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The Semiclassical Limit for $SU(2)$ and $SO(3)$ Gauge Theory on the Torus

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Abstract: We prove that for $SU(2)$ and $SO(3)$ quantum gauge theory on a torus, holonomy expectation values with respect to the Yang-Mills measure $d\mu_T(\omega) = N_T^{-1}e^{-S_{YM}(\omega)/T}[\mathscr{D}\omega]$ converge, as $T \downarrow 0$, to integrals with respect to a symplectic volume measure μ_0 on the moduli space of flat connections on the bundle. These moduli spaces and the symplectic structures are described explicitly.

1. Introduction

In this paper we prove that for $SU(2)$ and $SO(3)$ quantum gauge theory on a torus, the holonomy ("Wilson loop") expectation values for the Yang-Mills measure $d\mu_T(\omega) = N_T^{-1}e^{-S_{YM}(\omega)/T}[\mathscr{D}\omega]$ (the notation is explained in Subsect. 2.5 below) converge, as $T \downarrow 0$, to integrals with respect to a symplectic volume measure μ_0 on the moduli space of flat connections on the bundle. We also show that for the non-trivial $SO(3)$–bundle over the torus, the moduli space of flat connections consists of just one point and the limiting measure exists and is thus, of course, just the unit mass on this point (a similar situation exists in genus 0, which is treated from a slightly different point of view in [Se 1]). The proofs are by direct computation using the expectation value formulas derived from a continuum quantum gauge theory in [Se 2, 3] and by lattice theory in a number of works including [Wi 1] (the work [Wi 2] also contains results of related interest), and the description of the symplectic form obtained in [KS 1].

The most significant result related to the present work is the corresponding result by Forman [Fo] for gauge theory on compact orientable surfaces of genus > 1. Forman's proof relies on results of Witten [Wi 1]; a more direct proof of part of Forman's result has been obtained by C. King and the author in [KS 2]. The main case we work with in this paper, genus 1 and gauge group $SU(2)$, is singular in two ways (thereby making the method used in [Fo, Wi, KS 2] inapplicable): (1) the "partition function" goes to ∞, as $T \downarrow 0$, and (2) no flat connection is irreducible. The situation over the torus is singular for other gauge groups as well, but the case

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$SU(2)$ (or $SO(3)$) may deserve this separate study because the symplectic structure on the space of flat connections is, in this case, very simple and thus a completely "hands-on" study is possible.

The techniques used in this paper rely on the geometry of $SU(2)$. The author hopes to address the case of a general gauge group in a future work.

2. The Limiting Expectation Values

2.1. The torus as a quotient of the disk. Let Σ be a torus equipped with a Riemannian metric, scaled so that the total area of Σ is 1. The area of $A \subset \Sigma$ will be denoted $|A|$. As a manifold, Σ may be obtained from the closed planar disk $D = \{(x, y) \in R^2 : x^2 + y^2 \leqq 1\}$, centered at $O = (0,0)$, as follows. Let $x_t = (\cos(2\pi t), \sin(2\pi t))$, for $t \in R$, and let K_i be the arc $[\frac{i-1}{4}, \frac{i}{4}] \to \partial D : t \mapsto x_t$. Thus the $K_i\,(i = 1,2,3,4)$ split ∂D into four congruent arcs. Identify K_1 with K_3 reversed, and K_2 with K_4 reversed, linearly; i.e. x_t is identified with $x_{t'}$ whenever $t - \frac{i-1}{4} = \frac{i+2}{4} - t'$ with $i \in \{1,2\}$, $t \in [\frac{i-1}{4}, \frac{i}{4}]$ and $t' \in [\frac{i+1}{4}, \frac{i+2}{4}]$. The quotient space is a torus which we take to be Σ, and we denote the quotient map by $q : D \to \Sigma$. The two loops $S_1 = q(x_0O) \cdot q(K_1) \cdot q(Ox_0)$ and $S_2 = q(x_0O) \cdot q(K_2) \cdot q(Ox_0)$, where Ox_0 is radial, generate the fundamental group $\pi_1(\Sigma, q(O))$, with $\overline{S_2}\,\overline{S_1}S_2S_1$ being homotopic to the constant curve at $q(O)$. For technical convenience, we equip Σ with the orientation induced by q from the standard orientation of $D \subset R^2$.

2.2. Triangulation of Σ, and Lassos. We will work with a fixed triangulation $\mathscr{T}$ of Σ. In order that we can apply some of the results of [Se 2, 3] it is necessary to put some restrictions, which we describe below, on $\mathscr{T}$. We will not make any significant overt use of these restrictions, and the reader may choose to proceed by viewing Theorem 2.10 as a "definition." In any case, every triangulation of Σ has a subdivision which is isomorphic to a subdivision of a triangulation which satisfies the restrictions. The requirement on the triangulation of Σ is that it be the projection by q of a triangulation of D which satisfies : (i) each oriented 1–simplex is either radial or "cross-radial" (i.e. intersects each radius at most once), (ii) each K_i is the composite of 1–simplices, and (iii) each 0–simplex is an endpoint of a 1–simplex which is part of a sequence of radial 1–simplices going from O to a point on ∂D (in particular, $q(O)$ is a 0–simplex of the triangulation). If Δ is an oriented 2–simplex of $\mathscr{T}$ then there is a radial path Ox in D whose projection by q is a path B, consisting of oriented 1–simplices, from $q(O)$ to a 0–simplex on $\partial\Delta$. The corresponding loop $\overline{B}.\partial\Delta.B$ will be called a *lasso*. We will let $\{\Delta_1, \ldots, \Delta_n\}$ be the positively oriented two-simplices of the triangulation $\mathscr{T}$ of Σ, and $\mathscr{E} = \{e_1, \overline{e}_1, \ldots, e_M, \overline{e}_M\}$ the set of oriented one-simplices of the triangulation (wherein $\overline{e}_i$ is e_i with the opposite orientation). We will often work with loops $C_1, \ldots, C_k$ which are composites of 1–simplices.

2.3. The groups G and $\overline{G}$, the Lie algebra g, and the metric $\langle\ \cdot\ ,\ \cdot\ \rangle_g$. All through this paper G will denote the group $SU(2)$, while $\overline{G}$ will denote either $SU(2)$ or $SO(3)$. Thus there is a (universal) covering map $G \to \overline{G}$ (which is the identity map if $\overline{G} = G$). The Lie algebra g of G will be equipped with a fixed Ad–invariant metric $\langle\ \cdot\ ,\ \cdot\ \rangle_g$. We will write I to denote both the identity element in G and the identity element in $\overline{G}$.

2.4. Principal $\overline{G}$-bundle. Let $\pi : P \to \Sigma$ be a principal $\overline{G}$–bundle over Σ. This is classified, relative to the fixed orientation on Σ, up to bundle equivalence by an element $h \in ker(G \to \overline{G})$ (one may take Eq. (3.1) of Sect. 3 as defining h). Here this kernel is either $\{I\}$ or $\{\pm I\}$ (I being the 2×2 identity matrix); if $h = I$ then the bundle is trivial, while if $h = -I$ (which is possible only when $\overline{G} = SO(3)$) the bundle is non-trivial.

2.5. The Yang-Mills measure on the space $\mathscr{C}$. Let $\mathscr{A}$ be the set of all connections on P and let $\mathscr{G}_m$ be the group of all bundle automorphisms of P, covering the identity map $\Sigma \to \Sigma$, and fixing the fiber $\pi^{-1}(m)$ pointwise, where $m = q(O)$. The group $\mathscr{G}_m$ acts on $\mathscr{A}$ by pullbacks of connections : $(\phi, \omega) \mapsto \phi^*\omega$. The Yang-Mills measure μ_T is a probability measure on a certain "completion" $\mathscr{C}$ of the quotient space $\mathscr{A}/\mathscr{G}_m$. The construction of μ_T is carried out in [Se 2, 3]. Formally, $d\mu_T(\omega) = N_T^{-1}e^{-S_{YM}(\omega)/T}[\mathscr{D}\omega]$, where S_{YM} is the Yang-Mills action functional, $[\mathscr{D}\omega]$ is the pushforward on $\mathscr{C}$ of "Lebesgue measure" on $\mathscr{A}$, and N_T is a "normalization constant."

