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Fock Representations of Quantum Fields with Generalized Statistics

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Abstract: We develop a rigorous framework for constructing Fock representations of quantum fields obeying generalized statistics. The main features of these representations are investigated. Various aspects of the underlying mathematical structure are illustrated by means of explicit examples.

1. Introduction

The present paper concerns a generalization of the concept of statistic in quantum field theory. We investigate the Fock realization of N-component fields whose annihilation parts obey the relation

$$a_\alpha(x_1)a_\beta(x_2) = R^{\delta\gamma}_{\beta\alpha}(x_2, x_1)a_\gamma(x_2)a_\delta(x_1)\,. \tag{1.1}$$

Here $x_1, x_2 \in \mathbf{R}^s$ and the *exchange factor* R is a $N^2 \times N^2$ matrix function on $\mathbf{R}^s \times \mathbf{R}^s$, which satisfies certain consistency conditions to be specified in what follows. Our main task below is to construct a Fock representation $\mathscr{F}_R$ of the algebra (1.1). We also establish the basic properties of the fields $a_\alpha(x)$ as operators in $\mathscr{F}_R$ and derive the relative correlation functions. Finally, some aspects of the time evolution in the Fock space $\mathscr{F}_R$ are investigated.

The statistic of quantum fields is usually associated with the irreducible representations of the permutation group which lead to bosons (more generally para-bosons) and fermions (para-fermions). In the last two decades it has been discovered, however, that alternative, not permutation group statistics [15, 27, 30, 39, 42] appear actually in many different areas of quantum theory. An outstanding example is conformal field theory; the basic building blocks of conformal invariant models in $1 + 1$ space-time dimensions obey statistics corresponding in general to irreducible representations of the braid instead of the permutation group [10, 12, 13]. Charged sectors of $2 + 1$ dimensional gauge theories [6] also give rise to braid statistics [14]. The excitations associated with Abelian and non-Abelian representations of the braid group are called anyons and plektons respectively. It is generally believed [18, 28] that anyons occur as quasi-particles in the fractional quantum Hall effect and may be relevant for explaining some features of high T_c-superconductors [9, 25, 29]. Other striking physical phenomena as fractional charge [22, 41] and fractional spin [39] are also deeply related to generalized statistics. For all these reasons the subject has been extensively studied in the last few years (see [11, 40] and references therein). The methods which have been most frequently applied are the mean-field approach [1, 4] and the non-relativistic field theory formulation of the N-body quantum-mechanical anyon problem [21].

One of the main goals of this work is to show that the Fock representation of the algebra (1.1) provides a convenient basis and a unifying framework for the investigation of statistics in quantum field theory. Indeed, varying R among the admissible exchange factors, one gets a rich family of quantum fields. The x-independent R-matrices lead to permutation group statistics whereas piecewise-constant R-factors give rise to braid statistics. But the family in consideration is even larger because it involves also fields corresponding to R-matrices with more complicated x-dependence. From the mathematical point of view we find it instructive to study the whole family in general. Concerning physics, one should mention that deviations from the Bose-Fermi alternative are not expected for fundamental elementary particles [17]. There exist several indications, however, that generalized statistics are relevant for describing collective excitations in solid state physics [9, 18, 25, 28, 29]. Products of "order" and "disorder" variables [20, 24], which control the phase structure in quantum field theory and statistical mechanics, are also expected to have exotic statistics [19].

We end the introduction by recalling those universal structures of any Fock representation (see for example Sect. X.7 of [36]), which are used below. This allows both to fix the notation and to explain our strategy for constructing the Fock representation $\mathscr{F}_R$. Consider a separable Hilbert space $\{\mathscr{H}, (\cdot,\cdot)\}$ and its n-fold tensor power $\mathscr{H}^n = \mathscr{H}^{\otimes n}$ which in physical terms is the n-particle space. The direct sum

$$\mathscr{F}(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^n, \tag{1.2}$$

where $\mathscr{H}^0 = \mathbf{C}^1$ is called the Fock space over $\mathscr{H}$. The elements of $\mathscr{F}(\mathscr{H})$ can be represented by sequences $\{\varphi = (\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(n)}, \ldots) : \varphi^{(n)} \in \mathscr{H}^n\}$ and the finite particle subspace $\mathscr{F}^0(\mathscr{H}) \subset \mathscr{F}(\mathscr{H})$ is defined as follows: $\varphi \in \mathscr{F}^0(\mathscr{H})$ if and only if $\varphi^{(n)} = 0$ for n large enough. By construction $\mathscr{F}^0(\mathscr{H})$ is dense in $\mathscr{F}(\mathscr{H})$.

We turn now to the definition of annihilation and creation operators on $\mathscr{F}^0(\mathscr{H})$. Let us denote by $\mathscr{D}^n$ the set of decomposable vectors

$$\mathscr{D}^n = \{f_1 \otimes \ldots \otimes f_n : f_i \in \mathscr{H}\} \subset \mathscr{H}^n, \qquad \mathscr{D}^0 = \mathbf{C}^1 .$$

Recall that $\mathscr{D}^n$ is a total set, i.e. the set $\mathscr{L}(\mathscr{D}^n)$ of finite linear combinations of elements of $\mathscr{D}^n$ is dense in $\mathscr{H}^n$. Following [36], for each $f \in \mathscr{H}$ we introduce the operators

$$b(f) : \mathscr{D}^n \to \mathscr{D}^{n-1}, \qquad n \geq 1,$$
$$b^*(f) : \mathscr{D}^n \to \mathscr{D}^{n+1}, \qquad n \geq 0,$$

defined by

$$b(f) f_1 \otimes \ldots \otimes f_n = \sqrt{n}(f, f_1) f_2 \otimes \ldots \otimes f_n , \tag{1.3}$$
$$b^*(f) f_1 \otimes \ldots \otimes f_n = \sqrt{n+1} f \otimes f_1 \otimes \ldots \otimes f_n . \tag{1.4}$$

In addition, we set $b(f)\mathscr{H}^0 = 0$ and extend by linearity both $b(f)$ and $b^*(f)$ to $\mathscr{L}(\mathscr{D}^n)$. Now, one can easily prove [36] that for any $\varphi \in \mathscr{L}(\mathscr{D}^n)$ and $\psi \in \mathscr{L}(\mathscr{D}^{n+1})$ one has:

$$\|b(f)\varphi\| \leq \sqrt{n}\|f\| \, \|\varphi\| \,, \qquad \|b^*(f)\varphi\| \leq \sqrt{n+1}\|f\| \, \|\varphi\| \,, \tag{1.5}$$

$$(\psi, b^*(f)\varphi) = (b(f)\psi, \varphi) \,. \tag{1.6}$$

In what follows $b^\sharp$ denotes either b or b^*. From the estimates (1.5) one has that $b^\sharp(f)$ can be extended by continuity to $\mathscr{H}^n$ and finally, by linearity, to $\mathscr{F}^0(\mathscr{H})$. The extensions obviously obey the relation (1.6) for any $\varphi, \psi \in \mathscr{F}^0(\mathscr{H})$.

