p^{2n-2})\tilde{g}_{n+1} + p^{2l}(1-p^{2n-2})(\gamma_0 - \gamma_1 p^{2n})\tilde{g}_{n-1}\\
&\quad = ((\delta_0 - \delta_1 p^{2n})(\gamma_0 - \gamma_1 p^{2n-2}) + (\alpha_0 - \alpha_1 p^{2n-2})(\gamma_0 - \gamma_1 p^{2n}))\tilde{g}_n\,. \qquad \text{(A9b)}
\end{aligned}$$

We are going to compare (A9) with the recurrence relation (B8) for q-Jacobi polynomials (see Appendix B below). For general values of (α, β) they look quite different. However, for the particular values: I) $\beta = \alpha + 1$; II) $\beta = \alpha - 1$; III) $\beta = \alpha$, considerable simplifications occur in (B8). For example, in the case I) we have

$$\begin{aligned}
&(1-p^{2n+4\alpha+4})(1-p^{2n+2\alpha+1})R_{n+1}(x) + p^{2\alpha+1}(1-p^{2n})(1-p^{2n+2\alpha+3})R_{n-1}(x)\\
&\quad = -(1-p^2)(1-p^{2\alpha+1})p^{2n+2\alpha+1}R_n(x) + 2xp^{\alpha+1/2}(1-p^{2n+2\alpha+3})\\
&\quad\quad \times (1-p^{2n+2\alpha+1})R_n(x)\,, \qquad \text{(A10)}
\end{aligned}$$

whose l.h.s. is *identical* to that of (A9a) provided

$$\alpha = l - 1/2\,, \qquad \text{(A11)}$$

$$\beta_1 = \beta_0 p^{2l}\,, \qquad \text{(A12)}$$

$$\tilde{f}_n = R_{n-1}(x)\,. \qquad \text{(A13)}$$

What about the right-hand sides? A straightforward calculation shows that they are identical provided

$$\begin{cases} \alpha_0 + \delta_0 = 2xp^{\alpha+1/2}\,, & \text{(A14)} \\ \alpha_1 + \delta_1 = 2xp^{3\alpha+3/2}\,, & \text{(A15)} \\ \alpha_0 - p^{-2l}\alpha_1 = 1 - p^{2l}\,. & \text{(A16)} \end{cases}$$

One can see that all these conditions are mutually consistent. They determine the matrix elements of U. Solving (A11)–(A16) using (A8), we obtain the following simple relations:

$$2x = y + y^{-1} = p^{iu} + p^{-iu}\,, \tag{A17}$$

$$\frac{u_{12}}{u_{22}} = -y^{-1} = -p^{-iu}\,. \tag{A18}$$

The other matrix elements of U are not fixed yet because so far we have used only one equation from the pair (A9).

It turns out that the equation for $\tilde{q}_n$ (A9b) can be obtained from (B8) in a similar way. Namely, let us rewrite (B8) in the case II ($\beta' = \alpha' - 1$; the primes indicate that the values of α and β may be different from the case above) as equations for the polynomials $\tilde{R}_n(x)$ normalized as follows:

$$\tilde{R}_n(x) = p^{-n}(1 - p^{2n+2\alpha'-2})R_{n-1}^{(\alpha',\alpha'-1)}(x;p)\,. \tag{A19}$$

We arrive at

$$\begin{aligned}(1 - p^{2n+4\alpha'-2})&(1 - p^{2n+2\alpha'-3})\tilde{R}_{n+1}(x)\\ &+ p^{2\alpha'-1}(1 - p^{2n-2})(1 - p^{2n+2\alpha'-1})\tilde{R}_{n-1}(x)\\ = (1 - p^2)&(1 - p^{2\alpha'-1})p^{2n+2\alpha'-3}\tilde{R}_n(x)\\ &+ 2xp^{\alpha'-1/2}(1 - p^{2n+2\alpha'-1})(1 - p^{2n+2\alpha'-3})\tilde{R}_n(x)\,.\end{aligned} \tag{A20}$$