2.6. Stochastic holonomy. Fix once and for all, a point u on the fiber $\pi^{-1}(m)$, where $m = q(O)$. If C is a piecewise smooth closed curve on Σ, based at $q(O)$, then corresponding to a connection ω on P there is associated the *holonomy* $g_u(C;\omega) \in \overline{G}$ of ω around C with initial point u. The holonomy $g_u(C;\omega)$ remains invariant if ω is replaced by $\phi^*\omega$, for any $\phi \in \mathscr{G}_m$. If C is a closed curve on Σ, based at $q(O)$, which is the composite of oriented 1–simplices of $\mathscr{T}$, then (as shown in [Se 2, 3]) there is a μ_T–almost-everywhere defined measurable function $\omega \mapsto g(C;\omega) \in \overline{G}$ on $\mathscr{C}$. This function may be called the "stochastic holonomy around C" and is defined by reinterpreting the classical equation of parallel transport as a stochastic differential equation. If $C_1, \ldots, C_k$ are k such closed curves, and f is a bounded measurable function on G^k, then the expectation value

$$\int_{\mathscr{C}} f(g_u(C_1;\omega), \ldots, g_u(C_k;\omega))\, d\mu_T(\omega)$$

is of interest. We state in Theorem 2.10 below an explicit formula for the expectation value. Part of the goal of this paper is the determination of the limit of this expectation value, for continuous f, as $T \downarrow 0$.

2.7. Notation ($b \mapsto x_b$). As before, let $\{\Delta_1, \ldots, \Delta_n\}$ be the positively oriented two-simplices of the triangulation $\mathscr{T}$ of Σ, and $\mathscr{E} = \{e_1, \overline{e}_1, \ldots, e_M, \overline{e}_M\}$ the set of oriented one-simplices of the triangulation. We shall use maps $\mathscr{E} \to G : b \mapsto x_b$ for which $x_{\overline{b}} = x_b^{-1}$, where $\overline{b}$ denotes the orientation-reverse of the oriented 1–simplex b. If C is a closed curve which is the composite $b_r \ldots b_1$ of oriented 1–simplices, then we write $x(C)$ to mean $x_{b_r} \ldots x_{b_1}$. Thus $x(\partial\Delta_i) \in G$, where $\partial\Delta_i$ is the boundary of Δ_i with some choice of initial point. We write $\overline{x}_b$ for the element of $\overline{G}$ covered by $x_b \in G$; so $\overline{x}(C) \in G$ is covered by $x(C) \in G$.

2.8. The Brownian Density (or "Heat Kernel") $Q_t(x)$. Let $Q_t(x)$ be the density, with respect to the Haar measure dx on G (of total mass 1), at time $t(> 0)$ of standard Brownian motion on G (governed by the Ad-invariant metric $\langle \cdot , \cdot \rangle_g$ on g).

2.9. Facts about $Q_t(x)$. $Q_t(x)$ is a multiple of the fundamental solution of the heat equation on G. It can be expressed as $Q_t(x) = \Sigma_{n=1}^{\infty} e^{-C_n t/2} n\chi_n(x)$, where χ_n is the character of the n-dimensional irreducible representation of G, and $C_n = (n^2 - 1)/\kappa^2$ (wherein κ, specifying the metric $\langle \cdot , \cdot \rangle_g$, is the length of the vector $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in g$), the series being absolutely convergent. For any $\delta > 0$, $\lim_{t \downarrow 0} \sup_{|x-I|>\delta} Q_t(x) = 0$, wherein $|x - I|$ denotes the distance between x and I. If f is a bounded measurable function on G and is continuous near I, then $\lim_{t\downarrow 0} \int_G f(x)Q_t(x)dx = f(I)$.

We shall use the following result from [Se 2, 3] (Eq. (5.5) in [Se 2], and the Introduction in [Se 3]) :

2.10. Theorem. ([Se 2, 3]). *Let $\{\Delta_1, \ldots, \Delta_n\}$ be the positively oriented two-simplices of the triangulation $\mathcal{T}$ of Σ, and $\mathcal{E} = \{e_1, \bar{e}_1, \ldots, e_M, \bar{e}_M\}$ the set of oriented one-simplices of the triangulation. Suppose that $C_1, \ldots, C_k$ are loops on Σ, all based at m (which, recall, is assumed to be a 0–simplex of the triangulation), which are composites of oriented 1–simplices. Then, for every bounded measurable function f on $\overline{G}^k$,:*

$$\int_{\mathscr{C}} f(g(C_1;\omega), \ldots, g(C_k;\omega))d\mu_T(\omega)$$

$$= \frac{1}{Z_T^h} \int_{G^M} f(\bar{x}(C_1), \ldots, \bar{x}(C_k))Q_{T|\Delta_1|}(hx(\partial\Delta_1))\prod_{i=2}^{n} Q_{T|\Delta_i|}(x(\partial\Delta_i))\, dx_{e_1} \ldots dx_{e_M} ,$$

where $|\Delta_i|$ is the area of Δ_i (measured by means of the metric on Σ), and

$$Z_T^h = \int_{G^2} Q_T(hb^{-1}a^{-1}ba)da.db. \quad \blacksquare$$

2.11. The variables y_{Δ_i} and the function F. Given $b \mapsto x_b$, as above, it is possible, with an appropriate indexing of the 2–simplices as $\Delta_1, \ldots, \Delta_n$, to introduce $y_{\Delta_i} \in G$, one for each oriented 2–simplex Δ_i, and $a, b \in G$, such that : (i) $y_{\Delta_i} = x(B)^{-1}x(\partial\Delta_i)x(B)$ for the lassos $\overline{B}.\partial\Delta_i.B$ (for appropriate choices of B), $i \in \{2, \ldots, n\}$, while for the remaining simplex Δ_1 : $y_{\Delta_1} = hx(B)^{-1}x(\Delta_1)x(B) = hb^{-1}a^{-1}ba(y_{\Delta_2} \cdots y_{\Delta_n})^{-1}$, (ii) $x(S_1) = a$, and $x(S_2) = b$, and (iii) each $x(C_i)$ is a product of suitable powers of the y_{Δ_i} and a, b. The latter actually follows from (i) and (ii) using the fact that every loop of the type C_i can be obtained from the composite of lassos and reversed lassos and the basic loops S_1 and S_2 (and their reverses) by dropping certain 1–simplices which are traversed in opposite directions consecutively. These facts are proven in [Se 2] (Lemmas A2 and A3). Thus there is a function F such that

$$f(\bar{x}(C_1), \ldots, \bar{x}(C_k)) = F(\{\bar{y}_\Delta\}, \bar{a}, \bar{b}) \tag{2.1}$$

wherein we have written $F(\{\bar{y}_\Delta\}, \bar{a}, \bar{b})$ to mean $F(\bar{y}_{\Delta_2}, \ldots, \bar{y}_{\Delta_n}, \bar{a}, \bar{b})$.

Thus $F(\{\bar{y}_\Delta\}, \bar{a}, \bar{b})$ is obtained from $f(\bar{x}(C_1), \ldots, \bar{x}(C_k))$ by writing each $\bar{x}(C_i)$ as a product of powers of the $\bar{y}_{\Delta_i}$ and $\bar{a}, \bar{b}$. In particular, if f is continuous then so is F. Moreover,

$$\int_{G^M} f(\bar{x}(C_1), \ldots, \bar{x}(C_k))Q_{T|\Delta_1|}(hx(\partial\Delta_1))\prod_{i=2}^{n} Q_{T|\Delta_i|}(x(\partial\Delta_i))\, dx_{e_1} \ldots dx_{e_M}$$

$$= \int F(\{\bar{y}_\Delta\}, \bar{a}, \bar{b})Q_{T|\Delta_1|}(hb^{-1}a^{-1}ba(y_{\Delta_2} \cdots y_{\Delta_n})^{-1})$$

$$\prod_{i=2}^{n} Q_{T|\Delta_i|}(y_{\Delta_i})\, dadb.dy_{\Delta_2} \ldots dy_{\Delta_n} \tag{2.2}$$

This is proven in (Lemma 8.5 of) [Se 3] (and in [Se 2] in the case $h = I$).

