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Branching Rules for Conformal Embeddings

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Abstract: We give explicit formulas for the branching rules of the conformal embeddings $su(n(n+1)/2)_1 \supset su(n)_{n+2}$, $su(n(n-1)/2)_1 \supset su(n)_{n-2}$, $sp(n)_1 \supset so(n)_4 \oplus su(2)_n$, and $so(m+n)_1 \supset so(m)_1 \oplus so(n)_1$ with m and n odd.

Introduction

The theory of affine Lie algebras has found very useful applications in Theoretical Physics. Our work is related to the models found in Conformal Field Theory.

In [K-P] a set of functions called string functions were introduced to describe the branching rules of an integrable highest weight representation of an affine Lie algebra $\hat{g}$ with respect to its homogeneous Heisenberg subalgebra $\hat{h}$. There it was observed that those functions were modular functions with respect to a congruence subgroup of $Sl_2(Z)$. In [K-W], the problem of describing the branching rules for an arbitrary pair $\hat{g} \supset \hat{p}$ was considered, proving modular properties and finding the asymptotic behaviour of most of them.

A special case of pairs $\hat{g} \supset \hat{p}$ comes from the so-called coset construction, [G-K-O], given an irreducible highest weight representation $L(\Lambda)$ of $\hat{g}$ one constructs the Sugawara operators $T_m^{\hat{g}}$ that give a representation of the Virasoro algebra on $L(\Lambda)$, similarly for the restriction to $\hat{p}$ one obtains a representation of the Virasoro algebra by operators $T_m^{\hat{p}}$. Taking the difference of the Virasoro operators, a new representation of the Virasoro algebra is obtained and it commutes with $\hat{p}$. Thus we get the decomposition:

$$L(\Lambda) = \bigoplus_{\lambda} U(\Lambda, \lambda) \otimes L(\lambda) .$$

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The central charge of the Virasoro algebra acting on $U(\Lambda,\lambda)$ is $z_\Lambda(g) - z_\lambda(p)$, where

$$z_\Lambda(g) = \frac{k \dim g}{k + h(g)},$$

$h(g)$ being the dual Coxeter number, k the level of $L(\Lambda)$ and $z_\Lambda(g)$ (resp. $z_\lambda(p)$) the central charge of the Sugawara representation of the Virasoro algebra acting on $L(\Lambda)$ (resp. $L(\lambda)$). Let h and $\dot{h}$ denote Cartan subalgebras of g and p. One can choose them so that $\dot{h} \subset h$. Let $H = \{\tau \in C | \operatorname{Im} \tau > 0\}$ be the upper-half plane. The normalized character χ_Λ of $L(\Lambda)$ is the holomorphic function on $H \times h$:

$$\chi_\Lambda(\tau, z) = q^{-z_\Lambda(g)/24} \operatorname{tr}_{L(\Lambda)} \exp 2i\pi(\tau L_0 + z), \tag{0.1}$$

where as usual q denotes $\exp 2i\pi\tau$. Suppose that $z \in \dot{h}$, then from (0.1) we get:

$$\chi_\Lambda(\tau, z) = \sum_\lambda b_\lambda^\Lambda(\tau)\chi_\lambda(\tau, z), \tag{0.2}$$

where the branching function b_λ^Λ is

$$b_\lambda^\Lambda(\tau) = q^{-(z_\Lambda(g) - z_\lambda(p))/24} \operatorname{tr}_{U_{(\Lambda,\lambda)}} q^{L_0}. \tag{0.3}$$

The modular transformation properties of the characters are given by [KW]:

$$\chi_\Lambda(\tau + 1, z) = e^{2\pi i(h_\Lambda - z_\Lambda(g)/24)}\chi_\Lambda(\tau, z) \tag{0.4}$$

with

$$h_\Lambda = \frac{(\Lambda + 2\rho|\Lambda)}{2(k + h(g))}. \tag{0.5}$$

h_Λ is called the *trace anomaly*, and

$$\chi_\Lambda\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi(z|z)/\tau} \sum_{M \in P_+^k} a(\Lambda, M)\chi_M(\tau, z). \tag{0.6}$$

P_+^k is the set of dominant highest weights of level k, and

$$a(\Lambda, M) = i^{|\Delta_+|}|P/P^*|^{-1/2}(k + h(g))^{-n/2} \times \sum_{w \in W} \det(w) \exp \frac{-2i\pi}{k + h(g)}(\bar{\Lambda} + \bar{\rho}|w(\bar{M} + \bar{\rho})). \tag{0.7}$$

$|\Delta_+|$ is the number of positive roots of g, P is the weight lattice, P^* its dual, W the Weyl group, and $\bar{\Lambda}$ and $\bar{\rho}$ denote the "finite" parts of Λ and ρ, i.e. $\Lambda = k\Lambda_0 + \bar{\Lambda}$, $\rho = h(g)\Lambda_0 + \bar{\rho}$, see [K].

Set $a(\Lambda) = a(\Lambda, k\Lambda_0)$. By the Weyl denominator formula,

$$a(\Lambda) = |P/P^*|^{-1/2}(k + h(g))^{-n/2} \prod_{\gamma \in \Delta_+} 2 \sin \frac{\pi(\bar{\Lambda} + \bar{\rho}|\alpha)}{k + h(g)}. \tag{0.8}$$

Hence $a(\Lambda)$ is a positive real number, it is called the *asymptotic dimension* of $L(\Lambda)$, and appears in the asymptotic behavior of $\chi_\Lambda(\tau, 0)$ as $\tau \to 0$. It turns out to be:

$$\chi_\Lambda(\tau, 0) \sim a(\Lambda) e^{i\pi z_\Lambda(g)/12\tau} . \qquad (0.9)$$

From (0.2) one easily deduces the transformation law of the branching function:

$$b_\lambda^\Lambda\left(-\frac{1}{\tau}\right) = \sum_{M \in P_+^k,\, \mu \in \dot{P}_+^k} a(\Lambda, M) \dot{a}^*(\lambda, \mu) b_\mu^M(\tau) \qquad (0.10)$$

(dotted quantities refer to the subalgebra p).

We say that $g \supset p$ is a conformal embedding when $U(\Lambda, \lambda)$ is finite-dimensional, or equivalently when $z_\Lambda(g) = z_\lambda(p)$. This implies that the level of $\hat{g}$ is one. In this case $b_\lambda^\Lambda(\tau) = \dim U(\Lambda, \lambda) = b(\Lambda, \lambda)$ is a constant and (0.10) reads:

$$b(\Lambda, \lambda) = \sum_{M, \mu} a(\Lambda, M) \dot{a}^*(\lambda, \mu) b(M, \mu) , \qquad (0.11)$$

i.e. the rectangular matrix $b(\Lambda, \lambda)$ commutes with the action of the modular group on the characters of $\hat{g}$ and $\hat{p}$. This matrix obeys also the important identity:

$$a(\Lambda) = \sum_{\lambda \in \dot{P}_+^k} b(\Lambda, \lambda) \dot{a}(\lambda) , \qquad (0.12)$$

obtained by inserting (0.9) and its analog for p in (0.2).

