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$B \wedge F$ Theory and Flat Spacetimes

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Abstract: We propose a reduced constrained Hamiltonian formalism for the exactly soluble $B \wedge F$ theory of flat connections and closed two-forms over manifolds with topology $\Sigma^3 \times (0,1)$. The reduced phase space variables are the holonomies of a flat connection for loops which form a basis of the first homotopy group $\pi_1(\Sigma^3)$, and elements of the second cohomology group of Σ^3 with value in the Lie algebra $L(G)$. When $G = SO(3,1)$, and if the two-form can be expressed as $B = e \wedge e$, for some vierbein field e, then the variables represent a flat spacetime. This is not always possible: We show that the solutions of the theory generally represent spacetimes with "global torsion." We describe the dynamical evolution of spacetimes with and without global torsion, and classify the flat spacetimes which admit a locally homogeneous foliation, following Thurston's classification of geometric structures.

1. Introduction

The $B \wedge F$ theory was first considered by Horowitz [1] as an example of an exactly soluble theory in four dimensions, analogous to the Chern–Simons formulation of (2+1)-dimensional gravity [2,3]. The set of solutions was shown to be related to equivalence classes of flat $SO(3,1)$ connections and closed two-forms. When the four-manifold has the topology $\Sigma^3 \times (0,1)$, where Σ^3 is compact and orientable, there is a natural symplectic structure which is related to the Poincaré duality between the first and second homology groups of Σ^3: Roughly speaking, the flat connections are labeled by their holonomies around loops of $Z_1(\Sigma^3)$, and the two-forms are labeled by their integrals over elements of $Z_2(\Sigma^3)$. The symplectic structure on the set of holonomies and integrated two-forms would then be derived from the Poincaré duality between $H_1(\Sigma^3) = Z_1(\Sigma^3)/B_1(\Sigma^3)$ and $H_2(\Sigma^3) = Z_2(\Sigma^3)/B_2(\Sigma^3)$.

The purpose of this article is to elucidate further the physical content of this theory. We will first postulate a reduced constrained Hamiltonian formalism [4]

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which exploits the Poincaré duality explicitly, and can be solved for the dynamical evolution of the reduced phase space variables. Next, we will examine the relation between the solutions of two-form gravity and flat spacetimes, including a classification of flat spacetimes which admit a locally homogeneous foliation.

The article is organized as follows. We will begin with a review of the $B \wedge F$ field theory, derive the constrained Hamiltonian system for the reduced variables, and examine the dynamical evolution (Sect. 2). In Sect. 3, we will show that these solutions generally represent spacetimes with "global torsion," and propose vanishing torsion constraints. Finally, we will construct all flat spacetimes which admit a locally homogeneous slicing, by constructing the representations $\rho : \pi_1(\Sigma^3) \to ISO(3,1)$ for each of Thurston's eight geometric structures.

2. Two Forms and Flat Connections

2.1. The $B \wedge F$ Field Theory. The $B \wedge F$ theory describes a gauge connection A in G, and a two-form B with values in the Lie algebra $L(G)$, with the action

$$S = 6\int_M tr(B \wedge F) . \tag{2.1}$$

The field equations which derive from variations of B and A are, respectively, $F = 0$ and $D \wedge B = 0$, where F is the field strength and $D = \partial + A$ is the gauge-covariant derivative. This action is invariant under the gauge transformations $\delta A = D\tau, \delta B = [B, \tau]$, where τ is an arbitrary field on M with values in the Lie algebra $L(G)$, and under the translation of B by an exact form, $\delta B = D \wedge v$. When A satisfies its field equation, $D^2 = F = 0$, there is a cohomology of covariantly closed two-forms, modulo translations by covariantly exact forms. This implies that the physical information carried by B can be represented by a map form the second homology group $H_2(\Sigma^3)$ to the Lie algebra $L(G)$: When G is abelian, this map is simply the surface integral of the two-form over any representative of $H_2(\Sigma^3)$ in $Z_2(\Sigma^3)$.

If one sets $G = SO(3,1)$ and $B^{ab} = e^a \wedge e^b$, the action (2.1) becomes the Palatini action for vacuum gravity,

$$S_P = \int (e^a \wedge e^b \wedge F^{cd})\varepsilon_{abcd} . \tag{2.2}$$

The relation between the $B \wedge F$ theory and gravity will be examined further in Sect. 3.

As Horowitz has pointed out, if $M = \Sigma^3 \times (0, 1)$ the reduced phase space has a canonical structure which is related to the Poincaré duality between $H_1(\Sigma^3)$ and $H_2(\Sigma^3)$: The degrees of freedom of the connection are associated to non-contractible loops, while the inequivalent closed two-forms differ by the values of their integrals over non-contractible surfaces of $Z_2(\Sigma^3)$. Poincaré duality suggests that if the former are configuration space variables, then the latter should be the corresponding momentum variables. Before we can make this idea more explicit (Sect. 2.2), we first need to put the action (2.1) in canonical form.

Assuming that $M = \Sigma^3 \times (0, 1)$, one can separate the coordinates into "spatial" (i,j,k) and "time" (o) components, and write the action in the form

$$S = \int dt \int d\Sigma(\dot{A}^A_{[i} B_{jk]A} - F^A_{[ij} B_{k]oA} + A_o{}^A D_{[i} B_{jk]A}) . \tag{2.3}$$

(We will assume that G admits a non-degenerate Cartan metric; the index A is in the adjoint representation.)

The canonical formalism is obtained by computing $\pi = \frac{\partial L}{\partial A}$ and $\Pi = \frac{\partial L}{\partial B}$, and performing the Legendre transformation. One finds

$$[A_i{}^A, B_{jkB}] = \delta^A_B \Sigma_{ijk} \;, \tag{2.4}$$

$$\pi^0_A = \frac{\partial L}{\partial \dot{A}_0{}^A} \approx 0 \;, \tag{2.5}$$

$$\Pi^{ko}{}_A = \frac{\partial L}{\partial \dot{B}^A{}_{ko}} \approx 0 \;, \tag{2.6}$$

$$H = 3\int_\Sigma (F^A_{[ij} B_{k]oA} + A_o{}^A D_{[i} B_{jk]A}) d\Sigma \;, \tag{2.7}$$

and the secondary constraints [4], which express the consistency of (2.5) and (2.6) with time evolution, are

