Commun. Math. Phys. 169, 121–180 (1995)

Communications in Mathematical Physics

Mass Generation in the Large N Gross–Neveu-Model

C. Kopper[1], J. Magnen[2], V. Rivasseau[2]

[1] Institut für Theoretische Physik der Universität Göttingen, Bunsenstrasse 9, D-37073 Göttingen, Germany

[2] Centre de Physique Théorique, CNRS, UPR 14, Ecole Polytechnique, F-91128 Palaiseau Cedex, France

Received: 9 December 1993/in revised form: 1 June 1994

Abstract: We study the infrared behaviour of the Euclidean Gross–Neveu-Model with discrete chiral symmetry. Imposing a suitable UV-cutoff we prove that for a large (but finite!) number of fermion components the model has (at least) two pure phases, realized by suitable boundary conditions and that the fermion two-point function decays exponentially.

I. Introduction

We want to study the infrared behaviour of the two-dimensional Euclidean Gross–Neveu model [1] which is formally given through the Lagrangian

$$\mathscr{L} = \overline{\psi}(x)\, i\partial\!\!\!/\, \psi(x) - \frac{\lambda}{2N}(\overline{\psi}(x)\, \psi(x))^2 \tag{1}$$

for $N \gg 1$. Here N is the number of fermion flavours, i.e.

$$\psi = (\psi_1, \dots, \psi_N)^T, (\overline{\psi} = \overline{\psi}_1, \dots, \overline{\psi}_N)\,.$$

The coupling λ is supposed to be a constant of order 1. Since we will study the model with an UV-cutoff, it does not make much sense to choose $\lambda \gtrsim \pi$: we will show that the model is massive and that the mass approaches the size of the cutoff for large $\lambda > \pi$. If $\lambda \ll \pi$ there arises a technical difficulty: the mass decreases as $e^{-\pi/\lambda}$, and the correlations decay more and more slowly. But the cluster expansions can only be shown to converge for $N \gg m^{-1}$. So small λ requires (very) large N.

Our aim is to show that the mechanism of mass generation discovered by Gross and Neveu and analyzed by them to first order in $1/N$ persists in the full model for N sufficiently large. Being mainly interested in the IR behaviour of the model we will therefore study the model with an UV cutoff, the scale of which is put equal to one.

The UV limit will be postponed to another paper, and one should note in this respect that the UV limit of the two-dimensional massive (by hand!) [2, 3] and of the three-dimensional large N [4] Four-Fermion-Models have already been constructed.

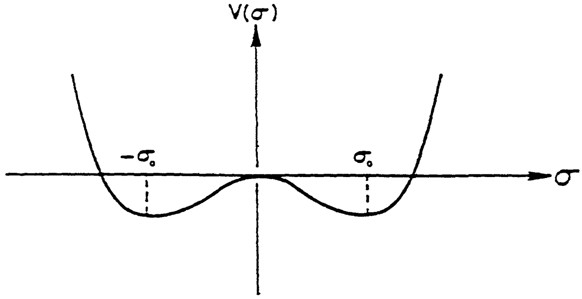

Fig. 1. Form of $V(\sigma)$ to leading order in $1/N$

In our study we will present the model in terms of an auxiliary bosonic field σ. Its action is obtained by formally integrating out the fermionic fields $\overline{\psi}, \psi$. This representation while providing the same perturbative expansion as the original fermionic one is better suited to the large N behaviour of the model. The thermodynamic limit and the Green functions of the model will be controlled in this paper with appropriate cluster expansion techniques and bounds to estimate its (nonlocal) interaction. Our main interest is to show that the model is massive, i.e. that the two-point function falls off exponentially

$$|S_2(x,y)| \leqq K\, e^{-m'|x-y|}$$

with some (in principle calculable) $m' > 0$ (see (209)), and that it has two phases at 0 temperature[1] which can be realized by imposing suitable boundary conditions before taking the thermodynamic limit. In the language of the σ field these 2 phases correspond to 2 opposite magnetizations (nonzero v.e.v. for the σ field). So the situation turns out to be quite analogous to that of the asymmetric φ^4 theory with an interaction $\lambda(\varphi^2 - \lambda^{-1})^2, \lambda \ll 1$ presented in [5], ch. 16.

As long as we don't take the UV limit the additional difficulties are mainly stemming from two facts:

1) Our model is massless in the beginning and the mass is physically present only for σ field values close to the minima of the action (depending in sign on which vacuum state is chosen) so that the treatment and the expansions depend on the "size" of the σ-field. If σ is close to one of the minima a local translation of the field variable σ will be performed.
2) The interaction of the model is non-polynomial and non-local in the σ-field. The analogy with $\lambda(\phi^2 - \lambda^{-1})^2$ implies that the model is indeed qualitatively well represented for large N by the "effective potential" $V(\sigma)$ which has been calculated by Gross and Neveu [1] (by effective potential we mean the value of the σ-Lagrangian as a function of σ for constant σ), see Fig. 1.

[1] Strictly speaking we prove the existence of at least two phases. In Ising models more than two phases can be ruled using correlation inequalities which, however, have not been established for our model.

II. Presentation of the Model

The aim of the paper is to show the existence and to derive bounds for the fermionic two-point function $\langle\overline{\psi}(x)\,\psi(y)\rangle$. The treatment of N point functions will be in obvious analogy. Our heuristic starting point is thus

$$S_2(x,y) \sim \int D(\psi,\overline{\psi})\,\overline{\psi}_i(y)\,\psi_i(x) e^{-\int(\overline{\psi} p\psi - 1/2 g^2(\overline{\psi}\psi)^2)}\,,$$

where

$g^2 = \frac{\lambda}{N}, \not p = p_0\gamma_0 + p_1\gamma_1$; γ_0, γ_1 are the two dimensional Euclidean γ matrices with

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}\,,\ \gamma_\mu^+ = -\gamma_\mu\,,\ \text{e.g.}$$

$$\gamma_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\,. \tag{2}$$

On introducing formally the σ field through

$$S_2(x,y) \sim \int D(\psi,\overline{\psi}) D\sigma\,\overline{\psi}_i(y)\psi_i(x)\exp\left\{-\int(\overline{\psi}\not p\,\psi + 1/2\,\sigma^2 + g\,\sigma\,\overline{\psi}\,\psi)\right\}\,, \tag{3}$$

we may integrate out the fermions arriving at

$$S_2(x,y) \sim \int D\sigma \left(\frac{1}{\not p + g\sigma}\right)_i (x,y) \det(\not p + g\sigma)\, e^{-1/2\int\sigma^2}\,, \tag{4}$$

where $\left(\frac{1}{\not p + g\sigma}\right)_i(x,y)$ is the position space kernel of $\frac{1}{\not p + g\sigma}$ sandwiched between projectors on the i^{th} flavour subspace. Expression (4) is still highly formal since the infinite-dimensional Lebesgue measure $D\sigma$ is ill-defined and since the terms written are plagued with infrared (IR) and ultraviolet (UV) divergences. We therefore introduce the following regularizations :

UVI. Let $u(p^2)$ be a smooth nonnegative function with $u(0) = 1$, $1/u \in \mathscr{L}^2(\mathbb{R}^2)$ (viewed as a function of p). To be definite we set

$$u(p^2) = e^{1/2 p^2}\,, \tag{5}$$

$$\not p_{rg} = \not p_u = \not p\ u(p^2)\ \text{(and analogously for functions of } \not p)\,, \tag{6}$$

u will thus regularize the fermion propagator. We have not explicitly introduced a cutoff scale in the definition of u which means that the cutoff scale is chosen equal to one.

Contrary to perturbation theory where the regularization of the fermion propagator suffices to regularize all diagrams since any σ-propagator is sandwiched between fermions, we also need a regularization of the σ-field here. Otherwise we would encounter (at least) considerable technical complications, in particular we could not prove Lemma 3 below.

The σ-field appears as an ultralocal field in (3), (4), but the σ-cutoff will be a new source of nonlocality which in turn leads to difficulties when the translation of the σ variable is performed. To minimize those we impose some conditions on the cutoff.

UVII. The regularizing function will be called $\hat{f}_\rho(p)$ and will depend on p^2 only. We write it in the form

$$\hat{f}_\rho = a\, f_\rho\, a \quad \rho \geqq 1\,, \tag{7}$$

where $a(p^2)$ is a smooth bounded strictly positive function of p^2 which will be specified later ($a(p^2) = \sqrt{\mu^2 + \pi_{\text{ren}}}$, see (83))

$$0 < \mu \leqq a(p^2) \leqq O(1)\,. \tag{8}$$

We first define f_1. Apart from Euclidean symmetry we demand

i) $f_1 \in C^\infty(\mathbb{R}^2)$.
ii) $f_1(p) \geqq 0\,; f_1(0) = 0\,; f_1(p) \leqq A|p|^{2L}, |p| < 1\,;$
$\alpha(p^2)^2 < f_1(p) < A(p^2)^2, |p| > 1.$ (9)
iii) With $g(p) = \dfrac{1}{1+f_1(p)}$ we also demand that $\tilde{g}(x) = \int g(p)\, e^{ipx} \dfrac{d^2p}{(2\pi)^2}$ has compact support, i.e. $\tilde{g}(x) \equiv 0$ for $|x| \geqq C$.

Here C, α, $A > \alpha$ are suitable positive constants. They may be chosen (arbitrarily) small. $L \in \mathbb{N}$ has to be fixed for later purposes not too small, to be definite we set

$$L = 32\,. \tag{10}$$

Lemma 1.

(i) *There exist functions f_1 fulfilling* (9).
(ii) *If f_1 fulfills* (9), *then, for $\rho > 0, f_\rho(p) := f_1\left(\frac{p}{\rho}\right)$ fulfills* (9) *on replacing $|p| \gtrless 1$ by $|p| \gtrless \rho$ in (9 ii) and C by C/ρ in (9 iii).*

Remark. We will fix $\rho > 1$ as a function of N for some given f later on (see (61)).

Proof. (i) We start from $\tilde{G}(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geqq 1 \end{cases}$.

Its Fourier transform is $G(p) = \int d^2x\, e^{-ipx}\tilde{G}(x) = 2\pi \int_0^1 x\, dx\, J_0(|p|\,x)$, where the Bessel function $J_0(z)$ is bounded by const $z^{-1/2}$ so that

$$|G(p)| \leqq C\frac{1}{1+|p|^{1/2}}\,.$$

Note that G depends on p^2 only. Choose an even integer $M \geqq 4L + 20$. We have

$$0 \leqq G^M(p) \leqq C^M \frac{1}{1+|p|^{M/2}}\,,$$

and the Fourier transform of G^M is in $C^{M/2-3}(\mathbb{R}^2)$ and supported in $|x| < M$. $G^M(p)$ is analytic in p and may be expanded around 0 as

$$G^M(p) = \sum_{n=0}^{\infty} a_n (p^2)^n\,,$$

where $a_0 = \int (\widetilde{G^M})(x)\, d^2x > 0$, and all coefficients are real.

We now set

$$G_1(p) = \frac{1}{a_0}\left(\sum_{n=0}^{L} b_n (p^2)^n\right) G^M(p)\,,$$

where $b_0 = 1$, and $b_n, 1 \leqq n < L$ are inductively fixed such that $G_1(p)$ has vanishing derivatives up to order $2L$ at zero ($b_1 = -a_1/a_0$, $b_2 = -1/a_0\,(a_2 + b_1 a_1), \ldots$); b_L is chosen sufficiently small (possibly < 0) as follows: $G_1(p)$ may be Taylor expanded at 0 as

$$G_1(p) = 1 + \sum_{n=L}^{\infty} c_n (p^2)^n\,,$$

and we choose b_L such that $c_L < 0$ and so small that

$$G_1(0) > G_1(p) \quad \forall\, p \neq 0\,.$$

This is possible, since G_1 has a local maximum at 0 and by decreasing b_L it decreases strictly for any finite p (note $G^M \geqq 0$).

The next step is to build $G_2(p)$ with

$$G_2(0) > |G_2(p)|, \quad p \neq 0\,. \tag{11}$$

If (11) is fulfilled for G_1 with suitably chosen b_L, we set $G_1 = G_2$. If it is not fulfilled for any b_L, we assume b_L has been chosen so small that

$$G_1 > 0 \text{ for } |p| < p_0\,, G_1 \leqq 0 \text{ for } |p| \geqq p_0 \text{ with some } p_0 > 0\,.$$

Then (11) is violated for those p for which

$$G_1(p) \leqq -1\,.$$

Since

$$|G_1| \leqq \frac{C'}{1 + |p|^{10}}\,,$$

this will happen only in some compact region. Set

$$d = -\inf_p G_1(p)\,.$$

We have $1 \leqq d < \infty$. Put

$$G_1^{(1)} = G_1 + \frac{1}{d} G_1^2\,.$$

Then

$$d_1 = -\inf_p G_1^{(1)}(p) = \frac{d}{4}\,,$$

and still

$$G_1^{(1)}(0) > G_1^{(1)}(p)\,,\ p \neq 0\,,$$

$$G_1^{(1)} > 0 \text{ for } |p| < p_0\,,\ G_1^{(1)} \leqq 0 \text{ for } |p| \geqq p_0\,.$$

If $G_1^{(1)}$ does not fulfill (11), we may continue until after j steps

$$G_1^{(j)} > -\frac{d}{4^j} > -1 \quad \text{for some } j > 1\,,$$

where inductively $G_1^{(n+1)} = G_1^{(n)} + \frac{1}{d_n}\left(G_1^{(n)}\right)^2$, $d_n := -\inf G_1^{(n)}$.

We then set

$$G_2 = G_1^{(j)} .$$

Equation (11) stays also true for

$$G_3(p) := G_2^2(p) \geqq 0 .$$

Since G_3 is a nontrivial analytic function, it has only finitely many zeroes in any compact region of space. We may thus choose $\mu \geqq 1$ such that

$$g_0(q) := \frac{1}{G_3(0)} G_3\left(\frac{q}{\mu}\right) > \frac{1}{2} \quad \text{for } q := |p| \leqq 2 .$$

Choosing $B \geqq 2$ we thus have for $1 \leqq q \leqq q_1$ (where $q_i = |p_i|$)

$$\frac{1}{Bq^4} < g_0(p), \quad \frac{1}{Bq_1^4} = g_0(p_1) \quad \text{for some} \quad q_1 > 2 .$$

Let

$$g_1(p) = \frac{1}{q_1^4}\, g_0\left(\frac{p}{q_1}\right) + g_0(p) .$$

Then

$$\frac{1}{Bq^4} < g_1(p), \frac{1}{Bq_2^4} = g_1(p_2) \text{ for } 1 \leqq q \leqq q_2 \text{ with } q_2 > q_1^2 .$$

Defining inductively q_n such that $g_{n-1}(q_n) = \frac{1}{Bq_n^4}$ and

$$g_n(p) = \frac{1}{q_n^4}\, g_0\left(\frac{p}{q_n}\right) + g_{n-1}(p) \quad q_n > q_1^n > 2^n ,$$

we find that

$$g_\infty(p) = \lim_{n\to\infty} g_n(p) > \frac{1}{Bq_n^4} \quad \text{for } q > 1 .$$

By standard uniform convergence arguments it is also smooth and has vanishing derivatives up to order $2L$ at 0 and it fulfills the analogue of (11). Its Fourier transform still vanishes for $|x| > (1/\mu)2^{j+1}M$. We also have an upper bound on g_∞ :

Choose $b < \left(\frac{1}{2}\right)^7$ such that

$$g_0 < \frac{1}{bq^8} \quad \forall\, q > 1 .$$

We then find for $q_n \leqq q < q_{n+1}$ (with $q_0 = 1$)

$$\begin{aligned} g_\infty(p) &< \frac{1}{b}\left(\frac{1}{q^8} + \frac{1}{q_1^4}\left(\frac{q_1}{q}\right)^8 + \ldots + \frac{1}{q_n^4}\left(\frac{q_n}{q}\right)^8 + \frac{1}{q_{n+1}^4} + \frac{1}{q_{n+2}^4}\cdots\right) \\ &< \frac{1}{bq^4}\left(\frac{1}{q^4} + \left(\frac{q_1}{q}\right)^4 + \ldots + \left(\frac{q_n}{q}\right)^4 + \left(\frac{q}{q_{n+1}}\right)^4 + \ldots\right) \\ &< \frac{1}{bq^4} 2\sum_{m=0}^{\infty}\left(\frac{1}{2^4}\right)^m < \frac{3}{bq^4} . \end{aligned}$$

We finally set

$$g(p) = \frac{g_\infty(p)}{g_\infty(0)}, \quad f_1 = \frac{1}{g} - 1\,.$$

Then all estimates on f_1 are immediate consequences of the established properties of g. On making once more a scale transformation of g, f_1 with a scale parameter $\mu > 1$ one finds that the constants may be scaled as $(\alpha, A) \to \mu^{-4}(\alpha, A), C \to 1/\mu C$ and thus may be chosen arbitrarily small.
(ii) is trivial. QED

As was already mentioned we also have to introduce an infrared (IR) regularization to make the expressions we handle meaningful. This regularization will be removed later on with the help of the cluster and Mayer expansions. There is, of course, a lot of arbitrariness in introducing such a regularization, in our case even more than e.g. for a $P(\varphi)$-theory, since our det-interaction is highly nonlocal. Roughly speaking the system will be enclosed in a (large) box: We choose a square

$$\Lambda \subset \mathbb{R}^2$$

centered at the origin with volume

$$|\Lambda| = 4n^2 \gg 1, \quad n \in \mathbb{N}\,.$$

More details will be given below (see (25), (26)).

We now use the UV regulator f_ρ to define a regularized σ-covariance. We will fix the choice of the parameter ρ in (7) later (see (61)) and call the functions f_ρ, $\hat{f}_\rho$ with that choice simply $f, \hat{f}$. We shall then denote by

$$d\mu_f(\sigma)$$

the Gaussian measure with mean zero and covariance

$$\left(\frac{1}{1+\hat{f}}\right)(x, y) \text{ or } \left(\frac{1}{1+\hat{f}}\right)(x-y)\,. \tag{12}$$

For shortness of notation we understand Fourier transform to position space when writing arguments x or y. $d\mu_f(\sigma)$ will replace in the rigorous definition of S_2 the term $\int D\sigma e^{-1/2 \int \sigma^2}$ from (4). Since we will always keep the UV-cutoffs the σ field may be viewed as an element of $\mathscr{L}^2(\Lambda)$ [6] (or even as a differentiable function if we restrict $g(p)$ in (9) to fall off more rapidly for large p. If we want to take away the cutoff, σ should be viewed as a distribution).

We now look at the determinant

$$\det(\not{p} + g\sigma)$$

in (4). Formally

$$\det(\not{p} + g\sigma) = (\det \not{p}) \det\left(1 + \frac{1}{\not{p}} g\sigma\right)\,.$$

We define χ_Λ to be the characteristic function of Λ and set $\sigma = \sigma\chi_\Lambda = \sigma_\Lambda$. Replacing also $\not{p} \to \not{p}_u$ (6) $\frac{\sigma}{\not{p}_u}$ is trace class, and the second determinant makes sense. The first will be omitted since it is a global normalization factor which drops out on dividing by the partition function or vacuum functional. (It may be interpreted as $\det^{-1}\left(\left(\frac{1}{\not p}\right)_{\Lambda}\right)$.) It turns out, however, that for our purposes it is advantageous to rewrite

$$\det\left(1+\frac{1}{\not p}g\sigma\right), \tag{13}$$

where $\not p$ from now on always is $\not p_u$ – in a different form, namely as a determinant of a Hermitian operator, which is more suitable for the subsequent estimates. We will then restrict this operator to the volume Λ, which frees us from certain annoying boundary terms (the reader will realize that there are still enough terms of that sort left). We think this justified since one volume cutoff is in principle as good as another, and simplicity is a reasonable criterion.