The expressions on the l.h.s. of (A9b) and (A20) coincide, provided

$$\alpha' = l + 1/2\,, \tag{A21}$$

$$\gamma_1 = \gamma_0 p^{2l}\,, \tag{A22}$$

$$\tilde{g}_n = \tilde{R}_n(x) \tag{A23}$$

Note that the values of α_i, δ_i are already fixed by (A14)–(A16). Substituting them into the r.h.s. of (A9b) and taking into account (A21)–(A23) one obtains exactly the r.h.s. of (A20). Equation (A22) gives an extra relation for the matrix elements of U:

$$\frac{u_{21}}{u_{11}} = y = p^{iu}\,. \tag{A24}$$

Now, recalling (A3), we find from (A18) and (A24) the desired linear combinations:

$$F_n^{\pm} = f_n \pm p^{-iu}g_n\,. \tag{A25}$$

The matrix equation for them is

$$2\frac{1-p^{2n+4l}}{1-p^{2n+2l}}\begin{pmatrix}F_{n+1}^{+}\\ F_{n+1}^{-}\end{pmatrix}$$
$$=\begin{pmatrix} p^{l}(p^{iu}+p^{-iu})+(1-p^{2l})\frac{1+p^{2n+2l}}{1-p^{2n+2l}} & -(1-p^{2l})-p^{l}(p^{iu}-p^{-iu}) \\ 1-p^{2l}-p^{l}(p^{iu}-p^{-iu}) & p^{l}(p^{iu}+p^{-iu})-(1-p^{2l})\frac{1+p^{2n+2l}}{1-p^{2n+2l}} \end{pmatrix}$$
$$\times\begin{pmatrix}F_{n}^{+}\\ F_{n}^{-}\end{pmatrix}. \qquad (A26)$$

Now to the case III ($\beta=\alpha$). The recurrence relation for $R_n^{(\alpha,\alpha)}$ has the form (B12) that is even simpler than (A10) or (A20). One can try to find another similarity transformation to fit the q-KZ equations (A1) to a pair of equations like (B12) with different values of α. Remarkably, it turns out to be possible, with the new linear combinations being

$$\tilde F_n^{\pm}=f_n\pm p^{\pm l-2iu}g_n\,. \qquad (A27)$$

The calculations are quite similar to those above. For $\tilde F_n^{\pm}$ one obtains the equation

$$\begin{pmatrix}\tilde F_{n+1}^{+}\\ \tilde F_{n+1}^{-}\end{pmatrix}=\begin{pmatrix} p^{iu}+p^{-iu} & p^{2l-iu}-p^{-iu} \\ \frac{1-p^{2n}}{1-p^{2n+4l}}(p^{-iu}-p^{2l+iu}) & \frac{p^{2l}(1-p^{2n})}{1-p^{2n+4l}}(p^{-iu}+p^{iu}) \end{pmatrix}\begin{pmatrix}\tilde F_{n}^{+}\\ \tilde F_{n}^{-}\end{pmatrix}. \qquad (A28)$$

Equations (A26) and (A28) are (different) discrete counterparts of Eqs. (2.7).

Appendix B

Here we collect some necessary formulas related to q-Jacobi polynomials and their asymptotic properties. As in the main text, we denote the base parameter as p rather than q to distinguish it from the deformation parameter of the quantum group.

It is convenient to introduce q-Jacobi polynomials as a particular case of a very general family of Askey–Wilson polynomials explicitly given by

$$p_n\left(\frac{1}{2}(z+z^{-1});\ a,b,c,d|p\right)$$
$$=(ab;p)_n(ac;p)_n(ad;p)_n a^{-n}\cdot {}_4\phi_3\left[\begin{matrix}p^{-n},\ abcdp^{n-1},\ \ az,\ az^{-1}\\ ab,\ ac,\ ad\end{matrix};p,p\right], \qquad (B1)$$

where

$${}_4\phi_3\left[\begin{matrix}\alpha,\beta,\gamma,\delta\\ \lambda,\mu,\nu\end{matrix};p,x\right]=\sum_{k=0}^{\infty}\frac{(\alpha;p)_k(\beta;p)_k(\gamma;p)_k(\delta;p)_k}{(\lambda;p)_k(\mu;p)_k(\nu;p)_k(p;p)_k}x^k \qquad (B2)$$

is the basic hypergeometric function. The p_n are polynomials in $(z+z^{-1})/2$ of degree n. An important property, not obvious from the definition, is that they are actually symmetric with respect to permutations of the parameters a,b,c,d. We suppose that all these parameters have moduli less than 1; in this case the Askey–Wilson polynomials are known to be orthogonal for z on the unit circle with a continuous measure. We need the asymptotics of the Askey–Wilson polynomials for large n. The formula is relatively simple though the derivation is extremely cumbersome [18, 26]:

$$p_n\left(\frac{1}{2}(z+z^{-1});\ a,b,c,d|p\right) n \overset{\sim}{\to} \infty$$

$$z^n B(z^{-1}) + z^{-n}B(z) + \text{exponentially small terms}\,, \tag{B3}$$

where

$$B(z) = \frac{(az;p)_\infty (bz;p)_\infty (cz;p)_\infty (dz;p)_\infty}{(z^2;p)_\infty} \tag{B4}$$