2.12. Remark. If ω is a connection on P, then by setting $\bar{y}_{\Delta_i} = g_u(\bar{B}.\partial\Delta_i.B;\omega)$ (notation as in Subsect. 2.11 above), and $a = g_u(S_1;\omega), b = g_u(S_2;\omega)$, we can take x_b to be the element of G describing ω–parallel transport along b (as measured by a suitable section of P over the bonds of the triangulation). Then $x(C_i) = g_u(C_i;\omega)$. In particular, if ω is flat, then Eq. (2.1) implies that :

$$f(g_u(C_1;\omega),\ldots,g_u(C_k;\omega)) = F(\{I\},g_u(S_1;\omega),g_u(S_2;\omega)) . \tag{2.1$'$}$$

2.13. Lemma. *For $T > 0$, we have :*

$$Z_T^h = \sum_{n=1}^{\infty} e^{-C_nT/2}\frac{\chi_n(h)}{n} , \tag{2.3}$$

where $C_n = (n^2-1)/\kappa^2$ (wherein $\kappa > 0$ specifies the metric on g, as in Facts 2.9), and $\frac{\chi_n(h)}{n}$ is 1 if $h = I$ (i.e. when P is trivial) and is $(-1)^{n+1}$ when $h = -I$ (i.e. when $G = SO(3)$ and the bundle is non-trivial). In particular, the series above is convergent. Moreover,

$$\lim_{T\downarrow 0} Z_T^h = \begin{cases} \infty & \text{if } h = I ; \\ \frac{1}{2} & \text{if} h = -I . \end{cases}$$

Proof. With χ_n as in Facts 2.9,

$$\int_G \chi_n(hy^{-1}x^{-1}yx)\,dxdy = \frac{1}{n^2}\chi_n(h) \tag{2.4}$$

(see Ex. 2.4.17.3 and Proposition 2.4.16 (iii) in [BrtD]), where the integration is with respect to Haar measure of total mass 1. Equation (2.4) is a consequence of Schur orthogonality (specifically Proposition 2.4.16 in [BrtD]). The expression (2.3) for Z_T^h follows upon integration term-by-term of the series for $Q_T(hy^{-1}x^{-1}yx)$ given in Facts 2.9. Term-by-term integration is valid by dominated convergence, since (see Facts 2.9),

$$\sum_{n=1}^{\infty} e^{-C_nT/2} n \int_G |\chi_n(hy^{-1}x^{-1}yx)|\,dxdy \leqq \sum_{n=1}^{\infty} e^{-C_nT/2} n.n = Q_T(I) < \infty .$$

Next, if $h = I$ then Eq. (2.3) gives $Z_T^I = \Sigma_{n=1}^{\infty} e^{-C_nT/2}$, and thus, by monotone convergence, $Z_T^I \to \infty$, as $T \downarrow 0$.

To evaluate $\lim_{T\downarrow 0} Z_T^{-I}$ we use the theta function identity :

$$1 + 2\sum_{n=1}^{\infty}\cos(2\pi nx)e^{-\pi n^2 t/2} = \sqrt{2}t^{-1/2}\sum_{m\in\boldsymbol{Z}} e^{-2\pi(x-m)^2/t} ,$$

wherein $\boldsymbol{Z}$ is the set of all integers. Setting $x = 1/2$ we obtain :

$$\sum_{n=1}^{\infty}(-1)^{n+1}e^{-\pi n^2 t/2} = \frac{1}{2} - \frac{1}{\sqrt{2}}t^{-1/2}\sum_{m\in\boldsymbol{Z}} e^{-2\pi(m-\frac{1}{2})^2/t}$$

The left side of this equation is $e^{-\pi t/2}Z_{\pi t\kappa^2}^{-I}$. Letting $t \downarrow 0$, we obtain $\lim_{T\downarrow 0} Z_T^{-I} = \frac{1}{2}$.

In Sect. 3 (following Theorem 3.15) we will calculate $\lim_{T\downarrow 0} Z_T^{-I} = \frac{1}{2}$ by a different method. ∎

2.14. Lemma. *Let $C_1,\ldots,C_k$ be a collection of loops as above, f a continuous function on G^k, and let F be the function described in Eq.* (2.1). *Then*:

$$\lim_{T\downarrow 0}\int_{\mathscr{C}} f(g(C_1;\omega),\ldots,g(C_k;\omega))\,d\mu_T(\omega)$$
$$=\lim_{T\downarrow 0}\frac{1}{Z_T^h}\int F(\{\overline{y}_\Delta = I\},\overline{a},\overline{b})Q_T(hb^{-1}a^{-1}ba)\,dadb\,,$$

provided the limit on the right-hand side exists.

Proof. By continuity, $F(\{\overline{y}_\Delta\},\overline{a},\overline{b})$ is uniformly close to $F(\{I\},\overline{a},\overline{b})$ when the y_{Δ_i} are close to I. Let $U_\delta = \{y_{\Delta_i} : |y_{\Delta_i} - I| > \delta, i = 2,\ldots,n\}$, for any $\delta > 0$, wherein $|y_{\Delta_i} - I|$ denotes the distance between y_{Δ_i} and I. In view of the expectation value formula given in Theorem 2.10 and Eq. (2.2), it will suffice to show that, for any $\delta > 0$,

$$\frac{1}{Z_T^h}\int_{U_\delta} Q_{T|\Delta_1|}(hb^{-1}a^{-1}ba(y_{\Delta_2}\ldots y_{\Delta_n})^{-1})\prod_{i=2}^{n} Q_{T|\Delta_i|}(y_{\Delta_i})\,dadb.dy_{\Delta_2}\ldots dy_{\Delta_n}$$

converges to 0, as $T\downarrow 0$.

The integral appearing above is dominated by

$$\prod_{i=2}^{n}\sup_{|y_i - I|>\delta} Q_{T|\Delta_i|}(y_i),$$

and by Facts 2.9, this goes to 0, as $T\downarrow 0$. The desired result now follows by taking into account Lemma 2.13. ■

2.15. Remark. In view of Theorem 2.10, Eq. (2.1) and (2.2), and Remark 2.12, we see that Lemma 2.14 says essentially that $\lim_{T\downarrow 0}\mu_T$, if it exists in a suitable weak sense, *lives on the space of flat connections* on P.

2.16. Remark. Lemma 2.14 also shows that the determination of the "weak limit" of μ_T, as $T\downarrow 0$, is essentially reduced to the determination of the limit

$$\lim_{T\downarrow 0}\frac{1}{Z_T^h}\int_{G^2} F(\overline{a},\overline{b})Q_T(haba^{-1}b^{-1})\,da.db\,,$$

for continuous functions F on $\overline{G}^2$.

2.17. Lemma. *If F is continuous on G^2, then*

$$\lim_{T\downarrow 0}\frac{1}{Z_T^I}\int_{G^2} F(a,b)Q_T(b^{-1}a^{-1}ba)\,dadb = \lim_{n\to\infty} n.\int_{G^2} F(a,b)\chi_n(b^{-1}a^{-1}ba)\,da.db\,,$$

provided the limit on the right-hand side exists (*as a finite number*).

Proof. This is an application of the L'Hopital rule. Using Facts 2.9, we have as in the proof of Lemma 2.13 :

$$\int_{G^2} F(a,b)Q_T(b^{-1}a^{-1}ba)\,dadb = \sum_{n=1}^{\infty} e^{-TC_n/2}\alpha_n\,,$$

where

$$\alpha_n = n. \int_{G^2} F(a,b)\chi_n(b^{-1}a^{-1}ba)\, da.db .$$

Taking F to be 1, we have, as in Lemma 2.13,

$$Z_T^I = \sum_{n=1}^{\infty} e^{-C_n T/2} .$$

Assume that $\alpha_n \to L$, a finite number, as $n \to \infty$. Let $\varepsilon > 0$, and choose N so that $|\alpha_n - L| < \varepsilon$ when $n \geqq N$. Then :

$$\begin{aligned}\frac{1}{Z_T^I} \int_{G^2} F(a,b)Q_T(b^{-1}a^{-1}ba)\, dadb =& \frac{\sum_{n<N} e^{-C_n T/2}\alpha_n}{Z_T^I} \\ &+ \frac{\sum_{n \geqq N} e^{-C_n T/2}\alpha_n}{\sum_{n \geqq N} e^{-C_n T/2}} \left\{ \frac{1 - \sum_{n<N} e^{-C_n T/2}}{Z_T^I} \right\} \\ =& O(T) + (L + \varepsilon')(1 - O(T)), \\ & \text{where } 0 < \varepsilon' < \varepsilon,\end{aligned}$$

where we have used Lemma 2.13 in the second equality.

So

$$\left|\frac{1}{Z_T^I} \int_{G^2} F(a,b)Q_T(b^{-1}a^{-1}ba)\, dadb - L\right| \leqq O(T) + L.O(T) + \varepsilon.(1 - O(T)) .$$

The required result follows upon taking lim sup as $T \downarrow 0$, since $\varepsilon > 0$ is arbitrary. ■

2.18. Notation (k_t, a_θ, K)/Some facts about $SU(2)$. Every element $g \in SU(2)$ can be written in the form :

$$g = k_\phi a_\theta k_\psi \tag{2.5a}$$

with $0 \leqq \phi < 2\pi$, $0 < \theta < \pi$, $0 \leqq \psi < 2\pi$ (these correspond to the "Euler angles" for $SO(3)$), where

$$k_t = k(e^{it}) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \tag{2.5b}$$

and

$$a_\theta = a(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} . \tag{2.5c}$$

We denote by K the maximal torus $\{k_t : t \in \boldsymbol{R}\}$ in G.