We proceed as follows. Introducing

$$\tau(x)=\sigma(x)-\sigma_0(x) \text{ with } 0 \leqq \sigma_0(x) \leqq \sigma_0 \in \mathbb{R}, \tag{14}$$

we may rewrite (13) as

$$\det\left(1+\frac{1}{\not p}g\sigma\right)=\det\left(1+\frac{1}{\not p}(g\tau+g\sigma_0)\right). \tag{15}$$

To make everything in (15) well-defined we suppose

$$\tau,\sigma_0 \in \mathscr{L}^1(\mathbb{R}^2).$$

We also suppose $\sigma_0(x)$ to be constant in Λ with value σ_0 and smoothly decreasing with $|x|$. We call

$$m(x)=g\,\sigma_0(x). \tag{16}$$

Under the preceding assumptions we now prove the following

Proposition 1.

$$\det\left(1+\frac{1}{\not p}(g\tau+m)\right)=\det{}^{1/2}(1+\widetilde{A}+\widetilde{B})\det\left(1+\frac{m}{\not p}\right), \tag{17}$$

where

$$\widetilde{A}=g\sigma\frac{1}{(-\not p+m)(\not p+m)}g\sigma-m\frac{1}{(-\not p+m)(\not p+m)}m,$$
$$\widetilde{B}=g\sigma\frac{1}{(-\not p+m)(\not p+m)}\not p-\not p\frac{1}{(-\not p+m)(\not p+m)}g\sigma. \tag{18}$$

Here and in the proof m stands for $m(x)$, the associated operator on $\mathscr{L}^2$.

Proof. Our assumptions guarantee $\frac{1}{\not p}(g\tau+m)$ to be trace class. We first assume g to be so small (for given τ and σ_0) that

$$\operatorname{Tr}\ln\left(1+\frac{1}{\not p}(g\tau+m)\right)$$

has a convergent expansion in g. Then

$$\det\left(1+\frac{1}{\not p}(g\tau+m)\right) = \exp \operatorname{Tr} \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}\left(\frac{1}{\not p}(g\tau+m)\right)^{n} . \tag{19}$$

Since det is an entire function of g [7] one may then convince oneself that (17) also holds for arbitrary g by analytic continuation, once it is established for sufficiently small g.

We find for g sufficiently small (remember $m = g\,\sigma_0$)

$$\begin{aligned}\det\left(1+\frac{1}{\not p}(g\tau+m)\right) &= \exp \operatorname{Tr} \sum_{n} \frac{(-)^{n+1}}{n}\left(\frac{1}{\not p}(g\tau+m)\right)^{n}\\ &= \exp \operatorname{Tr} \sum_{n} \frac{(-1)}{2n}\left(\frac{1}{\not p}(g\tau+m)\right)^{2n}\\ &= \exp \operatorname{Tr} \sum_{n} \frac{(-1)}{2n}\left((g\tau+m)\frac{1}{-\not p}\right)^{2n}\\ &= \det\left(1+(g\tau+m)\frac{1}{-\not p}\right) .\end{aligned} \tag{20}$$

Here we used the fact that the trace of an odd number of $\frac{1}{\not p}$'s vanishes. The last det is that of the adjoint of the first. We therefore know for g small and real

$$\det\left(1+\frac{1}{\not p}(g\tau+m)\right) > 0 . \tag{21}$$

This together with (20) and

$$\det\left(1+\frac{1}{\not p}(g\tau+m)\right) = \det\left(\frac{\not p+m}{\not p}\right)\det\left(1+\frac{1}{\not p+m}g\tau\right)$$

(and similarly for the adjoint) now tells us

$$\begin{aligned}\det\left(1+\frac{1}{\not p}(g\tau+m)\right) &= \det{}^{1/2}\left(\frac{\not p+m}{\not p}\right)\det{}^{1/2}\left(\frac{-\not p+m}{-\not p}\right)\\ \times \det{}^{1/2}&\left(1+\frac{1}{\not p+m}g\tau+g\tau\frac{1}{-\not p+m}+g\tau\frac{1}{-\not p+m}\,\frac{1}{\not p+m}g\tau\right) .\end{aligned} \tag{22}$$

The first two $\det^{1/2}$'s give the second term on the r.h.s. of (17). The last determinant may be reexpressed in terms of σ as

$$\begin{aligned}&\det{}^{1/2}\left(1+\frac{1}{\not p+m}(g\sigma-m)+(g\sigma-m)\frac{1}{-\not p+m}\right.\\ &\qquad\left.+\,(g\sigma-m)\frac{1}{-\not p+m}\,\frac{1}{\not p+m}(g\sigma-m)\right)\\ &= \det{}^{1/2}\left(1+(-\not p+m)\frac{1}{(-\not p+m)(\not p+m)}(g\sigma-m)\right.\\ &\qquad+(g\sigma-m)\frac{1}{(-\not p+m)(\not p+m)}(\not p+m)\end{aligned}$$

$$+ (g\sigma - m)\frac{1}{(-\not{p}+m)(\not{p}+m)}(g\sigma - m)\Bigg)$$

$$= \det{}^{1/2}\Bigg(1 + (-\not{p})\frac{1}{(-\not{p}+m)(\not{p}+m)}g\sigma + g\sigma\frac{1}{(-\not{p}+m)(\not{p}+m)}\not{p}$$

$$+g\sigma\frac{1}{(-\not{p}+m)(\not{p}+m)}g\sigma - m\frac{1}{(-\not{p}+m)(\not{p}+m)}m\Bigg)\ . \tag{23}$$

This establishes (17) for g small and real.

To establish the relation for any real and positive g note that det $(1+\widetilde{A}+\widetilde{B})$ is analytic in g and that for g real,

$$1+\widetilde{A}+\widetilde{B} = (1+F^*)(1+F) \geqq 0 \quad \text{with} \quad F = \frac{1}{\not{p}+m}g\tau\ . \tag{24}$$

Therefore det $(1+\widetilde{A}+\widetilde{B}) \geqq 0$, and any zero on the positive real g-axis has to be of even order. Then $\det^{1/2}(1+\widetilde{A}+\widetilde{B})$ is also analytic in a neighbourhood of the positive real axis and (17) holds for any $g > 0$ by analytic continuation, thus in particular for $g=\sqrt{\lambda/N}$ (which, in fact, is very small !). QED

With the aid of the proposition we now come to the definition of the IR-regularized determinant. We set

$$A = \chi_\Lambda \left(g\sigma\frac{1}{p^2+m^2}g\sigma - \frac{m^2}{p^2+m^2}\right)\chi_\Lambda\ ,$$
$$B = \chi_\Lambda \left(g\sigma\frac{1}{p^2+m^2}\not{p} - \not{p}\frac{m^2}{p^2+m^2}g\sigma\right)\chi_\Lambda\ . \tag{25}$$

Here and from now on m denotes $m(0) = g\sigma_0$, i.e. we have taken the limit of constant mass, which is possible after introducing the χ_Λ. The value of m will be fixed later.

The support of the field variable σ in the interaction determinant is restricted to Λ. At a later stage we will take $\Lambda \to \infty$. Denoting $\widehat{\Lambda} = \mathbb{R}^2\backslash\Lambda$ we include (for technical simplification later on) also a term

$$r(\sigma) := e^{-R\int_{\widehat{\Lambda}}\sigma^2(x)dx}, \quad R \gg 1\ , \tag{26}$$

and take $R \to \infty$ once we have performed the expansion. Equation (26) makes explicit that interactions with $\widehat{\Lambda}$ are suppressed. (Absorbing (26) in the covariance would replace, for $R \to \infty$,

$$\left(\frac{1}{1+\widehat{f}}\right)(x,y) \to \chi_\Lambda(x)\left(\frac{1}{1+\widehat{f}}\right)(x,y)\,\chi_\Lambda(y).)$$

Having fixed the way in which we introduce the finite volume we also have to say a word on the boundary conditions (b.c) on $\partial\Lambda$: According to the Peierls argument [8] the b.c. are decisive for the realization of one of the two phases in the two phase region. To fix them – and for later use – we introduce a lattice of (closed) unit squares with corner coordinates in $\mathbb{Z}^2$, called

$$\Delta(z_1, z_2) \text{ or } \Delta_j \text{ or } \Delta$$

with $\Delta \subset \Lambda, \cup_\Delta = \Lambda$ and demand:
If $\Delta \cap \partial\Lambda \neq \emptyset : \sigma(\Delta) := \int_\Delta \sigma \geqq 0$.
More precisely, all σ-functional integrals in the following will contain a term $b(\sigma)$ (b for boundary)

$$b(\sigma) := \sum_{\Delta \cap \partial\Lambda \neq \emptyset} \theta(\sigma(\Delta) - 1/4), \tag{27}$$

where $\theta(x)$ is (for technical reasons) a smoothed step function

$$\theta(x) = \begin{cases} 0 & x \leqq -1/4 \\ 1 & x \geqq 1/4 \end{cases}, \text{ monotonic and smooth.} \tag{27'}$$

Of course we could equally well study the case with opposite sign of $\sigma(\Delta)$. And we can also fix the value of σ near the boundary by different rules to obtain the same result, but we do not intend to discuss this problem here in full generality. Obviously the b.c. (27) favours positive expectation values of σ, and we shall indeed find out

$$\langle\sigma\rangle > 0 \quad \text{(see Proposition 4)}.$$

Using the regulators and the b.c. we may now make an attempt to define the two point function $S_2(x, y)$ in a rigorous way. For technical reasons it will be sometimes helpful (but not very important) to smear it out with test functions f_1, f_2. We assume them to be real $\mathscr{L}^2$-functions and (without restriction) to have their support only in a single square $\Delta(f_1)$ (resp. $\Delta(f_2)$) which are of course arbitrary and may be varied. Since we are working in a $2 \times N$-component space they strictly speaking will be assumed to be of the form

$$(0, \ldots, 0, f_j^{(1)}, f_j^{(2)}, 0, \ldots, 0), \quad j = 1, 2 \tag{28.1}$$

where the entries are in position $2\ i-1$, $2i$ of the $2N$-vector. Finally we assume (again without restriction) that

$$\langle f_j, f_j\rangle := \int f_j^*(x) f_j(x) d^2x = 1\,. \tag{28.2}$$

We thus now define the unnormalized UV-regularized two-point function in the volume Λ and the respective partition function as

$$S_{2\mathrm{un}}^{\Lambda}(f_1, f_2) = \int d\mu_f(\sigma) \left\langle f_1, \frac{1}{\not p + g\sigma} f_2 \right\rangle \det{}^{1/2}(1 + A + B)\, b(\sigma) r(\sigma),$$

$$Z^\Lambda = \int d\mu_f(\sigma) \det{}^{1/2}(1 + A + B)\, b(\sigma) r(\sigma) \tag{29}$$

(we used (12), (17), (25)–(27), formally starting from (4), and we formally divided S, Z by $\det(\not p + m)$). $\not p$ denotes $\not p_u(6)$, $p^2 = (-\not p_u)\not p_u$.

To estimate the finite volume quantities and to be able to take the infinite volume limit later on we have to perform the usual cluster and Mayer expansions. Since the regions where σ is far from either of the minima of the potential, are highly suppressed in probability, we shall estimate them directly without expanding the couplings between squares within a given connected component of such a region. This means that our expansion will be defined differently for different sets of σ-configurations. The Hilbert–Schmidt norm of the operator A (25) restricted to a given square Δ turns out to be an appropriate criterion to distinguish between the small field (near the minima) and large field (highly suppressed) regions. Thus for every square Δ in Λ we shall introduce in the functional integral a factor

$$1_\Delta = \theta\left(\frac{\|A_\Delta\|_2}{N^\alpha} - 1\right) + \left(1 - \theta\left(\frac{\|A_\Delta\|_2}{N^\alpha} - 1\right)\right)$$

(see (27′))
and then expand the product over the Δ's (see (47)). We use the following

Definition. *Given any σ-configuration with $\sigma \in \mathscr{L}^2(\Lambda)$ we call $\Delta \subset \Lambda$ a large field or l-square if it carries a factor*

$$\theta^l_\Delta(\sigma) = \theta\left(\frac{\|A_\Delta\|_2}{N^\alpha} - 1\right) \qquad \textit{where} \tag{30}$$

$\|A_\Delta\|_2 = \|\chi_\Delta A \chi_\Delta\|_2$ *(Hilbert–Schmidt-Norm), (in contrast to the operator norm $\|A_\Delta\|$), and we call it a small field or s-square, if it carries a factor*

$$\theta^s_\Delta(\sigma) = 1 - \theta\left(\frac{\|A_\Delta\|_2}{N^\alpha} - 1\right) = 1 - \theta^l_\Delta(\sigma)\,. \tag{31}$$

Note that if the Hilbert–Schmidt-norms $\|A_\Delta\|_2$ in (30), (31) are bounded by $O(1)N^\alpha$, then the corresponding operator norms are bounded by $O(1)N^{(\alpha-1/2)}$ due to the N-fold degeneracy, i.e. they are much smaller than 1 with the choice (34).

For any square Δ we write

$$1 = \theta(\sigma(\Delta)) + \theta(-\sigma(\Delta))\,, \tag{32}$$

where $\sigma(\Delta) = \int_\Delta \sigma(x) d^2x$.
And we call an s-square or an l-square a $\pm$-square ($s\pm, l\pm$-square)

$$\text{if it carries a factor } \theta(\pm\sigma(\Delta))\,. \tag{33}$$

α in (30), (31) has to be chosen between 0 and 1/6, for definiteness

$$\alpha = \frac{1}{10}\,. \tag{34}$$

For technical reasons it is (unfortunately) necessary to define several kinds of large field regions by successively enlarging the region of l-squares:

Definition. *Let l and s be the set of large and small field squares for a given $\sigma(x)$ (see (30)–(33))*

$$\begin{aligned} l \cup s &= \Lambda & l \cap s &= 0 \\ s &= s_+ \cup s_- & s_+ \cap s_- &= 0\,. \end{aligned} \tag{35}$$

0 in set-theoretic relations always denotes a set of measure 0 w.r.t. to the standard Lebesgue measure in $\mathbb{R}^2$. The next step is to define $l_1 \supset l$ through $\Delta \in l_1$, if $\Delta \in l$ or if $\Delta \in s_\pm$ and there exists $\Delta' \in s_\mp, \Delta' \neq \Delta$ such that Δ and Δ' have a common edge.

Accordingly

$$s_1 = \Lambda \backslash l_1\,,\; s_1 = s_{1+} \cup s_{1-}\,,\; s_{1\pm} \subset s_\pm\,. \tag{36}$$

Then we define $l_2 \supset l_1$ through:

$$\Delta \in l_2, \text{ if } \Delta \in l_1 \text{ or if } \operatorname{dist}(\Delta, l_1) = 0\,,$$

and accordingly

$$s_2 = \Lambda \backslash l_2\,,\; s_2 = s_{2+} \cup s_{2-}\,,\; s_{2\pm} \subset s_{1\pm}\,. \tag{37}$$

Now we define $L \supset l_2$,

$$\Delta \in L\,, \text{ if } \Delta \in l_2 \text{ or if } \operatorname{dist}(\Delta, l_2) = 0\,, \tag{38}$$

and correspondingly S, S_+ and S_-.
For given l_1 we define

$$\Gamma = \{\Delta \subset \Lambda |\; \operatorname{dist}(\Delta, l_1) \leqq M\}\,, \tag{39}$$

where we choose (for definiteness)

$$M = \frac{2}{m} \log N \gg 1\,. \tag{40}$$

The set l_1 is also split into connectivity components as follows:
For $\Delta_i, \Delta_j \in l_1$ we say there is a connectivity link between Δ_i, Δ_j if there exists $\Delta \subset \Lambda$ such that

$$\operatorname{dist}(\Delta_i, \Delta) + \operatorname{dist}\,(\Delta_j, \Delta) \leqq 2M\,. \tag{41}$$

By $l_{11}, \ldots, l_{1n}$ we denote the maximal subsets of l_1, connected by connectivity links, and call these connectivity components. This splitting induces a corresponding one of l_2 and L into $l_{21}, \ldots, l_{2n}$ and $L_1, \ldots, L_n$.

There is a one-to-one relation between $l_{11}, \ldots, l_{1n}$ and the corresponding subsets of Γ, called connectivity components of Γ:

$$l_{1a} \subset \Gamma_a \subset \Gamma\,, \text{with}$$
$$\Gamma_a = \{\Delta \mid \operatorname{dist}(l_{1a}, \Delta) \leqq M\}\,,$$

so that

$$\Gamma_a \cap \Gamma_b = 0\,,\; a \neq b\,,\; \bigcup_{a=1}^{n} \Gamma_a = \Gamma\,. \tag{42}$$

Finally we introduce

$$\begin{aligned} \gamma_a &= \left\{\Delta \subset \Lambda \mid \operatorname{dist}(\Delta, l_{1a}) \leqq \frac{M}{2}\right\} \cup \left\{x \in \widehat{\Lambda} \mid \operatorname{dist}(x, l_{1a}) \leqq \frac{M}{2}\right\} \\ \gamma &= \bigcup_a \gamma_a \end{aligned} \tag{43}$$

so that

$$\operatorname{dist}\,(\gamma, \Gamma') \geqq \frac{M}{2} - \sqrt{2}\,, \tag{44}$$

where we always denote for any set $E \subset \mathbb{R}^2$,

$$E' = \Lambda \backslash E\,,\; \widehat{E} = \mathbb{R}^2 \backslash E\,. \tag{45}$$

The full hierarchy of large field regions is thus

$$l \subset l_1 \subset l_2 \subset L \subset (\gamma \cap \Lambda) \subset \Gamma\,,\; l_{1a} \subset L_a \subset \gamma_a \cap \Lambda \subset \Gamma_a\,. \tag{46}$$

All squares Δ fulfill $\Delta \subset \Lambda$. $\widehat{\Lambda}$ will be viewed as a single (large field) block.

Now we go back to (29). We reexpress

$$S^{\Lambda}_{2un} = \sum_{l,s^+\subset\Lambda} \int d\mu_f(\sigma) \left\langle f_1, \frac{1}{\not p + g\sigma} f_2 \right\rangle \det{}^{1/2}(1+K) b(\sigma) r(\sigma)$$
$$\times \prod_{\Delta\in l} \theta^l_\Delta(\sigma) \prod_{\Delta\in s^+} \theta^{s^+}_\Delta(\sigma) \prod_{\Delta\in s^-} \theta^{s^-}_\Delta(\sigma) \tag{47}$$

(see (27′), (30)–(31)).

(The sum is over subsets of squares l, s^+ with $l \cap s^+ = 0, s^- = \Lambda\backslash\{l \cup \sigma^+\}$ (as set of Δ's)).

$$K = A + B \qquad (\text{see}(25))\,. \tag{48}$$

As mentioned above K can be viewed as an Hilbert–Schmidt-operator on $\mathscr{L}^2(\Lambda)$. In the following we will use the same symbol for an operator and its position space kernel, indicating arguments explicitly if necessary ($K(x, y), \ldots$).

The subsequent treatment of (47) will depend crucially on whether we are in the regions L or S. The representation in terms of K (48) is suited for L, but not for S. We therefore rewrite $\det^{1/2}(1+K)$ in an (unfortunately) more lengthy form.