Conformal embeddings were classified in [B-B, S-W and A-G-O]. The problem of finding the branching rules for them was considered in [K-P, K-W, K-S, W, V and A-B-I]. We will give the branching rules for the families:

$$su(n(n+1)/2) \supset su(n) \qquad \text{index} \quad n+2 ,$$
$$su(n(n-1)/2) \supset su(n) \qquad \text{index} \quad n-2 ,$$
$$sp(n) \supset so(n) \oplus su(2) \qquad \text{index} \quad (4, n) ,$$
$$so(2(m+n+1)) \supset so(2m+1) \oplus so(2n+1) \qquad \text{index} \quad (1, 1) .$$

The paper is organized as follows: we compute in the first four sections the decompositions corresponding to each of the cases mentioned above. Finally Sect. 5 contains some conclusions and remarks concerning modular invariant partition functions.

1. Branching Rules for $su(n) \subset su(n(n+1)/2)$

The description of $b(\Lambda, \dot{\lambda})$ will be obtained from the study of the conformal pair

$$u(1) \oplus su(n) \subset so(n(n+1)) , \qquad (1.1)$$

which comes from the symmetric space

$$Sp(n)/U(n) .$$

The link of (1.1) with

$$su(n)_{n+2} \subset su(n(n+1)/2)_1$$

is provided by

$$u(1) \oplus su(n(n+1)/2) \subset so(n(n+1)) , \tag{1.2}$$

which is also conformal, with known branching rules (see below).

To compute the branching functions $b(\Lambda, \dot{\lambda})$ in the cases (1.1) and (1.2), we use the following theorem from [A-B-I], which is a generalization for the reductive case of the main theorem in [K-P] (with a correction by Nahm, see [N]), and gives the decomposition of the half-spin representations s and t. It is in fact a generalization of the finite-dimensional analog, which was proved in [P].

Theorem 1.1. *Let h be a simple Lie algebra, $p \subset h$ a reductive subalgebra of the same rank, such that $h = p \oplus V$ defines a symmetric space, i.e. $[V,V] \subset p$ and $p \subset g = so(V)$. Then the decomposition of $s \oplus t$ of $\hat{g}$ into irreducible $\hat{p}$-modules is*

$$s \oplus t = \bigoplus_{w \in W/\dot{W}} L(w(\rho) - \dot{\rho}) , \tag{1.3}$$

where the dots refer to p, W is the affine Weyl group of h, ρ is the affine Weyl vector of h, and $W/\dot{W}$ is a set of coset representatives such that $w(\rho) - \dot{\rho}$ is a dominant weight of $\hat{p}$.

In the case (1.1) we have $h = sp(n)$, $p = su(n) \oplus u(1)$. The Weyl vector of $\hat{h}$ is given by $\rho = (n+1)\Lambda_0 + \bar{\rho}$ with the (finite) Weyl vector $\bar{\rho} = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n$ (where the ε_i are orthonormal vectors); in the case of p we have $\dot{\rho} = n\dot{\Lambda}_0 + \dot{\bar{\rho}}$ with $\dot{\bar{\rho}} = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1} - \frac{(n-1)}{2}\sum_{i=1}^{n} \varepsilon_i$ and the fundamental weights are $\dot{\Lambda}_i = \dot{\Lambda}_0 + \dot{\bar{\Lambda}}_i$ with $\dot{\bar{\Lambda}}_i = \sum_{j=1}^{i} \varepsilon_j - \frac{i}{n}\sum_{j=1}^{n} \varepsilon_j$. The Weyl group is given by $W = \bar{W} \ltimes T$, where T are the translations $\{t_\alpha\}_{\alpha \in L}$ with $L = \sum_{i=1}^{n} 2Z\varepsilon i = \sum_{i=1}^{n-1} 2Z\alpha_i + Z\alpha_n (\{\alpha_i\}_{i=1}^{n} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$ are the simple roots of h), and $\bar{W} = S_n \ltimes (Z_2)^n$, where S_n is the permutation group on the set $\{\varepsilon_i\}_{i=1}^{n}$ and $(Z_2)^n$ acts by $\varepsilon_i \to \pm\varepsilon_i$; in the case of p, $\dot{W} = S_n \ltimes \dot{T}$, where T are the translations by $\alpha \in \dot{L} = \sum_{i=1}^{n-1} Z\dot{\alpha}_i$. Observe that p is included in h in such a way that $\dot{\alpha}_k = \alpha_k, 1 \leqq k \leqq n-1$, see [K].

In order to get a dominant weight $w(\rho) - \dot{\rho}$ we have to use as representatives of $T/(T \cap \dot{T})$ in $W/\dot{W}$, not the translations $t_{k\alpha_n}$ by multiples of α_n, but the powers of $\sigma_0 t_{\alpha_n}$, where σ_0 is the permutation

$$1 \to 2 \to \cdots \to n \to 1 \tag{1.4}$$

since, if μ is the automorphism given by

$$\mu(\dot{\Lambda}_i) = \dot{\Lambda}_{i+1} \bmod n ,$$

then

$$\sigma_0 t_{\alpha_n}(w(\rho)) - \dot{\rho} = \mu(w(\rho) - \dot{\rho}) . \tag{1.5}$$

The restriction of $\lambda = \sum_{i=1}^{n} a_i\varepsilon_i$ a weight of h to p is $\dot{\lambda} = \sum_{i=1}^{n-1}(a_i - a_{i+1})\dot{\bar{\Lambda}}_i$ and $\dot{\lambda}$ is a (strictly) dominant weight iff $a_i > a_{i+1}$. Thus we see that a suitable choice for $\bar{W}/\dot{\bar{W}}$ is the following: for each $s \in (Z_2)^n$ we take σ_s the permutation which orders the coefficients of $s(\bar{\rho}) = \sum_{i=1}^{n} a_i\varepsilon_i$ decreasingly. Then we take $\bar{W}/\dot{\bar{W}} = \{\sigma_s s\}$.

For example, if $n=3$ and $s=(-1,1,1)$ then $s(\bar{\rho})=-3\varepsilon_1+2\varepsilon_2+1\varepsilon_3$, so $\sigma_s = 1\to 3\to 2\to 1$, and $\sigma_s s(\rho)-\dot{\rho}=2\dot{\Lambda}_0+0\dot{\Lambda}_1+3\dot{\Lambda}_2$.

Therefore, we get the explicit form of the decomposition (1.3) in the case (1.1):

$$s\oplus t=\bigoplus_{k\in Z}\bigoplus_{s\in(Z_2)^n} L(\mu^k(\sigma_s s(\rho)-\dot{\rho}))\otimes F(h_{s,k})\,. \tag{1.6}$$

$F(h_{s,k})$ is an irreducible Fock space representation of the $u(1)$ Heisenberg algebra with conformal weight $h_{s,k}$.

Recall that the decomposition (1.3) in the case (1.2) is (see [K-W]):

$$s\oplus t=\bigoplus_{\Lambda\in Z} L\left(\Lambda \bmod \frac{n(n+1)}{2}\right)\otimes F(h_\Lambda)\,, \tag{1.7}$$

where $h_\Lambda=(\Lambda-n(n+1)/4)^2/(n(n+1))$, and we identify the weights Λ with the corresponding elements of $Z_{\frac{n(n+1)}{2}}$.