For describing bosons and fermions in this context one introduces the subspaces $\mathscr{H}^n_+$ and $\mathscr{H}^n_-$ of $\mathscr{H}^n$, which are the n-fold symmetric and totally anti-symmetric tensor powers of $\mathscr{H}$ respectively. Then, the associated bosonic and fermionic Fock spaces are

$$\mathscr{F}_\pm(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^n_\pm \,. \tag{1.7}$$

Denoting by $P_\pm$ the orthogonal projectors on $\mathscr{F}_\pm(\mathscr{H})$, the bosonic and fermionic creation and annihilation operators are defined by

$$a^\sharp_\pm(f) = P_\pm b^\sharp(f) P_\pm \,. \tag{1.8}$$

At this point we have a sufficient background for formulating our approach to the Fock realization of the algebra (1.1). The main steps are essentially two:
(i) With any admissible exchange factor $R(x_1, x_2)$ we associate a distinguished subspace $\mathscr{F}_R(\mathscr{H}) \subset \mathscr{F}(\mathscr{H})$;
(ii) We generalize Eq. (1.8) by replacing $P_\pm$ with the orthogonal projection P_R on $\mathscr{F}_R(\mathscr{H})$.

The key points of this procedure are discussed in full details in the next two sections. Section 4 concerns the description of time evolution in our framework. In Sect. 5 we show that the Leinaas-Myrheim anyons represent just a particular example of the general scheme developed in Sects. 2–5. In Sect. 6 we establish the generalization to multicomponent fields. Finally, Sect. 7 is devoted to our conclusions.

2. Exchange Factors and Relative Fock Spaces

For simplicity we start by considering Eq. (1.1) for fields with a single component ($N = 1$). We take as one-particle space $\mathscr{H}$ the complex Hilbert space $L^2(\mathbf{R}^s, d^s x)$, the generalization to an arbitrary $L^2(\mathbf{X}, d\mu)$ being straightforward. With these assumptions, the exchange factor R is in general a complex-valued function on $\mathbf{R}^s \times \mathbf{R}^s$. We require R to be measurable and to satisfy

$$R(x_1, x_2)R(x_2, x_1) = 1 \,, \tag{2.1}$$

$$\bar{R}(x_1, x_2) = R(x_2, x_1) \,, \tag{2.2}$$

where the bar stands for complex conjugation. Equation (2.1) guarantees the consistency of Eq. (1.1) under the interchange of x_1 and x_2. The meaning of (2.2) will be clarified a few lines below.

Combining Eqs. (2.1, 2) one gets the parametrization

$$R(x_1, x_2) = \exp[ir(x_1, x_2)] \,, \tag{2.3}$$

where r is real-valued and obeys

$$r(x_1, x_2) + r(x_2, x_1) = 2\pi k_r \,, \qquad k_r \in \mathbf{Z} \,. \tag{2.4}$$

We call R *factorizable* if there exists a real-valued function p on $\mathbf{R}^s$ and $k_r \in \mathbf{Z}$, such that

$$r(x_1, x_2) = p(x_1) - p(x_2) + \pi k_r \,. \tag{2.5}$$

Notice that in general $R(x_1, x_2)^2 \neq 1$. For $N = 1$ the only possible *constant* exchange factors are ± 1, i.e. generalized statistics require x-dependent R.

Fixing any admissible exchange factor, we introduce for $n \geq 2$ the operators $\{S_i : i = 1, \ldots, n-1\}$ acting in $\mathscr{H}^n$ according to

$$[S_i \varphi](x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = R(x_i, x_{i+1}) \varphi(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \,. \tag{2.6}$$

As a simple consequence of the defining properties of R, one has

Proposition 1. $\{S_i : i = 1, \ldots, n-1\}$ *are bounded* ($\|S_i\| = 1$) *Hermitian operators on* $\mathscr{H}^n$ *satisfying:*

$$S_i S_j = S_j S_i \,, \qquad |i - j| \geq 2 \,,$$
$$S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1} \,, \qquad S_i^2 = 1 \,.$$

We observe that (2.2) ensures the hermiticity of S_i, which is actually the main motivation for this condition on R.

Let $\mathscr{P}_n$ be the permutation group of n elements, whose generators (elementary permutations) are denoted by $\{\sigma_i : i = 1, \ldots, n-1\}$. The above proposition has the following simple

Corollary. *The mapping*

$$S : \sigma_i \mapsto S_i \tag{2.7}$$

provides a representation of $\mathscr{P}_n$ *in* $\mathscr{H}^n$ *and*

$$P_R^{(n)} = \frac{1}{n!} \sum_{\sigma \in \mathscr{P}_n} S(\sigma) \,, \qquad n \geq 2 \tag{2.8}$$

is an orthogonal projection operator.

Setting $P_R^{(0)} = \mathbf{1}$ and $P_R^{(1)} = \mathbf{1}$, we define the n-particle space relative to R by

$$\mathscr{H}_R^n = P_R^{(n)} \mathscr{H}^n \,. \tag{2.9}$$

The set $\{P_R^{(n)} : n = 0, 1, \ldots\}$ determines a projection operator P_R on $\mathscr{F}(\mathscr{H})$, given by

$$[P_R \varphi]^{(n)} = P_R^{(n)} \varphi^{(n)} \,. \tag{2.10}$$

It is worth mentioning that P_R for $R = \pm 1$ coincides precisely with $P_\pm$. The R-subspace $\mathscr{F}_R(\mathscr{H}) \subset \mathscr{F}(\mathscr{H})$ we are looking for is defined by

$$\mathscr{F}_R(\mathscr{H}) = P_R \mathscr{F}(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}_R^n \,. \tag{2.11}$$

The associated finite particle subspace reads $\mathscr{F}_R^0(\mathscr{H}) = P_R \mathscr{F}^0(\mathscr{H})$ and is dense in $\mathscr{F}_R(\mathscr{H})$.

Consider now the operator C, defined on $\mathscr{H}^n$ by

$$(C\varphi)(x_1, \dots, x_n) = \bar{\varphi}(x_n, \dots, x_1). \tag{2.12}$$

One immediately verifies that C is antilinear, norm-preserving and satisfies $C^2 = \mathbf{1}$. Therefore C represents a *conjugation* on $\mathscr{H}^n$, which automatically extends to a conjugation on $\mathscr{F}(\mathscr{H})$. From Eqs. (2.2, 6) it follows that $CS_\imath C = S_{n-\imath}$. Consequently $C\mathscr{F}_R(\mathscr{H}) \subset \mathscr{F}_R(\mathscr{H})$, i.e. C provides a conjugation in any R-subspace $\mathscr{F}_R(\mathscr{H})$.

We conclude this section by establishing the relationship between any couple of Fock spaces $\mathscr{F}_{R_1}(\mathscr{H})$ and $\mathscr{F}_{R_2}(\mathscr{H})$ relative to two different exchange factors R_1 and R_2. Clearly, being infinite dimensional separable Hilbert spaces, $\mathscr{F}_{R_1}(\mathscr{H})$ and $\mathscr{F}_{R_2}(\mathscr{H})$ are isomorphic. Among others, there exist however some natural isomorphisms which play a distinguished role and can be constructed as follows. Let us consider the functional equation

$$T_\pm(x_1, x_2)R(x_1, x_2) = \pm T_\pm(x_2, x_1) \tag{2.13}$$

with the supplementary condition

$$\bar{T}_\pm(x_1, x_2)T_\pm(x_1, x_2) = 1. \tag{2.14}$$

Equations (2.13, 14) have several solutions; recalling Eqs. (2.3, 4), the most evident ones are

$$\exp\left[-\frac{i}{2}\,r(x_1, x_2)\right] = \begin{cases} T_+(x_1, x_2) & \text{if } k_r \in 2\mathbf{Z}\,; \\ T_-(x_1, x_2) & \text{if } k_r \in 2\mathbf{Z}+1\,. \end{cases} \tag{2.15}$$