$$\dot{\Pi}^0_A \approx 0 \Rightarrow D_{[i} B_{jk]A} \approx 0 \;, \tag{2.8}$$

$$\dot{\Pi}^{ko}_A \approx 0 \Rightarrow F^A_{ij} \approx 0 \;. \tag{2.9}$$

These constraints state that B^A_{jk} is a covariantly closed form on Σ^3, and that A^A_i (the connection restricted to Σ^3) is flat. Note that, given (2.8) and (2.9),

$$H \approx 0 \;. \tag{2.10}$$

The flow generated by H is

$$[H, A_i{}^A] = D_i A_0{}^A \;, \tag{2.11}$$

$$[H, B_{ijA}] = D_{[i} B_{k]0A} + [B_{ij}, A_0]_A \;. \tag{2.12}$$

As Horowitz noted, diffeomorphisms are part of the symmetry group when the constraints are satisfied. For any vector field ξ,

$$L_\xi A^A_j = D_{[j}\left(A^A_{i]}\xi^i\right) + \xi^i F_{ij}{}^A \;, \tag{2.13}$$

which reduces to the form (2.11) when $F_{ij}{}^A = 0$.

2.2. The Reduced Theory. Since the fundamental group is presented by a finite basis $\{\gamma(\mu), \mu = 1, \ldots, dim H_1(\Sigma^3)\}$, the representations $\rho : \pi_1(\Sigma^3) \to SO(3,1)$ are parametrized by the holonomies for a set of basis loops:

$$M(\mu) = P\exp\left\{ -i \oint_{\gamma(\mu)} A_i dl^i \right\} \;, \tag{2.14}$$

where P denotes the usual path ordering of the exponential. The other half of the phase space is parametrized by elements of the second cohomology group with values in the Lie algebra $L(G)$. In de Rahm cohomology, the equivalence classes of closed two-forms modulo translations by exact forms can be parametrized by their integrals over basis surfaces $\sigma(\mu)$, where each $\sigma(\mu)$ intersects only the basis loop $\gamma(\mu)$ only at the base-point of $\pi_1(\Sigma^3)$. Generalizing this idea to the covariant cohomology defined by $D = \partial + A$ when $F = 0$, one would naively consider the integrals

$$B^A(\mu) = \oint_{\sigma(\mu)} P^A{}_B(x) B_{ij}{}^B(x) dx^i \wedge dx^j \ , \tag{2.15}$$

where $P^A{}_B(x)$ parallel-transports the index B from the fiber at x to the fiber at the base-point P of the fundamental group $\pi_1(\Sigma^3)$.

The key problem is that this parallel-transport is not well-defined, since different paths from x to P can differ by non-trivial elements of $\pi_1(\Sigma^3)$. To define uniquely the parallel-transport between points of (Σ^3), one can instead remove the $dimH_2(\Sigma^3)$ basis surfaces $\sigma(\mu)\varepsilon Z_2(\Sigma^3)$ to form a topologically trivial open manifold Σ^3_0. The integrated two-form is then defined by approaching the removed surface from one side or the other:

$$B^A(\mu) = \lim_{\varepsilon \to 0} \oint_{\sigma+\varepsilon(\mu)} P^A{}_B(x) B^B_{ij}(x) dx^i \wedge dx^j \ , \tag{2.16}$$

$$B^A(-\mu) = \lim_{\varepsilon \to 0} \oint_{\sigma-\varepsilon(\mu)} P^A{}_B(x) B^B_{ij}(x) dx^i \wedge dx^j \ . \tag{2.17}$$

The difference between these expressions is precisely the parallel-transport around the loop $\gamma(\mu)$, and we have

$$B^A(\mu) = M^A{}_B(\mu) B^B(-\mu) \ . \tag{2.18}$$

To resolve the ambiguity in parallel-transport, we have been forced to introduce a dependence on the surfaces $\sigma(\mu)$. One can show that it is impossible to achieve *both* gauge-invariance and surface-invariance within a homology class, for any choice of parallel-transport $P^A{}_B(x)$. This fact appears to be related to Teitelboim's proof that it is impossible to achieve a reparametrization-invariant ordering of the expression

$$P\exp\left(\oint B_{ij} dx^i \wedge dx^j\right) \ , \tag{2.19}$$

analogous to the holonomy (2.14)[5]. Note that the logarithm of Teitelboim's two-form "holonomy" can be written in the form (2.15), but taking $P^A{}_B(x)$ as a functional of B, rather than A.

The expression (2.16) is coordinate-invariant if the surfaces $\sigma(\mu)$ are chosen in a coordinate-independent way. For example, one might choose a maximal surface among the surfaces $\sigma(\mu) \in Z_2(\Sigma^3)$ in a given homology class and which contain P. Any intrinsic criterion which selects a unique representative in $Z_2(\Sigma^3)$ of each element of $H_2(\Sigma^3)$, leads to an acceptable definition of the global variables $B^A(\mu)$. For example, in Sect. 3, we will choose a piecewise geodesic triangulation of $\sigma(\mu)$.

We will assume that a choice of surfaces $\sigma(\mu)$ has been made, and examine the dynamical behaviour of the global variables $M(\mu), B(\mu)$.

The Poisson bracket algebra can be deduced from the bracket $[B^A_{ij}(x), A_k{}^B(x')] = \varepsilon_{ijk} g^{AB} \delta(x - x')$, using a lattice regularization, following the same lines as in (2+1)-dimensional gravity [6]. One finds

$$[M^A_B(\mu), M^C{}_D(\nu)] = 0 \ , \tag{2.20}$$

$$[B^A(\mu), M^C{}_D(\nu)] = \delta_{\nu\mu} C^{AC}{}_E M^E{}_D(\mu) \ , \tag{2.21}$$

$$[B^A(\mu), B^B(\nu)] = \delta_{\mu\nu} C^{AB}{}_C B^C(\mu) \ , \tag{2.22}$$

where $C^A{}_{BC}$ are the structure constants of $L(G)$. The first bracket is deduced directly from $[A_i{}^A(x), A_j{}^B(x')] = 0$. The second can be derived from the expressions (2.14) and (2.16), by considering the contribution from the point $x = x' = P$, where $\gamma(\mu)$ intersects $\sigma(\mu)$. The bracket of $B^A(\mu)$ with $B^B(\mu)$ is ill-defined because the same surface $\sigma(\mu)$ appears on both sides of the bracket; Eq. (2.22) is the result of the lattice regularization mentioned above.