We introduce the operators

$$K_+ = P_+ K P_+\,,\; K_- = P_- K P_-\,,\; A_{L_a} = P_{L_a} A P_{L_a}\,, \tag{49}$$

where the P's denote the projections on S_+, S_-, L_a, i.e. in position space multiplication by $\chi_{S_+}, \chi_{S_-}, \chi_{L_a}$. Going back to $F(\tau)$ (24) and defining

$$F_+(\tau) := FP_+ = \frac{1}{\not p + m} g\tau_+\,,\; F_-(\tau) := FP_- := \frac{1}{\not p - m} g\tau_- \tag{50}$$

($\tau_\pm := (\sigma \mp \sigma_0)P_\pm$),

we have, (see (24)), using the cyclicity of the trace,

$$\operatorname{Tr} K_\pm = \operatorname{Tr}(F_\pm + F^*_\pm + F_\pm F^*_\pm)\,, \tag{51}$$

$$\operatorname{Tr} K^2_+ = \operatorname{Tr}\left(F^2_+ + F^{*^2}_+ + F_+F^*_+ + F^*_+F_+ + 2F^2_+F^*_+ + 2F_+F^{*^2}_+ + (F_+F^*_+)^2\right)$$

(analogously for $+ \to -$)

and

$$\operatorname{Tr}(F^2_+F^*_+ + F_+F^{*^2}_+) = 2g^3 \operatorname{Tr}\left(\tau_+ \frac{1}{p^2+m^2}\tau_+ \frac{m}{p^2+m^2}\tau_+\right)\,,$$

$$\operatorname{Tr}((F_+F^*_+)^2) = g^4 \operatorname{Tr}\left(\tau^2_+ \frac{1}{p^2+m^2}\tau^2_+ \frac{1}{p^2+m^2}\right)\,. \tag{52}$$

From this we obtain (using $\mathrm{Tr}F = \mathrm{Tr}F^*$, $\mathrm{Tr}F^2 = \mathrm{Tr}F^{*2}$ (20))

$$\begin{aligned}\det{}^{1/2}(1+K) = {} & e^{\mathrm{Tr}(F_+ + F_- - 1/2(F_+^2 + F_-^2))} \\ & \times e^{-1/2\,\mathrm{Tr}(F_+^2 F_+^* + F_+ F_+^{*2} + 1/2(F_+ F_+^*)^2 + (+\leftrightarrow -))} \\ & \times e^{-1/2\,\mathrm{Tr}(K_+ + K_- - 1/2(K_+^2 + K_-^2))} \\ & \times \det{}^{1/2}(1+K)\,. \end{aligned} \tag{53}$$

The last two terms may be rewritten as

$$\begin{aligned} & \det{}^{1/2}\left(1 + \sum_a A_{L_a} + K_+ + K_- + K'\right) e^{-1/2\mathrm{Tr}(K_+ + K_- - 1/2(K_+^2 + K_-^2))} \\ & = \prod_a \det{}^{1/2}(1 + A_{L_a})\, \det{}_3^{1/2}(1+K_+)\, \det{}_3^{1/2}(1+K_-)\, \det{}^{1/2}(1+Q)\,, \end{aligned} \tag{54}$$

where we set

$$K' = K - \sum_a A_{L_a} - K_+ - K_-\,,\ Q = \frac{1}{1 + \sum_a A_{L_a} + K_+ + K_-} K'\,. \tag{55}$$

In (54) we used cyclicity and the pairwise orthogonality of the A_{L_a}, K_+ and K_-. Q is well-defined since $A_{L_a} > -1$ (see Lemma 17 below), and since $||K_\pm|| \ll 1$ due to the small field condition. We also introduced the (standard) definition [7]

$$\det{}_{r+1}(1+K) := \exp\left(\sum_{n=1}^{r} (-1)^n \frac{1}{n}\ \mathrm{Tr}K^n\right) \det\,(1+K)\,.$$

So far we have only introduced the auxiliary covariance (12) made up of the ultralocal σ^2-term and the regulator. The true covariance should contain (up to corrections, which are small in the expansion parameter N) all terms quadratic in σ. Otherwise the expansion with respect to the covariance will not converge (or cannot be shown to do so). This does not apply to the large field region, however, since this region can be shown directly to be highly suppressed in probability.

As a first step towards the new covariance (which obviously then will depend on the configurations l, s) we define a new variable ζ instead of σ which is shifted by $\mp\sigma_0$ in the regions $S_\pm$. A suitable choice for σ_0 then guarantees that we expand around the minima of the interaction in both regions S_+ and S_-, i.e. small ζ implies small deviations from the minima.

Due to the cutoff for the σ-field we have, however, to smooth this shifting of variable, since otherwise we get hardly controllable contributions from the cutoff. We therefore introduce smoothed characteristic functions of the squares Δ through the following steps:

α) Choose in one dimension a smooth function $\varphi(x_1)$ with

$$\begin{aligned} & 0 \leqq \varphi(x_1) \leqq 1\,, \\ & \varphi(x_1) = \begin{cases} 1\,, & 1/4 \leqq x_1 \leqq 3/4 \\ 0\,, & x_1 \leqq -1/4\,, x_1 \geqq 5/4 \end{cases}\,, \\ & \varphi(x_1) + \varphi(x_1+1) = 1, \quad -1/4 \leqq x_1 \leqq 1/4\,, \\ & \varphi(1/2 + x_1) = \varphi(1/2 - x_1)\,, \end{aligned} \tag{56}$$

and set

$$\varphi_{n_1}(x_1) = \varphi(x_1 - n_1), \quad n_1 \in \mathbb{Z}\,. \tag{57}$$

β) Then we define for the square Δ with lower left corner $x_\Delta = (n_1, n_2) = n$ the smoothed characteristic function

$$\varphi_{(n_1,n_2)}(x_1,x_2) := \varphi_n(x) := \varphi_\Delta(x) := \varphi_{n_1}(x_1)\varphi_{n_2}(x_2)\,. \tag{58}$$

From (56)–(58) we find

$$\sum_{n=\mathbb{Z}^2} \varphi_n(x) \equiv 1\,. \tag{59}$$

By construction we have for the Fourier transforms

$$\widetilde{\varphi}_\Delta(p) = e^{-ipx_\Delta}\widetilde{\varphi}(p) \text{ and } \widetilde{\varphi}(p) \in \mathscr{S}(\mathbb{R}^2)\,.$$

We now go back to (7)–(9) and Lemma 1. We choose $f_\rho, \hat{f}_\rho$ with (7)–(9) such that $\widetilde{h}_\Delta(p) := \widetilde{\varphi}_\Delta(p)\hat{f}_\rho(p)$ fulfills

$$|\widetilde{h}_\Delta(p)| \leqq \left(\frac{1}{1+p^2}\right)^2 N^{-3/2} \tag{60}$$

This is possible as will be shown now:

Assume f_1 to be given as in (9), (10). Choose $C_3, C_4(L) \geqq 1$ such that

$$|\widetilde{\varphi}(p)| \leqq \begin{cases} C_3\,, & q \leqq C_4 \\ q^{-2L}\,, & q \geqq C_4 \end{cases} \quad (q = |p|)\,.$$

Since

$$f_1(p) \leqq \begin{cases} A\,q^{2L} & q < 1 \\ A\,q^4 & q > 1 \end{cases}\,,$$

we have for $\rho^{1/2} \geqq C_4$,

$$|\widetilde{\varphi} f_\rho| \leqq \begin{cases} C_3 A \left(\frac{q}{\rho}\right)^{2L}\,, & q \leqq \rho^{1/2} \\ A q^{-2L}\left(\frac{q}{\rho}\right)^4\,, & q \geqq \rho^{1/2}\,. \end{cases}$$

From this we find (60) to be true if

$$\rho = N^{3/(2(L-2))} = N^{1/20}\,, \tag{61}$$

if 2A is chosen smaller than 1 and C_3^{-1}.

The important result is

Lemma 2. *The cutoff function for σ, $f := f_\rho$ is fixed as follows:*
(i) *Choose some f_1 according to* (9)
(ii) *Define $f(p) = f_\rho(p) = f_1(p/\rho)$ (see* (61)*).*

Then we have the estimate (60) *and*

$$\int d^2p \frac{1}{1+f(p)}\,,\ \int d^2p \frac{1}{1+\hat{f}(p)} \leqq O(1)N^{1/10}\,. \tag{62}$$

Here and in the following O(1) always denotes an N-independent constant (which also does not exceed 1 by orders of magnitude...)

Remark. The x-space support of $\frac{1}{1+f}$ (9) will decrease with N proportional to $N^{-1/20}$ so that we may safely assume

$$\left(\frac{1}{1+f}\right)(x,y) \equiv 0 \quad \text{if } |x-y| > 1\,. \tag{63}$$

Proof. Equation (60) has just been established, (62) is a consequence of the lower bound on f_1 in (9) and (61). QED

Now we define $\zeta(x)$ as

$$\zeta(x) = \sigma(x) - \sigma_0\phi_{2+} + \sigma_0\phi_{2-} =: \sigma(x) - \sigma_{02+}(x) + \sigma_{02-}(x)\,, \tag{64}$$

where (see (37))

$$\phi_{2\pm}(x) := \sum_{\Delta\in s_{2\pm}} \varphi_\Delta(x) \quad \text{(multiplication operators in position space)}\,. \tag{65}$$

So $\phi_{2\pm}$ are not projectors as P_+, P_-, P_L (49). But due to (56)–(59) we have

$$\phi_{2\pm}(x) = \begin{cases} 1 & x \in S_\pm \\ 0 & x \notin s_\pm \end{cases}\,, \tag{66}$$

so that

$$\frac{1}{\not p \pm m} g\tau_\pm = \frac{1}{\not p \pm m} g\zeta_\pm \tag{67}$$

with $\zeta_\pm = \zeta_\pm P_\pm$, $\zeta_L = \zeta P_L$ and similarly for σ, τ.

The relation between ζ_L and σ_L is then

$$\sigma_L = \zeta_L + \sigma_0(\phi_{2+} - \phi_{2-})P_L\,. \tag{68}$$

Here the second term is sort of a boundary contribution which vanishes in l_1, but not in $L\backslash l_1$.

We may now reexpress $d\mu_f(\sigma)$ in terms of $d\mu_f(\zeta)$:

$$\begin{aligned} d\mu_f(\sigma) = d\mu_f(\zeta)\exp\bigg\{&-\frac{1}{2}\langle(\sigma_{02+}-\sigma_{02-}),(1+\hat{f})\zeta\rangle - \frac{1}{2}\langle\zeta,\,(1+\hat{f})(\sigma_{02+}-\sigma_{02-})\rangle \\ &-\frac{1}{2}\langle(\sigma_{02+}-\sigma_{02-}),\,(1+\hat{f})(\sigma_{02+}-\sigma_{02-})\rangle\bigg\}\,. \end{aligned} \tag{69}$$

Due to (64, 65) we find

$$\langle(\sigma_{02+}-\sigma_{02-}),\, \hat{f}\zeta\rangle = \sigma_0 \left\langle \sum_{\Delta\in s_{2+}} h_\Delta - \sum_{\Delta\in s_{2-}} h_\Delta\,,\, \zeta \right\rangle$$

$$= \sigma_0 \left(\sum_{\Delta\in s_{2+}} \langle h_\Delta\,, \zeta\rangle - \sum_{\Delta\in s_{2-}} \langle h_\Delta\,, \zeta\rangle \right). \tag{70}$$

As a result of (60) we find:

$$|\langle h_\Delta\,, \zeta\rangle| \leqq O(1) N^{-3/2} \langle \zeta, \zeta\rangle^{1/2}. \tag{71}$$

The second term in (69) gives

$$\langle \sigma_{02+}-\sigma_{02-},\, \hat{f}(\sigma_{02+}-\sigma_{02-})\rangle = \sigma_0^2 \langle \phi_{2+}-\phi_{2-},\, \hat{f}(\phi_{2+}-\phi_{2-})\rangle$$

$$= \sigma_0^2 \left\langle \phi_{2+}-\phi_{2-},\, \sum_{\Delta\in s_{2+}} h_\Delta - \sum_{\Delta\in s_{2-}} h_\Delta \right\rangle, \tag{72}$$

and using (60) and the definition of $\phi_{2\pm}$ we estimate

$$\left| \left\langle \phi_{2+}-\phi_{2-}\,, \sum_{\Delta\in s_{2+}} h_\Delta - \sum_{\Delta\in s_{2-}} h_\Delta \right\rangle \right| \leqq O(1) |s_2| N^{-3/2}. \tag{73}$$

Since $\sigma_0 \sim N^{1/2}$ (see (76)) this implies that the perturbations caused by f are small; and this also ends the deviations necessitated by the σ-cutoff.

We noted before that, for the expansions to converge, it is necessary to include in the integration measure for ζ all those terms of the interaction which are quadratic in ζ and not suppressed by a negative power of N—as long as we stay in the small field region. We therefore include the terms from (53) giving an interaction quadratic in ζ in the S region. The parameter m will be fixed such that the terms linear in ζ from $F_\pm$ cancel with the $\hat{f}$-independent linear terms from the r.h.s. of (69). The higher order terms in $\zeta_\pm$ turn out to be small for large N (in the small field region!) since they involve at least three powers of g and thus contain a factor $N^{-1/2}$ after taking into account the factor N from the flavour trace.

We thus proceed as follows. For the first exponential in (53) which contains the terms in question, we find

$$\mathrm{Tr}\, F_+ = \mathrm{Tr}\, \frac{1}{\not p + m} g\zeta_+ = gN \left(\mathrm{Tr}_2 \int \frac{1}{\not p + m} \frac{d^2p}{(2\pi)^2} \right) \int_{S_+} \zeta_+ d^2x$$

$$= 2gN \left(\int \frac{m}{p^2+m^2} \frac{d^2p}{(2\pi)^2} \right) \int \zeta_+ d^2x. \tag{74}$$

(Tr_2 is the trace in spinor space only.)

We fix m or σ_0 such that (74) compensates with the term $\int \sigma_{0+}\zeta = \int \sigma_0\zeta_+$ from (69), i.e.

$$2gN \int \frac{m}{p^2+m^2} \frac{d^2p}{(2\pi)^2} = \sigma_0 := \frac{m}{g}. \tag{75}$$

Due to the fermion UV regularization u we find

$$m = (1+\delta)e^{-\pi/\lambda}, \quad \sigma_0 \cong \sqrt{\lambda}^{-1} N^{1/2} e^{-\pi/\lambda}, \tag{76}$$

(remember that $\lambda \lesssim 1$ is fixed, and $g^2 = \frac{\lambda}{N}$).
For a sharp cutoff at $p^2 = 1$ we find for $1+\delta$ the value $(1 - e^{-\pi/\lambda})^{-1/2}$. In our case δ cannot be calculated explicitly but easily estimated as $|\delta| \ll 1$ for $\lambda < 1$. Equation (76) shows that $\lambda > \pi$ does not make much sense in the cutoff theory as was mentioned in the introduction.

Doing the analogous calculation for the ζ_--terms we find that (75) is also the condition for

$$\operatorname{Tr} \frac{1}{\not p - m} g\zeta_- + \int \sigma_0 \zeta_- = 0 . \tag{77}$$

Now we come to the terms quadratic in $\zeta_\pm$:

$$\begin{aligned} \frac{1}{2}\operatorname{Tr} F_+^2 = \frac{1}{2}\operatorname{Tr}\left(\frac{1}{\not p + m} g\zeta_+\right)^2 &= \frac{\lambda}{2}\int \frac{d^2p}{(2\pi)^2}\tilde{\zeta}_+^*(p) \\ &\quad \times \operatorname{Tr}_2\left(\int \frac{d^2q}{(2\pi)^2}\frac{1}{(\not q + m)(\not p + \not q + m)}\right)\tilde{\zeta}_+(p) \\ &= \frac{1}{2}\langle \zeta_+, \pi\zeta_+\rangle , \end{aligned} \tag{78}$$

where π denotes the fermion bubble

$$\pi(p) = \lambda \operatorname{Tr}_2 \int \frac{d^2q}{(2\pi)^2}\frac{1}{(\not q + m)(\not p + \not q + m)} . \tag{79}$$

We then define

$$\pi_{\text{ren}}(p) := \pi(p) - \pi(0) \tag{80}$$

and

$$\mu^2 := 1 + \pi(0) = 1 + 2\lambda \int \frac{d^2q}{(2\pi)^2}\frac{-q^2 + m^2}{(q^2 + m^2)^2} . \tag{81}$$

The term 1 stems from the original ultralocal boson interaction which entered in $d\mu_f(\sigma)$. The calculation for the ζ_- term gives as in (78) a contribution

$$\frac{1}{2}\langle \zeta_-, \pi\zeta_-\rangle . \tag{82}$$

We will prove in Lemma 4 that

$$\mu^2 > 0 \quad \text{and} \quad \mu^2 + \pi_{\text{ren}} > 0 . \tag{83}$$

This tells us that $\pi_{\text{ren}} + \mu^2$ is a positive operator and thus may be legitimately absorbed in the covariance for ζ, which we now define in a first attempt through its inverse

$$\widetilde{C}_{LS}^{-1} = {}_+(\pi_{\text{ren}} + \mu^2)_+ + {}_-(\pi_{\text{ren}} + \mu^2)_- + \hat{f} + \hat{\varepsilon}_\gamma , \tag{84}$$

where

$$\varepsilon := N^{-1}\,,\ \varepsilon_\gamma := \varepsilon\chi_\Lambda\chi_\gamma\,,\ \hat{\varepsilon}_\gamma = \sqrt{\pi_{\rm ren}+\mu^2}\varepsilon_\gamma\sqrt{\pi_{\rm ren}+\mu^2}\,. \tag{85}$$

The indices LS indicate that $\widetilde{C}^{-1}$ depends on the regions L, S_+, S_-. The first two terms are those isolated above (78), (82). The ε term ensures positivity of $\widetilde{C}^{-1}$ also in the large field region $L \subset \gamma$ (38), (43) ($\hat{f}(p)$ vanishes for $p \to 0$). The ε-term then has to be compensated by adding it to the γ-region interaction terms. That it is sufficiently small so as not to deteriorate the large field bounds given below, Prop. 2, (ii), follows from Eq. (129) below. We can afford values up to $\sim N^{-4/5}$.

Starting from $\widetilde{C}_{LS}^{-1}$ we will now define the final covariance by setting

$$C_\gamma^{-1} = \sqrt{\mu^2+\pi_{\rm ren}}(1-\chi_\gamma+\varepsilon_\gamma+f)\sqrt{\mu^2+\pi_{\rm ren}}\,. \tag{86}$$

The difference

$$\widetilde{C}_{LS}^{-1} - C_\gamma^{-1}$$

contains a lot of terms which, however, are all easy to control. We find

$$\widetilde{C}_{LS}^{-1} - C_\gamma^{-1} = \sum_{i=1}^{5} C_i\,,\ C_i = C_i(\gamma)\,,\ \text{with}$$

$$\begin{aligned}
C_1 &= {}_+\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_\gamma \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_+ + {}_-\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_\gamma \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_-\,,\\
C_2 &= {}_-\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_+ + {}_+\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_-\,,\\
C_3 &= {}_L\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_S + {}_S\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_L\,,\\
C_4 &= {}_L\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_L\,,\\
C_5 &= {}_{\hat\Lambda}\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_\Lambda + {}_\Lambda\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\Lambda}\\
&\quad + {}_{\hat\Lambda}\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} \left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\Lambda}\,.
\end{aligned} \tag{87}$$

Here we set ${}_+\!\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)_{\hat\gamma} = \chi_{S_+}\left(\sqrt{\mu^2+\pi_{\rm ren}}\right)\chi_{\hat\gamma}$ etc. (see (43),(45)), remember that $\chi_{\hat\gamma} = 1-\chi_\gamma$, and we put

$$\delta_{LS} := \sum_{i=2}^{4} C_i + \hat{\varepsilon}_\gamma\,.$$

We now write the expression for the two-point function using $d\mu_\gamma(\zeta)$ defined as the Gaussian measure with mean zero and covariance C_γ, normalized according to

$$\int d\mu_\gamma = 1\,. \tag{88}$$

In writing S_2 we may dispose of a global normalization for $S_{2\rm un}$ and Z. We choose it such that for $\gamma = \emptyset$ the normalization factor equals 1. Then for $\gamma \neq \emptyset$ we get a factor Z_γ due to the change of covariance from C_0 (for $\gamma = \emptyset$) to C_γ which is ([5], Ch. 9)

$$Z_\gamma = \operatorname{Det}^{1/2}(C_\gamma / C_0)\,. \tag{89}$$

(We write Det instead of det since according to our rules det involves a flavour and spinor trace, which is absent here.)

We would be satisfied if the attentive reader had become curious after all these preparations, what we are going to write for $S_2(f_1, f_2)$, the normalized two-point function:

we get

$$S_2(f_1, f_2) = \frac{\sum_{l,s^+} Z_\gamma \int d\mu_\gamma(\zeta) F_2(\zeta) G_\gamma(\zeta)}{\sum_{l,s^+} Z_\gamma \int d\mu_\gamma(\zeta) G_\gamma(\zeta)} =: \frac{\sum_{l,s^+} Z_\gamma I_\gamma^{(2)}}{\sum_{l,s^+} Z_\gamma I_\gamma^{(0)}} \qquad (\gamma = \gamma(l)) \tag{90}$$

with

$$F_2(\zeta) = \left\langle f_1 \, , \frac{1}{\not{p} + g\sigma(\zeta)} f_2 \right\rangle \tag{91}$$

and

$$G_\gamma(\zeta) = G_{1_\gamma}(\zeta) G'_{2(\gamma)}(\zeta) G_{3_\gamma}(\zeta) G'_{4(\gamma)}(\zeta) G_{5_\gamma}(\zeta) G_{6_\gamma}(\zeta) \tag{92}$$

with

$$G_{1_\gamma} = b(\sigma) \prod_{\Delta \in l} \theta^l_\Delta(\sigma) \prod_{\Delta \in s^\pm} \theta^{s^\pm}_\Delta(\sigma) e^{1/2\langle \sigma, C_5 \sigma \rangle} r(\sigma) \qquad (\text{where } \sigma = \sigma(\zeta))$$

(see (27), (47), (48)),

$$G'_{2\gamma} = \prod_a \det{}^{1/2}(1 + A_{L_a}) e^{-1/2\langle \sigma_L, \sigma_L \rangle} \tag{93}$$

(see (47), (53), (54)).