For any solution $T_\pm(x_1, x_2)$ of (2.13,14) and for any $\varphi \in \mathscr{H}^n$ we define the operators

$$[U_\pm^{(n)}(R)\varphi]\,(x_1, \dots, x_n) = \left[\prod_{\substack{i,j=1 \\ i<j}}^{n} T_\pm(x_i, x_j)\right] \varphi(x_1, \dots, x_n)\,, \qquad n \geq 2\,, \tag{2.16}$$

setting also $U_\pm^{(0)}(R) = \mathbf{1}$ and $U_\pm^{(1)}(R) = \mathbf{1}$. Due to Eq. (2.14), $U_\pm^{(n)}(R)$ are unitary operators on $\mathscr{H}^n$, which therefore extend to unitary operators $U_\pm(R)$ on $\mathscr{F}(\mathscr{H})$. Using (2.13), one immediately verifies that

$$U_\pm(R) : \mathscr{F}_R(\mathscr{H}) \to \mathscr{F}_\pm(\mathscr{H}). \tag{2.17}$$

Consequently the compositions

$$U_\pm(R_1, R_2) = U_\pm(R_1)^{-1}U_\pm(R_2) \tag{2.18}$$

provide two natural isomorphisms between $\mathscr{F}_{R_2}(\mathscr{H})$ and $\mathscr{F}_{R_1}(\mathscr{H})$. Notice that the operators $C_\pm = U_\pm(R)CU_\pm(R)^{-1}$ are conjugations on $\mathscr{F}_\pm(\mathscr{H})$; one has

$$[C_\pm\varphi]\,(x_1, \dots, x_n) = \left[\prod_{\substack{\imath,j=1 \\ \imath\neq j}}^{n} T_\pm(x_\imath, x_j)\right] \bar{\varphi}(x_n, \dots, x_1)\,, \tag{2.19}$$

which in general differ from C.

Summarising, for any admissible exchange factor R we have explicitly constructed an R-subspace $\mathscr{F}_R(\mathscr{H}) \subset \mathscr{F}(\mathscr{H})$. Any $\mathscr{F}_R(\mathscr{H})$ is equipped with a conjugation and we have established some isomorphisms between pairs of such Fock spaces. Our next step will be to define creation and annihilation operators on $\mathscr{F}_R(\mathscr{H})$.

3. Creation and Annihilation Operators

In analogy with Eq. (1.8), we introduce the creation and annihilation operators

$$a^\sharp(f) = P_R b^\sharp(f) P_R \,. \tag{3.1}$$

The estimates (1.5) imply that $a^\sharp(f)$ are densely defined (with domain $\mathscr{F}^0(\mathscr{H})$) linear operators, which satisfy

$$(\psi, a^*(f)\varphi) = (a(f)\psi, \varphi) \,. \tag{3.2}$$

Therefore $a^\sharp(f)$ are closable. Since $\mathscr{F}_R^0(\mathscr{H})$ is invariant under $a^\sharp(f)$, we shall concentrate below on the restrictions of $a(f)$ and $a^*(f)$ to $\mathscr{F}_R^0(\mathscr{H})$. Their action is given by

$$[a(f)\varphi]^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1} \int d^s x \bar{f}(x) \varphi^{(n+1)}(x, x_1, \ldots, x_n) \,, \tag{3.3}$$

$$\begin{aligned}[a^*(f)\varphi]^{(n)}(x_1, \ldots, x_n) &= \frac{1}{\sqrt{n}} f(x_1) \varphi^{(n-1)}(x_2, \ldots, x_n) \\ &+ \frac{1}{\sqrt{n}} \sum_{k=2}^{n} R(x_{k-1}, x_k) \ldots R(x_1, x_k) f(x_k) \varphi^{(n-1)}(x_1, \ldots, \hat{x}_k, \ldots, x_n) \,,\end{aligned} \tag{3.4}$$

where $\varphi \in \mathscr{F}_R^0(\mathscr{H})$ and $\hat{x}$ indicates that the argument x must be omitted. For deriving the commutation properties of $a^\sharp(f)$ it is convenient to introduce the operator-valued distributions $a^\sharp(x)$ defined by

$$a(f) = \int d^s x \bar{f}(x) a(x) \,, \qquad a^*(f) = \int d^s x f(x) a^*(x) \,.$$

Then from Eqs. (3.3,4) one gets

$$[a(x)\varphi]^{(n)}(x_1, \ldots, x_n) = \sqrt{n+1}\, \varphi^{(n+1)}(x, x_1, \ldots, x_n) \,, \tag{3.5}$$

$$\begin{aligned}[a^*(x)\varphi]^{(n)}(x_1, \ldots, x_n) &= \frac{1}{\sqrt{n}} \delta(x - x_1) \varphi^{(n-1)}(x_2, \ldots, x_n) \\ &+ \frac{1}{\sqrt{n}} \sum_{k=2}^{n} R(x_1, x_k) \ldots R(x_{k-1}, x_k) \delta(x - x_k) \varphi^{(n-1)}(x_1, \ldots, \hat{x}_k, \ldots, x_n) \,.\end{aligned} \tag{3.6}$$

Precisely as for bosons and fermions, $a(x)$ is a densely defined operator in $\mathscr{F}_R(\mathscr{H})$ whereas $a^*(x)$ makes sense only as a densely defined quadratic form in $\mathscr{F}_R(\mathscr{H}) \times \mathscr{F}_R(\mathscr{H})$. Using Eqs. (3.5, 6), one easily checks that $a^\sharp(x)$ satisfy the exchange relations

$$a(x_1)a(x_2) - R(x_2, x_1) a(x_2) a(x_1) = 0 \,, \tag{3.7}$$

$$a^*(x_1)a^*(x_2) - R(x_2, x_1) a^*(x_2) a^*(x_1) = 0 \,, \tag{3.8}$$

$$a(x_1)a^*(x_2) - R(x_1, x_2) a^*(x_2) a(x_1) = \delta(x_1 - x_2) \,. \tag{3.9}$$

Now we are going to establish some other basic features of the fields $a^\sharp(f)$. It is well known that $a_-^\sharp(f)$ are bounded operators, which is not the case of $a_+^\sharp(f)$. The following proposition generalizes this statement.

Proposition 2. *$a^\sharp(f)$ are bounded operators of norm*

$$\|a^\sharp(f)\| \leq \|f\| \,, \tag{3.10}$$

if and only if

$$\int d^s y \int d^s z \bar{g}(y) R(y,z) g(z) \leq 0 \tag{3.11}$$

for any $g \in B_0(\mathbf{R}^s)$ – the space of bounded functions with compact support.

Proof. Because of Eq. (3.2), it is sufficient to consider $a^*(f)$. Using Eqs. (3.3, 4), one finds the equality

$$\|a^*(g)\varphi\|^2 = \|g\|^2\|\varphi\|^2 + n \int d^s x_1 \ldots \int d^s x_{n-1} \int d^s y \int d^s z \bar{g}(z) R(y,z) g(y) \\ \times \varphi(z, x_1, \ldots, x_{n-1}) \bar{\varphi}(y, x_1, \ldots, x_{n-1}), \tag{3.12}$$

valid for any $g \in B_0(\mathbf{R}^s)$ and $\varphi \in \mathscr{H}_R^n$.