There are two sets of constraints on the global variables $\{M(\mu), B(\mu)\}$. The former are the cycle conditions for the representations $M : \pi_1(\sigma(\mu)) \to G$, presented by a subset of the matrices $\{M(v)\}$: for each surface $\sigma(\mu)$, there is a contractible loop (the "cycle"), which follows every basis loop of $\pi_1(\sigma(\mu))$ in both directions. If $\sigma(\mu)$ is a genus g surface and $\pi_1(\sigma(\mu))$ is presented by the basis $\{\gamma(\mu), \mu = \mu_1, \mu_2, \cdots, \mu_{2g}\}$, the cycle conditions are

$$\begin{aligned} W(\mu) =& \left(M(\mu_1)M^{-1}(\mu_2)M^{-1}(\mu_1)M(\mu_2)\right)\left(M(\mu_3)M^{-1}(\mu_4)M^{-1}(\mu_3)\right. \\ & \left.M(\mu_4)\right)\left(M(\mu_{2g-1})M^{-1}(\mu_{2g-1})M(\mu_{2g})\right) = I \ , \end{aligned} \tag{2.23}$$

since the connection is flat and the cycle is homotopically trivial. The conditions (2.23) are the remnants of the flatness conditions for the global variables:

$$F_{ij}{}^A \approx 0 \to W(\mu) - I \approx 0 \ . \tag{2.24}$$

The other constraints are a consequence of the closure conditions $D \wedge B = 0$, and of our definition of $B^A(\mu)$, based on removing the region bound by $2 \times dimH_2(\Sigma^3)$ basis surfaces $\sigma_{\pm\varepsilon}(\mu)$, to obtain a topologically trivial open manifold Σ^3_ε. The integral of the two-form over the boundary, in the limit $\varepsilon \to 0$, is a sum over μ of the expressions (2.16) and (2.17), and vanishes as a consequence of the definitions (2.16), (2.17) and the closure of $B^A{}_{ij}(x)$. Thus,

$$D \wedge B = 0 \to J^A \equiv \Sigma_\mu (1 - M^{-1}(\mu))^A{}_B B^B(\mu) \approx 0 \ . \tag{2.25}$$

The brackets (2.20)–(2.22), together with the constraints (2.24) and (2.25), define a dynamical system in the constrained Hamiltonian formulation, which we constructed to have the same physical content as the $B \wedge F$ field theory. Since some steps in this construction are non-trivial, particularly the regularization of the brackets $[B^A(\mu), B^B(\mu)]$, we will state this equivalence of physical content as a conjecture.

We will say that theories A and B are "equivalent" if there is a gauge transformation which maps solutions of A to solutions of B, and this map is bijective.

Conjecture 2.1. The constrained Hamiltonian system given by the brackets (2.20)–(2.22) and the constraints (2.24), (2.25), with $H \approx 0$, is equivalent to that which derives from the $B \wedge F$ field theory (2.1), for any coordinate-independent choice of representation of the basis of $H_2(\Sigma^3)$ in $Z_2(\Sigma^3), \sigma : \mu \to \sigma(\mu)$.

2.3. Solution of the Time-Evolution Problem. The constraints (2.24)–(2.25) have a first-class algebra with the following structure:

$$[J^A, J^B] = C^{AB}{}_C J^C \ , \tag{2.26}$$

$$[J^A, (W(\mu) - I)^B{}_C] = C^{AB}{}_D (W(\mu) - I)^D{}_C - (W(\mu) - I)^B{}_D C^{AD}{}_C \ , \tag{2.27}$$

$$[(W(\mu) - I)^A{}_B, (W(v) - I)^C{}_D] = 0 \ . \tag{2.28}$$

The closure conditions $J^A \approx 0$ generate global G-transformations:

$$[J^A, B^B(\mu)] = C^{AB}{}_C B^C(\mu) \ , \tag{2.29}$$

$$[J^A, M^B{}_C(\mu)] = C^{AB}{}_D M^D{}_C(\mu) - M^B{}_D(\mu) C^{AD}{}_C \ . \tag{2.30}$$

The cycle conditions $W^A_B(\mu) \approx I$ generate transformations of $B^A(\mu)$, which include timelike translations of the base-point P, at which all loops $\gamma(\mu)$ and surfaces $\sigma(\mu)$ intersect. The time evolution is generated by the Hamiltonian constraint.

$$H = \sum_\mu Tr\big(\xi(\mu)(W(\mu) - I)\big) \ , \tag{2.31}$$

where $\xi(\mu)$ are generalized "lapse-shift" functions. Note that, given (2.20) and (2.21),

$$\ddot{B}^A(\mu) = \big[H, [H, B^A(\mu)]\big] = 0 \ , \tag{2.32}$$

$$\dot{M}^A{}_B(\mu) = [H, M^A{}_B(\mu)] = 0 \ . \tag{2.33}$$

Thus, the integrated two-forms are linear functions of time and the holonomies are constants of the motion.

The number of physical degrees of freedom is equal to the number of configuration space variables minus the number of first-class constraints. There are $dimL(G)$ independent closure conditions (2.25), but the cycle conditions are not all independent: they are associated with surfaces $\sigma_\varepsilon(\mu)$ which, together with their partners $\sigma_{-\varepsilon}(\mu)$, form the boundary of a homotopically trivial manifold Σ^3_ε; this implies that the product of the cycle conditions over all faces $\sigma_\varepsilon(\mu)$ and $\sigma_{-\varepsilon}(\mu)$ is an identity, so that $dimL(G)$ of the cycle conditions are redundant, and the number of independent cycle conditions is $dimL(G) \times (dimH_2(\Sigma^3) - 1)$. Altogether, in terms of the Betti numbers $b_i = dimH_i(\Sigma)$, one has $dimL(G) \times (b_1)$ configuration variables $M(\mu)$, minus $dimL(G) \times (1 + b_2 - 1)$ constraints, or

$$dimL(G) \times (b_1 - b_2) = 0 \tag{2.34}$$

degrees of freedom, using Poincaré's identity $b_i = b_{3-i}$. Note that one could also count directly the degrees of freedom in the representations $M : \pi_1(\Sigma^3) \rightarrow G$ as $b_1 \times dimL(G)$ minus the number of independent cycle conditions, $(b_2 - 1) \times dimL(G)$, minus $dimL(G)$ for the overall G-conjugacy. Again, one finds zero degrees of freedom.