Moreover, the Haar measure dg on G of unit total mass can be expressed as :

$$dg = \frac{1}{8\pi^2} \sin\, \theta\, d\phi d\theta d\psi . \tag{2.5d}$$

To be precise, the map $(0,2\pi) \times (0,\pi) \times (0,2\pi) \to G : (\phi,\theta,\psi) \mapsto k_\phi a_\theta k_\psi$ is a two-to-one smooth local diffeomorphism onto a dense open subset of $SU(2)$, with

Jacobian $\frac{1}{4\pi^2}\sin\theta$ (the expression for dg has a further factor of $\frac{1}{2}$ because of the two-to-one nature of $(\phi,\theta,\psi)\mapsto k_\phi a_\theta k_\psi$).

If H is any bounded measurable function on $SU(2)$ which is central (in the sense that $H(gxg^{-1})=H(x)$ for every $x,g\in SU(2)$) then :

$$\int_{SU(2)} H(g)\,dg = \frac{2}{\pi}\int_0^\pi dt\ \sin^2 t\cdot H(k_t)\,. \tag{2.5e}$$

The reader may consult [BrtD] (1.5.20.6 and 2.5.2) or [Waw] for these facts. ∎

2.19. Proposition. *If F is any smooth function on $SU(2)\times SU(2)$, then :*

$$\lim_{n\to\infty} n\cdot \int_{SU(2)\times SU(2)} F(a,b)\chi_n(b^{-1}a^{-1}ba)\,da\,db = \int_{K\times K}\overline{F}(k,k')\,dk\,dk'\,, \tag{2.6}$$

where dk and dk' are the Haar measure of unit total mass on K, and $\overline{F}(a,b)$ is the "average" $\int_{SU(2)}F(gag^{-1},gbg^{-1})dg$.

Proof. Since the integral on the left in Eq. (2.6) is unaltered if F is replaced by $\overline{F}$, it will suffice to assume that F, in addition to being smooth, is a central function, i.e. that $F(gag^{-1},gbg^{-1})=F(a,b)$ for every $g\in SU(2)$. Denote by I_n the integral on the left of Eq. (2.6). Since $b\mapsto\int_{SU(2)}F(a,b)\chi_n(b^{-1}a^{-1}ba)\,da$ is central, we have by Facts 2.18 (specifically, Eqs. (2.5*e*, *a*, *d*)) :

$$nI_n = \frac{2n}{\pi}\int_0^\pi dt\ \sin^2 t\cdot \int_{[0,2\pi]\times[0,\pi]\times[0,2\pi]} \frac{d\phi\cdot\sin(\theta)d\theta\cdot d\psi}{8\pi^2} F(k_\phi a_\theta k_\psi,k_t)\chi_n(k_t a_\theta k_{-t}a_{-\theta})\,.$$

Now the character χ_n is given by :

$$\chi_n(g) = \frac{\sin[n\cdot\cos^{-1}\{\frac{1}{2}Tr(g)\}]}{\sin[\cos^{-1}\{\frac{1}{2}Tr(g)\}]}\quad\text{for every } g\in SU(2)\backslash\{\pm I\}\,,$$

where Tr denotes trace of a matrix, and the same range of $\cos^{-1}$ is used in numerator and denominator. Calculation shows that :

$$A(\theta,t)\stackrel{\text{def}}{=}\frac{1}{2}Tr(k_t a_\theta k_{-t}a_{-\theta}) = 1-2\left(\sin^2\frac{\theta}{2}\right)\cdot\sin^2 t\,.$$

From these formulas we have the following useful relationship (which is verified by working out the derivative in the right side):

$$n\cdot\sin^2(t)\cdot\sin\theta\cdot\chi_n(k_t a_\theta k_{-t}a_{-\theta}) = -\frac{\partial}{\partial\theta}\cos\left[n\cdot\cos^{-1}\{A(\theta,t)\}\right]\,.$$

In view of this, integration by parts yields:

$$nI_n = B.T. + \frac{2}{\pi}\int_0^\pi dt \int_{[0,2\pi]\times[0,\pi]\times[0,2\pi]}\frac{d\phi\cdot d\theta\cdot d\psi}{8\pi^2}\frac{\partial F(k_\phi a_\theta k_\psi,k_t)}{\partial\theta}$$

$$\cos\left[n\cdot\cos^{-1}A(\theta,t)\right]\,. \tag{2.7}$$

For the boundary term $B.T.$ we have:

$$
\begin{aligned}
B.T. &= -\frac{2}{\pi}\int_0^\pi dt \int_{[0,2\pi]^2} \frac{d\phi \cdot d\psi}{8\pi^2}\{F(k_\phi a_\pi k_\psi, k_t)\cos(2nt) - F(k_{\phi+\psi}, k_t)\} \\
&\to \frac{2}{\pi}\int_0^\pi dt \int_{[0,2\pi]^2} \frac{d\phi \cdot d\psi}{8\pi^2} F(k_{\phi+\psi}, k_t), \quad \text{as } n \to \infty, \text{(Riemann-Lebesgue lemma)} \\
&= \int_K dk \int_0^\pi \frac{dt}{\pi} F(k, k_t) \\
&= \int_{K\times K} F(k,k')dk \cdot dk' ,
\end{aligned}
$$

where in the last step we used the conjugation-invariance property of F and, for instance, $a_\pi k_t a_\pi^{-1} = k_{-t}$.

The rest of the argument is to show that the second term on the right in Eq. (2.7) goes to zero as $n \to \infty$. To this end it will suffice to show that the integral

$$
\int_0^\pi dt \cdot \int_0^{2\pi} d\theta \cdot H'(t,\theta) \cdot \cos[n \cdot \cos^{-1} A(\theta, t)]
$$

goes to 0 as $n \to \infty$, for every continuous function H' on $[0,\pi] \times [0,2\pi]$. The second factor in the integrand above is invariant under $t \mapsto \pi - t$ and $\theta \mapsto 2\pi - \theta$; thus we can reduce the integral to the form:

$$
J_n \stackrel{\text{def}}{=} \int_0^{\pi/2} dt \int_0^\pi d\theta\, H(t,\theta) \cdot \cos[n \cdot \cos^{-1} A(t,\theta)] \, .
$$

where H is continuous on $B \stackrel{\text{def}}{=} [0,\pi/2] \times [0,\pi]$. Thus, we wish to show that $J_n \to 0$, as $n \to \infty$. It will be convenient and sufficient to assume that H vanishes near the boundary of $[0,\pi/2] \times [0,\pi]$. (To see that it suffices to assume that H vanishes near ∂B, choose, for each integer $k \geqq 2$, a continuous function ϕ_k on B which equals 1 on $[\frac{1}{k}, \frac{\pi}{2} - \frac{1}{k}] \times [\frac{1}{k}, \pi - \frac{1}{k}]$ and decreases to 0 in a neighborhood of ∂B; then $|\int_B H \cos[n\cos^{-1} A(\cdot)] - \int_B H\phi_k \cos[n\cos^{-1} A(\cdot)]| \leqq \int_B |H|(1 - \phi_k) \leqq \|H\|_{\sup} \frac{3\pi}{k}$, and therefore $\limsup_{n\to\infty} |J_n| \leqq \|H\|_{\sup} \frac{3\pi}{k} + |\limsup_{n\to\infty} \int_B H\phi_k \cos[n\cos^{-1} A(\cdot)]|$. Observing that $H\phi_k$ vanishes near ∂B and letting $k \to \infty$, we conclude the argument.)

For $t \in [0, \frac{\pi}{2}]$ and $\theta \in [0,\pi]$, introduce $\xi \in [0, \frac{\theta}{2}] \subset [0, \frac{\pi}{2}]$ by:

$$
\sin \xi = \sin t \cdot \sin(\theta/2) \, .
$$

Then $(t,\theta) \mapsto (\xi,\theta)$ is a diffeomorphism of the open set $(0,\pi/2) \times (0,\pi)$ onto the open set $\{(\xi,\theta) : 0 < \xi < \frac{\theta}{2} < \frac{\pi}{2}\}$, and the inverse map is of the form $(\xi,\theta) \mapsto (t(\xi,\theta),\theta)$.

We have

$$
\begin{aligned}
\cos\xi\, d\xi \wedge d\theta &= d(\sin\xi) \wedge d\theta = \cos t \sin(\theta/2) dt \wedge d\theta \\
&= \sqrt{\sin^2(\theta/2) - \sin^2\xi}\, dt \wedge d\theta \, ,
\end{aligned}
$$

where the positive square-root is taken because $\cos t \geqq 0$ and $\sin \frac{\theta}{2} \geqq 0$, since $t, \frac{\theta}{2} \in [0,\pi/2]$.