In order to compare both decompositions we introduce some notations. For each $s=(s_1,\dots,s_n)\in(Z_2)^n$, $(s_i=\pm 1)$, we define

$$c(s)=\sum_{i\,;\,s_i=1}(n-i+1)\,,$$

i.e. $c(s)$ is the sum of the positive coefficients of $s(\bar{\rho})=\sum_{i=1}^n s_i(n-i+1)\varepsilon_i$; in the previous example $c(s)=3$.

We will need the following lemma:

Lemma 1.1. *The trace anomaly of the weights in* (1.6) *is*

$$h_{\mu^k(\sigma_s s(\rho)-\dot{\rho})}=\frac{(c(s)+k(n+1))(n(n+1)-2(c(s)+k(n+1)))}{2n(n+1)} \bmod Z\,.$$

Proof. Since $h_{\dot{\lambda}}=\frac{(\dot{\lambda}+2\dot{\rho}\,|\,\dot{\lambda})}{4(n+1)}$ and $\mu^k(\sigma_s s(\rho)-\dot{\rho})=(\mu^k(\sigma_s s(\rho))-\dot{\rho}$, it follows that

$$h_{\mu^k(\sigma_s s(\rho)-\dot{\rho})}=\frac{(\mu^k(\sigma_s s(\rho))+\dot{\rho}|\mu^k(\sigma_s s(\rho))-\dot{\rho})}{4(n+1)}=\frac{|\mu^k(\sigma_s s(\rho))|^2-(\dot{\rho}|\dot{\rho})}{4(n+1)}\,,$$

and it is easy to see that $(\dot{\rho}|\dot{\rho})=\frac{n(n^2-1)}{12}$. By definition we have $\sigma_s s(\rho)=(n+1)\Lambda_0+\sum_{i=1}^n a_i\varepsilon_i$ (where $\{a_i\}_{i=1}^n=\{s_i(n-i+1)\}_{i=1}^n$ in decreasing order). Now we restrict $\sigma_s s(\rho)$ to the Cartan subalgebra of $su(n)$,

$$\begin{aligned}\sigma_s s(\rho)&=2(n+1)\dot{\Lambda}_0+\sum_{i=1}^{n-1}(a_i-a_{i+1})\dot{\Lambda}_i\\&=(2(n+1)+a_n+a_1)\dot{\Lambda}_0+\sum_{i=1}^{n-1}(a_i-a_{i+1})\dot{\Lambda}_i\,.\end{aligned} \tag{1.8}$$

Thus we have

$$\mu^k(\sigma_s s(\rho)) = \sum_{i=0}^{k-1} (a_{n-k+i} - a_{n-k+i+1})\dot{\Lambda}_i + (2(n+1) + a_n + a_1)\dot{\Lambda}_k$$

$$+ \sum_{i=k+1}^{n-1} (a_{i-k} - a_{i-k+1})\dot{\Lambda}_i \ , \tag{1.9}$$

and writing $\dot{\Lambda}_i$ as $[\varepsilon_1 + \cdots + \varepsilon_i - \frac{i}{n}(\sum_{j=1}^n \varepsilon_j)]$ and taking $b = \sum_{i=1}^n a_i$, we obtain the following, modulo $\dot{\Lambda}_0$

$$= \sum_{i=1}^{k} \frac{1}{n}(na_{n-k+i} - b + 2(n+1)(n-k))\varepsilon_i + \sum_{i=k+1}^{n} \frac{1}{n}(na_{i-k} - b - 2(n+1)k)\varepsilon_i \ . \tag{1.10}$$

Therefore $|\mu^k(\sigma_s s(\rho))|^2 = \frac{1}{n^2}\{\sum_{i=1}^k [4(n+1)n(n(n+1) + na_{n-k+i} - b - 2(n+1)k)] + \sum_{i=1}^n (na_i - b - 2(n+1)k)^2\} = 4(n+1)[k(n+1) + \sum_{i=1}^k a_{n-k+i}] + \frac{1}{n}[4(n+1)k(k(n-1) - b) + n\sum_{i=1}^n i^2 - b^2]$, hence

$$h_{\mu^k(\sigma_s s(\rho) - \dot{\rho})} = k(n+1) + \sum_{i=1}^{k} a_{n-k+i} + \frac{1}{4n(n+1)}$$
$$\times \left(\frac{n^2(n+1)^2}{4} - b^2 - 4(n+1)k(b + k(1-n))\right) ,$$

and since $c(s) = \frac{n(n+1)}{4} + \frac{b}{2}$ we prove the lemma.

Observe, that since $h_s + h_t = n(n+1)/16$, with Lemma 1.1 we can determine $h_{s,k}$ in (1.6):

$$h_{s,k} = (c(s) + k(n+1) - n(n+1)/4)^2/(n(n+1)) \operatorname{mod} Z \ . \tag{1.11}$$

For each $s = (s_1, \ldots, s_n) \in (Z_2)^n$ and $\sigma \in S_n$, we denote by $\sigma(s)$ the action of S_n on $(Z_2)^n$. Let $-s$ be given by $(-s)_i = -s_i$. The following lemma obtains the repetitions of the weights in (1.6), with an additional condition:

Lemma 1.2. *Let σ_1 be the permutation defined by $\sigma_1(i) = n - i + 1$. Then*

a) $\sigma_s s(\rho) - \dot{\rho} = \mu^k(\sigma_{\sigma_1(-s)}\sigma_1(-s)(\rho) - \dot{\rho})$ *and* $c(s) \equiv c(\sigma_1(-s)) + k(n+1)$ mod $\frac{n(n+1)}{2}$, *with* $k = \#\{i : s_i = 1\}$.

b) *If* $\sigma_s s(\rho) - \dot{\rho} = \mu^k(\sigma_{\tilde{s}}\tilde{s}(\rho) - r)$, *with* $c(s) \equiv c(\tilde{s}) + k(n+1) \operatorname{mod} \frac{n(n+1)}{2}$, *then* $s = \tilde{s}$, $k = 0$ *or* $\tilde{s} = \sigma_1(-s)$ *with* $k = \#\{i : s_i = 1\}$.

Proof. a) Given $s \in (Z_2)^n$, let a_i be as in the proof of Lemma 1.1, i.e. $\sigma_s s(\rho) = (n+1)\Lambda_0 + \sum_{i=1}^n a_i\varepsilon_i$ (where $\{a_i\}_{i=1}^n = \{s_i(n-i+1)\}_{i=1}^n$ in decreasing order). We define $A = \{i : s_i = 1\} = \{i_1, \ldots, i_k\}$ with $i_1 < \cdots < i_k$ and take the complement $A^c = \{i_{k+1}, \ldots, i_n\}$ with $i_{k+1} > \cdots > i_n$. Then

$$a_l = \begin{cases} n - i_l + 1 & l = 1, \ldots, k \\ -(n - i_l + 1) & l = k+1, \ldots, n \end{cases} \ . \tag{1.12}$$

We also have that $\hat{A} = \{n - i_l + 1 : l = k+1, \ldots, n\}$ and $\hat{A}^c = \{n - i_l + 1 : l = 1, \ldots, k\}$ (where $\hat{}$ refers to $\sigma_1(-s)$), and

$$\hat{a}_{l-k} = i_l \qquad l = k+1, \ldots, n ,$$
$$\hat{a}_{n-k+l} = -i_l \quad l = 1, \ldots, k .$$