The "generic" counting which lead to (2.34) must be modified for some topologies, for which there are fewer independent cycle conditions. Non-contractible spheres in the set $\{\sigma(\mu)\}$ do not contribute any cycle conditions, and non-contractible tori provide only the $dim(L(G)) - rank(L(G))$ independent conditions

$$[M(\mu_1), M(\mu_2)] = 0 \ . \tag{2.35}$$

If one considers the orientable compact topologies which admit locally homogeneous structures; only the quotients of H^3 and S^3 have no non-contractible tori or spheres, as far as we know. The fact that the $B \wedge F$ theory has no degrees of freedom in these cases is related to Mostow's rigidity theorem on the discrete representations of $\pi_1(\Sigma^3)$ into semisimple Lie groups with trivial centers and no compact factors, not isomorphic to $SL(2, R)$ [7].

We conclude this section by counting the physical degrees of freedom when Σ^3 has the topology of any one of Thurston's eight classes of locally homogeneous orientable compact manifolds [8]. The results are summarized in Table 3.1 in the next section.

1. Type a: Quotients of E^3. The topologies are T^3, T^3/Z_2, T^3/Z_3, T^3/Z_4, T^3/Z_6 and $T^3/Z_2 \times Z_2$. The fundamental group $\pi_1(\Sigma^3)$ has three generators which are dual to three non-contractible tori. The cycle conditions require that the matrices $\{M(\mu), \mu = 1,2,3\}$ commute. The closure conditions are then $r = \mathrm{rank}(G)$ times redundant, since the matrices $(1 - M(\mu))$ in the adjoint representation have r common null eigenvectors. If we denote by $d = dimL(G)$ the dimension of the Lie algebra, there are $3 \times d$ variables, minus $2(d - r)$ cycle conditions and $d - r$ independent closure conditions, or $3 \times r$ physical degrees of freedom ($6r$ phase space degrees of freedom).

2. Type b: Quotients of Nil. Σ^3 is a non-trivial S^1-bundle over T^2. Two of the cycles are as in case (1) above, but the third fixes $M(3)$ as a function of the commutator of $M(1)$ and $M(2)$. For example, for type $b_{LR}/1(n)$,

$$M(1)M^{-1}(2)M^{-1}(1)M(2) = M(3)^n \ . \tag{2.36}$$

This relation and one of the commutators can be chosen as a set of $d + (d - r) = 2d - r$ independent cycle conditions. One easily shows that they imply $M(3)^n = I$. If one considers the "generic" solution, $M(3) = I$, then there are only $d - r$ independent closure conditions. Note that the holonomy generates a non-faithful representation $\pi_1(\Sigma^3) \to G$ in this case. The number of degrees of freedom is $3d - (2d - r + d - r) = 2r$.

3. Type c: Quotients of $H_2 \times R$. Σ^3 is finitely covered by a trivial S^1-bundle over Σ_g, a genus g Riemann surface. The cycle conditions are

$$[M(\mu), M(2g+1)] = 0 \ (\mu = 1, \cdots, 2g) \ , \tag{2.37}$$

$$\begin{aligned} &(M(1)M^{-1}(2)M^{-1}(1)M(2))(M(3)M^{-1}(4)M^{-1}(3)M(4)) \\ &\ldots M(2g-1)M^{-1}(2g)M^{-1}(2g-1) = I \ . \end{aligned} \tag{2.38}$$

If one chooses $M(2g+1) = I$, all Eq. (2.37) are satisfied and, with (2.38), one has a total of $2d$ independent conditions. Otherwise ($M(2g+1) \neq I$), all of the $M(\mu)$ commute among themselves and Eq. (2.39) is satisfied in a trivial way. In either case the representation $M : \pi_1(\Sigma^3) \to G$ is not faithful. For $M(2g+1) = I$, one has $3d$ constraints and $(2g - 2) \times d$ degrees of freedom, while in the other case the number of constraints is $(d - r) \times 2g + (d - r)$ and there are $r \times (2g + 1)$ degrees of freedom.

4. Type d: Quotients of $SL(2,R)$. Σ^3 is a non-trivial S^1-bundle over Σ_g. The cycle conditions are (2.37) and one relation which fixes $M(2g+1)$ as a function of the Σ_g-cycle (2.38). In the case $M(2g+1) = I$, one has $(2g - 2) \times d$ physical degrees of freedom. The representations $M : \pi_1(\Sigma^3) \to G$ are not faithful.

5. Type e: Quotients of H^3. There is an infinite set of compact quotients of H^3, such as the polyhedra discovered by Löbell [9] with two hexagonal faces and twelve pentagonal faces identified in pairs; the classification of these quotients is not complete. None of them are smoothly deformable, as a consequence of Mostow's theorem [7]. This theorem also tells us that a basis of $H_2(H^3/\Gamma)$ can have at most one non-contractible torus, since there are no free parameters in the quotienting group $\Gamma : \pi_1(\Sigma^3) \to G$, in accordance with the counting (2.34) (one of the cycles is redundant, and could be of the torus type, but as far as we know quotients of H^3 do not admit any non-contractible tori or spheres in $Z_2(H^3/\Gamma)$). Thus, Mostow's theorem implies that the $B \wedge F$ theory has no degrees of freedom for the hyperbolic manifolds.

6. Type f: Quotients of Sol. Σ^3 is finitely covered by a (non-trivial) T^2-bundle over S^1. Since the cycle is invariant under mapping class transformations (Dehn twists), the cycle conditions are the same as for type a topologies; so is the counting of degrees of freedom of the $B \wedge F$ theory.

7. Type g: Quotients of S^3. Σ^3 is one of the following [10]: S^3, S^3/Z_m, S^3/D_m, where D_m is the symmetry group of a regular m-gon in the plane, and S^3/T, S^3/O and S^3/I, where T, O and I are the symmetry groups of the regular tetrahedron, octahedron and icosahedron in R^3. The second holonomy group is trivial, and so is $\pi_1(\Sigma^3)$: the $B \wedge F$ theory has no physical degrees of freedom.

8. Type h: Quotients of $S^2 \times R$. Σ^3 is finitely covered by $S^2 \times S^1$. There are no cycle conditions, but $b_1 = b_2 = 1$: the phase space is spanned by a single matrix $M \in G$ and an "integrated two-form" B^A with values in $L(G)$. The closure conditions are

$$J^A = (I - M^{-1})^A{}_B B^B , \tag{2.39}$$

or $d - r$ independent conditions, leaving r physical degrees of freedom.

This completes the list of solutions of the $B \wedge F$ theory; in the next section we examine the possible relation between these solutions and flat spacetimes $\Sigma^3 \times (0, 1)$, when $G = SO(3, 1)$.