Another commonly used normalization of p_n's is obtained by disregarding z-independent factors in front of ${}_4\phi_3$ in (B1). Though this normalization is also convenient, the symmetry in a,b,c,d is lost in this case. We define continuous q-Jacobi polynomials (they are called "continuous" [28] because of their orthogonality with respect to a continuous measure, as opposed to the so-called "little" or "big" q-Jacobi polynomials which are different q-analogs of Jacobi polynomials, orthogonal with respect to discrete measures), using this asymmetric normalization and choosing a,b,c,d as follows:

$$a = p^{\alpha+1/2}, \qquad b = -p^{\beta+1/2}, \qquad c = p^{1/2}, \qquad d = -p^{1/2}\,. \tag{B5}$$

The parameters α and β are supposed to be real and greater than $-1/2$. In terms of the basic hypergometric series, we have from (B1):

$$R_n^{(\alpha,\beta)}\left(\frac{1}{2}(z+z^{-1});p\right) = {}_4\phi_3\left[\begin{matrix} p^{-n}, & p^{n+\alpha+\beta+1}, & p^{1/2}z, & p^{1/2}z^{-1} \\ p^{\alpha+1}, & -p^{\beta+1}, & -p & \end{matrix}\ ;p,p\right]\,. \tag{B6}$$

Our normalization (B6) is different from Rahman's [27] commonly used normalization. For the reader's convenience we give the explicit formula connecting our q-Jacobi polynomials $R_n^{(\alpha,\beta)}$ with $P_n^{(\alpha,\beta)}$ defined by Rahman:

$$R_n^{(\alpha,\beta)}\left(\frac{1}{2}(z+z^{-1});p\right) = \frac{(p;p)_n(-p;p)_n}{(p^{\alpha+1};p)_n(-p^{\beta+1};p)_n}\cdot P_n^{(\alpha,\beta)}\left(\frac{1}{2}(z+z^{-1});p\right)\,. \tag{B7}$$

These polynomials satisfy a three-term recurrence relation which can be written in the form:

$$\begin{aligned}
&(1-p^{2n+2\alpha+2\beta+2})(1-p^{2n+2\alpha+2})(1-p^{2n+\alpha+\beta})R_{n+1}^{(\alpha,\beta)}(x;p) \\
&\quad + p^{2\alpha+1}(1-p^{2n})(1-p^{2n+2\beta})(1-p^{2n+\alpha+\beta+2})R_{n-1}^{(\alpha,\beta)}(x;p) \\
&= (1+p)(p^{\beta}(1+p^{2\alpha})-p^{\alpha}(1+p^{2\beta}))p^{2n+\alpha+1}(1-p^{2n+\alpha+\beta+1})R_n^{(\alpha,\beta)}(x;p) \\
&\quad + 2xp^{\alpha+1/2}(1-p^{2n+\alpha+\beta})(1-p^{2n+\alpha+\beta+1}(1-p^{2n+\alpha+\beta+2})R_n^{(\alpha,\beta)}(x;p)\,.
\end{aligned} \tag{B8}$$

Their asymptotics at large n is given by (see (B3)):

$$R_n^{(\alpha,\beta)}\left(\frac{1}{2}(z+z^{-1});p\right) n \overset{\sim}{\rightarrow} \infty\, N_{\alpha\beta}\, p^{n(\alpha+1/2)}(z^n C_{\alpha\beta}(z^{-1})+z^{-n}C_{\alpha\beta}(z))\,, \qquad \text{(B9)}$$

where

$$C_{\alpha\beta}(z)=\frac{(zp^{\alpha+1/2};p)_\infty(-zp^{\beta+1/2};p)_\infty}{(z^2;p^2)_\infty}\,, \qquad \text{(B10)}$$

$$N_{\alpha\beta}^{-1}=(p^{2\alpha+2};p^2)_\infty(-p^{\alpha+\beta+1};p)_\infty\,. \qquad \text{(B11)}$$

In the special case $\beta=\alpha$ the polynomials (B6) become Macdonald polynomials [8] for the root system A_1. In the theory of q-hypergeometric functions they are known as Rogers–Askey–Ismail polynomials [17, 18]. The recurrence relation (B8) then simplifies to

$$\frac{1-p^{2n+4\alpha+2}}{1-p^{2n+2\alpha+1}}R_{n+1}^{(\alpha,\alpha)}(x)+p^{2\alpha+1}\frac{1-p^{2n}}{1-p^{2n+2\alpha+1}}R_{n-1}^{(\alpha,\alpha)}(x)=2xp^{\alpha+1/2}R_n^{(\alpha,\alpha)}(x)\,. \qquad \text{(B12)}$$