The last term stems from the original ultralocal interaction. In $S_\pm$ the analogous contributions have been absorbed in the definition μ^2 (81), (84) and thus in the measure $d\mu_\gamma$. The relation between ζ_L and σ_L (for which the large field condition was formulated) is in (68).

$$G_{3_\gamma} = \det{}^{1/2}(1 + Q) \quad (\text{see } (54))\,, \tag{94}$$

$$\begin{aligned} G'_{4\gamma} = \; & \exp\left\{ -\frac{1}{2} \langle \zeta, C_1 \zeta \rangle \right\} \\ & \times \exp\left\{ -\frac{1}{2} (\langle \sigma_{0+} \, , \sigma_{0+} \rangle + \langle \sigma_{0-} \, , \sigma_{0-} \rangle) \right\} \quad (\text{see } (69), (87))\,, \end{aligned} \tag{95}$$

$$\begin{aligned} G_{5\gamma} = \; \exp \Bigg\{ & -\frac{1}{2} \langle \zeta, \, \delta_{LS}, \zeta \rangle - \langle \sigma_{02+} - \sigma_{02-} \, , \hat{f} \zeta \rangle - \langle \sigma_{02+} - \sigma_{02-} \, , \\ & \hat{f}(\sigma_{02+} - \sigma_{02-}) \rangle \Bigg\} \end{aligned} \tag{96}$$

(see (85)–(87), (69))

$$G_{6\gamma} = \exp\left\{-(\mathrm{Tr}\,(F_+^2F_+^* + F_+F_+^{*2} + \frac{1}{2}(F_+F_+^*)^2 + (+\leftrightarrow -)\right\}$$
$$\times\ \det{}_3(1+K_+)\ \det{}_3(1+K_-) \tag{97}$$

(see (52)–(54)).

We have thus separated G_γ into 6 parts. The first collects restrictions on σ or ζ, the second the essential part of the large field contributions. $G_{3\gamma}$ contains the terms coupling the small and large field contributions. $G_{5\gamma}$ contains terms which are small in N (see Proposition 2). The second part of $G'_{4\gamma}$ is a normalization term (note that the contributions of the form $\langle\sigma_{02+}, \sigma_{02-}\rangle$ vanish due to (37), (55), (65) since dist $(s_{2+}, s_{2-}) \geqq 1$). $G'_{2\gamma}, G'_{4\gamma}$ carry a prime since their definition will be changed later (see (151), (152)). Finally $G_{6\gamma}$ contains the interaction terms in the small field region.

As a last preparation for the expansions we want to show that the covariance $C_\gamma(86)$ can be written as a sum over terms $C_{\gamma a}$ centered in the different connectivity components of γ. To do so we write the resolvent expansion for C_γ

$$C_\gamma = C_0 + C_0(C_0^{-1} - C_\gamma^{-1})C_\gamma$$
$$= C_0 + C_0(C_0^{-1} - C_\gamma^{-1})C_0 + C_0[(C_0^{-1} - C_\gamma^{-1})C_0]^2 + \dots\,. \tag{98}$$

Using

$$C_0^{-1} - C_\gamma^{-1} = \sqrt{\mu^2 + \pi_{\mathrm{ren}}}\,\chi_\gamma(1-\varepsilon)\sqrt{\mu^2 + \pi_{\mathrm{ren}}} \tag{99}$$

we deduce

$$C_\gamma = \frac{1}{\sqrt{\mu^2+\pi_{\mathrm{ren}}}}\frac{1}{\sqrt{1+f}}\sum_{m\geqq 0}\left(\frac{1}{\sqrt{1+f}}\chi_\gamma(1-\varepsilon)\frac{1}{\sqrt{1+f}}\right)^m$$
$$\times\frac{1}{\sqrt{1+f}}\frac{1}{\sqrt{\mu^2+\pi_{\mathrm{ren}}}}\,. \tag{100}$$

Note that the expansion converges in norm and that the lowest order term equals $C_0 = \dfrac{1}{\sqrt{\mu^2+\pi_{\mathrm{ren}}}}\dfrac{1}{1+f}\dfrac{1}{\sqrt{\mu^2+\pi_{\mathrm{ren}}}}$.

Since $\left(\dfrac{1}{1+f}\right)(x,y) \equiv 0$ for $|x-y| > 1$ and due to the definition of the connectivity components γ_a of $\gamma(43)$ we find from (100),

$$C_\gamma = C_0 + \frac{1}{\sqrt{\mu^2+\pi_{\mathrm{ren}}}}\frac{1}{1+f}\left(\sum_{a,r\geqq 1}\chi_{\gamma a}(1-\varepsilon)\left(\frac{1}{1+f}\chi_{\gamma a}(1-\varepsilon)\right)^r\right)$$
$$\times\frac{1}{1+f}\frac{1}{\sqrt{\mu^2+\pi_{\mathrm{ren}}}}\,, \tag{101}$$

which we write as $C_0 + C^{(\gamma)} = C_0 + \frac{1}{\sqrt{\mu^2 + \pi_{\text{ren}}}} \widehat{C}_\gamma \frac{1}{\sqrt{\mu^2 + \pi_{\text{ren}}}}$ with $C^{(\gamma)} = \sum_a C_a^{(\gamma)}$,

$$C_a^{(\gamma)} := \frac{1}{\sqrt{\mu^2 + \pi_{\text{ren}}}} \frac{1}{1+f} \left(\sum_{r \geqq 1} \chi_{\gamma a}(1-\varepsilon) \left(\frac{1}{1+f} \chi_{\gamma a}(1-\varepsilon) \right)^r \right) \frac{1}{1+f} \frac{1}{\sqrt{\mu^2 + \pi_{\text{ren}}}} ,$$

$$\widehat{C}_\gamma := \sum_a \widehat{C}_{\gamma a}, \widehat{C}_{\gamma a} = \frac{1}{1+f} \left(\sum_{r \geqq 1} \chi_{\gamma a}(1-\varepsilon) \left(\frac{1}{1+f} \chi_{\gamma a}(1-\varepsilon) \right)^r \right) \frac{1}{1+f} . \tag{102}$$

As a first application of this result we find that the normalization factor $Z_\gamma(89)$ factorizes over the connectivity components of γ

$$\begin{aligned} Z_\gamma &= \text{Det}^{1/2} \left(\frac{C_\gamma}{C_0} \right) = \text{Det}^{1/2} \left(1 + \frac{1}{C_0} \sum_a C_a^{(\gamma)} \right) \\ &= \text{Det}^{1/2} \left(1 + \sum_{a, r \geqq 1} \left((1-\varepsilon) \chi_{\gamma a} \frac{1}{1+f} \right)^r \right) \\ &= \prod_a \text{Det}^{1/2} \left(1 + \sum_{r \geqq 1} \left((1-\varepsilon) \chi_{\gamma a} \frac{1}{1+f} \right)^r \right) \\ &= \prod_a \text{Det}^{1/2} \left(1 + \frac{1}{C_0} C_a^{(\gamma)}) \right) = \prod_a Z_{\gamma a} \end{aligned} \tag{103}$$

(where we used the cyclicity of the determinant and the orthogonality of the $\chi_{\gamma a} \frac{1}{1+f}$). Now we bound $Z_{\gamma a}$

Lemma 3. $1 \leqq Z_{\gamma a} \leqq e^{O(1) N^{1/10} |\gamma_a|}$.

Proof.

$$\begin{aligned} Z_{\gamma a} &= \text{Det}^{1/2} \left(1 + \frac{(1-\varepsilon) \chi_{\gamma a} \frac{1}{1+f}}{1 - (1-\varepsilon) \chi_{\gamma a} \frac{1}{1+f}} \right) \\ &= \text{Det}^{-1/2} \left(1 - (1-\varepsilon) \chi_{\gamma a} \frac{1}{1+f} \right) \\ &= \exp \left\{ \left((-\frac{1}{2}) \operatorname{tr} \ln(1 - (1-\varepsilon) \chi_{\gamma a} \frac{1}{1+f} \right) \right\} \\ &= \exp \left\{ \frac{1}{2} \operatorname{tr} \sum_{n \geqq 1} \frac{1}{n} (1-\varepsilon)^n \left(\frac{1}{1+f} \chi_{\gamma a} \right)^n \right\} \end{aligned} \tag{104}$$

(tr denotes a flavour and spinor singlet trace).

The lower bound follows from the positivity of the trace of $\left(\frac{1}{1+f}\chi_{\gamma a}\right)^n$. Furthermore we have

Lemma 3′. *For any positive Hermitian trace class operator A and (orthogonal) projector P the following inequality holds:*

$$Tr(PAP)^n \leqq Tr\, PA^nP\,.$$

Proof. The inequality follows inductively from

$$\mathrm{Tr}\, PA^r(PAP)^s \leqq Tr\, PA^{r+1}(PAP)^{s-1}P. \tag{105}$$

To prove (105) evaluate the trace in an eigenbasis $\{\varphi_i\}$ of PAP with eigenvalues $\lambda_i \geqq 0$. The difference between the r.h.s. and the l.h.s. is given by

$$\mathrm{Tr}\, PA^r(1-P)A(PAP)^{s-1}P = \sum_i \lambda_i^{s-1}(P\varphi_i, A^r(1-P)AP\varphi_i)\,.$$

We rewrite $A^r = AA^{r-1}$, move A to the right and use $A = (1-P)A + PA$. Then the last expression equals:

$$\sum_i [\lambda_i^{s-1}(\varphi_i', A^{r-1}\varphi_i') + \lambda_i^s(\varphi_i, A^{r-1}(1-P)A\varphi_i)]\,.$$

Note that we used the fact that $P\varphi_i = \varphi_i$, if $\lambda_i \neq 0$, and we set $\varphi_i' = (1-P)AP\varphi_i$. Then the first term in the last sum is manifestly nonnegative and the second is so by induction on r, noting that $\lambda_i^n(\varphi_i, (1-P)A\varphi_i) = 0$ for $n \geqq 1$, to start the induction. QED

Applying (Lemma 3′) to (104) we obtain

$$\begin{aligned}\mathrm{tr}\left(\frac{1}{1+f}\chi_{\gamma a}\right)^n &\leqq \mathrm{tr}\left(\chi_{\gamma a}\left(\frac{1}{1+f}\right)^n \chi_{\gamma a}\right)\\ &\leqq O(1)\frac{1}{2^n}|\gamma_a|N^{1/10}\,.\end{aligned}$$

The last inequality follows from the lower bound on $f(9)$, (63) and from

$$\int d^2x\left(\frac{1}{1+x^4}\right)^n \sim \frac{1}{2^n}\,,$$

so that

$$Z_{\gamma a} \leqq \exp\left(\frac{1}{2}\sum_{n\geqq 1}\frac{1}{n}\frac{(1-\varepsilon)^n}{2^n}O(1)N^{1/10}|\gamma_a|\right) \leqq e^{O(1)N^{1/10}|\gamma_a|}\,. \tag{106}$$

QED

Remark. On inspection one finds that the constants may be arranged such that

$$Z_{\gamma a} \leqq e^{N^{1/10}|\gamma_a|}\,. \tag{106′}$$

III. Estimates

In this chapter we want to collect estimates which are independent of the cluster expansion techniques. They are required for the construction of the thermodynamic limit and for the proof of the exponential fall-off of the two-point function in the next chapter.

First we establish estimates on the fall-off of all non-local kernels and of the covariance. Then we present bounds on the Fredholm determinant and on the field variable in the small and large field regions. Using these bounds we show how to control the terms $G_{\imath\gamma}$ from (90).

III.1. Fall-Off Properties of Nonlocal Kernels and Consequences thereof.
We start by establishing properties of $\pi(p)$ appearing in (79),.... These properties imply that C_γ is positive and has exponential decay.

Lemma 4.
(i) $\mu^2, \mu^2 + \pi_{\text{ren}}(p) > 0$.
(ii) *The position space kernel of* $\pi_{\text{ren}}\ (p)$, *denoted* $\pi_{\text{ren}}(x-y)$ *decays as* $|\pi_{\text{ren}}\ (x-y)\ | \ \leqq O(1)\exp(-2m|x-y|)$.

Proof.
(i) For μ^2 we find from (81), (75) ,

$$\mu^2 = 1 + \pi(0) = 1 - 2\lambda \int \frac{d^2q}{(2\pi)^2}\left(\frac{1}{(q^2+m^2)} - \frac{2m^2}{(q^2+m^2)^2}\right)$$
$$= 1 + 2\lambda\left(-\frac{1}{2\lambda} + \frac{1}{2\pi}(1+O(m^2))\right) = \frac{\lambda}{\pi}(1+O(m^2)) > 0\ . \tag{107}$$

The correction term $O(m^2) \sim e^{-2\pi/\lambda} \ll 1$ stems from the fact that the propagators are UV regularized (see (5), (6)). Note that the first integral in (107) is *exactly* $\frac{1}{2\lambda}$ (75).

Now we introduce the operator V with kernel

$$V(x,y) := \int \frac{d^2q}{(2\pi)^2}\frac{e^{iq(x-y)}}{\not{q}+m}\zeta(y), \quad \zeta(y) \in \mathscr{L}^2(\Lambda)\ . \tag{108}$$

We find

$$\langle \zeta, (\mu^2 + \pi_{\text{ren}})\ \zeta\rangle = \lambda\ \mathrm{Tr}_2\ (V^2 + VV^*) + (\mu^2 - 2m^2)\langle\zeta,\zeta\rangle\ . \tag{109}$$

Since $\mu^2 \gg m^2$ the second term is positive, and the first is nonnegative (note that $\mathrm{Tr}_2\ V^2$ is real since ζ is a real function), which proves

$$\mu^2 + \pi_{\text{ren}} > 0\ . \tag{110}$$

(ii) $\pi_{\text{ren}}(p)$ and thus also $\pi_{\text{ren}}(x-y)$ are in the Schwartz space $\mathscr{S}(\mathbb{R}^2)$ due to the UV cutoff.

Our specific choice of cutoff even assures analyticity of $\pi_{\text{ren}}\ (p)$ for

$$(\mathrm{Im}\ p)^2 \leqq 4m^2(1 + m^2 + O(m^4))\ , \tag{111}$$

where the correction terms come again from the cutoff.

This implies the assertion. QED

Lemma 5.

$$1) \qquad 0 < \left(\frac{1}{p^2+m^2}\right)(x,y) \leqq O(1)\, e^{-m|x-y|}\,.$$

$$2) \qquad |(\mu^2+\pi_{\text{ren}})\,(x,y)| \leqq O(1)\, e^{-2m|x-y|}\,, \quad (x \neq y)\,. \tag{112}$$

$$3) \qquad |(\mu^2+\pi_{\text{ren}})^{\pm 1/2}(x,y)| \leqq O(1)\, e^{-\sqrt{2}\,m|x-y|}\,, \quad (x \neq y)\,,$$

(where again all propagators are regularized : $\not{p} \;\rightarrow\; \not{p}\, e^{\frac{1}{2}p^2}$).

Proof. 1) The upper bound follows again from the analyticity of $(p^2e^{p^2}+m^2)^{-1}$ in the strip $(\mathrm{Im} p)^2 \leqq m^2(1+m^2+O(m^4))$. The lower bound says that the kernel of $(p^2e^{p^2}+m^2)^{-1}$ is pointwise positive. To prove this we expand

$$\begin{aligned}\frac{1}{p^2e^{p^2}+m^2} &= e^{-p^2}\frac{1}{p^2+1-(1-m^2\,e^{-p^2})}\\ &= e^{-p^2}\frac{1}{p^2+1}\sum_{n=0}^{\infty}\left(\frac{1}{p^2+1}-m^2\frac{1}{p^2+1}e^{-p^2}\right)^n,\end{aligned}$$

$\frac{1}{p^2+1}$ and e^{-p^2} and thus also products of both have pointwise positive kernels in position space (for $\frac{1}{p^2+1}$ note that

$$\int e^{\mathrm{ipx}}\frac{1}{p^2+1}d^2p = \int d^2p\; e^{\mathrm{ipx}}\int_0^{\infty} d\alpha\, e^{-\alpha(p^2+1)})\,.$$

Finally we may write

$$\frac{1}{p^2+1} - m^2\frac{1}{p^2+1}\,e^{-p^2} = \frac{1}{p^2+1}(1-m^2e) + m^2e\int_0^1 ds\; e^{-s(p^2+1)}\,.$$

Both terms have manifestly positive kernels in position space for $m^2e < 1$ (which is assumed).

2) was proven in the previous lemma. The third statement follows from the following observations. a) First regard the unregularized expression for π_{ren} (which is given by a well-defined absolutely convergent integral). Using Feynman-parameters one gets

$$\pi_{\text{ren, unreg}}\,(p) = \frac{\lambda}{4\pi}p^2\int_0^1\frac{1-4x(1-x)}{m^2+p^2x(1-x)}dx\,. \tag{113}$$

From this expression one reads off

$$\mathrm{Re}\,\pi_{\text{ren, unreg}}\,(p) \;\geqq\; \mathrm{Re}\;\pi_{\text{ren, unreg}}\,(ip_2)\ (\text{for } p = p_1 + ip_2\,, p_2 = \mathrm{Im}\,p)\,. \tag{114}$$

And for $|p_2| \leqq \sqrt{2}\,m$ we find

$$\mathrm{Re}\,\pi_{\text{ren, unreg}}\,(ip_2) \geqq \;-\frac{\lambda}{3\pi}\,. \tag{115}$$

From these statements we find that $\sqrt{\mu^2 + \pi_{\mathrm{ren,\,unreg}}}^{\,\pm 1}$ is analytic for $|p_2| \leqq \sqrt{2}m$, since $\mu^2 = \frac{\lambda}{\pi}(1 + O(m^2))$.

b) Now since for $|p| \leqq O(1)m$

$$\pi_{\mathrm{ren}}\ (p) = \pi_{\mathrm{ren,\,unreg}}\ (p)(1 + O(m^2))\,, \tag{116}$$

due to the quadratic convergence of the renormalized integral we obtain for $|\mathrm{Im} p| \leqq \sqrt{2}\,m$ using also (114)

$$\mathrm{Re}\,\pi_{\mathrm{ren}}(p) \geqq -\frac{\lambda}{3\pi}(1 + O(m^2))\,, \tag{117}$$

and therefore statement 3) (where, of course, $\sqrt{2}\,m$ is not optimal). QED

Using the fall-off properties established in Lemma 3, Lemma 4 we can now prove estimates on the terms in δ_{LS} from $G_{5\gamma}$ (96), (87).

Lemma 6. *For the operators C_i with kernels $C_i(x,y)$, $i = 1,\dots,5$ defined in* (87) *we have the following estimates*:

(i) $C_1 \geqq 0$ *as operator on* $\mathscr{L}^2(\mathbb{R}^2)$.

(ii) $|C_j\ (x,y)\ | \leqq O(1)\ \inf\ (e^{-\sqrt{2}\,m|x-y|}, N^{-\sqrt{2}})\quad j = 2, 3, 4$. (118)

(iii) $\|C_5\| \leqq O(1)$.

Proof.

(i) follows from the positivity of $\mu^2 + \pi_{\mathrm{ren}}$ (Lemma 4) ,

(ii) follows from Lemma 5 and the definitions (35)–(43) which imply (for C_3, C_4) dist $(\hat{\gamma}, L) > \frac{M}{2} - 2$ (see (43)) .

For C_2 we proceed as follows: If $x \in S_+$ and $y \in S_-$ (or vice versa) we have either dist$(x,y) \geqq \frac{M}{2}$ or dist$(x,y) < \frac{M}{2}$. In the second case there exists some $\Delta \in l_1$ such that dist(x,Δ) + dist$(y,\Delta) \leqq \frac{M}{2}$ (see (36)–(38)).