Assume first that $a^*(f)$ is bounded and satisfies (3.10). Taking $\varphi \in \mathscr{H}$ in Eq. (3.12), one gets

$$\|g\|^2\|\varphi\|^2 + \int d^s y \int d^s z \bar{g}(z) R(y,z) g(y) \varphi(z) \bar{\varphi}(y) \\ = \|a^*(g)\varphi\|^2 \leq \|a^*(g)\|^2\|\varphi\|^2, \tag{3.13}$$

which in view of Eq. (3.10) leads to

$$\int d^s y \int d^s z \bar{\varphi}(y) g(y) R(y,z) \varphi(z) \bar{g}(z) \leq 0 \tag{3.14}$$

for arbitrary $g \in B_0(\mathbf{R}^s)$ and $\varphi \in \mathscr{H}$. This proves (3.11).

Suppose now that (3.11) holds and consider

$$\chi(x_1, \ldots, x_{n-1}) = \int d^s y \int d^s z \bar{g}(z) R(y,z) g(y) \\ \times \varphi(z, x_1, \ldots, x_{n-1}) \bar{\varphi}(y, x_1, \ldots, x_{n-1}). \tag{3.15}$$

For any $g \in B_0(\mathbf{R}^s)$ one has that $\chi \in L^1(\mathbf{R}^{s(n-1)})$. Moreover, from (3.11) it follows that $\chi(x_1, \ldots, x_{n-1}) \leq 0$ almost everywhere in $\mathbf{R}^{s(n-1)}$. Therefore

$$\int d^s x_1 \ldots \int d^s x_{n-1} \chi(x_1, \ldots, x_{n-1}) \leq 0,$$

which, combined with Eq. (3.12) gives

$$\|a^*(g)\varphi\| \leq \|g\| \, \|\varphi\|. \tag{3.16}$$

This estimate holds obviously also for any $\varphi \in \mathscr{F}_R^0(\mathscr{H})$ and implies (3.10) because $B_0(\mathbf{R}^s)$ and $\mathscr{F}_R^0(\mathscr{H})$ are dense in $\mathscr{H}$ and $\mathscr{F}_R(\mathscr{H})$ respectively.

We observe that any factorizable exchange factor (see Eq. (2.5)) with odd values of k_r obeys (3.11) and consequently leads to bounded creation and annihilation operators. In general $a^\sharp(f)$ are not bounded, but the technical difficulties stemming from this fact can be avoided (at least partially) by considering bounded functions of $a^\sharp(f)$. Following for instance the standard treatment [36] of the boson field $a_+^\sharp(f)$, one can introduce the Segal-type operator

$$\Phi(f) = \frac{1}{\sqrt{2}} [a(f) + a^*(f)] \tag{3.17}$$

and prove exactly in the same way

Proposition 3. *The operator $\Phi(f)$ is essentially self-adjoint on $\mathscr{F}_R^0(\mathscr{H})$.*

Therefore, in spite of the fact that in general $\Phi(f)$ is unbounded, the operator $\exp[i\Phi(f)]$ built with the self-adjoint closure of (3.17), is a unitary operator on $\mathscr{F}_R(\mathscr{H})$. Notice that the Segal field (3.17) is not a linear functional in f since $a(f)$ is antilinear.

We now turn to the construction of the correlation functions of the fields $a^\sharp(f)$. Denoting the vacuum state by $\Omega = (1, 0, \dots, 0, \dots)$ and using that $\mathscr{D}^n$ is a total set in $\mathscr{H}^n$, one can show that $\{a^*(f_1)\dots a^*(f_n)\Omega : f_i \in \mathscr{H}\}$ is a total set in $\mathscr{H}_R^n$. Therefore, all non-trivial correlation functions (in the form of distributions) are

$$\begin{aligned} &w_n(x_1, \dots, x_n; y_1, \dots, y_n) \\ &\quad = (a^*(x_1)\dots a^*(x_n)\Omega, a^*(y_1)\dots a^*(y_n)\Omega), \qquad n \geq 0. \end{aligned} \tag{3.18}$$

Applying Eq. (3.2,9) one derives the recursive relation

$$\begin{aligned} w_n(x_1, \dots, x_n; y_1, \dots, y_n) = {}& \delta(x_1 - y_1) w_{n-1}(x_2, \dots, x_n; y_2, \dots, y_n) \\ & + \sum_{k=2}^{n} R(x_1, y_1)\dots R(x_1, y_{k-1})\delta(x_1 - y_k) \\ & \times w_{n-1}(x_2, \dots, x_n; y_1, \dots, \hat{y}_k, \dots, y_n), \end{aligned} \tag{3.19}$$

which permits to compute w_n. One finds

$$w_0 = (\Omega, \Omega) = 1, \qquad w_1(x_1; y_1) = \delta(x_1 - y_1),$$
$$w_2(x_1, x_2; y_1, y_2) = \delta(x_1 - y_1)\delta(x_2 - y_2) + R(x_1, y_1)\delta(x_1 - y_2)\delta(x_2 - y_1)$$

and so on. As expected the R-dependence shows up for $n \geq 2$.

4. Time Evolution

We first recall the notion of *second quantization* $d\Gamma(A)$ of an operator A acting in the one-particle space $\mathscr{H}$. Assume that A has a dense domain $D \subset \mathscr{H}$. Then the subset $D(A) = \{\varphi \in \mathscr{F}^0(\mathscr{H}) : \varphi^{(n)} \in D^{\otimes n} \text{ for } n \geq 1\}$ is dense in $\mathscr{F}(\mathscr{H})$ and $d\Gamma(A)$ is defined on $D(A)$ by [35]

$$[d\Gamma(A)\varphi]^{(0)} = 0, \tag{4.1}$$

$$\begin{aligned} [d\Gamma(A)\varphi]^{(n)} = {}& (A \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes A \otimes \dots \otimes \mathbf{1} + \dots \\ & + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes A)\varphi^{(n)}. \end{aligned} \tag{4.2}$$

We furthermore introduce the operator

$$d\Gamma_R(A) = P_R d\Gamma(A) P_R. \tag{4.3}$$

Provided that

$$P_R D(A) \subset D(A), \tag{4.4}$$

Eq. (4.3) defines an operator in $\mathscr{F}_R(\mathscr{H})$ with dense domain $D_R(A) = P_R D(A)$. For $R = \pm 1$ the condition (4.4) is automatically satisfied and Eq. (4.3) gives rise to the operators $d\Gamma_\pm(A) = P_\pm d\Gamma(A) P_\pm$ on $D_\pm(A) = P_\pm D(A)$. These operators are familiar from the Bose- and Fermi-quantization.