Thus

$$J_n = \int_0^{\pi/2} d\xi \left\{ \int_{2\xi}^{\pi} d\theta\, \tilde{H}(\xi,\theta) \right\} \cdot \cos(2n\xi) ,$$

where $\tilde{H}$ is the continuous function on $\{(\xi,\theta) : 0 \leqq \xi \leqq \frac{\theta}{2} \leqq \frac{\pi}{2}\}$, zero near the boundary of the domain, and given in the interior by:

$$\tilde{H}(\xi,\theta) = \frac{\cos\xi \cdot H(t(\xi,\theta),\theta)}{\sqrt{\sin^2\frac{\theta}{2} - \sin^2\xi}} .$$

Thus $\xi \mapsto \int_{2\xi}^{\pi} \tilde{H}(\xi,\theta)d\theta$ is also continuous. Therefore, again by the Riemann-Lebesgue lemma, it follows that $\lim_{n\to\infty} J_n = 0$.

Putting all this together, we obtain Eq. (2.5). ■

Combining Lemma 2.17 and Proposition 2.19 we obtain (after uniformly approximating continuous F by smooth ones):

2.20. Proposition. *If F is continuous on G^2 then:*

$$\lim_{T\downarrow 0} \frac{1}{Z_T^I} \int_{G^2} F(a,b)Q_T(b^{-1}a^{-1}ba)dadb = \int_{K\times K} \overline{F}(k,k')dkdk' ,$$

where dk and dk' are the Haar measure of unit total mass on K, and $\overline{F}(a,b)$ is the 'average' $\int_{SU(2)} F(gag^{-1},gbg^{-1})dg$. (If $F(gag^{-1},gbg^{-1}) = F(a,b)$ for every $g \in SU(2)$, which is the case of main interest, then $\overline{F} = F$.)

3. The Symplectic Structure on Flat Connections

3.1. Tangent vectors to $\mathscr{A}$. Let $\mathscr{A}$ be the space of all connections on a principal $\overline{G}$–bundle $\pi : P \to \Sigma$. The space of tangent vectors to $\mathscr{A}$ is naturally identified as the space of $\underline{g}$–valued 1–forms A on P satisfying: (i) $A((R_g)_*X) = Ad(g^{-1})A(X)$ for every $g \in \overline{G}, p \in P$ (we have written $R_g p$ to denote the action of $g \in \overline{G}$ on $p \in P$ arising from the principal $\overline{G}$–bundle structure of P), and $X \in T_pP$, and (ii) $A(Y) = 0$ whenever $\pi_* Y = 0$.

Recall that $\underline{g}$ is equipped with an Ad–invariant metric $\langle \cdot , \cdot \rangle_{\underline{g}}$. If ξ and η are $\underline{g}$–valued 1–forms on a space we denote by $\langle \xi \wedge \eta \rangle$ the 2–form defined by:

$$\langle \xi \wedge \eta\rangle(X,Y) = \langle \xi(X), \eta(Y)\rangle_{\underline{g}} - \langle \xi(Y), \eta(X)\rangle_{\underline{g}} .$$

3.2. The symplectic form on $\mathscr{A}$. If $A^{(1)}$ and $A^{(2)}$ are tangent vectors to $\mathscr{A}$ then, as is easily verifiable, $\langle A^{(1)} \wedge A^{(2)}\rangle$ is π^* of a smooth 2–form on Σ; the integral of this 2–form over (the oriented surface) Σ will be denoted $\Theta(A^{(1)},A^{(2)})$. Thus Θ itself is a 2–form on $\mathscr{A}$. In fact, it is a symplectic form.

3.3. Notation Ω^ω (curvature) and $\mathscr{A}^0$ (flat connections). Recall that the curvature of ω is the $\underline{g}$–valued 2–form Ω^ω specified by $\Omega^\omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$, for every $X,Y \in T_pP$ and every $p \in P$. We denote by $\mathscr{A}^0$ the space of *flat connections* on P, i.e. those which have zero curvature.

3.4. Symplectic form on $\mathscr{A}^0$ and $\mathscr{C}^0$. Let $\mathscr{G}$ be the group of all bundle automorphisms of P which cover the identity on Σ, i.e. $\mathscr{G}$ consists of diffeomorphisms $\phi : P \to P$ for which $\phi(pg) = \phi(p)g$ for every $p \in P$ and $g \in \overline{G}$, and $\pi\phi = \pi$. Then $\mathscr{G}$ acts on $\mathscr{A}$ by pullbacks of connections : $(\phi, \omega) \mapsto \phi^*\omega$. The symplectic form Θ induces a "symplectic form" $\overline{\Theta}$ on the quotient space $\mathscr{C}^0 = \mathscr{A}^0/\mathscr{G}$ in the sense that if $A^{(1)}$ and $A^{(2)}$ are vectors in $\mathscr{A}$ tangent to $\mathscr{A}^0$ (i.e. they are tangent to paths in $\mathscr{A}^0$) then:

$$\Theta(A^{(1)}, A^{(2)}) = \Theta(\phi^*A^{(1)}, \phi^*A^{(2)})$$

for every $\phi \in \mathscr{G}$.

Let Σ be the torus, as before, and recall from Subsect. 2.1 the two basic loops S_1 and S_2 which generate the fundamental group $\pi_1(\Sigma, m)$ subject to the relation $\overline{S}_2\overline{S}_1S_2S_1 = 1$, in homotopy. Recall (Notation 2.4) that principal $\overline{G}$–bundles over Σ are classified by $h \in ker(G \to \overline{G})$.

3.5. Notation ($\mathscr{F}, \mathscr{F}'$, $\mathscr{F}_{sing}, \mathscr{F}'/\overline{G}$). Recall (Notation 2.3) that $G = SU(2)$, $\overline{G}$ is either $SU(2)$ or $SO(3) = SU(2)/\{\pm I\}$, and $G \to \overline{G} : x \mapsto \overline{x}$ is the covering map. Let:

$$\mathscr{F} = \{(\overline{a}, \overline{b}) \in \overline{G}^2 \; : \; b^{-1}a^{-1}ba = h^{-1}\} \, . \tag{3.1}$$

The group $\overline{G}$ acts on $\mathscr{F}$ by : $(g, (x, y)) \mapsto (gxg^{-1}, gyg^{-1})$.
Thus we have a quotient $p : \mathscr{F} \mapsto \mathscr{F}/\overline{G} : (a, b) \mapsto [a, b]$.
If $h = I$ (i.e. the bundle P is trivial) then we write

$$\mathscr{F} = \mathscr{F}' \cup \mathscr{F}_{sing} \, ,$$

where $\mathscr{F}_{sing} = \mathscr{F} \cap (\{\sqrt{I}\} \times \{\sqrt{I}\})$ (here $\{\sqrt{I}\} = \{x \in \overline{G} : x^2 = I\}$), and $\mathscr{F}' = \mathscr{F} \setminus \mathscr{F}_{sing}$. Thus if $\overline{G} = G = SU(2)$, then $\mathscr{F}_{sing} = Z(G) \times Z(G)$, while if $\overline{G} = SO(3)$, then $\mathscr{F}_{sing}$ is the union of $Z(\overline{G}) \times Z(\overline{G}) = \{(I, I)\}$ and the two-dimensional manifolds consisting of all points (I, gxg^{-1}), all points (gxg^{-1}, I), and all points (gxg^{-1}, gxg^{-1}), as g runs over $SO(3)$ and x is any fixed $180°$ degree rotation. Note that the $\mathscr{F}_{sing}/\overline{G}$ consists of a finite number of points. If $h = -I$, we set $\mathscr{F}' = \mathscr{F}$. ■

3.6. Fact/ Notation ($\Theta, \overline{\Theta}$). The map $\mathscr{A}^0/\mathscr{G}_m \to \mathscr{F} : [\omega] \mapsto (g_u(S_1; \omega), g_u(S_2; \omega))$ is a bijection (see 2.5 for $\mathscr{G}_m$), and induces a bijection:

$$\mathscr{C}^0 \to \mathscr{F}/\overline{G} : [\omega] \mapsto [g_u(S_1; \omega), g_u(S_2; \omega)] \, . \tag{3.2}$$

Using these bijections it is possible (as described in Theorem 3.7 below) to transfer the form Θ to a 2–form on $\mathscr{F}'$, which descends to a 2–form on $\mathscr{F}'/\overline{G}$. In general, $\mathscr{F}$ and $\mathscr{F}/\overline{G}$ are not naturally smooth manifolds and so care needs to be taken in working with "2–forms" on these spaces. For the case we are interested in, where Σ is the torus and $G = SU(2)$, we will describe the structure of $\mathscr{F}$ and $\mathscr{F}/\overline{G}$ and the corresponding 2–form explicitly.

The following description of Θ is a simple special case of Eq. (2.23) in [KS1]. (A group cohomological approach to the symplectic structure of the space of conjugacy classes of representations of $\pi_1(\Sigma)$ was developed in [Go] and, aside from questions of smoothness and bundle topology, this is equivalent to the symplectic structure on the moduli space of flat connections; the description of Θ given below should thus also be a consequence of the results, specifically Eq. (3.4), in [Go]).