This implies

$$\mathrm{dist}\,(x,\hat{\gamma}) + \mathrm{dist}\,(y,\hat{\gamma}) \geqq \frac{M}{2}\ . \tag{119}$$

Collecting the distance factors in $C_2 - C_4$ we obtain (118) using $M = \frac{2}{m}\log N$. (iii) is trivial. QED

Lemma 7. *For the covariance $C_\gamma = C_0 + C^{(\gamma)}$* (101) *we have the following estimates: For $\gamma \neq \emptyset$*

$$|C^{(\gamma)}(x,y)| \leqq O\left(\frac{1}{m^2}\right) N^{1/10}\ \exp^{\{-\sqrt{2}\,m\ (\mathrm{dist}\,(x,\gamma) + \mathrm{dist}\,(y,\gamma))\}}\ . \tag{120}$$

If $x,y \in \Gamma'$ or $x \in \Gamma_a$, $y \in \Gamma_b$, $a \neq b$ we find $|C^{(\gamma)}(x,y)| \leqq O(N^{-2})$.

If $x \in \Gamma$, $y \in \Gamma'$: $|C^{(\gamma)}(x,y)| \leqq O(N^{-1})$,

$$|C_0(x,y)| \leqq O(1)\,e^{-\sqrt{2}\,m|x-y|}\ . \tag{121}$$

Proof. (120) follows Lemmas 5,4 and from the properties of f (63) which imply that $\hat{C}_\gamma$ is supported in $\{(x,y)\,|\mathrm{dist}\,(y,\gamma), \mathrm{dist}\,(y,\gamma) \leqq 1\}$ and from a calculation similar to that in the proof of Lemma 3. Let Δ_1, Δ_2 be two squares in γ with characteristic functions χ_1, χ_2. Then

$$|\langle \chi_1, \widehat{C}_\gamma \chi_2 \rangle| \leqq \langle \chi_1, \widehat{C}_\gamma \chi_1 \rangle^{1/2} \langle \chi_2, \widehat{C}_\gamma \chi_2 \rangle^{1/2}$$

and

$$\langle \chi_i, \widehat{C}_\gamma \chi_i \rangle \leqq O(1) \sum_n (1-\varepsilon)^n \operatorname{Tr}\left(\chi_i \left(\frac{1}{1+f} \right)^n \chi_i \right) \leqq O(1) N^{1/10} \sum_n \left(\frac{1}{2} \right)^n .$$

On summing over the squares from γ and using the fall-off of $\left(\sqrt{\mu^2 + \pi_{\text{ren}}}\right)^{-1}$ this leads to (120). The subsequent statements follow from the fall-off of $\left(\sqrt{\mu^2 + \pi_{\text{ren}}}\right)^{-1}$ and the definitions of γ, Γ. Equation (121) follows using Lemma 5 and the properties of f (9), (63). QED

III.2. Estimates on the Fredholm Determinant in the Small and Large Field Blocks

III.2.1. The large field region.

A large field square (lfs) Δ was defined in (30), (31) to be one carrying a factor

$$\theta\left(\frac{\|A_\Delta\|_2}{N^\alpha} - 1 \right) ,$$

which vanishes unless

$$\|A_\Delta\|_2 > \frac{3}{4} N^\alpha, \quad \alpha = \frac{1}{10} . \tag{122}$$

$\|A_\Delta\|_2$ was defined as the HS norm of the operator A_Δ. Let U be a union of squares Δ. Define $\|A_U\|_2, A_U$ analogously to $\|A_\Delta\|_2, A_\Delta$.

Lemma 8. *ln det* $(1 + A_U) \leqq \operatorname{Tr} A_U - \frac{1}{2} \operatorname{Tr} \frac{A_U^2}{2 + A_U} \leqq \operatorname{Tr} A_U$.

Proof. Since A_U is traceclass, selfadjoint and > -1, the last inequality is true and we may write

$$\ln \det (1 + A_U) = \operatorname{Tr} \ln (1 + A_U) = \sum_{i=1}^{\infty} \ln (1 + \lambda_i) , \tag{123}$$

where $\lambda_i = \lambda_i(\sigma_U)$ are the eingenvalues of $A_U, \lambda_1 \geqq \lambda_2 \geqq \ldots$ (which are (at least) $2N$-fold degenerate). Note that σ_U as usual may be assumed to be in $\mathscr{L}^2(U)$. Then the assertion follows from $\ln(1+x) \leqq x - \frac{1}{2} \frac{x^2}{2+x^2}$ for $x > -1$. QED (124)

Now we regard $U = l$ (30), (35).

Lemma 9. *If* $\|A_\Delta\|_2 > \frac{3}{4} N^{\frac{1}{10}}\ \forall \Delta \subset l$, *we have* $\operatorname{Tr} \frac{A_l^2}{2 + A_l} \geqq \frac{1}{4} |l| N^{1/5}$.

Proof. To evaluate the trace we choose the following complete orthonormal system: In any square Δ we choose a complete system of orthonormal eigenfunctions of A_Δ with eigenvalues $\lambda_{i\Delta}(\sigma \Delta)$.

We find

$$\mathrm{Tr}\,\frac{A_l^2}{2+A_l}=\sum_{\Delta,i}\frac{\lambda_{i\Delta}^2}{2+\lambda_{i\Delta}}=\sum_{\Delta}\left(\sum_{\lambda_{i\Delta}>\frac{1}{10}}\frac{\lambda_{i\Delta}^2}{2+\lambda_{i\Delta}}+\sum_{\lambda_{i\Delta}\leqq\frac{1}{10}}\frac{\lambda_{i\Delta}^2}{2+\lambda_{i\Delta}}\right)$$

$$\geqq\sum_{\Delta}\left(\sum_{>\frac{1}{10}}\frac{\lambda_{i\Delta}}{21}+\frac{10}{21}\sum_{\leqq\frac{1}{10}}\lambda_{i\Delta}^2\right)\geqq\frac{10}{21}\frac{9}{16}|l|N^{1/5}\,. \tag{125}$$

The last inequality is true for a single Δ, if the first sum in the brackets does not vanish, due to the $2N$ fold degeneracy. If it vanishes, the second sum fulfills this bound due to (122). QED

At a late stage of our estimations we shall also require a distinction between the lfs. A lfs is called a very large field square (vlfs) if

$$\langle\varphi_\Delta,A_\Delta\varphi_\Delta\rangle>1\,, \tag{126}$$

where for given σ_Δ we define

$$\varphi_\Delta(\sigma_\Delta)=\frac{\sigma_\Delta}{\|\sigma_\Delta\|}\quad\left(\|\sigma_\Delta\|=(\textstyle\int_\Delta\sigma^2d^2x)^{1/2}\right)\,. \tag{127}$$

Lemma 10. *For any set V of vlfs Δ we have*

$$\frac{1}{2}\mathrm{Tr}\frac{A_V^2}{2+A_V}\geqq\frac{3}{50}N|V|+\frac{1}{20}\int_V\sigma^2(x)\,d^2x\,. \tag{128}$$

If $\Delta\in l/V$ we have

$$g^2\int_\Delta\sigma^2\leqq4\lambda\left(1+\frac{m^2}{2\lambda}\right)\,. \tag{129}$$

Proof. As in the proof of Lemma 9 we have

$$\mathrm{Tr}\,\frac{A_V^2}{2+A_V}\geqq\frac{3}{7}\sum_{\Delta\in V}\sum_i^{(1)}\langle\varphi_{i\Delta},A_\Delta\,\varphi_{i\Delta}\rangle\,, \tag{130}$$

where $\sum_i^{(1)}$ is over a normalized basis $\varphi_{i\Delta}$ of that subspace of $\mathscr{L}^2(\Delta)$, where $A_\Delta\geqq\frac{1}{3}$. By assumption then

$$\sum_i^{(1)}\langle\varphi_{i\Delta},A_\Delta\varphi_{i\Delta}\rangle\geqq2N\langle\varphi_\Delta^{(1)},A_\Delta\varphi_\Delta^{(1)}\rangle\geqq2N\left(\langle\varphi_\Delta,A_\Delta\varphi_\Delta\rangle-\frac{1}{3}\right)\,, \tag{131}$$

where $\varphi_\Delta^{(1)}$ is the projection of φ_Δ onto that subspace. Now

$$\begin{aligned}\langle\varphi_\Delta,A\,\varphi_\Delta\rangle-\frac{1}{3}&=2.\,\frac{1}{2}\langle\varphi_\Delta,A\,\varphi_\Delta\rangle-\frac{1}{3}\\&\geqq\frac{1}{2}+\frac{1}{2}\frac{1}{\|\sigma_\Delta\|^2}\int_{\Delta x\Delta}\{g^2\sigma^2(x)F(x-y)\,\sigma^2(y)-m^2\,\sigma(x)F(x-y)\,\sigma(y)\}-\frac{1}{3}\\&\geqq\left(\frac{1}{2}-\frac{1}{3}-\frac{m^2}{4\lambda}\right)+\frac{1}{2}\frac{1}{\|\sigma_\Delta\|^2}\int_{\Delta x\Delta}(g^2\sigma^2(x)F(x-y)\sigma^2(y))\\&\geqq\left(\frac{1}{6}-\frac{m^2}{4\lambda}\right)+\frac{1}{8N}\left(\int_\Delta\sigma^2\right)\end{aligned} \tag{132}$$

with the following explanations: we set $F(x-y)=\int \frac{e^{ip(x-y)}}{p^2e^{p^2}+m^2}\frac{d^2p}{(2\pi)^2}$.
$F(x)$ is smooth, $F(0) = \frac{1}{2\lambda}$ (75) and $F(x) \geqq \frac{1}{4\lambda}$ for $|x| \leqq \sqrt{2}$, since $m^2 \ll 1$. Furthermore

$$\int_{\Delta x\Delta} \sigma^2 F(x-y)\sigma^2 \geqq \frac{1}{4\lambda}\left(\int \sigma^2\right)^2 . \tag{133}$$

From this one directly verifies (128) for $\frac{1}{6}-\frac{m^2}{4\lambda} > \frac{1}{7}$ (which we assume). The same estimates as used previously then show that $\langle\varphi_\Delta, A\varphi_\Delta\rangle < 1$ implies

$$\frac{1}{4\lambda}g^2\left(\int_\Delta \sigma^2\right) \leqq 1+\frac{m^2}{2\lambda} , \tag{134}$$

which is (129). QED

From Lemma 8, 9, 10 we find directly

Lemma 11. (i) *For a set of lfs l we have*

$$\det(1+A_l) \leqq \exp\left(\operatorname{Tr}A_l - \frac{1}{8}|l|N^{1/5}\right) . \tag{135}$$

(ii) *For a set of vlfs V we have*

$$\det(1+A_V) \leqq \exp\left(\operatorname{Tr}A_V - \frac{3}{50}|V|N - \frac{1}{20}\int_V \sigma^2(x)\,d^2x\right) . \tag{136}$$

III.2.2. The small field region.
Δ is a small field square (sfs) if it carries a factor $1-\theta\left(\frac{\|A_\Delta\|_2}{N^\alpha}-1\right)$, which vanishes unless

$$\|A_\Delta\|_2 < \frac{5}{4}N^{1/10} . \tag{137}$$

We want to show that this condition implies bounds on the field variable itself if integrated over small regions. First we prove

Lemma 12. *For a sfs Δ with $\int_\Delta \sigma > 0$ we have*

$$0 < g\int_\Delta \sigma \leqq g\int_\Delta |\sigma| < \sqrt{2}\,m\,|\Delta'| \tag{138}$$

for any rectangle $\Delta' \subset \Delta$ with $|\Delta'| \geqq \delta, 0<\delta \ll 1$ suitably chosen, and the corresponding statement for $\int_\Delta \sigma < 0$.

Proof. The proof is by contradiction.
(α) Assume $g\int_{\Delta'}\sigma \geqq \sqrt{2}\,m\,|\Delta'|$ for some suitable Δ'. Then with $\Delta_+ = \{x \in \Delta' | \sigma(x) \geqq 0\}, \Delta_- = \Delta'\backslash\Delta_+$ we find for

$$\varphi_+ := \frac{1}{|\Delta_+|^{1/2}}\chi_{\Delta +} , \tag{139}$$

$$\begin{aligned}|\langle\varphi_+ A\varphi_+\rangle| &\geqq \frac{1}{|\Delta_+|}\frac{1}{4\lambda}\left(\int_{\Delta+} g\varphi_+\right)^2 - \frac{m^2}{2\lambda}|\Delta_+| \\ &\geqq \frac{1}{4\lambda|\Delta_+|}\left(\sqrt{2}m|\Delta'|\right)^2 - \frac{m^2}{2\lambda}|\Delta'| \geqq \frac{1}{3}\frac{m}{\lambda}\end{aligned} \tag{140}$$

(see (132), (133) for the first inequality and note $m \ll 1$), so that

$$\|A_\Delta\|_2^2 \geqq 2N\left(\frac{1}{3}\frac{m}{\lambda}\right)^2 , \tag{141}$$

which contradicts (137) for $N^{4/5} \geqq 8\frac{\lambda^2}{m^2}$ (which we assume). The bound on $\int |\sigma|$ is proven analogously by changing $\varphi_+ \to \frac{\operatorname{sign}\sigma}{|\Delta'|^{1/2}}\chi_{\Delta'}$.

(β) assume $\int_{\Delta'} \sigma \leqq 0$ for some suitable Δ'. Since $\int_\Delta \sigma > 0$ there exists some $\Delta'' \subset \Delta, \Delta' \subset \Delta''$ such that $\int_{\Delta''} \sigma = 0$; and we can find (possibly new) squares Δ', Δ'' with arbitrarily small sidelengths such that all relations still hold. Calling $\varphi := \frac{1}{|\Delta''|^{1/2}}\chi_{\Delta''}$ we estimate

$$|\langle\varphi, A\varphi\rangle| \geqq m^2\frac{1}{2\lambda}(1 - |\Delta|'')|\Delta''| - g^2\big(\int_{\Delta''}|\sigma|^2\big) \geqq |\Delta''|\frac{m^2}{2\lambda}(1 - 3|\Delta''|) \,. \tag{142}$$

In the first inequality we used

$$\begin{aligned}\int_{\Delta'' x \Delta''} \sigma(x)F(x-y)\sigma(y) &= \int_{\Delta'' x \Delta''} \sigma(x)(F(x-y) - F(0))\sigma(y) \\ &\leqq \sup_{x,y\in\Delta''} |F(x-y) - F(0)|\Big(\int_{\Delta''}|\sigma|\Big)^2 \\ &\leqq |\Delta''|\Big(\int_{\Delta''}|\sigma|\Big)^2 ,\end{aligned} \tag{143}$$

using that $|F(x-y) - F(0)| \leqq |\Delta''|$ for $x, y \in \Delta''$ and $|\Delta''| < \frac{1}{4}$ (which in turn follows by inspection of F). This again contradicts (137) if

$$|\Delta''|\frac{m^2}{2\lambda}(1 - 3|\Delta''|)(2N)^{1/2} > \frac{5}{4}N^{1/10} ,$$

which we assume to be true for $\frac{1}{5} > |\Delta''| \geqq \delta > 0$. QED

With the aid of Lemma 12 we can show now that the mean of the translated field variable ζ in a sfs is small when multiplied by g.

Lemma 13. *For a sfs Δ with $\int_\Delta \sigma > 0$ we find*

$$g\int_\Delta|\zeta| \leqq \frac{15}{2}\frac{\lambda}{m}N^{-\frac{2}{5}}, \quad \textit{where } g\zeta = g\sigma - m$$

$$g^2\int_\Delta\zeta^2 \leqq O\left(\frac{1}{m^2}\right)N^{-4/5} \tag{144}$$

(*similarly for* $\int_\Delta \sigma < 0$ *and* $g\zeta = g\sigma + m$).

Proof. If $g\int_\Delta|\zeta| > \frac{15}{2}\frac{\lambda}{m}N^{-\frac{2}{5}} = a$, then

$$\text{(i) } g\int_\Delta\zeta_+ > \frac{1}{2}a \text{ or } \text{(ii) } g\int_\Delta\zeta_- > \frac{1}{2}a,$$

where $\zeta_{(\pm)} = \sup(0, (\pm)\zeta(x))$. Set also $\Delta_+ = \{x \in \Delta | \zeta(x) > 0\}$, $\Delta_- = \Delta \backslash \Delta_+$. We assume (ii) to be true ((i) is simpler). We have

$$\|A_\Delta\|_2 = \left\| \left[g\,\zeta \frac{1}{p^2 + m^2}\,(g\,\zeta + m) \right]_\Delta \right\|_2 . \tag{145}$$

Choosing $\varphi_- = \frac{-\chi(\Delta_-)}{|\Delta_-|^{1/2}}$ we obtain from (137) (See (132) pp.)

$$\frac{1}{|\Delta_-|}\frac{1}{4\lambda}\Big(g\int_\Delta \zeta_-\Big)^2 + \frac{1}{4\lambda}\Big(g\int_\Delta \zeta_-\Big)\, m \leqq \frac{5}{\sqrt{2}\,4} N^{-\frac{4}{10}} . \tag{146}$$

This implies

$$\Big(g\int_\Delta \zeta_-\Big) \leqq \frac{5}{\sqrt{2}}\frac{\lambda}{m} N^{-\frac{2}{5}} , \tag{147}$$

which contradicts (ii) and thus proves (144), first part. The second inequality is proven similarly on replacing φ_- by $\frac{\zeta_\Delta}{\|\zeta_\Delta\|}$.
Then we obtain instead of (146)

$$\frac{g^2}{4\lambda}\|\zeta_\Delta\|^2 + \frac{g}{4\lambda} m \|\zeta_\Delta\| \leqq N^{-4/10} ,$$

which implies

$$g\|\zeta_\Delta\| \leqq O\left(\frac{1}{m}\right) N^{-4/10}$$

or

$$g^2\Big(\int_\Delta \zeta^2\Big) \leqq O\left(\frac{1}{m^2}\right) N^{-4/5} .$$

QED

Now let Δ_+ and Δ_- be two small field squares (sfs) with a common edge belonging to s_+ and s_- (35) respectively. Equation (33) then implies that there exists a square $\Delta \subset \Delta_+ \cup \Delta_-$ such that

$$\int_\Delta \sigma = 0 \tag{148}$$

(since $\int_{\Delta(s)} \sigma$ changes continuously from negative into positive values, if we move continuously $\Delta(s)$ from Δ_- into Δ_+ $(\Delta(0) = \Delta_-, \Delta(1) = \Delta_+)$). Now

Lemma 14.

$$-\frac{1}{2} \leqq \int_\Delta \sigma \leqq \frac{1}{2} \Rightarrow \|A_\Delta\|_2 \geqq \frac{5}{4} N^\alpha . \tag{149}$$

Proof. The proof is identical to the reasoning in the proof of Lemma 12 (β): one shows (142) (with a small correction $\sim N^{-1/2}$ if $\int_\Delta \sigma \neq 0$) to be true for a subsquare Δ' of Δ which implies the statement.

QED

Taking together Lemma 8, 9, 14 we easily find for two squares Δ_+, Δ_- as above, i.e. belonging to $l_1 \backslash l$,

$$\det(1 + A_{\Delta_+ \cup \Delta_-}) \leqq \operatorname{Tr} A_{\Delta_+ \cup \Delta_-} - \frac{1}{4} N^{1/5} . \tag{150}$$

III.3. Bounds on the Interaction Terms in the Functional Integral.
We bound the interaction terms present in (90). First we perform a change of normalization. As can be seen in (95) the interaction in the small field region is normalized such that it does not vanish for $\zeta \equiv 0$, but is of order $-N(|S_+| + |S_-|)$. Our presentation, hinging on the fact that the large field region is suppressed in probability, would thus go completely astray if such a term were not present in L. But we have

Lemma 15.

$$e^{\frac{1}{2}\mathrm{Tr}\, A_L - \frac{1}{2}\langle\sigma_L,\sigma_L\rangle} = e^{-\frac{1}{2}\langle\sigma_{0L},\sigma_{0L}\rangle} \,. \tag{151}$$

Remark. Equation (151) shows that exactly the same normalization factor as in (95) is also present per large field square. It just drops out if we divide numerator and denominator in (90) by this factor so that we may redefine

$$G_{2\gamma} = \prod_a \det{}_2^{1/2}(1 + A_{L_a}) \,,$$

$$G_{4\gamma} = \exp\left(-\frac{1}{2} < \zeta, C_1\zeta >\right) ,$$

$$(\text{note that } \mathrm{Tr}\, A_L = \sum_a \mathrm{Tr}\, A_{L_a}) \,, \tag{152}$$

and we use henceforth $G_{2\gamma}, G_{4\gamma}$ instead of $G'_{2\gamma}, G'_{4\gamma}$.