Let us consider now the time evolution in $\mathscr{F}_R(\mathscr{H})$ starting with the free one-particle hamiltonian

$$h = -\frac{1}{2}\,\Delta\,, \tag{4.5}$$

defined on the space of C^∞-functions with compact support $C_0^\infty(\mathbf{R}^s) \subset L^2(\mathbf{R}^s)$. It is well known that h is essentially self-adjoint. In this section we concentrate on smooth exchange factors $R \in C^\infty(\mathbf{R}^s \times \mathbf{R}^s)$. For such factors $P_R D(h) \subset D(h)$ and therefore

$$H_R = d\Gamma_R(h) = P_R d\Gamma(h) P_R \tag{4.6}$$

is a densely defined Hermitian operator in $\mathscr{F}_R(\mathscr{H})$ with domain $D_R(h)$. Using that P_R and $d\Gamma(h)$ commute with the conjugation C, which in turn leaves invariant $D_R(h)$, one concludes that H_R admits at least one self-adjoint extension [36]. We shall prove now that H_R has a unique self-adjoint extension. For this purpose we introduce the operators $\tilde{H}_\pm$ related to H_R by the isospectral transformation

$$\tilde{H}_\pm = U_\pm(R) H_R U_\pm(R)^{-1}\,. \tag{4.7}$$

$\tilde{H}_\pm$ act in $\mathscr{F}_\pm(\mathscr{H})$ but, as shown in what follows, are different and should be distinguished from the free bosonic and fermionic hamiltonians $H_\pm = P_\pm d\Gamma(h) P_\pm$. In order to determine the domains of $\tilde{H}_\pm$, one can use the operator identity

$$U_\pm(R) P_R = P_\pm U_\pm(R)\,, \tag{4.8}$$

valid on $\mathscr{F}(\mathscr{H})$. Applying Eq. (4.8), one obtains the chain of equalities

$$U_\pm(R) D_R(h) = U_\pm(R) P_R D(h) = P_\pm U_\pm(R) D(h) = P_\pm D(h) = D_\pm(h)\,,$$

which show that $\tilde{H}_\pm$ are well defined on $D_\pm(h)$ – the domains of $H_\pm$.

The n-particle hamiltonian following from Eq. (4.6) reads

$$H_R^{(n)} = -\frac{1}{2}\, P_R^{(n)} \left(\sum_{k=1}^{n} \Delta_k \right) P_R^{(n)}\,, \tag{4.9}$$

where Δ_k operates on the k^{th} variable. Therefore

$$\tilde{H}_\pm^{(n)} = -\frac{1}{2}\, U_\pm^{(n)}(R) P_R^{(n)} \left(\sum_{k=1}^{n} \Delta_k \right) P_R^{(n)} U_\pm^{(n)}(R)^{-1}\,,$$

which according to Eq. (4.8) can be written in the form

$$\tilde{H}_\pm^{(n)} = -\frac{1}{2}\, P_\pm^{(n)} U_\pm^{(n)}(R) \left(\sum_{k=1}^{n} \Delta_k \right) U_\pm^{(n)}(R)^{-1} P_\pm^{(n)}. \tag{4.10}$$

This general result holds for $U_\pm^{(n)}(R)$ constructed (see Eq. (2.16)) in terms of any solution $T_\pm$ of Eqs. (2.13, 14). For obtaining more explicit expressions for $\tilde{H}_\pm^{(n)}$ one should fix some $T_\pm$. At this stage it is convenient to distinguish two cases; we consider $\tilde{H}_+^{(n)}$ if $k_r \in 2\mathbf{Z}$ and $\tilde{H}_-^{(n)}$ if $k_r \in 2\mathbf{Z}+1$. Then one can adopt for $T_\pm$ the simple expression (2.15). Moreover, without loss of generality one can assume that $r \in C^\infty(\mathbf{R}^s \times \mathbf{R}^s)$, because R is smooth. With these choices one easily derives

$$\tilde{H}_\pm^{(n)} = -\frac{1}{2} P_\pm^{(n)} \sum_{k=1}^n \left[\nabla_k + \frac{i}{2} \sum_{\substack{i,j=1 \\ i<j}}^n (\nabla_k r_{ij}) \right]^2 P_\pm^{(n)}, \tag{4.11}$$

where the short notation

$$r_{ij} = r(x_i, x_j)$$

has been introduced. The idea now is to move the projections $P_\pm^{(n)}$ in front of the r.h.s. of Eq. (4.11) to the right, taking into account at the end that $[P_\pm^{(n)}]^2 = P_\pm^{(n)}$. After some algebra one finds

$$\tilde{H}_\pm^{(n)} = \tilde{H}^{(n)} P_\pm^{(n)}, \tag{4.12}$$

where

$$\tilde{H}^{(n)} = -\frac{1}{2} \sum_{k=1}^n \Delta_k + V^{(n)}, \tag{4.13}$$

with

$$V^{(n)} = \frac{1}{24} \sum_{k=1}^n \left\{ 2 \sum_{\substack{i=1 \\ i\neq k}}^n (\nabla_k r_{ki})^2 + \left[\sum_{\substack{i=1 \\ i\neq k}}^n (\nabla_k r_{ki}) \right]^2 \right\}. \tag{4.14}$$

Observing that $V^{(n)} \geq 0$ and $V^{(n)} \in L^2(\mathbf{R}^{ns})_{\text{loc}}$, by Theorem X.28 of reference [36] one concludes that $\tilde{H}^{(n)}$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^{ns})$. Since $P_\pm^{(n)}$ commute with $\tilde{H}^{(n)}$, the same conclusion holds for $\tilde{H}_\pm^{(n)}$ on $D_\pm^n(h) = D_\pm(h) \cap \mathscr{H}_\pm^n$. Then, a standard argument (cf. Sect. VIII.10 of [35]) implies that $\tilde{H}_\pm$ are essentially self-adjoint on $D_\pm(h)$, which combined with Eq. (4.7) proves the validity of

Proposition 4. *H_R has a unique self-adjoint extension.*

It is useful for physical considerations to analyze the type of interactions corresponding to the potential $V^{(n)}$. One easily derives

$$V^{(n)} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^n V_{ij} + \frac{1}{6} \sum_{\substack{i,j,k=1 \\ i\neq j, i\neq k, j\neq k}}^n V_{ijk}. \tag{4.15}$$

Therefore $V^{(n)}$ involves non-trivial two-body

$$V_{ij} = \frac{1}{8} (\nabla_i r(x_i, x_j))^2 + \frac{1}{8} (\nabla_j r(x_j, x_i))^2 \tag{4.16}$$

and three-body interactions

$$V_{ijk} = \frac{1}{12} (\nabla_i r(x_i, x_j)) (\nabla_i r(x_i, x_k)) + \text{cyclic perm}. \tag{4.17}$$

Summarising, the unique self-adjoint extension of H_R defines a time evolution in $\mathscr{F}_R(\mathscr{H})$. By means of $U_\pm(R)$ this evolution can be equivalently transferred to $\mathscr{F}_\pm(\mathscr{H})$

and in the n-particle spaces $\mathscr{H}_\pm^n$ gives rise to the hamiltonian $\tilde{H}^{(n)}$. The explicit form of $\tilde{H}^{(n)}$ (see Eqs. (4.13, 15–17)) shows that due to the non-trivial exchange properties (encoded in the projection $P_R^{(n)}$ in Eq. (4.9)), the time evolution corresponding to the free one-particle hamiltonian (4.5) is in general a complicated dynamical problem for $n \geq 2$. In this respect the appearance of three-body interactions is worth stressing.

Along the above lines one can treat also the problem with external potential, namely

$$h = -\frac{1}{2}\,\Delta + W(x)\,. \tag{4.18}$$

The one-particle hamiltonian (4.18) gives rise to

$$\tilde{H}_\pm^{(n)} = -\frac{1}{2}\,P_\pm^{(n)}U_\pm^{(n)}(R)\left(\sum_{k=1}^{n}\Delta_k\right)U_\pm^{(n)}(R)^{-1}P_\pm^{(n)} + \sum_{k=1}^{n}W_k\,, \tag{4.19}$$

where $W_k = W(x_k)$.