3. Flat Spacetimes

If $B = e \wedge e$ and $G = SO(3, 1)$, then $F = R$ is the Riemann curvature and the action 2.1 becomes the Palatini action

$$S = \int e \wedge e \wedge R. \tag{3.1}$$

This begs the question: When do the solutions of the $B \wedge F$ theory represent flat spacetimes? Clearly, any flat spacetime gives a solution of the $B \wedge F$ theory, by $B = e \wedge e$. The converse is not true, as we will see shortly. We will first review the global properties of flat spacetimes, then see under what conditions the global variables $\{B(\mu), M(\mu)\}$ of the $B \wedge F$-theory describe flat spacetimes.

3.1. $ISO(3, 1)$ Holonomy and Flat Spacetimes. Given a flat spacetime $M = \Sigma \times (0, 1)$, where Σ is a compact, orientable, spacelike three-manifold, the holonomy injects $\pi_1(\Sigma)$ into $ISO(3, 1)$. Following Mess [11], one can classify the flat spacetimes by the rank of the kernel of the linear holonomy. If the rank is equal to three (three independent translations), then Σ is a three-torus and the maximal development is all of R^{3+1} (Thurston type a universe; see Conjecture 3.1 below). If the rank is 2, then Σ is finitely covered by T^3 and the maximal development is again R^{3+1} (Thurston type b, f). If the rank is 1, Σ is closed Seifert bundle with rational Euler class zero. The universal cover of the maximal development is the direct product of R and a domain W in R^{2+1}, which is the universal cover of the maximal development of a (2+1)-dimensional flat spacetime $\Sigma^{(2)} \times (0,1)$, where $\Sigma^{(2)}$ is a genus g Riemann surface (Thurston type c and d universes). If the rank is zero, then the holonomy embeds $\pi_1(\Sigma)$ into $SO(3,1)$ as a discrete compact subgroup, which is rigid by Mostow's theorem. These spacetimes are quotients of the interior of the future light-cone by this subgroup, have an orbifold-singularity at the origin and admit a natural foliation by the hyperbolic surfaces $H^3 \equiv t^2 - x^2 - y^2 - z^2 = \tau^2$ (Thurston type e universes).

We will denote the holonomy for $\gamma(\mu)$ by $\{M(\mu), b(\mu)\}$, where $M(\mu)$ is the $SO(3,1)$ projection of the holonomy in the usual 4×4 matrix representation, and $b(\mu)$ is the four-vector

$$b^a(\mu) = \oint_{\gamma(\mu)} M^a{}_b(s) e^b{}_i(s) ds^i \ , \tag{3.2}$$

where

$$M(s) = P \exp\left\{ -i \int_{\gamma(\mu)}^{(s)} A_i dl^i \right\} \ . \tag{3.3}$$

A flat spacetime $\Sigma^3 \times (0,1)$ can be represented by an open subset of R^{3+1} with points identified, $\mathscr{P} \times (0,1)$, where $\mathscr{P}$ is a polyhedron with $2 \times dimH_2(\Sigma^3)$ faces which are identified in pairs by the isometries $\{M(\mu), b(\mu), \mu = 1, \cdots, dim(H_2(\Sigma^3))\}$. The edges of the polyhedron are given by the vectors $b(\mu)$ and their images under the $SO(3,1)$ components of the identifications, $M(\nu)b(\mu)$. All corners of the polyhedron are identified to a single point of Σ^3, which is the base point P of the homotopy group $\pi_1(\Sigma^3)$.To reconstruct the representation of the spacetime $\mathscr{P} \times (0,1)$, from the polyhedron $\mathscr{P}_0$ given by $\{M(\mu), b(\mu)\}$, one constructs a one-parameter family of polyhedra by displacing one corner of $\mathscr{P}_0$ along a timelike segment $S(\tau), \tau \in (0,1)$, and the other corners along the identified segments ($M(\mu)S(\tau)$, etc.). One can show that different choices of the timelike segment $S(\tau)$ lead to representations of the same spacetime $\Sigma^3 \times (0,1)$. Likewise, the geometry of the polyhedral faces can be chosen arbitrarily: It is sufficient to give the relative positions of the corners, $b(\mu)$, and the $SO(3,1)$ holonomies $M(\mu)$, to specify completely the spacetime. Roughly speaking, a polyhedron $\mathscr{P}$ can be obtained from Σ^3 by chosing a base point $P \in \Sigma^3$, cutting out $dimH_2(\Sigma^3)$ basis surfaces and "unfolding" the resulting three-manifold into a polyhedron. The point P, and the basis surfaces, correspond to different choices of segments $S(\tau)$ and polyhedral faces $\sigma(\mu)$.

We will list all flat spacetimes, by Thurston type, and in each case count the number of parameters needed to describe them. We are using Thurston's classification as a list of topologies which are likely to have some relevance in cosmology, regardless of whether or not there exists a foliation of these spacetimes in locally homogeneous leaves. If such a foliation exists for any flat spacetime with topology $\Sigma^3 \times (0,1)$, where Σ^3 is a compact, orientable three-manifold, then the spacetimes given below are locally homogeneous cosmologies, and the list is complete: Any flat spacetime $\Sigma^3 \times (0,1)$ belongs to one of the categories listed below. We conjecture that a locally homogeneous foliation always exists:

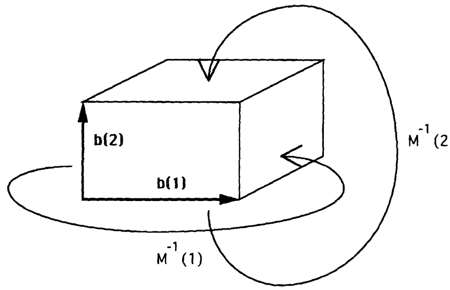

Fig. 3.1. A T^3-universe is represented as a hexahedron in $\mathbb{R}^{3+1}$ with opposite faces identified. As one transports a vector from the face $\sigma(\mu)$ to its identified partner $\sigma(-\mu)$, the vector is Lorentz-transformed by the matrix $M^{-1}(\mu)$. The closure of each face is a condition on the basis vectors $b(\mu)$ and the matrices $M(\mu)$; for instance the closure of the forward face requires that $b(1)+M^{-1}(1)b(2)-M^{-1}(2)b(1)-b(2)=0$.