Proof.

$$\mathrm{Tr}\, A_L = 2N \int (g^2\sigma_L(x)F(0)\,\sigma_L(x) - m^2\chi_L(x)F(0))\,d^2x$$

$$= \frac{2Ng^2}{2\lambda}\langle\sigma_L,\sigma_L\rangle - 2N\,\frac{m^2}{2\lambda}|L|$$

$$= \langle\sigma_L,\sigma_L\rangle - \sigma_0^2|L| \,.$$

QED

Now we bound $G_{3\gamma}$:

Lemma 16.

$$\det(1+Q) = \det(1+Q^*) = \det{}^{1/2}(1+Q+Q^*+QQ^*) \,, \tag{153}$$

$$0 < \det(1+Q) \leqq 1 \,.$$

Proof. $Q = \frac{1}{1+\widehat{K}}K'$, where $\widehat{K} = \sum_a A_{L_a} + K_+ + K_-$ (see (55)).
By cyclicity we find

$$\det(1+Q) = \det(1+Q^*), \text{ noting that } Q^* = K'\frac{1}{1+\widehat{K}} \,,$$

so that

$$\det{}^2(1+Q) = \det(1+Q+Q^*+QQ^*) \,.$$

Since Tr $Q^{2n+1} = 0$, we also find

$$\det(1+Q) = \det{}^{1/2}(1-Q^2) = \det{}^{1/2}(1-\tilde{Q}) \tag{154}$$

with

$$\tilde{Q} = \frac{1}{\sqrt{1+\widehat{K}}} K' \frac{1}{1+\widehat{K}} K' \frac{1}{\sqrt{1+\widehat{K}}} .$$

$\tilde{Q}$ is selfadjoint and positive. $\det(1+Q)$ does not vanish, since $\det(1+K)$ does not either (see (53)–(55)). QED

Going back to (90) (after the change (152)), we can show now (for N sufficiently large):

Proposition 2.

(i)
$$|G_{1\gamma}| \leqq r(\sigma)|e^{-\frac{1}{2}\langle\sigma,C_5\sigma\rangle}| .$$

(ii)
$$|G_{2\gamma}| \leqq exp\left(-\frac{1}{16}(|l| + \frac{1}{4}|l_1\backslash l|)N^{1/5}\right) \leqq exp(-b|\Gamma|N^{1/5})$$

with $b = \frac{1}{16}\frac{1}{4(M+1)^2}$ *and also*

$$|G_{2\gamma}| \leqq exp\left(-\frac{1}{16}|l\backslash V|N^{1/5} - \frac{3}{100}|V|N - \frac{1}{40}\int_V \sigma^2(x)d^2x\right) . \tag{155}$$

(iii)
$$|G_{3\gamma}| \leqq 1 .$$

(iv)
$$|G_{4\gamma}| \leqq 1 .$$

(v)
$$|G_{5\gamma}| \leqq \exp O(1)\left\{\frac{1}{N}(|L|^{1/2}\|\zeta_L\|_2 + |S| + \|\zeta\|_2^2 + \|\zeta\|_2) + N^{-1/2}|s_2|\right\} .$$

(vi)
$$|G_{6\gamma}| \leqq \exp\{N^{-1/10}|S|\} .$$

(vii)
$$|Z_\gamma| \leqq e^{|\gamma|N^{1/10}} .$$

Remark. It is important to note that all these bounds also hold, if we restrict the support of the appearing operators to some subset of squares $Y \subset \Lambda$ (of course then $l, \gamma \cap \Lambda$ etc..., have to be restricted to Y too). And they still hold when we will have introduced the cluster expansion parameters h in the next section, which leave invariant :

1. kernels restricted to a single square or block (see (165), (168)),
2. fall-off properties of all kernels (Lemma 5),
3. Positivity properties of all kernels (see (166)).

1.3. imply the previous assertion.

Proof. (i) is trivial, (ii) follows from Lemmas 11, 14 and from (35)–(43) and simple geometric considerations which also show that b can be chosen much larger, apart from the case, where l consists of widely isolated lfs.
(iii) follows from Lemma 16.

(iv) follows from Lemma 6.
(v) From the bounds in Lemmas 5, 4, and Lemma 6 as well as Lemma 13 we find for C_3,

$$|\langle\zeta_L, C_3\zeta\rangle| \leqq O(1)\frac{1}{m}N^{1/10}\Big(\int_{(\mathrm{dist}(x,y)\geqq\frac{1}{m}\log N)} |\zeta_L(x)|e^{-m\sqrt{2}|x-y|}d^2x\,d^2y$$

$$+\left(\int|\zeta_L|d^2x\right)N^{-\sqrt{2}}\frac{1}{m}\log N\Big) \leqq O(1)\frac{1}{m^5}N^{-1.3}\|\zeta_L\|_1 \leqq$$

$$\leqq O\left(\frac{1}{m^5}\right)N^{-1.3}|L|^{1/2}\|\zeta_L\|_2 \leqq \frac{|L|^{1/2}}{N}\|\zeta_L\|_2\,. \tag{156}$$

C_4 is positive. Similarly we find for C_2

$$|\langle\zeta, C_2\zeta\rangle| \leqq O(1)\frac{1}{m^2}N^{\frac{2}{10}}N^{-\sqrt{2}}\frac{1}{m^4}|S| \leqq N^{-1}|S|\,. \tag{157}$$

This implies (v) on using $C_7 \geqq 0, \|\widehat{\varepsilon}_y\| \leqq O(1)\frac{1}{N}$ and (70)–(73).
(vi) Using again the Localization Lemma 13 and Lemma 4 we obtain

$$|\mathrm{Tr}(F_+^2F_+^* + F_+F_+^{*2})| \leqq O\left(\frac{1}{m^3}\right)N^{1-6/5}|S_+|\,,[1]$$

$$|\mathrm{Tr}(F_+F_+^*)^2| \leqq O\left(\frac{1}{m^4}\right)N^{1-8/5}|S_+|\,, \tag{158}$$

and similarly for the expansion terms in $\det_3(1+K_\pm)$

$$\sum_{n\geqq 3}\left|\frac{1}{n}\mathrm{Tr}K_+^n\right| \leqq O(1)\sum_{n\geqq 3}\frac{1}{n}\left(\frac{\lambda}{m}\right)^n N^{1-\frac{2n}{5}}|S_+|\,, \tag{159}$$

which implies (vi)
(vii) is Lemma 3. QED

The following corollaries will be useful in the next section.

Corollary 1. *Restricting the volume Λ in the functional integral to consist of a single small field square Δ in s_+ (or in s_-) we find*

$$\int d\mu_0^\Delta(\zeta)G_0(\zeta) = 1 + o(N^{-\frac{1}{10}})\,. \tag{160}$$

$d\mu_0^\Delta$ is the normalized Gaussian measure with covariance C_0 restricted to Δ.

[1] An additional factor of m may be gained on writing

$$\mathrm{Tr}(F_+^2F_+^* + F_+^2F_+) = \mathrm{Tr}\left(g\zeta_+\frac{1}{p^2+m^2}g\zeta_+(g\zeta_+ + m)\frac{1}{p^2+m^2}g\zeta_+ \;-(g\zeta_+\frac{1}{p^2+m^2}g\zeta_+)^2\right)$$

and using (145), the sf condition and Lemma 13 to estimate

$$\mathrm{Tr}\left(\left(g\zeta_+\frac{1}{p^2+m^2}g\zeta_+\right)(g\zeta_+ + m)\frac{1}{p^2+m^2}g\zeta_+\right)$$
$$\leqq O(1)N\,O\left(\frac{1}{m^2}\right)N^{-4/5}N^{-2/5}|S_+|$$

Proof. Statements (i), (iii)–(iv) and in particular (vi) of Proposition 2 serve to bound $G_o(\zeta)$ in Δ. $G_{30} \equiv G_{40} \equiv 1$ in our case. The integration in (160) is restricted to ζ's fulfilling the small field condition. The remaining ζ's for which (in case of s_+)

$$\left\| g\,\zeta_\Delta \frac{1}{p^2+m^2}\,(g\zeta_\Delta + m_\Delta)\right\|_2 \geqq \frac{3}{4} N^{\frac{1}{10}} ,$$

fulfill

$$\langle \zeta_\Delta, C_o^{-1}\zeta_\Delta\rangle \geqq O(1) m^2 N^{1/5} \geqq N^{1/10+\varepsilon} ,$$

which implies the assertion. QED

Corollary 2. *For a given large field block Γ_a we have*

$$|G_\gamma(\zeta_{\Gamma_a})| \leqq e^{-b/2|\Gamma_a|N^{1/5}} . \tag{161}$$

Proof. The statement follows from (i)–(vii) : Using (ii), (vi) and assuming

$$\ln 2\frac{|\gamma_a|}{|\Gamma_a|}N^{1/10} - bN^{1/5} \leqq -\frac{b}{2}N^{1/5} - \delta N^{1/10} , \tag{162}$$

we have to show in l_a/V_a :

$$|G_{5\gamma a}| \leqq e^{\delta N^{1/10}|\Gamma_a|} .$$

This follows from (v) on using (129), (144) to estimate $\|\zeta\|$ in l_a/V_a or S. In V_a we are safe due to (136) anyway (note $\sigma \equiv \zeta$ in V_a). QED

Using the bounds of this chapter we can show in the next that the cluster expansion of our model converges for large N. The restrictions on N come essentially from the structure of the previous bounds: N has to be such that the small field quantities (e.g. (158), (159)) give a small factor per volume unit Δ which has to be small enough to beat all combinatoric factors of the cluster expansion. As we mentioned in the beginning of the paper, N has to be chosen sufficiently large for *given m*. From (159) we have e.g. to demand

$$O\left(\frac{1}{m^3}\right) N^{-1/5} \ll 1 ,$$

which is true if

$$\frac{1}{m^3}N^{-0.1} \ll 1 . \tag{163}$$

Going through the previous lemmas we see that (163) is the most restrictive condition on N and will be assumed true henceforth (see also (207)).

It goes without saying that we did not optimize our expansion techniques with respect to the conditions on N for the sake of (relative) simplicity.

IV. The Expansions, Proof of Mass Generation

We come now to the description of the cluster expansion which allows to control the spatial correlations of the model, and combined with a subsequent Mayer expansion permits to take the thermodynamic limit and to bound the decay of the two point function. For earlier references on similar low-temperature expansions applied to different models see [20, 21]. A pedagogical introduction to the subject can be found in [5]. [9, 10] are closest to the present adaptation.

IV.1. The Cluster Expansion. The cluster expansion is a technique to select explicit connections between different spatial regions. The best formulas for the clusters involve trees, which are the minimal way to connect abstract objects together. Tree formulas such as those of Brydges, Battle and Federbush used in [9] still rely on some arbitrary choices. A conceptual advance was achieved in the tree formulas of [23, 24] (which we prefer to call "forest formulas"), since no arbitrary choices were involved.[2] The original argument of Brydges and Kennedy is based on a Hamilton–Jacobi equation, but equivalently one can derive them in an inductive algebraic way [25]. Let us start with an abstract introduction to these forest formulas and their proof.

Let I be a finite index set and $P(I)$ the set of all unordered pairs $(i,j) \in I \times I$, $i \neq j$. A forest $\mathcal{F}$ on I is a subset of $P(I)$ which does not contain loops $(i_1, i_2) \ldots (i_n, i_1)$. Any such forest splits as a single union of disjoint trees, and it gives also a decomposition of I into $|I| - |\mathcal{F}|$ clusters (some of them possibly singletons). The non-trivial clusters are connected by the (non-empty) trees of the forest.

Let H be a function of variables x_{ij}, $(i,j) \in P$. A typical example for a forest formula is

Lemma 17. (The Brydges.2 forest formula)

$$H(1, \ldots, 1) = \sum_{\mathcal{F}} \left(\prod_{l \in \mathcal{F}} \int_0^1 dh_l \right) \left(\left(\prod_{l \in \mathcal{F}} \frac{d}{dx_l} \right) H \right) (h_{ij}^{\mathcal{F}}(h)), \tag{164}$$

where

$$h_{ij}^{\mathcal{F}}(h) = \inf\{h_l, \, l \in L_{\mathcal{F}}(i,j)\}, \tag{165}$$

and $L_{\mathcal{F}}(i,j)$ is the unique path in the forest $\mathcal{F}$ connecting i to j. If no such path exists, by convention $h_{ij}^{\mathcal{F}}(h) = 0$.

Proof. A detailed proof of (164) and various extensions or applications will appear elsewhere [25]. Here we sketch how these formulas can all be proven by induction on the number k of elements in the forest; it seems to us slightly more transparent than the differential equation (Hamilton–Jacobi) originally used in [23] to prove a similar formula

$$e^{\sum_{(i,j) \in P} u_{ij}} = \sum_{\mathcal{F}} \left(\prod_{l \in \mathcal{F}} u_l \int_0^1 dh_l \right) e^{\sum_{(i,j) \in P} h_{ij}^{\mathcal{F}}(h) u_{ij}} \tag{164b}$$

(Eq. (164) corresponds to (164b) but with $\frac{\partial}{\partial x_{ij}}$ formally substituted to u_{ij}).

To prove e.g. (164) one considers the symmetric matrix $X = (x_{ij})$, with $x_{ii} \equiv 1$, and the function $\boldsymbol{H}(X)$ (so that $H(1, \ldots, 1) = \boldsymbol{H}(\mathbb{1})$) and one interpolates repeatedly. The first interpolation is

$$X = X(h_1)|_{h_1=1} \; ; \; X(h_1) = (h_1 X + (1 - h_1) X_{D,1}),$$

[2] V. Rivasseau thanks D. Brydges for introducing him to this type of formulas.

where in $X_{D,1}$ we suppress all the non-diagonal terms in X. Then we perform one Taylor step $\boldsymbol{H}(X) = \boldsymbol{H}(X(h_1 = 0)) + \int_0^1 dh_1 \boldsymbol{H}'(X(h_1))$. The first term corresponds to the (completed) empty forest. Expanding the error term $\boldsymbol{H}'$ we generate a link $\sum_{l_1=(i_1,j_1)\in P}(\frac{d}{dx_l}\boldsymbol{H})(X(h_1))$. Then we interpolate again the first part of the matrix above (not the second part)

$$h_1 X = (h_2 X + (h_1 - h_2)X_{D,2}^{(l_1)})|_{h_2=h_1} ,$$

where in $X_{D,2}^{(l_1)}$ we suppress all the non-diagonal terms in X coupling clusters, which are defined as the previous single indices plus the union of i_1 and j_1 from now on considered as a single cluster. Then we rewrite a Taylor expansion step for h_2, and we iterate, until all indices are exhausted. At step k we have completed forests with less than k links, and "forests in being" with k links, hence $|I| - k$ clusters. We apply a new interpolation, and in the remainder term a new link is expanded between two indices; it must link together two different clusters, hence it reduces by one unit the total number of clusters.

In the end we have naturally a decomposition of the whole space of indices as a union of clusters. The function $H(1,\ldots,1)$ is written as a sum of integrals over h parameters but with a natural ordering $h_1 \geqq \ldots \geqq h_n$, and there is an attribution of a pair of indices to each such parameter, such that all pairs form a forest. Now we fix this forest structure on P and group together all corresponding orderings, and we rename the parameters, attributing to them the index of the line in the forest that they created. This reindexing is the main subtlety: we call h_l no longer the parameter with number l in the decreasing list $h_1 \geqq \ldots \geqq h_n$, but the parameter whose Taylor expansion creates a given line l of the fixed forest. This reconstructs a sum over forests, and an integral for parameters h_l associated to the forest lines all from 0 to 1 (because all orderings are possible and strictly equivalent for building the forest). It remains only to check the formula (165) for the interpolated matrix X. However remark that our inductive interpolation rule is such that the coefficient x_l of the matrix X, where l is a link $l = (i,j)$, becomes an element of the diagonal matrix $X_{D,k}$ for the first time when i and j get first connected by the forest; after this time the parameter h which multiplies it is no longer changed. Since our parameters h are ordered so that the first line created has highest value, and so on, it follows that the parameter h multiplying x_l is the smallest parameter on the path in $\mathcal{F}$ from i to j. This property is intrinsic, so when we regroup all orderings creating a given forest to reconstruct the full range of integration from 0 to 1 for each h_l, it remains true. This achieves the proof of (164)[3]. ∎

We start from a given large field region l with $\Gamma(l) = \cup_a \Gamma_a$, and a fixed set of closed connected contours $\boldsymbol{C} = \cup_b \boldsymbol{C}_b$. These contours are made of all edges between pairs of squares (Δ, Δ') which carry a factor $\Theta(\sigma(\Delta))$ and $\Theta(-\sigma(\Delta'))$ (33). We recall that these contours are known from the decomposition (32), performed for each square, and that by rule they are entirely imbedded in the large field region, so for every connected $\boldsymbol{C}_b$ there is a single large field connected component Γ_a such that $\boldsymbol{C}_b \subset \Gamma_a$.

[3] It is also easy to check that these interpolated matrices preserve positivity if X is positive; the key is to check that in this case for any ordering of the h parameters the interpolated matrix is a positive combination of positive matrices of the type PXP, where P is a projector, but *this combination varies with the ordering.*

Given these data, we perform a cluster expansion in three steps, which constructs connected sets of squares. The outcome is a forest formula with three different kind of links. This forest formula can be made totally explicit.

– The first cluster expansion links together all the squares in each connected component Γ_a of the large field region by drawing explicit trees made of "neighbor" links. It is a kind of symmetric solution of the "Königsberg bridge (should it rather be Brydges?) problem," and leads to a first forest formula which depends on Γ.

– The second cluster expansion links together the previous clusters by interpolating all the non-local kernels in the theory. It gives a forest formula which is an extension of the first one.

– The third cluster expansion introduces a constraint on the links of the second: If two squares are (directly) linked by the second cluster expansion and if the distance of their centres is bigger than $2kM$, $k \in \boldsymbol{N}$, then we add inductively as many intermediate squares as necessary so as to diminish this distance below $2M$. A precise rule is given below. This ensures that the clusters that are built, although not connected in the ordinary sense of the word, cannot "jump" over contours. This is due to the fact that the length of a link jumping over a contour must be at least $2M + 2$ (see (42)). Without this last expansion there would be "non-local sign constraints" when a cluster jumps over a contour, hence the sum over sign assignments for $\sigma(\Delta)$ would not factorize into subsums associated with each cluster. The third cluster expansion gives a final forest formula which is an extension of the second one. In fact even in this final third formula, sign assignments for $\sigma(\Delta)$ do not completely factorize; there is a single global sign per cluster which does not factorize, but disappears thanks to the $\boldsymbol{Z}_2$ invariance of the theory. The rest of the sums and factors which make up the cluster amplitudes do factorize.

At the end of the cluster expansion we can perform a standard Mayer expansion, which frees the clusters from their hard core constraints. It is again given by a forest formula of the same type (187).

The First Cluster Expansion. Let $P(L)$ be the set of all pairs (i,j) of distinct squares in Γ. We define $\varepsilon_{ij} = 0$ if $\Delta_i \cap \Delta_j$ is empty (not even a single point), $\varepsilon_{ij} = 1$ otherwise, and $\eta_{ij} = 1 - \varepsilon_{ij}$. Hence $\varepsilon_{ij} = 1$ means that Δ_i and Δ_j have a common edge or point, and $\eta_{ij} = 1$ means the contrary.

Our first forest formula is simply

$$1 = \sum_{\mathscr{F}_1} \prod_{l \in \mathscr{F}_1} \left(\varepsilon_l \int_0^1 dh_l \right) \prod_{l \notin \mathscr{F}_1} (\eta_l + \varepsilon_l h_l^{\mathscr{F}_1}(h)) . \tag{166}$$

Proof. Apply the forest formula to $H(\{x_{ij}\}) = \prod_{(i,j) \in P} (x_{ij}\varepsilon_{ij} + \eta_{ij})$ (and remark that $H(1,\dots,1) = 1$).

The only non-zero terms in this formula are those for which the clusters associated to the forest $\mathscr{F}_1$ are exactly the set of connected components Γ_a of the large field region. Indeed they cannot be larger because of the factor $\prod_{l \in \mathscr{F}_1} \varepsilon_l$, nor can they be smaller because of the factor $\prod_{l \notin \mathscr{F}_1} (\eta_l + \varepsilon_l h_l^{\mathscr{F}}(h))$ which is zero if there are some neighbours (for which $\eta_{ij} = 0$) belonging to different clusters (for which $h_{ij}^{\mathscr{F}}(h) = 0$). Therefore this formula simply associates connecting trees of "neighbour links" to each such connected component, but in a symmetric way without arbitrary choices.