In conclusion we consider as examples two particular exchange factors which lead to relatively simple hamiltonians. For factorizable exchange factors (2.5) one obtains from Eqs. (4.13,14)

$$\tilde{H}^{(n)} = -\frac{1}{2}\sum_{k=1}^{n}\Delta_k + \frac{1}{24}\,(n^2-1)\sum_{k=1}^{n}(\nabla p(x_k))^2\,. \tag{4.20}$$

Therefore, in this case the two- and three-body interactions give rise to a sort of effective external potential.

An exactly solvable example with non-factorizable exchange factor is the following one. Let Ω be an antisymmetric real $s \times s$ matrix. Take

$$r(x_1, x_2) = \Omega_{\mu\nu}x_1^\mu x_2^\nu + \pi k_r\,, \qquad \mu,\nu = 1,\ldots,s,\ k_r \in \mathbf{Z}\,, \tag{4.21}$$

where hereafter the summation over repeated upper and lower indices is always understood. Inserting (4.21) in (4.16, 17) one finds

$$\begin{aligned} V_{ij} &= -\frac{1}{8}\,(\Omega)^2_{\mu\nu}(x_i^\mu x_i^\nu + x_j^\mu x_j^\nu)\,,\\ V_{ijk} &= -\frac{1}{12}\,(\Omega)^2_{\mu\nu}(x_i^\mu x_j^\nu + x_j^\mu x_k^\nu + x_k^\mu x_i^\nu)\,. \end{aligned} \tag{4.22}$$

In this case the three-body potential is actually a superposition of two-body interactions and $\tilde{H}^{(n)}$ takes the form:

$$\tilde{H}^{(n)} = -\frac{1}{2}\sum_{k=1}^{n}\Delta_k - \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}(\Omega)^2_{\mu\nu}x_i^\mu x_j^\nu \tag{4.23}$$

with

$$M_{ij} = \begin{cases} \frac{1}{4}\,(n-1) & \text{if } i=j\,; \\ \frac{1}{12}\,(n-2) & \text{if } i \neq j \end{cases} \quad i,j = 1,\ldots,n\,. \tag{4.24}$$

Being symmetric, M can be diagonalized by an orthogonal matrix; the relative eigenvalues are:

$$m_1 = m_2 = \ldots = m_{n-1} = \frac{2n-1}{12}\,, \qquad m_n = \frac{n^2-1}{12}\,. \tag{4.25}$$

Consequently, the hamiltonian (4.23) describes sn harmonic oscillators with the frequencies $|\omega_\mu|\sqrt{m_i}$, where ω_μ are the eigenvalues of Ω.

5. The Leinaas-Myrheim Anyon

We already mentioned in the introduction that braid statistics are incorporated in the above general framework as particular cases. As an example we consider in this section the Leinaas-Myrheim (L–M) anyon field. In order to make contact with the L–M approach [30], we set $s = 2$ and consider the relation between two-anyon L–M wave functions $\psi(x_1, x_2)$ and two-boson wave functions $\tilde{\psi}(x_1, x_2) \in \mathscr{H}_+^2$. Neglect for a moment the center of mass coordinates and introduce in the relative space polar coordinates (ϱ, ϕ). Then, according to reference [30], one has

$$\tilde{\psi}(\varrho, \phi) = \exp(-i\vartheta\phi)\psi(\varrho, \phi)\,, \tag{5.1}$$

where ϑ is called *statistical* parameter.

Let us denote by $\arg(x; u) \in [-\pi, \pi)$ the oriented angle between x and an arbitrary but fixed unit vector u. Usually one takes $u = (1, 0)$. Restoring the coordinates x_1 and x_2, Eq. (5.1) reads

$$\tilde{\psi}(x_1, x_2) = T_+(x_1, x_2)\psi(x_1, x_2) \tag{5.2}$$

with

$$T_+(x_1, x_2) = \exp[-i\vartheta \arg(x_1 - x_2; u)]\,. \tag{5.3}$$

Now, using Eq. (2.13), one obtains for the exchange factor of the L–M anyon field

$$R(x_1, x_2) = T_+(x_2, x_1)T_+(x_1, x_2)^{-1} = \exp(-i\pi\vartheta \operatorname{sgn}[(x_1 - x_2)^\mu \tilde{u}_\mu])\,, \tag{5.4}$$

where "sgn" is the sign function and $\tilde{u}_\mu = \varepsilon_{\mu\nu}u^\nu$ is the vector dual to u. In deriving Eq. (5.4) we have used the identity

$$\arg(x; u) - \arg(-x; u) = -\pi \operatorname{sgn}(x^\mu \tilde{u}_\mu)\,. \tag{5.5}$$

The expression (5.4) defines a piecewise-constant admissible exchange factor which is already familiar [31, 32, 34]; in $2 + 1$ dimensional gauge theories it describes the exchange properties of charged fields localized on strings. So, one can apply our general procedure and reconstruct the L–M anyon fields $a(f; u)$ and $a^*(f; u)$ as operators acting in the Fock space $\mathscr{F}_R(L^2(\mathbf{R}^2))$. In particular, using Eq. (3.19) one obtains in explicit form all equal-time anyon correlation functions.

Let us consider now the time evolution of L–M anyons associated with the one-particle hamiltonian (4.18). Strictly speaking, one can not apply directly the results of the previous section, because R is discontinuous on

$$\gamma_{12} = \{x_1, x_2 \in \mathbf{R}^2 : (x_1 - x_2)^\mu \tilde{u}_\mu = 0\}\,. \tag{5.6}$$

In order to avoid this difficulty we introduce the space $\tilde{D}_+^n$ of C_0^∞-functions in $\mathscr{H}_+^n$, which vanish with all derivatives on the union of γ_{ij} with $1 \leq i < j \leq n$. Employing Eq. (5.3) for constructing $U_+^{(n)}(R)$, from Eq. (4.19) one derives the following n-particle anyon hamiltonian:

$$\tilde{H}_+^{(n)} = -\frac{1}{2}\sum_{k=1}^{n}\left\{\nabla_k + i\vartheta\sum_{\substack{i=1\\ i\neq k}}^{n}[\nabla_k \arg(x_k - x_i; u)]\right\}^2 + \sum_{k=1}^{n} W_k\,, \tag{5.7}$$

which is Hermitian on $\tilde{D}_+^n$. This domain is invariant under the conjugation operator C_+ (see Eq. (2.19)), which in the case under consideration takes the form

$$[C_+\varphi](x_1,\ldots,x_n) = \left\{ \exp\left[-i\vartheta \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \arg(x_i - x_j; u) \right] \right\} \bar{\varphi}(x_n,\ldots,x_1)\,. \qquad (5.8)$$

One can also verify that the hamiltonian (5.7) commutes with C_+ and therefore admits self-adjoint extensions. It is known [26, 38], that there is actually a whole family of such extensions. The physical meaning of this phenomenon has not been however fully clarified.