Thus (a) follows from (1.8) and (1.9).

b) With the same notation as the previous lemmas, $c(s) \equiv c(\tilde{s}) + k(n+1) \bmod \frac{n(n+1)}{2}$ implies $\frac{b}{2} \equiv \frac{\tilde{b}}{2} + k(n+1) \bmod \frac{n(n+1)}{2}$. Then, using that $\mu^k(\sigma_s s(\rho) - \dot{\rho}) = \mu^k(\sigma_s s(\rho)) - \dot{\rho}$.

$$\sigma_s s(\rho) = 2(n+1)\dot{\Lambda}_0 + \sum_{i=1}^{n} \frac{1}{n}(na_i - b)\varepsilon_i ,$$

and from (1.10), we obtain that for some integer j

$$a_i = \begin{cases} \tilde{a}_{n-k+i} + (j+2)(n+1) & i = 1, \ldots, k \\ \tilde{a}_{i-k} + j(n+1) & i = k+1, \ldots, n \end{cases} . \tag{1.13}$$

Using that $\{a_i\}_{i=1}^n = \{s_i(n-i+1)\}_{i=1}^n$ in decreasing order, it is easy to see that k must be the number of 1's in s and that

$$j = \begin{cases} -1 & \text{if } k \neq 0 \\ 0 \text{ or } 1 & \text{if } k = 0 \end{cases} .$$

Thus the lemma follows from (1.13).

By Lemma 1.2.(b) it is natural to define the relation $s \sim \tilde{s}$ if $\tilde{s} = s$ or $\tilde{s} = \sigma_1(-s)$ and take S a set of representatives in $(Z_2)^n$ for the quotient of $(Z_2)^n$ by this relation. Finally, we take $n_s = n$ if $s \neq \sigma_1(-s)$ and $n_s = n/2$ otherwise. The next lemma gives the relation between the asymptotic dimensions.

Lemma 1.3. *Let Λ_i be the fundamental weights of $su(n(n+1)/2)$. Then*

$$\sum_{i=0}^{\frac{n(n+1)}{2}-1} a(\Lambda_i) = \sum_{s \in S} \sum_{k=0}^{n_s - 1} a(\mu^k(\sigma_s s(\rho) - \dot{\rho})) . \tag{1.14}$$

Proof. Using the decompositions (1.6) and (1.7) and taking the character on both sides, we obtain

$$\begin{aligned} \sum_{\Lambda \in Z} \chi_{L(\Lambda \bmod \frac{n(n+1)}{2})} \chi_{F(h_\Lambda)} &= \sum_{k \in Z} \sum_{s \in (Z_2)n} \chi_{\mu^k(\sigma_s s(\rho) - \dot{\rho})} \chi_{F(h_{s,k})} , \\ \sum_{k=0}^{\frac{n(n+1)}{2}-1} \sum_{j \in Z} \chi_{\Lambda_k} \chi_{F(h_{k+jn(n+1)/2})} &= \sum_{k=0}^{n-1} \sum_{s \in (Z_2)^n} \sum_{j \in Z} \chi_{\mu^k(\sigma_s s(\rho) - \dot{\rho})} \chi_{F(h_{s,k+jn})} , \\ \sum_{k=0}^{\frac{n(n+1)}{2}-1} \chi_{\Lambda_k} \left(\sum_{j \in Z} \chi_{F(h_{k+jn(n+1)/2})} \right) &= \sum_{k=0}^{n-1} \sum_{s \in (Z_2)^n} \chi_{\mu^k(\sigma_s s(\rho) - \dot{\rho})} \left(\sum_{j \in Z} \chi_{F(h_{s,k+jn})} \right) . \end{aligned} \tag{1.15}$$

Recall that, in general,

$$\chi_{F(h_\Lambda)}(\tau) = q^{h_\Lambda}/\eta(\tau)\,,$$

where as usual q stands for $e^{2\pi i\tau}$, with $\tau \in C$, $\operatorname{Im}\tau > 0$, and η is the Dedekind η-function.

Given $a,\ b \in \frac{1}{2}Z,\ a > 0$, let (see [K, p. 259])

$$f_{a,b}(\tau) = \sum_{j\in Z} q^{a(j+b/2a)^2}\,.$$

Then we obtain from (1.11) and (1.7),

$$\sum_{j\in Z}\chi_{F(h_{k+jn(n+1)/2})} = f_{\frac{n(n+1)}{4},\,k-\frac{n(n+1)}{4}}/\eta$$

and

$$\sum_{j\in Z}\chi_{F(h_{s,k+jn})} = f_{n(n+1),\,\tilde{b}}/\eta$$

with $\tilde{b} = 2c(s) + 2k(n+1) - n(n+1)/2$.

Noting that, as $\tau \to 0$, we have

$$\eta(\tau)^{-1} \sim (-i\tau)^{1/2}e^{\pi i/12\tau}\,,$$

$$f_{a,b}(\tau) \sim (-i\tau)^{-1/2}(2a)^{-1/2}\,,$$

$$\chi_\Lambda(\tau) \sim a(\Lambda)e^{\pi i z_\Lambda/12\tau}\,,$$

we compute the asymptotic behavior of both sides of (1.15),

$$\text{L.H.S.} \sim (n(n+1)/2)^{-1/2}\sum_{k=0}^{\frac{n(n+1)}{2}-1} a(\Lambda_k)e^{\pi i z_{\Lambda_k}/12\tau+\pi i/12\tau}\,,$$

$$\text{R.H.S.} \sim (2n(n+1))^{-1/2}\sum_{k=0}^{n-1}\sum_{s\in(Z_2)^n} a(\mu^k(\sigma_s s(\rho)-\dot{\rho}))e^{\pi i z_{\mu^k(\sigma_s s(\rho)-\dot{\rho})}/12\tau+\pi i/12\tau}\,.$$

Therefore we obtain the formula

$$2\sum_{i=0}^{\frac{n(n+1)}{2}-1} a(\Lambda_i) = \sum_{k=0}^{n-1}\sum_{s\in(Z_2)^n} a(\mu^k(\sigma_s s(\rho)-\dot{\rho}))\,. \tag{1.16}$$

In order to prove the lemma, we have to see that each one of the weights in the right-hand side of (1.14), appears twice in (1.16). First observe that, by Lemma 1.2.(a),

$$\{\mu^k(\sigma_s s(\rho)-\dot{\rho})\}_{k=0}^{n-1} = \{\mu^k(\sigma_{\sigma_1(-s)}\sigma_1(-s)(\rho)-\dot{\rho})\}_{k=0}^{n-1}\,. \tag{1.17}$$

Therefore, when $s \neq \sigma_1(-s)$, $n_s = n$ and the asymptotic dimension of the weights in (1.17) are repeated in the right-hand side of (1.16). Finally, if $s = \sigma_1(-s)$ then n must be even and (with the notation of Lemma 1.2) $k = \frac{n}{2}$, because $-s$ must have the same number of -1's as s. Thus we have

$$\sigma_s s(\rho) - \dot{\rho} = \mu^{\frac{n}{2}}(\sigma_s s(\rho) - \dot{\rho})\,.$$

So in the set (1.17) the weights $\mu^k(\sigma_s s(\rho)-\dot\rho)$, $0 \leqq k < \frac{n}{2} = n_s$ appear twice, completing the proof.