The famous Macdonald parameters q_M, t_M are then

$$q_M=p^2\,, \qquad \text{(B13)}$$

$$t_M=p^{2\alpha+1}\,. \qquad \text{(B14)}$$

In our normalization we have

$$R_0^{(\alpha,\alpha)}(x)=R_n^{(\alpha,\alpha)}\left(\frac{1}{2}(p^{\alpha+1/2}+p^{-\alpha-1/2})\right)=1\,. \qquad \text{(B15)}$$

Note that unlike in the case of usual boundary conditions for three term recurrence relations ($R_{-1}=0$, $R_0=1$), we dot *not* have to fix R_{-1}, because the coefficient in front of it in (B12) is zero. The same is true for the general q-Jacobi polynomials (B6).

References

1. Freund, P.G.O., Zabrodin, A.V.: Phys. Lett. **B294**, 347 (1992)
2. Freund, P.G.O., Zabrodin, A.V.: Phys. Lett. **B311**, 103 (1993)
3. Frenkel, I., Reshetikhin, N.Yu.: Commun. Math. Phys. **146**, 1 (1992)
4. Knizhnik, V.G., Zamolodchikov, A.B.: Nucl. Phys. **B247**, 83 (1984)
5. Bargmann, V., Wigner, E.P.: Proc. Nat. Acad. Sci. USA **34**, 211 (1946)
6. Faddeev, L.D., Takhtajan, A.L.: Zapiski Nauchn. Sem. LOMI **109**, 134 (1981)
7. Kirillov, A.N., Reshetikhin, N.Yu.: J. Physics **A20**, 1565 (1987)
8. Macdonald, I.G.: In: Nevai, P. (ed.) Orthogonal Polynomials: Theory and Practice. Dordrecht: Kluwer Academic, 1990; Macdonald, I.G.: Queen Mary College preprint 1989
9. Freund, P.G.O.: In: Clavelli, L., Harms, B. (eds.) Superstrings and Particle Theory. Singapore: World Scientific, 1990
10. Zabrodin, A.V.: Mod. Phys. Lett. **A7**, 441 (1992)
11. Freund, P.G.O., Zabrodin, A.V.: Commun. Math. Phys. **147**, 277 (1992)
12. Matsuo, A.: Invent. Math. **110**, 95 (1992)
13. Cherednik, I.: Preprint RIMS-776 (1991); Duke Math. J. **68**, 171 (1992)
14. Erdélyi, A.: Higher Transcendental Functions, v. II. New York: McGraw-Hill, 1953

15. Davies, B., Foda, O., Jimbo, M., Miwa, T., Nakayashiki, A.: Commun. Math. Phys. **151**, 89 (1993); Idzumi, M., Iohara, K., Jimbo, M., Miwa, T., Nakashima, T., Tokihiro, T.: RIMS preprint 1992
16. Rogers, L.J.: Proc. Lond. Math. Soc. **26**, 318 (1895)
17. Askey, R., Ismail, M.E.H. In: Erdös, P. (ed.) Studies in Pure Math. Basel: Birkhäuser, 1983
18. Gasper, G., Rahman, M.: Basic hypergeometric series. Cambridge: Cambridge University Press, 1990
19. Cherednik, I.: Commun. Math. Phys. **150**, 109 (1992)
20. Noumi, M., Mimachi, K.: Proc. Japan Acad. Ser. A **66**, 146 (1990)
21. Freund, P.G.O., Zabrodin, A.V.: Phys. Lett. **B284**, 283 (1992)
22. Koornwinder, T.: J. Math. Anal. and Appl. **148**, 44 (1990)
23. Mautner, F.: Am. J. Math. **80**, 441 (1958); Cartier, P.: Proc. Symp. Pure Math., vol **26**, Providence, RI: A.M.S., 1973
24. Bruhat, F., Tits, J.: Publ. Math. IHES **41**, 5 (1972)
25. Matsuo, A.: Commun. Math. Phys. **151**, 263 (1993)
26. Rahman, M.: SIAM J. Math. Anal. **17**, 1280 (1986)
27. Rahman, M.: Canad. J. Math. **33**, 255 (1981)
28. Askey, R., Wilson, J.A.: Memoirs Amer. Math. Soc. **319** (1985)

Communicated by G. Felder