Introducing the potential

$$A_\mu(x_k; x_1,\ldots,\hat{x}_k,\ldots,x_n) = \varepsilon_{\mu\nu} \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{(x_k - x_i)^\nu}{(x_k - x_i)^2}\,,$$

Eq. (5.7) can be rewritten in the form

$$\tilde{H}_+^{(n)} = -\frac{1}{2} \sum_{k=1}^{n} [\nabla_k - i\vartheta A(x_k; x_1,\ldots,\hat{x}_k,\ldots,x_n)]^2 + \sum_{k=1}^{n} W_k\,. \qquad (5.9)$$

This is precisely the expression one usually encounters in the physical literature. Notice that the operator (5.9) is actually well defined on the symmetric C_0^∞-functions on $\mathbf{R}^{2n}\backslash\delta$, where δ is the subset of points $(x_1,\ldots,x_n)$ in which two or more coordinates x_i coincide. Recently, there is some interest in the spectral problem associated with the hamiltonian (5.9). The case $n = 2$ (with $W = 0$ and a harmonic potential in $|x_1 - x_2|$) is solved exactly in the L–M pioneering work [30]. Non-trivial three-body interactions appear for $n \geq 3$ and the derivation of an exact solution is problematic in that case. Some partial results [5, 8] and perturbative (in the parameter ϑ) computations [33] are however available. A rigorous analysis of the spectral problem on bounded domains in $\mathbf{R}^2$ has been performed in [2].

6. Multicomponent Fields

Along the same lines, though with slight modifications, one can treat the Fock realization of N-component fields satisfying Eq. (1.1). As one-particle Hilbert space we take

$$\mathscr{H} = \bigoplus_{\alpha=1}^{N} L^2(\mathbf{R}^s, d^s x)\,. \qquad (6.1)$$

The elements $f \in \mathscr{H}$ will be represented as columns with N components. The scalar product is

$$(f,g) = \int d^s x f^{\dagger\alpha}(x) g_\alpha(x) = \sum_{\alpha=1}^{N} \int d^s x \bar{f}_\alpha(x) g_\alpha(x)\,, \qquad (6.2)$$

where $\dagger$ stands for Hermitian conjugation. In this notation an n-particle wave function $\varphi \in \mathscr{H}^n$ is a column whose entries are $\varphi_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n)$.

The main ingredient in constructing $\mathscr{F}_R(\mathscr{H})$ is the exchange factor which for N-component fields is an $N^2 \times N^2$ matrix-valued function on $\mathbf{R}^s \times \mathbf{R}^s$. We assume that the entries of $R(x_1, x_2)$ are measurable functions and impose the following additional requirements:

$$R_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(x_1, x_2) R_{\gamma_1\gamma_2}^{\beta_1\beta_2}(x_2, x_1) = \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2}\,, \tag{6.3}$$

$$\bar{R}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(x_1, x_2) = R_{\beta_1\beta_2}^{\alpha_1\alpha_2}(x_2, x_1)\,, \tag{6.4}$$

$$\begin{aligned} &R_{\alpha_1\alpha_2}^{\gamma_1\gamma_2}(x_1, x_2) R_{\gamma_2\alpha_3}^{\gamma_3\beta_3}(x_1, x_3) R_{\gamma_1\gamma_3}^{\beta_1\beta_2}(x_2, x_3) \\ &\quad = R_{\alpha_2\alpha_3}^{\gamma_2\gamma_3}(x_2, x_3) R_{\alpha_1\gamma_2}^{\beta_1\gamma_1}(x_1, x_3) R_{\gamma_1\gamma_3}^{\beta_2\beta_3}(x_1, x_2)\,. \end{aligned} \tag{6.5}$$

The first two equations are the natural generalizations of Eqs. (2.1, 2) and imply that $R(x_1, x_2)$ is a unitary matrix. Equation (6.5) is the *spectral* quantum Yang-Baxter equation [3, 43], where $\mathbf{R}^s$ plays the role of spectral set. The counterpart of Eq. (6.5) for $N = 1$ is always satisfied and for this reason has not been mentioned in our previous discussion. On the contrary, for multi-component fields Eq. (6.5) is a crucial constraint which has its origin in the associativity of the operator algebra generated by a_α.

The system (6.3–5) admits non-trivial solutions, but it is a hard task to solve it in general. Since a complete description of all solutions is presently lacking, it is instructive to give some explicit examples. One particular solution, which can be interpreted as a generalization of Eq. (2.3), is

$$R(x_1, x_2) = \sum_{\alpha,\beta=1}^{N} \exp[i r_{\alpha\beta}(x_1, x_2)] E_{\alpha\beta} \otimes E_{\beta\alpha}\,, \tag{6.6}$$

where $E_{\alpha\beta}$ are the Weyl matrices and $r_{\alpha\beta}$ are real-valued functions obeying

$$r_{\alpha\beta}(x_1, x_2) + r_{\beta\alpha}(x_2, x_1) \in 2\pi\mathbf{Z}\,. \tag{6.7}$$

Another solution $R(x_1, x_2) = \mathscr{R}(x_1 - x_2)$ of Eqs. (6.3–5) for $x_1, x_2 \in \mathbf{R}^1$ is given by

$$\begin{aligned} \mathscr{R}(x) = &\sum_{\alpha=1}^{N} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + (\mathrm{e}^{ix} - \mathrm{e}^{-ix})(q\mathrm{e}^{ix} - q^{-1}\mathrm{e}^{-ix})^{-1} \\ &\times \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} E_{\alpha\beta} \otimes E_{\beta\alpha} + (q - q^{-1})(q\mathrm{e}^{ix} - q^{-1}\mathrm{e}^{-ix})^{-1} \\ &\times \left(\mathrm{e}^{ix} \sum_{\substack{\alpha,\beta=1 \\ \alpha>\beta}}^{N} E_{\alpha\alpha} \otimes E_{\beta\beta} + \mathrm{e}^{-ix} \sum_{\substack{\alpha,\beta=1 \\ \alpha<\beta}}^{N} E_{\alpha\alpha} \otimes E_{\beta\beta} \right), \end{aligned} \tag{6.8}$$

where $q \in \mathbf{R}^1$. The exchange matrix (6.8) stems from the quantum deformation [7, 23] of the affine Lie algebra $A_N^{(1)}$ and has an obvious generalization to $\mathbf{R}^s$ by replacing x with the scalar product $x^\mu u_\mu$, $u \in \mathbf{R}^s$ being an arbitrary but fixed (unit) vector. Notice also that for generic $q \in \mathbf{R}^1$ the entries of $\mathscr{R}(x)$ are smooth functions of x. Using the results of reference [23], it is not difficult to see that the quantum deformation of the remaining types of affine Lie algebras give rise to solutions of Eqs. (6.3–5) as well.

Given any admissible exchange matrix, we introduce for $n \geq 2$ the operators $\{S_i : i = 1, \ldots, n-1\}$ acting on $\mathscr{H}^n$ according to

$$\begin{aligned}[S_i\varphi]_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) \\ = [R_{i\,i+1}(x_i,x_{i+1})]^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n}\varphi_{\beta_1\ldots\beta_n}(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)\,,\end{aligned} \tag{6.9}$$

where

$$[R_{ij}(x_i,x_j)]^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n} = \delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2}\ldots\widehat{\delta^{\beta_i}_{\alpha_i}}\ldots\widehat{\delta^{\beta_j}_{\alpha_j}}\ldots\delta^{\beta_n}_{\alpha_n}R^{\beta_i\beta_j}_{\alpha_i\alpha_j}(x_i,x_j)\,. \tag{6.10}$$

Employing Eqs. (6.3–5), one can prove the N-component analog of Proposition 1, namely

Proposition 5. *$\{S_i : i = 1,\ldots,n-1\}$ given by Eq. (6.9) are bounded ($\|S_i\| = 1$) Hermitian operators on $\mathscr{H}^n$ and the mapping $S : \sigma_i \mapsto S_i$ is a representation of the permutation group $\mathscr{P}_n$ in $\mathscr{H}^n$.*