Now we are in a position to state our first basic result

Theorem 1.2. *Let* $\Lambda \in Z_{\frac{n(n+1)}{2}}$ *denote a level one highest weight of* $su(n(n+1)/2)$. *Let* $\dot\lambda \in \dot P_+^{(n+2)}$. *Then the multiplicity* $b(\Lambda,\dot\lambda)$ *of* $L(\dot\lambda)$ *in* $L(\Lambda)$ *satisfies*:

$$b(\Lambda,\dot\lambda)\neq 0 \quad \text{iff} \quad \dot\lambda = \mu^k(\sigma_s s(\rho)-\dot\rho) \quad \text{for some} \quad k\in Z, \quad s\in (Z_2)^n$$
$$\text{and } \Lambda \equiv c(s)+k(n+1) \bmod \frac{n(n+1)}{2}$$

and in this case $b(\Lambda,\dot\lambda)=1$.

Proof. The center Z_n of $su(n)$ is embedded in the center $Z_{\frac{n(n+1)}{2}}$ of $su(n(n+1)/2)$ by the map $v\to v(n+1)$ in the additive notation. Since the elements of Z_n and $Z_{\frac{n(n+1)}{2}}$ act as automorphisms of the corresponding algebras, this implies that

$$b(\Lambda+v(n+1), v\dot\lambda) = b(\Lambda,\dot\lambda)\,. \tag{1.18}$$

Observe, that since $h_\Lambda = \frac{\Lambda(n(n+1)-2\Lambda)}{2n(n+1)}$, then

$$h_\Lambda = h_{\Lambda'} \bmod Z \quad \text{iff} \quad \Lambda'=\Lambda \quad \text{or} \quad \Lambda' = \frac{n(n+1)}{2}-\Lambda\,. \tag{1.19}$$

If $b(\Lambda,\dot\lambda)\neq 0$, one derives (see, e.g., [K-W]) that

$$h_\Lambda - h_{\dot\lambda}\in Z\,, \tag{1.20}$$

and by the decompositions (1.6) and (1.7), $\dot\lambda = \mu^k(\sigma_s s(\rho)-\dot\rho)$ for some $k\in Z$, $s\in (Z_2)^n$. We have to see that $\Lambda \equiv c(s)+k(n+1) \bmod \frac{n(n+1)}{2}$. We prove this in two steps. If $k=0$, (i.e. $\dot\lambda = \sigma_s s(\rho)-\dot\rho$) by Lemma 1.1, (1.19) and (1.20) we have

$$\Lambda = c(s) \quad \text{or} \quad \Lambda = \frac{n(n+1)}{2} - c(s)\,.$$

Observe that from (1.18) and (1.20) we get

$$h_{\Lambda+(n+1)} - h_{\mu(\sigma_s s(\rho)-\rho)} \in Z$$

then, due to Lemma 1.1 and (1.19) we have

$$\Lambda+(n+1) = c(s)+(n+1) \quad \text{or} \quad \Lambda+(n+1) = \frac{n(n+1)}{2} - (c(s)+(n+1))\,,$$

therefore $\Lambda = c(s)$. And the case $k\neq 0$ follows immediately from the previous case and (1.18).

Conversely, if $\dot\lambda = \mu^k(\sigma_s s(\rho)-\dot\rho)$ for some $k\in Z$, $s\in(Z_2)^n$, the decompositions (1.6) and (1.7) imply that there exist Λ such that $b(\Lambda,\dot\lambda)\neq 0$, and by the previous considerations we have that Λ must satisfy $\Lambda\equiv c(s)+k(n+1) \bmod \frac{n(n+1)}{2}$.

Finally, it follows from the asymptotic behavior (see Lemma 1.3), Lemma 1.2.(b) and (0.12), that $b(\Lambda,\dot\lambda)=1$ if $b(\Lambda,\dot\lambda)\neq 0$, completing the proof.

Example. To illustrate the use of the theorem, we will compute in the case $n = 3$, i.e. the conformal pair

$$su(3)_5 \subseteq su(6),$$

the decomposition of $L(\Lambda_1)$. By Theorem 1.2 we are interested in $s = (s_1, s_2, s_3) \in (Z_2)^3$ and $0 \leqq k < 3$ such that $1 \equiv c(s) + 4k \bmod (6)$, where $c(s) = \sum_{i:s_i=1}(4 - i)$. If $k = 0$ then $c(s) = 1$ and the only possibility is $s = (-1, -1, 1)$; if $k = 1$ then $c(s) = 3$, and we have $s = (-1, 1, 1)$ or $s = (1, -1, -1)$. Finally, if $k = 2$ then $c(s) = 5$, and $s = (1, 1, -1)$. But $(-1, -1, 1) \sim (-1, 1, 1)$ and $(1, -1, -1) \sim (1, 1, -1)$. Therefore we must take $\sigma_s s(\rho) - \dot{\rho}$ $(k = 0)$ with $s = (-1, -1, 1)$, and $\mu(\sigma_s s(\rho) - \dot{\rho})$ $(k = 1)$ with $s = (1, -1, -1)$. So we obtain

$$L(\Lambda_1) = L(3\dot{\Lambda}_0 + 2\dot{\Lambda}_1) + L(2\dot{\Lambda}_1 + 3\dot{\Lambda}_2).$$

2. Branching Rules for $su(n) \subset su(n(n-1)/2)$

The idea is essentially the same as in Sect. 1. Here the description of $b(\Lambda, \lambda)$ will be obtained from the study of the conformal pair

$$u(1) \oplus su(n) \subset so(n(n-1)) \tag{2.1}$$

which comes from the symmetric space

$$SO(2n)/U(n).$$

And in this case the link of (2.1) with

$$su(n)_{n-2} \subset su(n(n-1)/2)_1$$

is provided by

$$u(1) \oplus su(n(n-1)/2) \subset so(n(n-1)), \tag{2.2}$$

which is also conformal.

We use Theorem 1.1 to obtain the decomposition of s and t in the case (2.1). We have $h = so(2n)$ and $p = su(n) \oplus u(1)$. The fundamental weights of h are $\bar{\Lambda}_i = \sum_{j=1}^{i} \varepsilon_j$ $(1 \leqq i \leqq n-2)$, $\bar{\Lambda}_{n-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n)$, $\bar{\Lambda}_n = \frac{1}{2}(\varepsilon_1 + \ldots + \varepsilon_n)$, and the Weyl vector is given by $\rho = (2n-2)\Lambda_0 + \bar{\rho}$ with the (finite) Weyl vector $\bar{\rho} = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \cdots + \varepsilon_{n-1}$ (where the ε_i are orthonormal vectors). The Weyl group is given by $W = \bar{W} \ltimes T$, where T are the translations $\{t_\alpha\}_{\alpha \in L}$ with $L = \sum_{i=1}^{n} Z\alpha_i$ ($\{\alpha_i\}_{i=1}^{n} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$), and $\bar{W}$ the semidirect product of the permutation group and $(Z_2)^n_{\text{even}} = \{(s_1, \ldots, s_n) \in (Z_2)^n : \#\{i : s_i = -1\}$ is even$\}$. In the case of $p, \dot{\Lambda}_i, \dot{\bar{\Lambda}}_i, \dot{\bar{\rho}}, \dot{\rho}$ and $\dot{W}$ are as in Sect. 1.