The Second Cluster Expansion. We consider all non-local kernels in our theory, that is if we remember that $C_\gamma = C_0 + S\hat{C}_\gamma S$ (see (101–102)),

$$C_0 \ , \ S = \frac{1}{\sqrt{\mu^2 + \pi_{\text{ren}}}} \ , \ \sqrt{\mu^2 + \pi_{\text{ren}}} \ , \ \frac{1}{\not{p} \pm m} \text{ and } \frac{1}{p^2 \pm m^2} \ . \tag{167}$$

Note that we need not introduce the decoupling parameters $h_{\mathscr{F}_2 - \mathscr{F}_1}$ in $\hat{C}_\gamma$ due to our definition of the "blocks" Γ_a, γ_a and the compact support of $\frac{1}{1+f}$ (63). But they are introduced in each $S(h)$ and also in each square root in (87). Otherwise we would not achieve factorization for the amplitudes of the clusters in $\mathscr{F}_2$ which could be coupled to different clusters via the inner kernel $\hat{C}_\gamma$.

All kernels in (167) are generically called R. We apply a cluster expansion to the vacuum normalization or two point functions that we want to compute in the thermodynamic limit. This second expansion takes into account the connections built by the first, i.e. it interpolates only the links.

$$R_l(x,y) = R_{ij}(x,y) = \Delta_i(x)R(x,y)\Delta_j(y) \tag{168}$$

for squares which belong to *different* clusters of the first forest. At this stage we add to the list of squares Δ also $\hat{\Lambda} = \boldsymbol{R}^2 - \Lambda$.

Let $Z(R,\Gamma)$ be a generic name for the quantities we want to compute. Then the second forest formula gives:

$$Z(R,\Gamma) = \sum_{\mathscr{F}_1} \prod_{l \in \mathscr{F}_1} \left(\varepsilon_l \int_0^1 dh_l \right) \prod_{l \notin \mathscr{F}_1} (\eta_l + \varepsilon_l h_l^{\mathscr{F}_1}(h))$$

$$\sum_{\mathscr{F}_2 \supset \mathscr{F}_1} \prod_{l \in \mathscr{F}_2 - \mathscr{F}_1} \left(\int_0^1 dh_l \right) \prod_{l \in \mathscr{F}_2 - \mathscr{F}_1} \left(\frac{d}{dx_l} \right) Z(R(\{h_{\mathscr{F}_2 - \mathscr{F}_1}\}) \ , \tag{169}$$

where $Z(R(\{h_{\mathscr{F}_2 - \mathscr{F}_1}\}))$ is a functional integral with interpolated kernels $R(\{h_{\mathscr{F}_2 - \mathscr{F}_1}\})$. These interpolated kernels are defined by $R(\{h_{\mathscr{F}_2 - \mathscr{F}_1}\})_l(x,y) = h_l^{\mathscr{F}_1,\mathscr{F}_2}(h) R_l(x,y)$, where $h_l^{\mathscr{F}_1,\mathscr{F}_2}(h)$ is the inf of the h parameters of the lines of $\mathscr{F}_2 - \mathscr{F}_1$ on the unique path in $\mathscr{F}_2$ joining Δ_i to Δ_j (if $l = (i,j)$). If no such path exists, by convention $h_l^{\mathscr{F}_1,\mathscr{F}_2}(h) = 0$, and if the path exists but is made solely of lines of $\mathscr{F}_1$, by convention $h_l^{\mathscr{F}_1,\mathscr{F}_2}(h) = 1$. In other words the path is computed with the full forest, but only the parameters of the forest $\mathscr{F}_2 - \mathscr{F}_1$ are taken into account for the interpolated non-local kernels.

The product $\prod_{l \in \mathscr{F}_2 - \mathscr{F}_1} (\frac{d}{dx_l})$ is a short notation for an operator which takes derivatives with respect to a parameter x_l multiplying all of the nonlocal kernels R_l and then takes x_l to 1. Therefore the action of $\prod_{l \in \mathscr{F}_2 - \mathscr{F}_1} (\frac{d}{dx_l})$ creates the product $\prod_{l \in \mathscr{F}_2 - \mathscr{F}_1} R_l$ (with summation over the finite set of possible R's). In Sect. IV.3 we give a list of possible contributions generated this way. The important fact to be shown is that to each of these derivatives is associated a factor which tends to zero as $N \to \infty$.

We remarked already that it is a crucial property of the forest formulas of this type that they preserve positivity properties, so that in the sense of operators if R is a positive operator, $R(\{h_{\mathscr{F}_2 - \mathscr{F}_1}\})$ is also positive. Recall that this is not obvious at first sight from the infimum rule of (165), but it is true [23–25] because for any ordering of the h parameters (say $h_1 \leqq \cdots \leqq h_n$) there is a way (*which* *varies with the ordering*) to rewrite the interpolated $R(h)$ as an explicit sum of positive operators: $R(h) = \sum_p (h_p - h_{p-1}) \sum_{q=1}^{p} \chi_{p,q} R_{\chi_{p,q}}$ (the functions $\chi_{p,q}$ being the characteristic functions of the p connected clusters built with the part of the forest made of lines $p,\ p+1, \ldots, n$).

The Third Cluster Expansion . In this last cluster expansion step we "hook" all the non-trivial contours $\boldsymbol{C}_b$ crossed by the links of the second expansion to the endpoints of these links. This is not automatic because in the second cluster expansion the kernels can "jump" far away, in contrast with the "Erice" type of cluster expansion. This last step is absolutely necessary for the sign assignments in a cluster to become *completely local* except for one global sign.

To implement this last step in a way similar in spirit to the first step, we introduce a factor which is equal to 1 and we interpolate it inductively again according to the same forest formula. We consider any link $l \in \mathscr{F}_2 - \mathscr{F}_1$. Let the coordinates of the two squares Δ_i, Δ_j, $i < j$ linked by $l \in \mathscr{F}_2 - \mathscr{F}_1$ be (i_1, i_2) and (j_1, j_2). If $2k\,M < \sqrt{(j_1 - i_1)^2 + (j_2 - i_2)^2} \leqq 2(k+1)M$, we call $I_l = \{\Delta'_1 = \Delta_i, \Delta'_2, \ldots, \Delta'_{n_l} = \Delta_j\}$ the set of squares with centre coordinates (i_1, j_1), $(i_1 + [\frac{j_1 - i_1}{k+1}], i_2 + [\frac{j_2 - i_2}{k+1}]), \ldots, (i_1 + [k\frac{j_1 - i_1}{k+1}], i_2 + [k\frac{j_2 - i_2}{k+1}])$, (i_2, j_2), on which there is a natural ordering induced by the link (since the link is an unordered pair, we order it arbitrarily as an ordered pair (i,j) so that $\Delta_i = \Delta'_1$ is the first square).

Note that our prescription is not minimal from the point of view of adding only the minimal number of squares. For avoiding jumps over contours it would be sufficient only to add squares if some Δ_i has distance $> 2M$ from *all* squares within its $\mathscr{F}_2$-cluster.

Remark then the following trivial inequality:

$$\operatorname{dist}(\Delta'_{j'}, \Delta'_{j'+1}) \leqq 2M\,. \tag{170}$$

We define the index set I' as the union for all the links $l \in \mathscr{F}_2 - \mathscr{F}_1$ of all the squares in I_l. Then we consider the set $P(I')$ of all pairs built on this index set; on $P(I')$ we define the function $\varepsilon'_{ij} = 1$ if the squares Δ_i and Δ_j have consecutive indices on I_l for some $l \in \mathscr{F}_2 - \mathscr{F}_1, \varepsilon'_{ij} = 0$ if they don't, and we put $\eta'_{ij} = 1 - \varepsilon'_{ij}$. Then we apply the forest formula to

$$1 = \prod_{(i,j) \in P(I')} (\varepsilon'_{ij} + \eta'_{ij}) \tag{171}$$

just as in the first cluster expansion, but taking of course into account the connections of the previous forest $\mathscr{F}_2$. We obtain a third forest formula which is an extension of the second one. It is equal to (169) except for addition of a third factor

$$\sum_{\mathscr{F}_3 \supset \mathscr{F}_2} \prod_{l \in \mathscr{F}_3 - \mathscr{F}_2} \left(\varepsilon'_l \int_0^1 dh_l \right) \prod_{l \in P(I'), l \notin \mathscr{F}_3} (\eta'_l + \varepsilon'_l h_l^{\mathscr{F}_2, \mathscr{F}_3}(h))\,, \tag{172}$$

where again $h_l^{\mathscr{F}_2, \mathscr{F}_3}(h)$ is the infimum of the h parameters of the lines of $\mathscr{F}_3 - \mathscr{F}_2$ on the unique path in $\mathscr{F}_3$ joining Δ_i to Δ_j (if $l = (i,j)$) ($h_l^{\mathscr{F}_2, \mathscr{F}_3}(h) = 0$ if no such path exists, $h_l^{\mathscr{F}_2, \mathscr{F}_3}(h) = 1$ if such a path exists but contains no lines of $\mathscr{F}_3$). In other words the path is computed with the full forest, but only the parameters of the forest $\mathscr{F}_3 - \mathscr{F}_2$ are taken into account for the interpolated ε'_l links, $l \in P(I')$.

The clusters deduced from this third and final forest formula are not connected in the ordinary sense, but they satisfy some geometric constraints (otherwise their contribution is zero). This constraint is expressed by the rule that no link l of the second type in a cluster Y can cross a distance $> 2M$ without enforcing that all squares in I_l belong to that same cluster. Indeed if such a situation occurs, there must be a first pair $\Delta'_{i'}$, $\Delta'_{j'}$ in I_l such that $\Delta'_{i'} \in Y$, $\Delta'_{j'} \in Y' \neq Y$. Then the contribution contains the factor $(\eta'_{l'} + \varepsilon'_{l'} h_{l'}^{\mathscr{F}_2,\mathscr{F}_3}(h))$ for $l' = (i',j') \in P(I')$, which is zero because $\eta'_{l'} = 0$ and $h_{l'}^{\mathscr{F}_2,\mathscr{F}_3}(h)) = 0$.

Therefore, combining with the first rule that each large field connected component Γ_a is contained in a single cluster and with the fact that any contour lies entirely in the large field region, hence in a single Γ_a, which has "thickness" $\geqq 2M + 2$, we obtain that each cluster of $\mathscr{F}_3$ which has squares both in the interior and the exterior of any contour completely contains this contour. This is crucial to factorize the cluster amplitudes as shown below.

We remark finally that in formulas (166) or (172) the interpolated factors $\prod_{l \notin \mathscr{F}_1} (\eta_l + \varepsilon_l h_l^{\mathscr{F}_1}(h))$ and $\prod_{l \in P(I'), l \notin \mathscr{F}_3} (\eta'_l + \varepsilon'_l h_l^{\mathscr{F}_2,\mathscr{F}_3}(h))$, after giving the necessary constraints on the clusters used below (each Γ_a included in a single cluster, no jump over contours) can be bounded simply by 1.

The Cluster Amplitudes. Factorization. The result of the cluster expansion is a formula for e.g. the two point function (90):

$$S_2(f_1, f_2) = \frac{\sum_{\ell,s} \prod_a Z_{\gamma_a} \sum_{\substack{q, Y_l^{\ell,s} \\ Y_i \cap Y_j = 0, \cup_l Y_l = \Lambda}} A^{\ell,s}(Y_1, f_1, f_2)(1/(q-1)!) \prod_{i=2}^q A^{\ell,s}(Y_i)}{\sum_{\ell,s} \prod_a Z_{\gamma_a} \sum_{\substack{q, Y_l^{\ell,s} \\ Y_l \cap Y_j = 0, \cup_l Y_l = \Lambda}} (1/q!) \prod_{i=1}^q A^{\ell,s}(Y_i)} . \tag{173}$$

Remark.

1) The amplitudes for the clusters depend on the choice l of the large field region, and the choice of all sign assignments, which we denote as $\boldsymbol{s}$. They include a sum over the choices as to which kinds of links may connect the squares within a cluster through a tree. Choices not fulfilling the above mentioned constraints are dismissed (give 0).

2) The difference between the numerator and the denominator in (173) is that in the numerator there is one external polymer depending on the sources f_1 and f_2. Since by the rule of our cluster expansion, each connected component Γ_a of the large field region is contained in exactly one cluster Y, we may absorb each normalization factor Z_{γ_a} into its cluster, defining

$$\widetilde{A}(Y) := A(Y) \prod_{a/\gamma_a \subset Y} Z_{\gamma_a} . \tag{174}$$

3) The simplest possibility for a cluster is the case of a single small field square Δ in s_+ or s_-. By $\boldsymbol{Z}_2$ invariance and (160) we find that the amplitude is the same in the $+$ and $-$ case, and is

$$A_0(\Delta) = 1 + o(N^{-1/10}) .$$

Therefore it is convenient to cancel out the background of trivial single square small field clusters, hence to introduce for a polymer Y the normalized amplitude

$$a(Y) = \frac{\widetilde{A}(Y)}{\prod_{\Delta\subset Y} A_0(\Delta)} . \tag{175}$$

Then we obtain the usual dilute polymer representation:

$$S_2(f_1,f_2) = \frac{\sum_{\ell,s}\sum_{\substack{q,Y_i^{\ell,s}\\ Y_i\cap Y_j=0}} a^{\ell,s}(Y_1,f_1,f_2)\frac{1}{(q-1)!}\prod_{i=2}^{q} a^{\ell,s}(Y_i)}{\sum_{\ell,s}\sum_{\substack{q,Y_i^{\ell,s}\\ Y_i\cap Y_j=0}} (1/q!)\prod_{i=1}^{q} a^{\ell,s}(Y_i)} . \tag{176}$$

To get factorization we must analyze how the choice of ℓ and the sign assignments s affect the cluster amplitudes.

The choice over large field regions is clearly a local one that can be factorized and absorbed in the value of each amplitude. Indeed we can replace the global sums over ℓ and s by local ones:

$$\sum_{\ell,s}\sum_{\substack{q,Y_i^{\ell,s}\\ Y_i\cap Y_j=0}} a^{\ell,s}(Y_1,f_1,f_2)\frac{1}{(q-1)!}\prod_{i=2}^{q} a^{\ell,s}(Y_i)$$

$$= \sum_{\substack{q,Y_i\\ Y_i\cap Y_j=0}} b^{\varepsilon(Y_1,s_1,\ldots,Y_q,s_q)}(Y_1,f_1,f_2)\frac{1}{(q-1)!}\prod_{i=2}^{q} b^{\varepsilon(Y_1,s_1,\ldots,Y_q,s_q)}(Y_i) , \tag{177a}$$

$$\sum_{\ell,s}\sum_{\substack{q,Y_i^{\ell,s}\\ Y_i\cap Y_j=0}} (1/q!)\prod_{i=1}^{q} a^{\ell,s}(Y_i) = \sum_{\substack{q,Y_i\\ Y_i\cap Y_j=0}} (1/q!)\prod_{i=1}^{q} b^{\varepsilon(Y_1,s_1,\ldots,Y_q,s_q)}(Y_i) \tag{177b}$$

with the explanations:

(i) The right sum is over all sets $\{Y_1,\ldots,Y_q\}$, where the Y_i are sets of Δ's, a single Δ being excluded (except if it were an external square containing the sources f_1 and f_2). One has the disjointness or hard core constraints $Y_i\cap Y_j=0$ for $i\neq j$. (One Y_j contains $\hat{\Lambda}$).

(ii) $b^{\varepsilon(Y_1,s_1,\ldots,Y_q,s_q)}(Y)$ is computed from $a(Y)$ through

$$b^{\varepsilon(Y_1,s_1,\ldots,Y_q,s_q)}(Y) = \sum_{(\ell,s)\subset Y} a^{\ell,s}(Y) , \tag{178}$$

where the sum is over all assignments of large field regions *included* in Y and all sign assignments restricted to the squares of Y, subject to certain constraints of compatibility. The sum over large field regions is submitted only to local constraints. More precisely by choosing the factors (30)–(33) we define $l_1(Y)$ as in (35)–(36) but with Λ replaced by Y. Then any assignment for which there exists some $\Delta\in l_1(Y)$ with

$$\operatorname{dist}(\Delta,(\partial Y-\partial\Lambda)) \leqq M$$

is forbidden. For the sign assignments, hence the contour assignments, this implies that all closed contours lie entirely within Y since they are enclosed within one large field connected component. In particular these contours cannot touch the border of

Y as expressed by the previous inequality. Since we know that links from Y cannot jump over contours disjoint from Y we know that once a local configuration of closed contours is chosen entirely within Y, there remain exactly two assignments compatible with it. Now *only one* of them is compatible with the rest of the contour configurations that surround Y and with our $+$ boundary condition. The amplitude $b^{\varepsilon(Y_1,s_1,\dots,Y_q,s_q)}(Y)$ is defined with this particular sign assignment. Because this global sign choice is non-local and depends on all the contour assignments within the other clusters we write it as $\varepsilon(Y_1,s_1,\dots,Y_q,s_q) = \pm$ (indeed it does not only depend on the supports $Y_1,\dots,Y_q$ of the other polymers but also on the contour assignments $s_1,\dots,s_q$ within them).

We conclude that the only non-factorized dependence for the cluster amplitudes in formula (178) is through this global sign $\varepsilon(Y_1,s_1,\dots,Y_q,s_q)$ affecting each cluster, which measures the parity of the number of contours crossed from the global boundary condition to say the outermost point in the cluster. However thanks to the $\boldsymbol{Z}_2$ invariance of the theory and of all our expansion steps we have

$$b^+(Y) = b^-(Y) \equiv b(Y)\,. \tag{179}$$

This sort of "discrete Ward identity" allows complete factorization:

$$S_2(f_1,f_2) = \frac{\sum_{\substack{q,Y_i \\ Y_i\cap Y_j=0}} b(Y_1,f_1,f_2)\frac{1}{(q-1)!}\prod_{i=2}^{q} b(Y_i)}{\sum_{\substack{q,Y_i \\ Y_i\cap Y_j=0}} (1/q!)\prod_{i=1}^{q} b(Y_i)}\,. \tag{180}$$

The Mayer Expansion. Convergence. Equation (180) has now the form required for the application of the Mayer expansion in a standard way. The hard core interaction between the clusters or polymers Y is $V(X,Y)=0$ if $X\cap Y=0$, and $V(X,Y)= +\infty$ if $X\cap Y \neq 0$, and the disjointness constraint for the polymers can be replaced by the inclusion of an interaction $e^{-V(Y_i,Y_j)}$ between each pair of polymers. We use the notations of [9], ch.III.1. A configuration M is an ordered sequence of polymers. We define $b^T(M)$ by

$$b^T(M) = T(M)\left(\frac{1}{q!}\prod_{i=1}^{q} b(Y_i)\right)\,, \tag{181}$$

where the connectivity factor $T(M)$ is defined by

$$T(M) \equiv \sum_{G \text{ connected on } M}\ \prod_{ij\in G} (e^{-V(Y_i,Y_j)}-1)\,. \tag{182}$$

Then we can divide by the vacuum functional to obtain (in the notation of [9])

$$S_2(f_1,f_2) = \sum_{M\ (f_1,f_2)-\text{configuration}} b^T(M)\,, \tag{183}$$

where M is a sequence of overlapping polymers $Y_1,\dots,Y_q$, the first of which contains the support of f_1 and f_2 and includes the factor $F_2(\zeta)$ from (90).

The sufficient condition for the convergence of (183) in the thermodynamic limit is well known: it is a particular bound on the sum over all clusters containing a given square or point to break translation invariance. We state it as

Proposition 3.

$$\Big|\sum_{Y,0\in Y} b(Y)e^{|Y|}\Big| \leqq 1/2 \tag{184}$$

for N sufficiently large, uniformly in Λ, $|Y|$ being the number of squares in Y.

For N large enough, Proposition 3 in fact holds if one replaces the number e in (184) by any other constant.