3.7. Theorem [(KS 1]). *Let* $(\bar{a},\bar{b}) \in \mathscr{F}$, *and let* $A^{(1)},A^{(2)},B^{(1)},B^{(2)} \in g$ *be such that* $(\bar{a}A^{(1)},\bar{b}B^{(1)})$ *and* $(\bar{a}A^{(2)},\bar{b}B^{(2)})$ *are tangent to* $\mathscr{F}$ *at* $(\bar{a},\bar{b})$; *i.e. they are the initial tangent vectors of smooth paths in* $\overline{G}^2$ *lying on* $\mathscr{F}$ *and passing through* $(\bar{a},\bar{b})$ *initially. Then each of these paths corresponds to a path* $\epsilon \mapsto \omega_\epsilon$ *in* $\mathscr{A}^0$ *such that* $(\epsilon,x) \mapsto \omega_{\epsilon|_x}$ *is smooth (here* $x \in P$*). Moreover,* Θ *evaluated on the "tangent vectors" (i.e. the corresponding 1–forms* $\frac{\partial \omega_\epsilon}{\partial \epsilon}|_{\epsilon=0}$*) to these paths in* $\mathscr{A}^0$ *equals*

$$\Theta_{(\bar{a},\bar{b})}\left((A^{(1)},B^{(1)}),(A^{(2)},B^{(2)})\right) \stackrel{\text{def}}{=} \langle A^{(1)},B^{(2)}\rangle_{\underline{g}} - \langle A^{(2)},B^{(1)}\rangle_{\underline{g}} \,. \tag{3.3}$$

The following result describes the smooth structure of $\mathscr{F}'$ and $\mathscr{F}'/\overline{G}$. (See also Sect. 2.2 of [EMSS] and Sect. 5a of [AdPW] for a discussion of the moduli space $(L \times L)/W$ for simply connected gauge groups.)

3.8. Theorem. *Suppose* $h = I$ *(i.e. the bundle P is trivial). Let L be a maximal torus in* $\overline{G}$*, to be definite:* $L = \overline{K}$ *the corresponding Weyl group is* $W = N(L)/L = \{L,aL\}$*, where* $a \in N(L)\backslash L$. *Let* $L' = L\backslash\{\sqrt{I}\}$*, and let* $(L \times L)' = (L' \times L) \cup (L' \times L) = (L \times L) \cap \mathscr{F}'$.
We consider the action of W on
$(L \times L)' \times \overline{G}/L$ *specified by* $aL.(l_1,l_2,gL) = (al_1a^{-1},al_2a^{-1},ga^{-1}L)$; *the image of* (l_1,l_2,xL) *in the quotient* $\left((L \times L)' \times \overline{G}/L\right)/W$ *will be denoted* $[l_1,l_2,xL]_W$. *Then:*
(i) $\overline{G}$ *acts smoothly on* $\left((L \times L)' \times \overline{G}/L\right)/W$ *by* $g \cdot [l_1,l_2,xL]_W = [l_1,l_2,gxL]_W$;
(ii) $\mathscr{F}'$ *is a connected smooth 4–dimensional submanifold of* $\overline{G}^2$*, and the map* $\Psi : \left((L \times L)' \times \overline{G}/L\right)/W \to \mathscr{F}' : [l_1,l_2,gL]_W \mapsto (gl_1g^{-1},gl_2g^{-1})$ *is a well-defined* $\overline{G}$*–equivariant diffeomorphism;*
(iii) $p' : \left((L \times L)' \times \overline{G}/L\right)/W \to (L \times L)'/W : [l_1,l_2,gL]_W \mapsto [l_1,l_2]_W$ *is a smooth fiber bundle over the smooth manifold* $(L \times L)'/W$ *with fiber* $\overline{G}/L$ *and structure group W;*
(iv) *the map* $\psi : (L \times L)'/W \to \mathscr{F}'/\overline{G} : [l_1,l_2]_W \mapsto [l_1,l_2]$ *is a well-defined homeomorphism;*
(v) *the map* $p : \mathscr{F}' \to \mathscr{F}'/\overline{G}$ *is a smooth submersion if and only if* $\mathscr{F}'/\overline{G}$ *is equipped with the differentiable structure making* ψ *a diffeomorphism;*
(vi) $p\Psi = \psi p'$*, and thus, by transferring smooth bundle structure by means of* Ψ *and* ψ*,* $p : \mathscr{F}' \to \mathscr{F}'/\overline{G}$ *is a smooth fiber bundle with fiber* $\overline{G}/L$ *and structure group W, and* (Ψ,ψ) *is an isomorphism of such bundles.*

Proof. (i) The mapping $\overline{G} \times ((L \times L)' \times \overline{G}/L) \to (L \times L)' \times \overline{G}/L : (g,(l_1,l_2,xL)) \mapsto (l_1,l_2,gxL)$ is smooth and W–equivariant. Since $(L \times L)' \times \overline{G}/L$ is a cover of $((L \times L)' \times \overline{G}/L)/W$, the latter equipped with the smooth structure making the covering map a local diffeomorphism, it follows that the induced action of $\overline{G}$ on $((L \times L)' \times \overline{G}/L)/W$ is smooth.

(ii) The mapping Ψ_L: $L \times L \times \overline{G}/L \to \overline{G}^2 : (l_1,l_2,gL) \mapsto (gl_1g^{-1},gl_2g^{-1})$ is a well-defined $\overline{G}$–equivariant smooth map with image $\mathscr{F}$, and maps $(L \times L)' \times \overline{G}/L$ onto $\mathscr{F}'$. Its derivative at $(l_1,l_2,gL) \in L \times L \times \overline{G}/L$ is specified by the map (wherein $\underline{l}$ is the Lie algebra of L)

$$\begin{aligned} &\underline{l} \times \underline{l} \times \underline{l}^{\perp} \to \Psi_L(l_1,l_2,gL)^{-1} T_{\Psi_L(l_1,l_2,gL)}\overline{G}^2 \\ &(A,B,X) \mapsto \left(Ad(g)\{A + (Ad(l_1^{-1}) - 1)X\}, Ad(g)\{B + (Ad(l_2^{-1}) - 1)X\}\right). \end{aligned}$$

Examination of this shows that Ψ_L on $(L \times L)' \times \overline{G}/L$ is an immersion.

We show that Ψ_L restricted to $(L \times L)' \times \overline{G}/L \to \mathscr{F}'$ is a closed map. Let C be a relatively closed subset of $(L \times L)' \times \overline{G}/L$, and let $\overline{C}$ be its closure in $L \times L \times \overline{G}/L$. Then $\overline{C} \cap ((L \times L)' \times \overline{G}/L) = C$. Moreover, $\Psi_L^{-1}(\mathscr{F}_{sing}) = \{(L \times L)\backslash(L \times L)'\} \times \overline{G}/L$. It follows that $\Psi_L(C) = \Psi_L(\overline{C}) \cap \mathscr{F}'$. Thus since $\Psi_L(\overline{C})$ is closed (being compact) in $\overline{G}^2$, $\Psi_L(C)$ is closed in $\mathscr{F}'$.

The map $\Psi : ((L \times L)' \times \overline{G}/L)/W \to \mathscr{F}'$ is a continuous bijection. Since Ψ_L takes relatively closed subsets of $(L \times L)' \times \overline{G}/L$ into relatively closed subsets of $\mathscr{F}'$, Ψ is also a closed map. Therefore Ψ is a homeomorphism. Since Ψ_L is an immersion, it follows that Ψ is smooth and is an immersion. Since Ψ is a homeomorphism onto its image $\mathscr{F}'$, and Ψ is also an immersion, $\mathscr{F}'$ is a submanifold of $\overline{G}^2$ and Ψ is a diffeomorphism. From $\dim((L \times L)' \times \overline{G}/L) = 4$, we conclude that $\dim \mathscr{F}' = 4$. Since Ψ_L is $\overline{G}$–equivariant, so is Ψ.