In order to get a dominant weight $w(\rho) - \dot{\rho}$ we have to use as representatives of $T/\dot{T}$ in $W/\dot{W}$, not the translations $t_{k\alpha_n}$ by multiples of α_n, but the powers of $\tilde{\sigma}_0 t_{\alpha_n}$, where $\tilde{\sigma}_0$ is the permutation σ_0^2 (see (1.4)), since, if μ is the automorphism given in Sect. 1, then

$$\tilde{\sigma}_0 t_{\alpha_n}(w(\rho)) - \dot{\rho} = \mu^2(w(\rho) - \dot{\rho}).$$

As in Sect. 1, the restriction of $\lambda = \sum_{i=1}^{n} a_i \varepsilon_i$ a weight of h to p is $\dot{\lambda} = \sum_{i=1}^{n-1}(a_i - a_{i+1})\dot{\bar{\Lambda}}_i$ and $\dot{\lambda}$ is a (strictly) dominant weight iff $a_i > a_{i+1}$. Thus we see that a suitable choice for $\bar{W}/\dot{\bar{W}}$ is the following: for each $s \in (Z_2)^n_{\text{even}}$ we take σ_s the permutation which orders the coefficients of $s(\bar{\rho}) = \sum_{i=1}^n a_i\varepsilon_i$ decreasingly. For example, if $n = 3$ and $s = (-1, 1, -1)$ (even number of sign changes), then $s(\bar{\rho}) = -2\varepsilon_1 + 1\varepsilon_2 - 0\varepsilon_3$, so $\sigma_s = 1 \to 3 \to 2 \to 1$.

Remark. 2.1. In this case the coefficients of the ε_i in ρ are different from those of Sect. 1, and ε_n has coefficient 0. Since we are taking $s(\rho)$, then we can think that an $s \in (Z_2)^n$ with an even number of sign changes, belongs to $(Z_2)^{n-1}$ and acts in $\{\varepsilon_1, \ldots, \varepsilon_{n-1}\}$.

Therefore, we get the explicit form of the decomposition (1.3) in the case (2.1):

$$s \oplus t = \bigoplus_{k\in Z} \bigoplus_{s\in(Z_2)^n_{\text{even}}} L(\mu^{2k}(\sigma_s s(\rho) - \dot{\rho})) \otimes F(h_{s,k})\,. \tag{2.3}$$

As in Sect. 1, the decomposition (1.3) in the case (2.2) is:

$$s \oplus t = \bigoplus_{\Lambda\in Z} L\left(\Lambda \bmod \frac{n(n-1)}{2}\right) \otimes F(h_\Lambda)\,, \tag{2.4}$$

where $h_\Lambda = (\Lambda - n(n-1)/4)^2/(n(n-1))$, and we identify the weights Λ with the corresponding elements of $Z_{\frac{n(n-1)}{2}}$.

In this case the numbers $c(s)$ are defined as follows. For each $s = (s_1, \ldots, s_n) \in (Z_2)^n_{\text{even}}$, $(s_i = \pm 1)$, we define

$$c(s) = \sum_{i:s_i=1} (n-i)\,,$$

i.e. $c(s)$ is the sum of the positive coefficient of $s(\bar{\rho}) = \sum_{i=1}^{n-1} s_i(n-i)\varepsilon_i$.

We will need the following result, which is similar to Lemma 1.1:

Lemma 2.1. *The trace anomaly of the weights in* (2.3) *is*

$$h_{\mu^{2k}(\sigma_s s(\rho)-\rho))} = \frac{(c(s) + 2k(n-1))(n(n-1) - 2(c(s) + 2k(n-1)))}{2n(n-1)} \bmod Z\,.$$

Proof. See Lemma 1.1.

For each $s \in (Z_2)^n_{\text{even}}$, we define $\beta(s) = \sigma_1(-s_1, \ldots, -s_{n-1}, s_n)$. The following lemma obtains the repetitions of the weights in (2.3), with an additional condition:

Lemma 2.2. a) *If n is even and $0 \leqq k \leqq \frac{n}{2}$, then $\sigma_s s(\rho) - \dot{\rho} = \mu^{2k}(\sigma_{\tilde{s}}\tilde{s}(\rho) - \dot{\rho})$, with $c(s) \equiv c(\tilde{s}) + 2k(n-1) \bmod \frac{n(n-1)}{2}$ iff $s = \tilde{s}$. $k = 0$ or $\tilde{s} = \sigma_1(-s)$ with $2k = \#\{i : s_i = 1\}$.*

b) *If n is odd and $0 \leqq k < n$, then $\sigma_s s(\rho) - \dot{\rho} = \mu^{2k}(\sigma_{\tilde{s}}\tilde{s}(\rho) - \dot{\rho})$, with $c(s) \equiv c(\tilde{s}) + 2k(n-1) \bmod \frac{n(n-1)}{2}$* iff *$\tilde{s} = \beta^r(s)$ with $r = 0, 1, 2$ or 3, and $2k + s_n = \#\{i : s_i = 1\}$ if $r = 1$.*

The proof is similar to that of Lemma 1.2. Now by this result, it is natural to define that $s \sim \tilde{s}$, in the case n even, if $s = \tilde{s}$ or $\tilde{s} = \sigma_1(-s)$, and we take $n_s = n/2$ if $\tilde{s} \neq \sigma_1(-s)$, $n_s = n/4$ otherwise; and in the case n odd, $s \sim \tilde{s}$ if $\tilde{s} = \beta^r(s)$ with $r = 0, 1, 2$ or 3 and we put $n_s = n$. Finally, we define S a set of representatives in $(Z_2)^n_{\text{even}}$ for the quotient of $(Z_2)^n_{\text{even}}$ by this relation. The next lemma gives the relation between the asymptotic dimensions and the proof is basically the same as that of Lemma 1.3.

Lemma 2.3. *Let Λ_i be the fundamental weights of $su(n(n-1)/2)$. Then*

$$\sum_{i=0}^{\frac{n(n-1)}{2}-1} a(\Lambda_i) = \sum_{s\in S}\sum_{k=1}^{n_s} a(\mu^{2k}(\sigma_s s(\rho)-\dot{\rho}))\,.$$

Now we are in a position to state our second basic result

Theorem 2.1. *Let $\Lambda \in Z_{\frac{n(n-1)}{2}}$ denote a level one highest weight of $su(n(n-1)/2)$. Let $\dot{\lambda} \in \dot{P}_+^{(n-2)}$. Then the multiplicity $b(\Lambda,\dot{\lambda})$ of $L(\dot{\lambda})$ in $L(\Lambda)$ satisfies:*

$$b(\Lambda,\dot{\lambda})\neq 0 \quad \textit{iff} \quad \dot{\lambda}=\mu^{2k}(\sigma_s s(\rho)-\dot{\rho}) \quad \textit{for some} \quad k\in Z,\ s\in (Z_2)^{n-1}$$
$$\textit{and}\ \Lambda \equiv c(s)+2k(n-1)\operatorname{mod}\frac{n(n-1)}{2}\,,$$

and in this case $b(\Lambda,\dot{\lambda})=1$.