Conjecture 3.1. Let M be a flat spacetime with topology $\Sigma^3 \times (0,1)$, where Σ^3 is a compact, orientable three-manifold with orientable normal bundle. There exists a foliation of M such that the metric on each leaf, which is inherited from the flat metric on M, is locally homogeneous.

1. Type a_1 Cosmologies. The universe is a three-torus. The holonomies $\{M(\mu), b(\mu); \mu = 1,2,3\}$ must satisfy the $SO(3,1)$ cycle conditions, which state that the matrices $M(\mu)$ commute, and the following polyhedron closure relations [Fig. 3.1]

$$b(1)+M^{-1}(1)b(2)-M^{-1}(2)b(1)-b(2)=0\ , \tag{3.4}$$

$$b(1)+M^{-1}(1)b(3)-M^{-1}(3)b(1)-b(3)=0\ , \tag{3.5}$$

$$b(2)+M^{-1}(2)b(3)-M^{-1}(3)b(2)-b(3)=0\ , \tag{3.6}$$

$$[M(1),M(2)]=[M(1),M(3)]=0\ . \tag{3.7}$$

The most general solution to the cycle conditions (3.7) is given, in the appropriate frame, by

$$M(\mu)=\exp\big(\alpha(\mu)J_{xt}+\beta(\mu)J_{yz}\big)\ . \tag{3.8}$$

In the generic case, when all six coefficients $\{\alpha(\mu),\beta(\mu);\ \mu=1,2,3\}$ are nonzero and different from each other, the only solutions to (3.4)–(3.6) are polyhedra which collapse to a point singularity at a finite proper time in the past or future: at the singularity $b(\mu)=0$, and at time τ one has

$$b(\mu)=(M^{-1}(\mu)-I)N\tau\ , \tag{3.9}$$

where N is a timelike normal.

The largest family of solutions is found when $\beta(\mu)=0$, for all μ. In this case there is again an initial or final singularity (unless $\alpha(\mu)=0$ as well), but one where the singular polyhedron collapses to the y–z plane, which is invariant under $M(\mu)$. The polyhedron at time τ is then given by

$$b(\mu)=b_0(\mu)+(M^{-1}(\mu)-I)N\tau\ , \tag{3.10}$$

where $b_0^x(\mu) = b_0^t(\mu) = 0, \forall\mu$. One needs five scalar parameters to specify $\{b_0(\mu)\}$ up to a rotation in the plane, on top of the three independent parameters $\alpha(\mu)$, altogether eight parameters to specify the spacetime.

It is interesting to compare these spacetimes to the Kasner solutions [12]

$$ds^2 = -dt^2 + \sum_i t^{2b_i} dx_i^2 \ , \tag{3.11}$$

which are flat if one chooses $b = (1, 0, 0)$. They are then specified by six parameters, which define the identifications in $\mathbb{R}^3, \Gamma \ : \ \vec{x} \sim \vec{x} + n_1\vec{a} + n_2\vec{b} + n_3\vec{c}$ and the quotient $T^3 = \mathbb{R}^3/\Gamma$. At $\tau = 0$, the spatial part of the metric (3.11) collapses to a parallelogram in the y–z plane, comparable to (3.10) but with $b_0(3) = 0$. The vanishing of these two parameters in the Kasner solution is due to the requirement that the T^3 universes be surfaces of instantaneity: this requirement is not satisfied in the solution (3.10) when $b_0(3) \neq 0$.

2. *Type $a_n, n > 1$.* For $\Sigma^3 = T^3/Z_2$, the two images of the surface $\sigma(3)$ are identified after a twist by 180°. This is consistent with the closure of the polyhedron's faces only if $b(1)$ and $b(2)$ are orthogonal to $b(3)$. With these two new conditions, the largest set of spacetimes with Type a_2 slices is parametrized by 6 numbers. The polyhedron is given, as for Kasner's solution, by Eq. (3.10) with $b_0(3) = 0$

For T^3/Z_3 and T^3/Z_6, the vectors $b(\mu)$ must have the same lengths and equal bihedral angles (4 conditions). The flat spacetimes with Type a_3 and a_5 sections are labeled by 4 independent parameters. Finally, the flat spacetimes with a_4 and a_6 sections are labeled by 2 parameters and three parameters, respectively.

3. *Type b Cosmologies.* Σ^3 is a non-trivial S^1-bundle over T^2, and has negative sectional curvature. There are seven subclasses, all of which can be represented as a hexahedron with identifications. We will consider only the type $b_{LR}/1(n)$. The cycle conditions are

$$[M(1), M(3)] = 0 \ , \tag{3.12}$$

$$[M(2), M(3)] = 0 \ , \tag{3.13}$$

$$M(1)M^{-1}(2)M^{-1}(1)M(2) = M(3)^n \ . \tag{3.14}$$

There are three constraints on the polyhedron vectors $b_0(\mu)$ in the solution (3.10), on top of $\alpha(3) = 0$ from (3.14) (note, however, that the identification rules are not the same as for T^3). This gives the largest set of spacetimes in this class; they are parametrized by the variables $\alpha(1), \alpha(2)$ and the vectors $b_0(\mu)$ which satisfy the constraints, altogether four parameters.

Type c Cosmologies. The universe is finitely covered by $\Sigma_g \times S^1$, where Σ_g is a genus g surface. The cycle conditions were given in (2.37)–(2.38):

$$[M(\mu), M(2g+1)] = 0 \ , \tag{3.15}$$

$$W(2g+1) = I \ , \tag{3.16}$$

and the face closure conditions are [Fig. 3.2]:

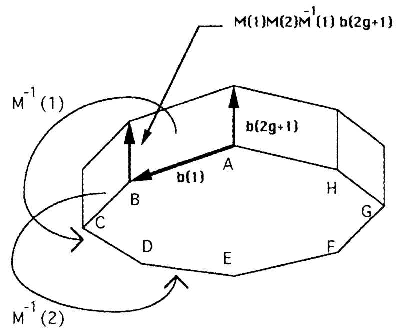

Fig. 3.2. A universe with topology $\Sigma_2 \times S^1$, where Σ_2 is a genus $g = 2$ surface, is represented as a decahedron in $\mathbb{R}^{3+1}$ with the "top" and "bottom" octagons identified by $M(2g+1)$, and the $4g$ "latteral" faces identified in pairs. The vector $b(2g+1)$ at A is identified to $M^{-1}(1)b(2g+1)$ at D, to $M(2)M^{-1}(1)b(2g+1)$ at C and finally $M(1)M(2)M^{-1}(1)b(2g+1)$ at B, so the face $B(1)$ closes if $(I - M^{-1}(2g+1)b(1) + (M(1)M(2)M^{-1}(1) - I)b(2g+1) = 0$.