(iii) The smooth manifold $(L \times L)'$ is a two-fold cover of $(L \times L)'/W$, and so the latter has the smooth manifold structure induced from that of $(L \times L)'$. If $(l_1, l_2) \in (L \times L)'$ we can choose a neighborhood U of $[l_1, l_2]_W$ in $(L \times L)'/W$ which is covered once by a neighborhood $\tilde{U}$ of $(l_1, l_2) \in (L \times L)'$. Consider $f : U \times \overline{G}/L \to ((L \times L)' \times \overline{G}/L)/W : ([j_1, j_2]_W, xL) \mapsto [j_1', j_2', xL]_W$, where $(j_1', j_2') \in \tilde{U}$ covers $(j_1, j_2) \in U$. The map f is a smooth $\overline{G}$–equivariant mapping onto $(p')^{-1}(U)$. Moreover, f^{-1} is also smooth since it is obtained by means of the smooth map $(L \times L)' \times \overline{G}/L \to (L \times L)'/W : (l_1, l_2, xL) \mapsto [l_1, l_2]_W$. Taking the maps f as local trivializations, p' becomes a smooth bundle with fiber $\overline{G}/L$. Now W acts on $\overline{G}/L$ by : $(aL, xL) \mapsto xa^{-1}L$. If f_1 and f_2 are local trivializations of the type f, defined on the same domain, then $f_2^{-1} f_1$ is smooth and $f_2^{-1} f_1([l_1, l_2]_W, xL)$ equals either itself or $([l_1, l_2]_W, xaL)$. Thus the bundle specified by p' has structure group W.

(iv) It is clear that ψ is well-defined, and that $\psi p' = p\Psi$. Since p is open, and since p' has continuous local sections, it follows that ψ is continuous. We show now that ψ is injective. Suppose $[l_1, l_2] = [l_3, l_4]$ in $\mathscr{F}/\overline{G}$. Then there is a $g \in \overline{G}$, for which $gl_1g^{-1} = l_3$ and $gl_2g^{-1} = l_4$. Since l_1 or l_2 is in L', it follows from the structure of the elements of $L = \overline{K}$, that either $g \in L$ or that $g = \overline{h}$ for some $h \in G$ of the form $\begin{pmatrix} 0 & \beta \\ -\overline{\beta} & 0 \end{pmatrix}$ with $|\beta| = 1$. In either case, $g \in N(L)$. Therefore, $[l_1, l_2]_W = [l_3, l_4]_W$. So ψ is one-to-one. Since Ψ is closed and p is closed (this is a general fact about quotients of Hausdorff spaces by compact groups) it follows that ψ is closed. Therefore ψ is a homeomorphism.

(v) If p is a smooth submersion, thereby having smooth local sections p_{loc}^{-1}, the map ψ^{-1}, being expressible locally as $p'\Psi^{-1}p_{loc}^{-1}$, is also smooth. Conversely, if ψ is a diffeomorphism then $p = \psi p' \Psi^{-1}$ is a smooth submersion.

(vi) It is readily verified that $p\Psi = \psi p'$. ■

3.9. Remark. The bundle $p' : ((L \times L)' \times \overline{G}/L)/W \to (L \times L)'/W$ is, as is seen directly from the definition, the bundle with fiber $\overline{G}/L$ associated to the principal W–bundle $(L \times L)' \to (L \times L)'/W : (l_1, l_2) \mapsto [l_1, l_2]_W$; for this, note that W acts on the left on $\overline{G}/L$ by : $yL \cdot xL = xy^{-1}L$ (for every $yL \in W = N(L)/L$, and $xL \in \overline{G}/L$). Since $(L \times L)' \to (L \times L)'/W$ is non-trivial (because $(L \times L)'$ is connected and W is discrete and has more than one element), it follows that p' is non-trivial. Hence *the bundle $p : \mathscr{F}' \to \mathscr{F}'/\overline{G}$ is non-trivial*, and is a fiber bundle, with fiber $\overline{G}/L$, associated to the principal W–bundle $(L \times L)' \to (L \times L)'/W$. ■

3.10. Proposition. *Let $(K \times K)' = \{(x, y) \in K \times K : (\overline{x}, \overline{y}) \in (L \times L)'\}$, and let ρ denote the composite of the covering projection $(K \times K)' \to (L \times L)'$ with the* *covering projection $(L \times L)' \to (L \times L)'/W$. Recall from Theorem 3.8(iv) the diffeomorphism $\psi : (L \times L)'/W \to \mathscr{F}'/\overline{G}$. Then the form $(\psi\rho)^*\overline{\Theta}$ on $(K \times K)'$ is given by*:

$$(\psi\rho)^*\overline{\Theta}\Big((X_1, Y_1), (X_2, Y_2)\Big) = \langle X_1, Y_2\rangle_{\underline{g}} - \langle X_2, Y_1\rangle_{\underline{g}} .$$

In particular, the corresponding volume measure $|(\psi\rho)^*\overline{\Theta}|$ is the restriction to $(K \times K)'$ of the Riemannian measure on $K \times K$ induced by the (restriction to $K \times K$ of the) metric $\langle \cdot , \cdot \rangle_{\underline{g}}$. Thus, if f is a bounded measurable function on $K \times K$ such that $f(wx, wy) = f(x, y)$, for all $(w, x, y) \in W_K \times K^2$, wherein W_K is the Weyl group of K, then:

$$\int_{(K\times K)'} f\, d|(\psi\rho)^*\overline{\Theta}| = |K|^2 \int_{K\times K} f(k, k')\, dk dk' ,$$

where dk and dk' are the Haar measure on K of unit total mass, and $|K|$ is the "volume" of K as measured by the restriction of the metric $\langle \cdot , \cdot \rangle_{\underline{g}}$ to K.

Proof. The expression for $(\psi\rho)^*\overline{\Theta}$ follows from Theorem 3.7 and the definitions of ψ and ρ. ■

3.11. Lemma. *If f is a continuous function on $\overline{G} \times \overline{G}$ which is invariant under the conjugation action of $\overline{G}$, then*:

$$\lim_{T\downarrow 0} \frac{1}{Z_T^I} \int_{G\times G} f(\overline{x}, \overline{y}) Q_T(y^{-1}x^{-1}yx)\, dx dy = \frac{1}{\mathrm{vol}_{\overline{\Theta}}(\mathscr{F}'/\overline{G})} \int_{\mathscr{F}'/\overline{G}} \tilde{f}\, d|\overline{\Theta}| ,$$

where on the right we have written $\tilde{f}$ to denote the function on $\mathscr{F}'/\overline{G}$ (corresponding to $h = I$ in Notation 3.5, 3.6) induced by the conjugation-invariant function f, $|\overline{\Theta}|$ is the measure corresponding to the "volume form" $\overline{\Theta}$, and (with $|K|$ the "volume" of K as measured by the metric induced from G):

$$\mathrm{vol}_{\overline{\Theta}}\left(\mathscr{F}'/\overline{G}\right) = \begin{cases} \frac{1}{2}|K|^2 & \text{if } \overline{G} = SU(2) ; \\ \frac{1}{8}|K|^2, & \text{if } \overline{G} = SO(3) . \end{cases}$$

Proof. This follows from Proposition 3.10 and Proposition 2.20, and the observation that $\psi\rho$ is a two-to-one cover of $\mathscr{F}'/\overline{G}$ in case $\overline{G} = G = SU(2)$, while if $\overline{G} = SO(3)$ then $\psi\rho$ is an eight-to-one cover. The factor $\mathrm{vol}_{\overline{\Theta}}(\mathscr{F}'/\overline{G})$ is the volume of $\mathscr{F}'/\overline{G}$ as measured by the symplectic form $\overline{\Theta}$. ■

3.12. Theorem. *Let μ_T denote the* YM *measure, described formally by $d\mu_T(\omega) = \frac{1}{Z_T} e^{-S_{YM}(\omega)/T}[\mathscr{D}\omega]$, on the "moduli space" $\mathscr{C}$ of (generalized) connections on a principal SU(2)-bundle or the trivial SO(3)-bundle, over the torus Σ. Let $C_1, \dots, C_k$ be loops on Σ (which are composites of 1-simplices of a triangulation as in Subsect.* 2.2*). Then, for any continuous function f on $\overline{G}^k$, invariant under the conjugation action of $\overline{G}$,*:

$$\lim_{T\downarrow 0} \int_{\mathscr{C}} f(g(C_1; \omega), \dots, g(C_k; \omega))\, d\mu_T(\omega)$$

$$= \frac{1}{\mathrm{vol}_{\overline{\Theta}}\left(\mathscr{F}'/\overline{G}\right)} \int_{\mathscr{F}'/\overline{G}} f(g(C_1; \omega), \dots , g(C_k; \omega))\, d|\overline{\Theta}|(\omega) ,$$

wherein $\overline{G}$ is $SU(2)$ or $SO(3)$, as before, and we have identified the space of flat connections with $\mathscr{F}$.

Proof. This follows directly from Lemma 2.14, Lemma 3.11, and Eq. (2.1′) in Remark 2.12. ■

3.13. Lemma. *Suppose $a, b \in SU(2)$, and $b^{-1}a^{-1}ba = -I$. Then there is a $g \in SU(2)$ with*

$$gag^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and

$$gbg^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

An element $x \in SU(2)$ commutes with both a and b if and only if $x \in \{\pm I\}$.