Proof. See Theorem 1.2.

3. The Branching Rules for $sp(n) \supset so(n)_4 \oplus su(2)_n$

We consider Cartan subalgebras $h, \dot{h}$ and $\ddot{h}$ of $\widehat{sp}(n), \widehat{so}(n)$ and $\widehat{su}(2)$ respectively, such that, $h \supset \dot{h}\oplus\ddot{h}$. We take a triangular decomposition $\widehat{s}p(n)=n_- + h + n_+$ and in the same way $\dot{n}_\pm$ and $\ddot{n}_\pm$, such that they are contained in $n_\pm$. In this section, single dots refer to $so(n)$ and double dots to $su(2)$. With respect to these Cartan subalgebras we have the systems of simple roots. Let $\{\Lambda_i\}_0^n$, $\{\dot{\Lambda}_i\}_0^{n/2}$ and $\{\ddot{\Lambda}_i\}_0^1$ the respective dual root basis.

When $n=2m$ let

$$\begin{aligned}
&\dot{\lambda}_0 = 2\dot{\Lambda}_0 && \dot{\lambda}_0' = 2\dot{\Lambda}_1\\
&\dot{\lambda}_m = 2\dot{\Lambda}_m && \dot{\lambda}_m' = 2\dot{\Lambda}_{m-1}\\
&\dot{\lambda}_i = \dot{\Lambda}_{i-1}+\dot{\Lambda}_i && i\in\{1,m-1\}\\
&\dot{\lambda}_i = \dot{\Lambda}_i && 1<i<m-1\\
&\dot{\lambda}_i = \dot{\lambda}_{n-i} && 0<i<m\,.
\end{aligned}\tag{3.1}$$

For $n=2m+1$ let

$$\begin{aligned}
&\dot{\lambda}_0 = 2\dot{\Lambda}_0 && \dot{\lambda}_0' = 2\dot{\Lambda}_1\\
&\dot{\Lambda}_1 = \dot{\Lambda}_0+\dot{\Lambda}_1 && \dot{\lambda}_m = 2\dot{\Lambda}_m\\
&\dot{\lambda}_i = \dot{\Lambda}_i && 1<i<m-1\\
&\dot{\lambda}_i = \dot{\lambda}_{n-i} && 1\leqq i\leqq m\,.
\end{aligned}\tag{3.2}$$

Now we can state our third main result:

Theorem 3.1. *For the conformal embedding* $sp(n) \supset so(n)_4 \oplus su(2)_n$ *we have the following decompositions of the representations of level one of* $sp(n)$:

a) *If* $n = 2m$ *and* $j \neq m$, *then*

$$L(\Lambda_j) = \sum_{j \leqq 2i \leqq n+j} L(\dot{\lambda}_i + \dot{\lambda}_{|i-j|}) \otimes L((n-2i+j)\ddot{\Lambda}_0 + (2i-j)\ddot{\Lambda}_1)$$

$$\oplus L(\dot{\lambda}_j + \dot{\lambda}_0') \oplus L((n-j)\ddot{\Lambda}_0 + j\ddot{\Lambda}_1)$$

$$\oplus L(\dot{\lambda}_m' + \dot{\lambda}_{|m-j|}) \otimes L(j\ddot{\Lambda}_0 + (n-j)\ddot{\Lambda}_1)\,, \tag{3.3}$$

$$L(\Lambda_m) = \sum_{m \leqq 2i \leqq n+m} L(\dot{\lambda}_i + \dot{\lambda}_{|m-i|}) \otimes L((n-2i+m)\ddot{\Lambda}_0 + (2i-m)\ddot{\Lambda}_1)$$

$$\oplus L(\dot{\lambda}_m' + \dot{\lambda}_0') \otimes L(m\ddot{\Lambda}_0 + m\ddot{\Lambda}_1)\,.$$

b) *If* $n = 2m+1$ *and* $j = 2k$ *or* $j = 2k-1$, *then*

$$L(\Lambda_j) = \sum_{i=k}^{m+k} L(\dot{\lambda}_i + \dot{\lambda}_{|i-j|}) \otimes L((n-2i+j)\ddot{\Lambda}_0 + (2i-j)\ddot{\Lambda}_1)$$

$$\oplus L(\dot{\lambda}_j + \dot{\lambda}_0') \otimes L((n-j)\ddot{\Lambda}_0 + j\ddot{\Lambda}_1)\,. \tag{3.4}$$

Remark. 3.1. If $n = 2m$, the sum is over the integers between $j/2$ and $m + j/2$, so we will have another term if j is even. Observe that all the weights in the decomposition are different, so the multiplicities are one.

In order to prove the theorem, we will need the following lemma:

Lemma 3.1. *The trace anomalies of the weights in* (3.3) *and* (3.4) *are the following*:

a) $h_{\Lambda_j} = \frac{j(2n+2-j)}{4(n+2)}$,

b) $h_{(n-j)\ddot{\Lambda}_0 + j\ddot{\Lambda}_1} = \frac{j(j+2)}{4(n+2)}$,

c) $h_{\dot{\lambda}_i + \dot{\lambda}_k} = \frac{i(n-i)+k(n-k+2)}{2(n+2)}, \quad 0 \leqq k \leqq i \leqq m\,.$

Proof. Given $\Lambda \in P_+^{(m)}(g)$, the number h_Λ can be calculated as follows. Let $\bar{\Lambda} = \sum_{i=1}^{l} k_i \bar{\Lambda}_i$ and let $(\tilde{a}_{ij})$ be the inverse of the Cartan matrix of g, then

$$h_\Lambda = \sum_{i,j=1}^{l} \tilde{a}_{ij} k_i (k_j + 2)/2(m + h(g))\,.$$

Now, the lemma follows from this formula.

Proof of Theorem 3.1. First we show that the right-hand side of (3.3) and (3.4) is contained in $L(\Lambda_j)$. For this we use the decompositions (see [K-W], p. 212)

$$L(\Lambda_j) = \sum_{k=0}^{n-1} \sum_{s \in Z} L(\dot{\Lambda}_k + \dot{\Lambda}_{k-j}) \otimes F(2k - j - 2sn) \tag{3.5}$$

of $\widehat{sp}(n) \supset \widehat{su}(n)_2 \times \hat{u}(1)$, in this case $\dot{\Lambda}_k$ are the fundamental weights of $\widehat{su}(n)$. Also, we have $\widehat{su}(n) \supset \widehat{so}(n)_2$, and the restrictions of the fundamental weights of $\widehat{su}(n)$ to $\widehat{so}(n)$ are given by the $\dot{\lambda}_i$ in (3.1) and (3.2). From (3.5), there are weight vectors in $L(\Lambda_j)$ that are highest weight vectors in $L(\Lambda_j)$ as $\widehat{su}(n)$-module, with weights $\dot{\Lambda}_k + \dot{\Lambda}_{k-j}$, with $0 \leqq k \leqq n-1$. And therefore, they are highest weight vectors in $L(\Lambda_j)$ as $\widehat{so}(n)$-module, with weights $\dot{\lambda}_i + \dot{\lambda}_{|i-j|}$, with $j/2 \leqq i \leqq (n+j)/2$, which are all the different weights that appear in the restrictions. Since the action of $\widehat{su}(2)$ commutes with the action of $\widehat{so}(n)$, applying elements of $\ddot{n}_+$, we get a highest weight vector for $\widehat{so}(n) \times \widehat{su}(2)$, with weight $\dot{\lambda}_i + \dot{\lambda}_{|i-j|}$ and $(n-k)\ddot{\Lambda}_0 + k\ddot{\Lambda}_1$ for some k. Now using Lemma 3.1, we see that the only possibility for k that satisfies (1.20) is $(2i-j)$. Finally, using the automorphism that comes from the Dynkin diagram, we obtain the terms involving $\dot{\lambda}'_k$ in (3.3) and (3.4).