$$\begin{cases} \big(I - M^{-1}(2g+1)\big)b(1) \\ \quad +\big(M(1)M(2)M^{-1}(1) - I\big)b(2g+1) = 0 & (3.17) \\ \big(I - M^{-1}(2g+1)\big)b(2) \\ \quad +\big(M(2)M^{-1}(1) - M(1)M(2)M^{-1}(1)\big)b(2g+1) = 0 & (3.18) \\ etc. \end{cases}$$

If $M(2g+1) = I$ and $M(1), \cdots, M(2g)$ generate a faithful representation $M: \pi_1(\Sigma_g) \to SO(3,1)$, there are $2g \times 4$ closure conditions and 12 independent conditions on the matrices $M(\mu)$. If one subtracts the overall equivalence by $ISO(3,1)$ conjugacy, one is left with $(2g+1) \times 10 - 8g - 12 - 10 = 12g - 12$ parameters to describe these spacetimes. This is also the number of parameters which describe the flat (2+1)-dimensional spacetimes with the topology $\Sigma_g \times (0,1)$. Indeed, we have set $M(2g+1) = I$, so these spacetimes are just the direct product of (2+1)-dimensional spacetimes with S^1. There are other solutions to (3.15)–(3.18) besides the ones we have just described, for $M(2g+1) \neq I$; as in Sect. 2, these have fewer free parameters.

Type d Cosmologies. The universe is finitely covered by a non-trivial S^1-bundle over Σ_g; the cycle conditions and closure conditions are as above, except for (3.16), which now sets $M(2g+1)$ as a function of the Σ_g-cycle $W(2g+1)$ (Eq. 2.38):

$$W(2g+1) = M(2g+1)^n \ . \tag{3.19}$$

Since $[M(\mu), M(2g+1)] = 0$ for $\mu = 1, \cdots, 2g$, the generic solution satisfies $W(2g+1) = M(2g+1) = I$, which we have discussed in Type e above, and again there is a $(12g-12)$-parameter family of solutions. Note, however, that the identifications of points on the boundary of the polyhedra are different. Roughly speaking, by following the cycle on Σ_g one goes around S^1 n times.

Type e: Hyperbolic Spaces. The spacetime is a quotient of the interior light-cone by a discrete subgroup of $SO(3,1)$. There is a natural foliation in terms of the Lorentz-invariant hyperbolic surfaces $t^2 = \tau^2 - x^2 - y^2 - z^2$. The discrete subgroups are not completely classified to this date, and they have no free parameters by Mostow's theorem. One may check this fact in our formalism, for any given topology, by counting the constraints and showing that these are sufficient to fix all the variables $\{M(\mu), b(\mu)\}$, up to an overall $ISO(3,1)$ transformation.

Type f Cosmologies. The universe can be represented as a hexahedron, as T^3, but where one face is identified to the opposite face by means of a Dehn twist. The cycle conditions are unchanged (from T^3), since the cycle is mapping-class invariant, but one must impose three constraints on the vectors $\{b(1), b(2), b(3)\}$, which require that the faces which are identified with a Dehn twist, say $\mu = 1$, can be cut diagonally into two similar isoceles triangles, and lie in a plane orthogonal to the vector $b(1)$:

$$b(1) \cdot b(2) = 0 \ , \tag{3.20}$$

$$b(1) \cdot b(3) = 0 \ , \tag{3.21}$$

$$b^2(2) + 2(b(1) \cdot b(2)) = 0 \ . \tag{3.22}$$

The largest set of spacetimes in this class is parametrized by 8–3 = 5 numbers.

Type g: Spherical Spaces. There are no flat spacetimes with a foliation in spherical locally homogeneous slices, S^3/Γ. Indeed, if there were such a spacetime, one could map Σ^3 as a sphere embedded in Minkowski space, with points identified. This is impossible because S^3 cannot be embedded in $\mathbb{R}^{3+1}$.

Type h: Kantowski-Sachs Spaces. There are no flat spacetimes with foliations in $S^2 \times S^1$ leaves, i.e. the topology of the Kantowski-Sachs solutions. One proves this as for the quotients of the three-sphere (Type g), by noting that it is impossible to embed $S^2 \times (0,1)$ in $\mathbb{R}^{3+1}$ in such a way that S^2 is spacelike and the segment (0,1) is timelike.

The results of this section are summarized in Table 3.1 and compared to the counting of solutions of the $B \wedge F$ theory and to the Teichmüller parameters of locally homogeneous structures given in [13]. The number of spacetime parameters is generally less than twice the number of Teichmüller parameters, because arbitrary initial conditions in the cotangent bundle to Teichmüller space generally do not lead to flat spacetimes.

3.2. Global Torsion. Given a set of global variables $\{M(\mu), B(\mu)\}$, we wish to interpret the matrices $M(\mu)$ as the $SO(3,1)$ holonomies for loops $\gamma(\mu)$ in a flat spacetime, and each $B(\mu)$ as the area bivector of the face $\sigma(\mu)$ of a polyhedron $\mathscr{P}$ in $\mathbb{R}^{3+1}$, which represents a section of the spacetime as explained in Sect. 3.1. First of all, note that the area bivectors of a polyhedron in $\mathbb{R}^{3+1}$ depend only on the boundary segments of the faces, and not on their local metric properties. We denote the boundary segments of an n-gon $\sigma(\mu)$ by $b_i(\mu), i = 1, \cdots, n$, with $\Sigma_i b_i(\mu) = 0$. The area bivectors can be computed from these vectors by triangularizing the n-gon and summing the contributions of each triangle. For example, for a quadrilateral face, $B(\mu) = 1/2(b_1(\mu) \wedge b_2(\mu) + b_3(\mu) \wedge b_4(\mu))$. One would like to invert these expressions, to compute the polygon vectors which lead to given bivectors $B(\mu)$. It is not always possible to find such vectors in such a way that the polygon faces close: we will define the global torsion for a "polyhedron" given by $b_i^a(\mu)$, by

$$T^a(\mu) = \sum_{i=1}^{n} b_i^a(\mu) \ . \tag{3.23}$$

Table 3.1. The number of parameters which label the solutions of the $B \wedge F$ theory and the flat spacetimes $\Sigma^3 \times (0,1)$ are given, when Σ^3 is one of the Thurston geometries. The rank of the algebra L(G) is r and its dimension is d; for $SO(3,1)$, $r = 2$ and $d = 6$