Proof. Without loss of generality, assume that $a = k_\phi$, with $\phi \in [0, \pi]$. Then, calculating $b^{-1}a^{-1}ba$, we see by direct computation that this commutator equals $-I$ if and only if b is of the form $\begin{pmatrix} 0 & e^{i\psi} \\ -e^{i\psi} & 0 \end{pmatrix}$ and ϕ is $\frac{\pi}{2}$. Now let $g = k_{\psi/2}^{-1}$. Then $gag^{-1} = a$, and, as calculation shows, $gbg^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

If $x \in G$ commutes with $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ then $x \in K$, i.e. $x = \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix}$, for some α with $|\alpha| = 1$. Direct calculation shows that if this x commutes with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $\alpha^2 = 1$, and so $x = \pm I$. ■

Combining Fact 3.5 and Lemma 3.13 we obtain:

Proposition 3.14. *There is, up to bundle automorphism, exactly one flat connection on the non-trivial SO(3)–bundle over the torus.* ■

3.15. Theorem. *Let μ_T be the Yang-Mills measure for the non-trivial $SO(3)$–bundle over Σ. Then:*

$$\lim_{T \downarrow 0} \frac{1}{Z_T^{-I}} \int_{\mathscr{C}} f\left(g(C_1;\omega), \ldots, g(C_k;\omega)\right) d\mu_T(\omega) = f\left(g(C_1;\omega^0), \ldots, g(C_k;\omega^0)\right)$$

for any continuous $SO(3)$–invariant function f on $SO(3)^k$ (i.e. $f(gx_1g^{-1}, \ldots, gx_kg^{-1}) = f(x_1, \ldots, x_k)$ for every $g, x_1, \ldots, x_k \in SO(3)$) and loops $C_1, \ldots, C_k$ on Σ (satisfying the conditions of 2.2), and ω^0 is "the" flat connection on the bundle.

Proof. As in the proof of Theorem 3.12, it will suffice to show that for every continuous $SU(2)$–invariant function F on $SU(2) \times SU(2)$:

$$\lim_{T \downarrow 0} \frac{1}{Z_T^{-I}} \int_{SU(2)\times SU(2)} F(x,y) Q_T(y^{-1}x^{-1}yxh) dx dy = F(a,b) , \tag{3.4}$$

wherein $h = -I$, and $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let $M : G \times G \to G : (x,y) \mapsto y^{-1}x^{-1}yx$. We write $M'_{(x,y)} : \underline{g} \times \underline{g} \to \underline{g}$ for the "translated derivative" of M at $(x,y) \in G^2$; i.e.

$$M'_{(x,y)}(X, Y) \stackrel{\text{def}}{=} M(x,y)^{-1} dM_{(x,y)}(xX, yY) .$$

Thus:

$$M'_{(x,y)}(X,Y) = \{1 - Ad(x^{-1}y^{-1}x)\}X + \{Ad(x^{-1}) - Ad(x^{-1}y^{-1}xy)\}Y\,. \quad (3.5)$$

So if $(x,y) \in M^{-1}(z)$, wherein $z \neq I$, then $M'_{(x,y)}$ is surjective; for, as some algebraic manipulation shows, any Z orthogonal to the image of $M'_{(x,y)}$ satisfies both $Ad(x)Z = Z$ and $Ad(y)Z = Z$, which, since x and y do not commute, imply that Z is zero (recall that we are working in the Lie algebra of $SU(2)$). Therefore, for every $z \in SU(2)\backslash\{I\}, M^{-1}(z)$ is a smooth 3–dimensional closed submanifold of $SU(2) \times SU(2)$.

For the sake of computational convenience it will also be convenient to assume that the metric $\langle \cdot , \cdot \rangle_{\underline{g}}$ on $\underline{g}$ is scaled so that G has unit total volume; in this case, the Haar measure dx is the same as the Riemannian volume measure (this rescaling does not alter either side of Eq. (3.4)). We claim that:

$$\int_{G^2} F(x,y)Q_T(y^{-1}x^{-1}yxh)dxdy = \int_{G\backslash\{I\}} dz \cdot Q_T(zh)\left[\int_{M^{-1}(z)} \frac{F(x,y)}{J(x,y)}dv_z(x,y)\right], \quad (3.6)$$

where dv_z is the Riemannian volume measure on $M^{-1}(z)$ corresponding to the metric induced by means of $\langle \cdot , \cdot \rangle_{\underline{g}}$, and $J(x,y)$ is the Jacobian factor $|det\{M'_{(x,y)}(M'_{(x,y)})^*\}|^{1/2}$. As we have seen above, $J > 0$ off $M^{-1}(I)$. Since $M^{-1}(I)$ has measure zero (by Fubini's theorem, since for every $x \in G\backslash Z(G)$, the set $\{y \in G : M(x,y) = I\}$, being a two-dimensional torus in G, has Haar measure zero) we can use a monotone limit argument to see that it suffices to prove (3.6) under the assumption that F vanishes in a neighborhood N of $M^{-1}(I)$ (actually this, in essence, is the only case we really need for our purposes). Consider any $(x,y) \in G^2\backslash N$, and let $M(x,y) = z \neq I$; then by the inverse function theorem, there is a neighborhood $W(\subset G^2\backslash N)$ of (x,y) in G^2 and a diffeomorphism $\Phi : W \to V \times U$, where V is a neighborhood of (x,y) in $M^{-1}(z)$ and U is a neighborhood of z in G such that Φ, on W, is of the form $(*, M(\cdot , \cdot))$. To prove Eq. (3.6) it will be sufficient (by a partition of unity argument) to assume that F has support in W. However, for F supported in W, Eq. (3.6) is just a "change-of-variable" formula, and $\int_{M^{-1}(z)} \frac{F(x,y)}{J(x,y)}dv_z(x,y)$ depends continuously on $z \in G\backslash\{I\}$. Thus (3.6) is proved for all continuous F, and the integrand $\int_{M^{-1}(z)} \frac{F(x,y)}{J(x,y)}dv_z(x,y)$ depends continuously on $z \in G\backslash\{I\}$.

Therefore, Eq. (3.6) and Facts 2.9 imply (by splitting the integral on the left below into a sum of two integrals, one outside a small neighborhood of $M^{-1}(I)$ and another, contributing zero in the limit, over the small neighborhood):

$$\lim_{T\downarrow 0} \int_{G^2} F(x,y)Q_T(y^{-1}x^{-1}yxh)dxdy = \int_{M^{-1}(h^{-1})} \frac{F(x,y)}{J(x,y)}dv_{h^{-1}}(x,y) \quad (3.7)$$

wherein $h = -I$. Setting $F = 1$, we see again that $\lim_{T\downarrow 0} Z_T^{-1}$ exists and is finite; dividing both sides of Eq. (3.7) by this we obtain Eq. (3.4) by means of Lemma 3.13. ∎

Another proof of the second part of Lemma 2.13. The aim is to calculate $\lim_{T\downarrow 0} Z_T^{-1}$. We will use the notation and conventions introduced in the proof of Theorem 3.15 above. By Lemma 3.13 and the observation that $J(x,y) = J(gxg^{-1}, gyg^{-1})$ for every $(g,x,y) \in G^3$, we see that the right-hand side of Eq. (3.7) equals $F(a,b)J(a,b)^{-1}\int_{M^{-1}(-I)} dv_{-I}(x,y)$. Thus

$$\lim_{T\downarrow 0}\int_{G^2} Q_T(y^{-1}x^{-1}yxh)dxdy = \frac{1}{J(a,b)}\mathrm{vol}\Big(M^{-1}(-I)\Big),$$

where the volume $\mathrm{vol}\Big(M^{-1}(-I)\Big)$ is with respect to dv_{-I}. By Lemma 3.13, for $(a,b)\in M^{-1}(-I)$, the smooth map $\Psi : G\to M^{-1}(-I) : x\mapsto(xax^{-1}, xbx^{-1})$ is surjective and two-to-one. Let $\Psi'_x : \underline{g}\to\underline{g}\times\underline{g} : X\mapsto\Psi(x)^{-1}d\Psi_x(xX)$. Then calculation shows that $(\Psi'_x)^*(\Psi'_x) = M'_{(a,b)}(M'_{(a,b)})^*$ (both have the same diagonal matrix, with diagonal entries $(4,4,8)$, relative to the basis $\left\{\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}\right\}$ of $\underline{g}$). Therefore, $\mathrm{vol}\Big(M^{-1}(-I)\Big)$ equals $\frac{1}{2}J(a,b)$. Therefore,

$$\lim_{T\downarrow 0}\int_{G^2} Q_T(y^{-1}x^{-1}yxh)dxdy = \frac{1}{2}\,. \quad \blacksquare$$

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