In order to finish the proof we show that the asymptotic dimensions of both sides of (3.3) and (3.4) coincide. For this we make use of the formulas:

$$a((n-k)\ddot{\Lambda}_0 + k\ddot{\Lambda}_1) = \sqrt{\frac{2}{n+2}}\sin\frac{(k+1)\pi}{n+2},$$

$$a(\Lambda_j) = \sqrt{\frac{2}{n+2}}\sin\frac{(j+1)\pi}{n+2}.$$

If $n = 2m$
$$a(\dot{\lambda}_i + \dot{\lambda}_j) = \frac{4}{n+2}\sin\frac{(i+j+1)\pi}{n+2}\sin\frac{(j-i+1)\pi}{n+2}$$
$$0 < i \leqq j < m,$$

$$a(\dot{\lambda}_i + \dot{\lambda}_j) = \frac{2}{n+2}\sin^2\frac{(i+j+1)\pi}{n+2} \qquad i \in \{0, m\} \quad 0 \leqq j \leqq m.$$

If $n = 2m+1$
$$a(\dot{\lambda}_i + \dot{\lambda}_j) = \frac{4}{n+2}\sin\frac{(i+j+1)\pi}{n+2}\sin\frac{(j-i+1)\pi}{n+2}$$
$$0 < i \leqq j < m,$$

$$a(\dot{\lambda}_0 + \dot{\lambda}_j) = \frac{2}{n+2}\sin^2\frac{(j+1)\pi}{n+2} \qquad 0 \leqq j \leqq m,$$

which are proven by induction from the definition. Recall that $a(\Lambda) = a(\sigma \cdot \Lambda)$ for any automorphism σ of the Dynkin diagram, then we obtain the asymptotic dimensions of the weights involving $\dot{\lambda}'_k$.

So we must show, in the case $n = 2m+1$,

$$\sum_{i=k}^{m+k} a(\dot{\lambda}_i + \dot{\lambda}_{|i-j|})a((n-2i+j)\ddot{\Lambda}_0 + (2i-j)\ddot{\Lambda}_1) + a(\dot{\lambda}_j + \dot{\lambda}'_0)a((n-j)\ddot{\Lambda}_0 + j\ddot{\Lambda}_1)$$

$$= \sum_{i=k}^{m+k} \frac{4}{n+2}\sin\frac{(2i-j+1)\pi}{n+2}\sin\frac{(j+1)\pi}{n+2}\sqrt{\frac{2}{n+2}}\sin\frac{(2i-j+1)\pi}{n+2}$$

$$= \sqrt{2}(n+2)^{-3/2} 4\sin\frac{(j+1)\pi}{n+2}\sum_{i=k}^{m+k}\sin^2\frac{(2i-j+1)\pi}{n+2}$$

$$= \sqrt{2}(n+2)^{-3/2}4\left(\sin\frac{(j+1)\pi}{n+2}\right)\frac{(n+2)}{4}$$

$$= \sqrt{\frac{2}{n+2}}\sin\frac{(j+1)\pi}{n+2} = a(\Lambda_j)\,.$$

Notice that in the first equality the dimensions corresponding to $L(\dot{\lambda}_0' + \dot{\lambda}_j)$ and $L(\dot{\lambda}_0 + \dot{\lambda}_j)$ have already been added, which accounts for the coefficient 4. The third equality is classical, see [K-W], p. 179.

In the case $n = 2m$, the proof that the asymptotic dimensions of both sides coincide is similar.

4. Branching Rules for $so(2m + 1) \oplus so(2n + 1) \subset so(2(m + n + 1))$

As in [K, p. 213], all the decompositions are easily derived by using (1.20) and asymptotics (0.11), (0.12):

$$L(\Lambda_0) = L(\dot{\Lambda}_0)\otimes L(\ddot{\Lambda}_0) + L(\dot{\Lambda}_1)\otimes L(\ddot{\Lambda}_1)\,,$$

$$L(\Lambda_1) = L(\dot{\Lambda}_0)\otimes L(\ddot{\Lambda}_1) + L(\dot{\Lambda}_1)\otimes L(\ddot{\Lambda}_0)\,,$$

$$L(\Lambda_{m+n-1}) = L(\dot{\Lambda}_m)\otimes L(\ddot{\Lambda}_n)\,,$$

$$L(\Lambda_{m+n}) = L(\dot{\Lambda}_m)\otimes L(\ddot{\Lambda}_n)\,.$$

5. Conclusion

We list in the following table the infinite families of conformal embeddings together with their index and the references where the corresponding branching rules were computed.

Embedding	Index	References
$su(m) \times su(n) \times u_1 \subset su(n+m)$	(1,1,–)	[K-W]
$so(m) \times so(n) \subset so(n+m)$	(1,1)	[K-W], this paper
$su(n) \times u_1 \subset so(2n)$	(1,–)	[K-W]
$so(n) \subset su(n)$	2	[K-W]
$u(n) \subset sp(2n)$	2	[K-W]
$h \subset so(\dim h)$	h(h)	[K-W]
$su(n) \subset su(n(n+1)/2)$	n+2	this paper
$su(n) \subset su(n(n-1)/2)$	n–2	this paper
$su(m) \times su(n) \subset su(nm)$	(n,m)	[A-B-I], [W]
$sp(2m) \times sp(2n) \subset so(4nm)$	(n,m)	[K-P], [V]
$so(m) \times so(n) \subset so(nm)$	(n,m)	[K-P], [V] (nm even)
$so(n) \times su(2) \subset sp(2n)$	(4,n)	this paper

All the cases when $\bar{g}$ is exceptional were computed in [K-S].

Now it is possible to apply some well known methods to construct modular invariant partition functions. Using the branching rules found in & 1 and & 2, we get by restricting a partition function built from the level one characters of $SU(N(N \pm 1)/2)_1$ partition functions for $SU(N)$ of level $N \pm 2$ respectively. Notice that from the classification of level one partition functions for $SU(N)$ we have that in $SU(N(N+1)/2)$ there are always off-diagonal representatives, since $N(N+1)/2$ is not prime for $N > 2$.

Using the decompositions from Sect. 3, we can restrict a partition function attached to level one characters of $Sp(N)$ and then contract with a level N partition function of $SU(2)$ and in this way we obtain partition functions for $SO(N)$ of level four.

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