Universal cover	Compact quotient	Solutions of $B \wedge F$ theory	Flat spacetimes
a	a1	6r	8
	a2		6
	a3		4
	a4		2
	a5		4
	a6		3
$b_{L,R}$	$b_{L,R}/1(n)$	4r	4
c	(orientable Seifert b.)	(4g+2)r	12g–12
d	(orientable Seifert b.)	(4g–4)d	12g–12
e	(hyperbolic manifolds)	0 (discrete)	0 (discrete)
f	(sol / G)	6r	5
g	(quotients of the sphere)	0 (discrete)	0 (no solutions)
h	(Kantowski Sachs)	0 (discrete)	0 (no solutions)

We will show with a simple example how one can construct solutions of the $B \wedge F$ theory which cannot be represented as torsion-free polyhedra. Consider the spacetime $T^3 \times (0,1)$, with trivial $SO(3,1)$ holonomies: a static three-torus cosmology. If one modifies this solution by taking $M(1)$ to be a rotation in the plane of the face $\sigma(1)$, then the bivectors $B(1)$ and $M^{-1}(1)B(1)$ do not change but the two identified faces are twisted with respect to each other. One easily shows that the other faces no longer close, regardless of the choice of $\{b(\mu)\}$ consistent with the bivectors $\{B(\mu)\}$. Yet, all of the constraints of the reduced $B \wedge F$ theory are satisfied.

One can identify a set of constraints on the variables $\{M(\mu), B(\mu)\}$ which guarantee that the global torsion vanishes. We will give these constraints below with a brief explanation; the reader is referred to analogous work in the exact lattice formulation of the $B \wedge F$ theory for a more detailed derivation [14].

The torsion constraints state that the pairs of faces which intersect at an edge of the polyhedron intersect transversally, and the various identified edges respect the identification rules of the polyhedron. For the intersection of faces (μ) and (ν), the former conditions (transversality) are

$$\varepsilon_{abcd} B^{ab}(\mu) B^{cd}(\nu) = 0 \ . \tag{3.24}$$

The latter are found by considering all sets of three faces which intersect along two identified edges: Let $B(\mu)$ and $B(\nu)$ represent two faces which intersect along edge

No. 1, and let $B(\nu)$ and $B(\rho)$ intersect along edge No. 2, which is identified to edge No. 1 by means of an $ISO(3,1)$ transformation with the $SO(3,1)$ component M. To reconstruct the triple intersection at edge No. 1, one transports $B(\rho)$ from edge No. 2 to edge No. 1, and considers the three bivectors $B(\mu)$, $B(\nu)$ and $MB(\rho)$, which intersect transversally in pairs by (3.5). They intersect along the same edge if

$$C_{ABC}B^A(\mu)B^B(\nu)M^C{}_DB^D(\rho) = 0 \ , \tag{3.25}$$

where C_{ABC} are the structure constants of $SO(3,1)$ and M is in the adjoint representation.

4. Conclusion

We have investigated two theories, the $B \wedge F$ gauge theory and the reduction of Einstein's equations to flat spacetimes, and examined the relation between them.

The $B \wedge F$ theory was reduced to a finite constrained Hamiltonian system, where the phase space is spanned by holonomies $M : \pi_1(\Sigma^3) \to G$ and "integrated two-forms," $B : H_2(\Sigma^3) \to L(G)$. This reduced theory could then be solved explicitly when Σ^3 is one of the orientable topologies listed by Thurston in his classification of geometric structures, and the number of degrees of freedom was given in all cases.

The flat spacetimes were constructed from their $ISO(3,1)$ holonomies $\{M(\mu), b(\mu)\}$ for a basis set of loops of $\pi_1(\Sigma^3)$, as a subset $\mathscr{P} \times (0,1)$ of $\mathbb{R}^{3+1}$ with points identified on the boundary faces of the polyhedron $\mathscr{P}$. The vectors $b(\mu)$ give the displacements between corners of $\mathscr{P}$, while the matrices $M(\mu) \in SO(3,1)$ are the $SO(3,1)$ holonomies for loops which cross through identified faces of the polyhedron. The number of parameters which label the flat spacetimes $\Sigma^3 \times (0,1)$ were given when Σ^3 is one of Thurston's manifolds. This completes Ellis' list of topologically non-trivial cosmological models in the case of vanishing spacetime curvature [15]. The solutions of the $B \wedge F$ theory with $G = SO(3,1)$ can be interpreted as representing spacetimes with global torsion, which we define as the failure of the polyhedra's faces to close. The variables $B^{ab}(\mu)$ represent the area bivectors of the faces of a polyhedron, when they do close. We also gave the vanishing torsion constraints as equations on the bivectors $B^{ab}(\mu)$.

The solution of the time evolution problem for the reduced $B \wedge F$ theory can be exploited to give information on the global dynamics of flat spacetimes, when the torsion constraints are satisfied. In particular, one has the result that the area bivectors of non-contractible surfaces, in an intrinsically chosen set of representatives of basis elements of $H_2(\Sigma^3)$ in $Z_2(\Sigma^3)$, have a linear evolution in time.

The derivation of the reduced theory from the $B \wedge F$ field theory required some non-trivial steps, and we could only conjecture that the reduction is exact. It would be of great interest to prove this conjecture, perhaps as in the polygon representation of (2+1)-dimensional gravity [16] or, by means of the exact lattice theory [6,14]. Also, the polyhedron representation of flat spacetimes is based on several assumptions, in particular it assumes the existence of non-contractible geodesic loops which become the polyhedron's edges $b(\mu)$ in the Minkowski space representation. If this construction can be formalized, it would give a generalization of Poincaré's polygon in H^2 [16], which labels the conformal structures on Riemann surfaces, to polyhedra in $\mathbb{R}^{3+1}$ which label the flat spacetimes.

Finally, the existence of a finite constrained Hamiltonian formulation of the $B \wedge F$ theory could be useful in developing the corresponding quantum theory [17].

The linear time-evolution of the classical variable $B(\mu)$ indicates that the quantum theory may have a representation in terms of "free particles" in $L(G)$, but with identifications given by the mapping class transformations, or "large diffeomorphisms" [18]. The Green's function could then be constructed by the method of images, as a sum of freely propagating amplitudes over the mapping-class images of the source. The quantum effects would likely appear as interference terms in this sum, as in (2+1)-dimensional quantum gravity [19].

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