At this point, the projections $P^{(n)}_R$, the n-particle spaces $\mathscr{H}^{(n)}_R$ and the Fock space $\mathscr{F}_R(\mathscr{H})$ for multicomponent fields are introduced exactly as in Eqs. (2.8, 9, 11). The creation and annihilation operators are defined by Eq. (3.1); $a(f)$ and $a^*(f)$ satisfy Eq. (3.2) and act on $\mathscr{F}^0_R(\mathscr{H})$ as follows:

$$\begin{aligned}&[a(f)\varphi]^{(n)}_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n) \\ &\quad = \sqrt{n+1}\int d^sx f^{\dagger\alpha_0}(x)\varphi^{(n+1)}_{\alpha_0\alpha_1\ldots\alpha_n}(x,x_1,\ldots,x_n)\,,\end{aligned} \tag{6.11}$$

$$\begin{aligned}&[a^*(f)\varphi]^{(n)}_{\alpha_1\ldots\alpha_n}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n}}f_{\alpha_1}(x_1)\varphi^{(n-1)}_{\alpha_2\ldots\alpha_n}(x_2,\ldots,x_n) \\ &\quad + \frac{1}{\sqrt{n}}\sum_{k=2}^{n}[R_{k-1\,k}(x_{k-1},x_k)\ldots R_{1\,2}(x_1,x_k)]^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n} \\ &\quad \times f_{\beta_1}(x_k)\varphi^{(n-1)}_{\beta_2\ldots\beta_n}(x_1,\ldots,\hat{x}_k,\ldots,x_n)\,.\end{aligned} \tag{6.12}$$

The multicomponent counterpart of Proposition 2 reads

Proposition 6. *The operators $a^\sharp(f)$ defined by Eqs.* (6.11, 12) *satisfy* (3.10) *if and only if*

$$\int d^sy\int d^sz\, g^{\dagger\alpha}(y)h^{\dagger\beta}(z)R^{\gamma\delta}_{\alpha\beta}(y,z)g_\gamma(z)h_\delta(y) \leq 0 \tag{6.13}$$

for any g_α, $h_\alpha \in B_0(\mathbf{R}^s)$.

The proof of this statement is an obvious generalization of the argument implying the validity of Proposition 2.

Introducing the operator-valued distributions $a_\alpha(x)$ and $a^{*\alpha}(x)$ defined by

$$a(f) = \int d^sx f^{\dagger\alpha}(x)a_\alpha(x)\,, \qquad a^*(f) = \int d^sx f_\alpha(x)a^{*\alpha}(x)\,,$$

from Eqs. (6.11, 12) one gets:

$$a_\alpha(x_1)a_\beta(x_2) - R^{\delta\gamma}_{\beta\alpha}(x_2,x_1)a_\gamma(x_2)a_\delta(x_1) = 0\,, \tag{6.14}$$

$$a^{*\alpha}(x_1)a^{*\beta}(x_2) - R^{\alpha\beta}_{\gamma\delta}(x_2,x_1)a^{*\gamma}(x_2)a^{*\delta}(x_1) = 0\,, \tag{6.15}$$

$$a_\alpha(x_1)a^{*\beta}(x_2) - R^{\beta\delta}_{\alpha\gamma}(x_1,x_2)a^{*\gamma}(x_2)a_\delta(x_1) = \delta^\beta_\alpha\delta(x_1-x_2)\,. \tag{6.16}$$

The R-algebra (6.14–16) should not be confused with the so-called quon algebra [16]. The conditions (6.3–5) imply in fact that, apart from bosons and fermions, the set of R-fields and the set of quon fields do not intersect at all.

Applying the general formalism developed in this section to concrete exchange factors, one obtains explicit realization of the corresponding Fock representations. Take for example Eq. (6.6) with $N = 2$ and

$$r_{11}(x_1, x_2) = r_{22}(x_1, x_2) = r(x_1, x_2)\,,$$
$$-r_{12}(x_1, x_2) = r_{21}(x_2, x_1) = r(x_1, x_2)\,,$$

where r satisfies (2.4). Then the algebra (6.14–16) can be expressed entirely in terms of $R(x_1, x_2)$ given by Eq. (2.3) and as generators one can take the Wightman-type fields

$$\phi_1(f) = \frac{1}{\sqrt{2}}\,[a_1(\bar{f}) + a^{*2}(f)]\,, \qquad \phi_2(f) = \frac{i}{\sqrt{2}}\,[a^{*1}(f) - a_2(\bar{f})]\,, \tag{6.17}$$

$$\phi_1^*(f) = \frac{1}{\sqrt{2}}\,[a^{*1}(f) + a_2(\bar{f})]\,, \qquad \phi_2^*(f) = \frac{-i}{\sqrt{2}}\,[a_1(\bar{f}) - a^{*2}(f)]\,. \tag{6.18}$$

Differently from the Segal operator (3.17), the fields (6.17, 18) are by definition linear functionals in f. Assuming that $R(x_1, x_2)$ is continuous for $x_1 = x_2$, one derives the exchange relations

$$\phi_1(x_1)\phi_1(x_2) = R(x_2, x_1)\phi_1(x_2)\phi_1(x_1)\,,$$
$$\phi_1^*(x_1)\phi_1^*(x_2) = R(x_2, x_1)\phi_1^*(x_2)\phi_1^*(x_1)\,,$$
$$\phi_1(x_1)\phi_1^*(x_2) - R(x_1, x_2)\phi_1^*(x_2)\phi_1(x_1) = \begin{cases} 0 & \text{if } R(x,x) = 1\,, \\ \delta(x_1 - x_2) & \text{if } R(x,x) = -1\,, \end{cases}$$

and similar equations involving also $\phi_2(x)$ and $\phi_2^*(x)$. The resulting algebra generalizes the known equal-time commutation relations of boson ($R = 1$) and fermion ($R = -1$) Wightman fields.

Analogously, the matrix (6.8) leads to an explicit Fock representation for the quantized field associated with the quantum deformation of the affine Lie algebra $A_N^{(1)}$.

7. Conclusions

We have analysed in the present paper some aspects of generalized statistics in quantum field theory. In particular, we have demonstrated that a quantum field can be associated with any solution of the spectral Yang-Baxter equation (6.5), satisfying the supplementary conditions (6.3, 4). The field in question admits a Fock representation with positive metric. We have explicitly constructed the underlying Fock space $\mathscr{F}_R$ and have studied its main features. The set of all admissible exchange factors gives rise to whole family of fields with various statistics; piecewise-constant (constant) R-matrices lead to braid (permutation) group statistics, but the family in consideration involves also fields corresponding to more general space-dependent exchange factors. The physical interpretation and possible applications of the latter need further investigation. Another point, which also deserves a more detailed analysis is the implementation of space-time symmetries in the above framework.

It is worth mentioning that the explicit realization of the algebra (6.14–16) provides a new insight into the S-matrix theory of integrable quantum systems (see e.g. [37]).

In fact, replacing the coordinate $x \in \mathbf{R}^s$ by the rapidity, one can apply our technique for reconstructing the scattering states of $1 + 1$ dimensional integrable models from their S-matrix.

Finally, the Fock representations introduced in this article suggest some new and interesting areas of research. Among others, we have in mind the study of concrete hamiltonian systems on $\mathscr{F}_R$ and the associated quantum statistical mechanics. This subject is currently under investigation.

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