To deduce convergence of (183) under condition (184) is standard but requires to reorganize the connectivity factor $T(M)$ according to a tree formula. In [9] this is done by choosing the tree in a particular algebraic way, but we can also use again the forest formula (164) to obtain a more symmetric sum over all trees. We define $v_{ij} = (e^{-V(Y_i,Y_j)} - 1)$ for $i \neq j$. We call P the set of pairs $1 \leqq i < j \leqq q$. Expanding $\prod_{(i,j)\in P}(1+v_{ij})$ with the basic forest formula (164) one gets still another forest formula, on which one can read the connectivity factor

$$T(M) = \sum_{T} \prod_{l\in T} \left(v_{i_l j_l} \int_0^1 dh_l\right) \prod_{(i,j)\notin T} (1 + h_T(i,j) v_{ij}) , \tag{185}$$

where $h_T(i,j)$ is the inf of all parameters in the unique path in the tree T joining i to j. This formula is then used e.g like in [9] to derive the convergence of (183). (Remark that every tree coefficient forces the necessary overlaps and is bounded by 1.)

It remains to prove Proposition 3. This is done in Sect. IV.4, using the results of Sect. IV.3, but let us now reduce the proof to certain bounds on functional derivatives generated by the links of the second forest $\mathcal{F}_2 - \mathcal{F}_1$. Because the amplitude $b(Y)$ is given by a tree formula (which is a piece of the forest formula) we will sum over all squares in Y by following the natural ordering of the tree, from the leaves towards the root (the particular square containing 0). The factorial of the Cayley theorem is compensated in the usual way by the symmetry factor $1/|Y|!$ that one naturally gets when summing over all positions of labelled squares. Then the only requirements to complete the proof of (184) are

(i) summable decay of the factor associated to each tree link. This is obvious for the ε_{ij} links of $\mathcal{F}_1$ because these extend only over neighbours, so have (very) bounded range. For the links of $\mathcal{F}_2 - \mathcal{F}_1$, it follows from the decay of the corresponding kernels (167) (see Lemmas 5, 6 and 7). For the links of $\mathcal{F}_3 - \mathcal{F}_2$ the situation is similar as for $\mathcal{F}_1$. The ε'_l exactly prescribe the positions of the Δ'_j in I_l, once those of Δ_i, Δ_j linked by $l \in \mathcal{F}_2 - \mathcal{F}_1$ are known. So there is no more freedom left and for the new squares there are no independent sums over their positions. (It does not matter whether a link prescribes the position of the new square as being that of an adjacent neighbour or whether it keeps it at a prescribed distance and direction).

(ii) A small factor for each tree link, or equivalently for each square of Y. This will compensate for lazy bounds, for factors to choose whether a square is large field or not, etc.... For tree links of $\mathcal{F}_1$ this small factor comes from the one associated to the large field squares, hence from Proposition 2, namely we have a factor $e^{-bN^{1/5}}$ per square in Γ. For the tree links of $\mathcal{F}_2 - \mathcal{F}_1$, it comes from the small factors attached at the ends of these links ("vertices"), and these small factors are described in more detail in the following section. Remark that these two types of small factors tend to zero as $N \to \infty$.

Finally the links of $\mathcal{F}_3 - \mathcal{F}_2$ join together clusters of the $\mathcal{F}_2$ forest. Per such link l we have at our disposal a distance factor from the exponential decay of the kernels in $\mathcal{F}_2 - \mathcal{F}_1$ bounded by

$$e^{-m2M} = N^{-4} , \tag{186}$$

since otherwise these links are not introduced. On replacing $m \longrightarrow m(1-\delta)$ in the exponential we therefore gain

$$e^{-m\delta 2M} = N^{-1/5} \text{ for } \delta = 1/20 . \tag{187}$$

This factor is smaller than what we obtain in the worst case for the links in $\mathscr{F}_2 - \mathscr{F}_1$ (see (202) below). Remark that this last argument would fail if we were to add all squares along $l \in \mathscr{F}_2 - \mathscr{F}_1$, hence if we wanted to build clusters connected in the ordinary sense.

Therefore to complete the proof of Proposition 3 we have simply to look in more detail at the outcome of the cluster expansion derivatives generated in the functional integral by the second type links (of $\mathscr{F}_2 - \mathscr{F}_1$), to extract from them a small factor. This is the content of the next sections.

IV.3 The Outcome of the Derivatives. From now on all h-parameters and links will be exclusively those associated with $\mathscr{F}_2 - \mathscr{F}_1$. The derivatives w.r.t. the h-parameters generated by the non-local kernel links may apply to $d\mu_\gamma(h)$ or to $F_2(h)G_\gamma(h)$. Application with respect to $d\mu_\gamma(h)$ is evaluated by partial integration ([5], chap. 9):

$$\partial_{h_i} \int d\mu_\gamma^Y(h,\zeta)\ldots = \int d\mu_\gamma^Y(h,\zeta)\left\langle \frac{\delta}{\delta\zeta(x)},\ \partial_{h_i} C_\gamma(h)\frac{\delta}{\delta\zeta(y)}\right\rangle \ldots . \tag{188}$$

The supports of the derived kernels, here $\partial_{h_i}S(h)$, are by construction restricted to the squares linked by the h_i derivative (which as we recall adds a link to the previous forest). Therefore the ζ functional derivatives are either directly localized in these squares (in the case where e.g. ∂_{h_l} applies to the first kernel $S(h)$ in $C^{(\gamma)}$, and we consider the $\frac{\delta}{\delta\zeta}$ derivative on the left, or they are essentially localized (when e.g. ∂_{h_l} applies to the first kernel $S(h)$ in $C^{(\gamma)}$ and we consider the $\frac{\delta}{\delta\zeta}$ derivative on the right). In this last case this means that the $\frac{\delta}{\delta\zeta}$ functional derivative is linked to its localization square by the second kernel $S(h)$ in $C^{(\gamma)}$, which has exponential decay. To fix the language we say the $\frac{\delta}{\delta\zeta}$ derivative is

(i) strictly localized (first case) *or* (ii) essentially localized (second case) (189)

in its localization square. In the second case summation over all squares linking to the localization square costs (at most) an additional $O(1/m^2)$, using the exponential decay of $S(h)$.

We apply the h- or ζ- derivatives to the various G_γ from (90). We first shortly comment on $G_{1\gamma}, G_{4\gamma}, G_{5\gamma}$, which are relatively simple to handle and of a technical nature. The remaining terms are treated more extensively.

G1γ. We start with the boundary terms which factorize over the squares Δ and are therefore independent of the h-parameters. Applying ζ-derivatives we find

(i) for $b(\sigma)$ a factor $\theta'(\int_\Delta \sigma)$ which is bounded by $O(1)$ if ζ is an l-square and vanishes for s-squares due to Lemma 14.
(ii) Deriving

$$\frac{\delta}{\delta\zeta(x)}\theta\left(\frac{\|A_\Delta\|_2}{N^\alpha}-1\right) = \theta'\left(\frac{\|A_\Delta\|_2}{N^\alpha}-1\right)N^{-\alpha}\frac{\delta}{\delta\zeta}\|A_\Delta\|_2 \leqq \mathrm{O}(1)N^{-\frac{1}{10}}\left(\alpha=\frac{1}{10}\right).$$

(The first factor restricts Δ to have $3/4\ N^{\alpha} \leqq \|A_{\Delta}\|_2 \leqq 5/4\,\mathrm{N}^{\alpha}$. So by Lemma 13 the last factor is easily bounded by $O(1)$ if the first does not vanish.)

Now we come to $r(\sigma)e^{-\frac{1}{2}\langle\sigma,C_5\sigma\rangle}$. These terms may be replaced by 1 unless the polymer Y contains $\widehat{\Lambda}$. In the latter case the h- or ζ- derivative(s) associated to $\widehat{\Lambda}$ will always produce at least one ζ-field "localized" in $\widehat{\Lambda}$. At this stage we may then take the limit $R \to \infty$ (26) to see that such a polymer gives a vanishing contribution, and we therefore assume from now on

$$\mathrm{Y} \subset \Lambda.$$

C_5 restricted to Y then vanishes as does the restriction of $(\sigma_{\widehat{\Lambda}})^2$, and we may from now on forget about these terms.

$G_{4\gamma}, G_{5\gamma}$ (95, 96, 152). Both terms are at most quadratic in ζ. Derivatives of the quadratic terms with respect to h or ζ generate at most two ζ-fields and a distance factor. This factor is bounded by N^{-1} for $G_{5\gamma}$ due to Lemma 6. We also get N^{-1} for $G_{4\gamma}$ if the derivative is associated to a small field square Δ since (see (44)) dist $(\Delta,\gamma) \geqq \frac{\mathrm{M}}{2}$. Only if a ζ-derivative is associated to some $\Delta \in \Gamma_a$ this distance factor may be $O(1)$. The ζ-fields generated by the derivatives are linked to the squares of the corresponding derivatives by exponentially decreasing kernels, so again are (essentially) localized in these squares. If $\delta_{\zeta(x)}$ hits the linear term in G_5 it generates $\sigma_0\, h_{\Delta}(x)$, which on integrating over the respective square is bounded by $\sim N^{-1}|D|$ (see (60), (70), (71)). ($\mathrm{h}_{\Delta}(x)$ should not be confused with the cluster expansion h's). Successive ζ-derivatives may also hit ζ-fields already descended by derivation. This leads to a factorially increasing number of terms, even more so when treating the subsequent G_{γ}'s(see below). This problem is already present in $P(\varphi)$-theories. It is solved by the so-called local factorial principle (LFP) : the descended ζ's are essentially localized in their squares. These squares have to be different from each other, apart from the case where the derivatives are associated to links in the respective tree which all join the same square. So the maximal number of derivatives localized in the same square is given by the coordination number d of the tree at that square. This implies that the distances to be covered by the subsequent links from this square to their new endpoints have to grow more and more for large d, in two dimensions up to $O(\sqrt{d})$, so that the distance factor of order

$$\prod_{l=1}^{O(\sqrt{d})} e^{-\eta m l} \sim e^{-O(1)\eta m d^{3/2}} \tag{190}$$

becomes smaller than any power of $(d!)$, even if we use only a fraction η of the decay of the kernels. If d is not large many ζ's may still accumulate in one square different from their localization squares, namely if they are only essentially localized there. Then a factor as (190) arises from the exponential decrease of $S(h)$ (167) between these squares and the former one. This latter case, technically not much different from the first, appears only due to our twofold expansion of the covariance $C^{(\gamma)}$ in case of nonempty large field contributions.

If a ζ-derivative hits some already descended ζ-field from G_4 or G_5, the new derivative does not generate a new small factor. So the net small factor per small field derivative is not $\sim N^{-1}$ but $\sim N^{-1/3}$ (one h and two subsequent ζ-derivatives in the worst case). This is similar as in $g\varphi^4$, where the gain per derivative is not g but $g^{1/4}$.

Among the remaining terms we now look at

$$\exp\left\{-\frac{1}{2}\operatorname{Tr}(F_+^2F_+^* + F_+F_+^{*2} + \frac{1}{2}(F_+F_+^*)^2 + (+\leftrightarrow -)\right\}\ (\text{see}\,(52))\ : \tag{191}$$

A ζ- or h- derivative will bring down a factor $\leqq O(\frac{1}{m^3})N^{-1/5} = o(N^{-1/10})$ (see (158)). All descended ζ-fields ($\zeta \equiv \tau$ in $S_\pm$!) are joined via exponentially decreasing kernels to the respective square of the derivative. So the LFP works as before. New derivatives hitting descended ζ's generate an additional $N^{-1/10}$ on replacing (in the estimates) $\int_\Delta g|\zeta| \leqq O(\frac{1}{m})N^{-2/5}$ by $g \sim N^{-1/2}$. The term $(F_+F_+^*)^2$ is suppressed by an additional $N^{-2/5}$ (see (158)).

Now it remains to discuss the determinants $\det_3^{1/2}(1+K_\pm)$, $\det^{1/2}(1+Q)$, $\det_2^{1/2}(1+A_{L_a})$ (see (50)–(55), (94), (97), (152), (155)). For the two $\det_3$-terms we have in principle two choices for the treatment: since the ζ-fields are in the small field region we may expand in powers of $\operatorname{Tr} K_\pm^n$, $n \geqq 3$ and then perform the derivatives. The outcome can be estimated using (159), Lemma 13 and the LFP. This does not work for the other terms since K', A_{L_a} are not bounded. So we use a different method. We will first deal with $\det^{1/2}(1+Q)$, since this is the most complicated term. As remarked above we introduce the h-parameters for every nonlocal kernel and thus also for those appearing in Q (see (167)).

$\det^{1/2}(1+Q)$. Using Lemma 16 we write $\det^{1/2}(1+Q) = \det^{1/4}(1+Q)\det^{1/4}(1+Q^*)$ and apply the h-derivatives generated by the second step of the cluster expansion to the r.h.s. Note that the organization of the expansion terms in trees requires that the derivatives are always applied to symmetric expressions (here the product of the two determinants). Once this is assured there is however no difference in the discussion of Q and Q^*, so we only look at $\det(1+Q)$ and we also forget about the power 1/4, for simplicity. It only corresponds to replace $N \to N/4$ in the following. (If $2N$ does not divide by 4 the purist might treat the remaining fractional power separately...).

Applying the h- and ζ-derivatives to $\det(1+Q)$ reproduces a determinant structure which may also be written using antisymmetric tensor products [13, 14, 15, 4]. This structure, which allows for improved estimates, traces back to the anticommuting fermionic variables or the Pauli principle.

We first regard the case where only h- derivatives are applied. ζ-derivatives are more difficult to describe though obeying sharper bounds. We find

$$\partial_{h_n}\cdots\partial_{h_1}\det(1+Q)$$
$$= \left[n!\operatorname{Tr}\left(\frac{1}{1+Q}Q_1\Lambda\ldots\Lambda\frac{1}{1+Q}Q_n\right) + \text{rd}\right]\det(1+Q) \tag{192}$$

with the explanations :

(i) Λ denotes the antisymmetric tensor product.
(ii) $Q_i := \partial_{h_i}Q$.
(iii) rd stands for rederived terms, i.e. where ∂_{h_j}, $j > i$ applies again to Q_i producing $Q_{ij} = \partial_{h_j}\partial_{h_i}Q$ etc. These terms may also be written as antisymmetric tensor products with less than n entries. A term as in (192) may be estimated as in [14] :

$$\left|n!\operatorname{Tr}\mathop{\Lambda}_{i=1}^{n}\left(\frac{1}{1+Q}Q_i\right)\right|\det(1+Q) \leqq \left(\prod_{j=1}^{n}(1+\lambda_j)^{-1}\prod_{i=1}^{n}\operatorname{Tr}|Q_i|\right)\det(1+Q), \tag{193}$$

where
$$\mathrm{Tr}\,|Q_i| = \mathrm{Tr}\,(Q_i\,Q_i^*)^{1/2}$$
and λ_j, $1 \leqq j \leqq n$, are the n smallest eigenvalues of Q (due to the antisymmetric product we do not get the n^{th} power of the smallest eigenvalue). Then we find

$$\prod_{j=1}^{n} (1+\lambda_j)^{-1} \det(1+Q) = \prod_{j=n+1}^{\infty} (1+\lambda_j) \leqq e^n \,. \tag{194}$$

Note that $\lambda_j > -1$ in finite volume, but the estimate also holds on passing to the infinite volume limit. The last bound in (194) is proven on noting that $\sum_{j=1}^{\infty} \lambda_j = 0$ so that $\sum_{j=n+1}^{\infty} \lambda_j < n$, and using $\ln(1+x) \leqq x$, $x > -1$. The fact that $\mathrm{Tr}\, Q = 0$ was already used in Lemma 16.

Now we look at the expressions for the Q_i. In particular we have to verify that a small factor per small field derivative (sfd) is generated. ∂_h applied to Q does not act on the term $(1+\widehat{K})^{-1}$ unless h connects squares within S_+ or S_-, and it acts only on K' if h connects different regions. $\sum_a A_{L_a}$ is h-independent. So if h connects two S_+-squares we have

$$\partial_h\, Q = [\partial_h (1+\widehat{K})^{-1}]\; K' = \frac{1}{1+K_+}\,(\partial_h K_+)\,\frac{1}{1+K_+}\,P_+ K' (P_- + P_L)\,, \tag{195}$$

due to the support properties of $\partial_h K^+$ and K'. Thus K' has to bridge the gap between $+$ and $-$ or between $+$ and L_a, which is at least of size M. The exponential decay of K then produces a small factor $N^{-2} = e^{-mM}$ through K'. This together with $O(\frac{1}{m})N^{-2/5}$ from (144) leaves a net $\sim N^{-4/3}$ on taking into account N from $\mathrm{Tr}\,|Q_i|$. If ∂_h acts on K' we get the small factor directly from $\partial_h K'$. We need however the decay of $\partial_h K'$ also to sum over the (squares of) Y for given tree T. It is therefore essential that we keep a factor

$$e^{-\eta . m . l(T,Y)}\,, \tag{196}$$

where $0 < \eta < 1$ and
$$l = \sum_{(i,j)} \mathrm{dist}\,(\Delta_i, \Delta_j)\,, \tag{197}$$
the sum being over the links (i,j) of the tree linking the squares from Y. The simplest way of doing is thus to keep only $\mathrm{N}^{-1/3}$ per small field derivative and choose $\eta \leqq 1/2$, e.g. $\eta = 1/3$, so as not to use up the decay twice. The reader should note that the factor (196) can always be extracted to wherever the derivatives apply (and always by the same reason). But we only mention it here when treating $\det(1+Q)$. Another fraction of the decay is used up on application of the LFP (see (190) and below). Whereas the latter comes into play for all parts of the interaction, the previous splitting is only necessary in those cases, where the small factor per sfd is generated by the large gap between different regions (see $G_{4\gamma}\, G_{5\gamma}$ above).

Now we look at the rd-terms: the (severe) restrictions on the supports of the (derived) kernels already effective in (195) will force many of these rd-terms to vanish (a fact which do not exploit completely). Any new sfd produces a new $O(\frac{1}{m})N^{-2/5}$ as before (and maybe further small factors). Note that there is no new $\mathrm{Tr}\,|Q_i| \sim N$ to beat! Still one realises that the number of terms, when taking into account all rd-terms, may grow as

$$\sim\; C^n\, n!$$

Again we prepare to use the LFP noting first

(i) the support restrictions.
(ii) the exponential decay of the kernels of $(1+K_\pm)^{-1}$. This decay is due to the small field condition (Lemma 13) which enforces rapid convergence of the geometric series. The decay constant is of order $m - O(\frac{1}{m})N^{-2/5}$.
(iii) the kernels $\frac{1}{1+\sum_a A_{L_a}}$ reduce to $\frac{1}{1+A_{L_a}}$ if joined to some $\partial_h K$ with support in L_a (to 1 otherwise).

Taking this into account one realizes that the sum over all possibilities of rederiving propagators is controlled by increasing distance factors as in (190). The situation is exactly the same as in (190) if we neglect the terms $(1+\widehat{K})^{-1}$ in a first step. Rederiving also those we even get a factorially increasing number of non-vanishing contributions when the coordination numbers of T do not grow large. Then (ii), (iii) come into play: If $(\partial_{h_{i+1}} K_\pm^{(\prime)})$ is to be grouped together with $\partial_{h_1} K_\pm^{(\prime)}, \dots, \partial_{h_i} K_\pm^{(\prime)}$ from previous derivatives, its support has to be more and more distant from that of most of the $\partial_{h_1} K_\pm^{(\prime)}, \dots, \partial_{h_i} K_\pm^{(\prime)}$ from previous derivatives for i large, and we get again an estimate as in (190).

We shortly mention that derivatives linking to lfs and thus applying to K' may generate large numbers of ζ-fields in lfs. These are estimated with the help of (129), or with (136) and the LFP, if they accumulate in a single square. A finer analysis would also display improved estimates due to the inverted operators, compensating effects of these fields.

The last point to mention concerns the complications caused by the ζ-derivatives. They give a smaller contribution per sfd, since $O(\frac{1}{m})N^{-2/5}$ may be replaced by $g \sim N^{-1/2}$. The complication comes again from the fact that they are attached to the derived covariance $\partial_h C_\gamma^Y$ (see (167)). Since

$$\partial_{h_i} C^{(\gamma)} = \left(\frac{\partial S}{\partial_{h_i}}\right) \widehat{C}_\gamma S(h) + S(h)\, \widehat{C}_\gamma \frac{\partial S}{\partial_{h_i}} \tag{198}$$

(restricted to the polymer Y in question) we get two terms if the derivative is associated to large field squares, i.e. to the suppressed region. Regarding e.g. the first we thus have to replace the kernels $Q_i(y,x)$ from (192) by

$$\widehat{Q}_i\,(y,x,z_i) = \int \frac{\delta Q(y,x)}{\delta\zeta(y')}\, \frac{\partial S}{\partial h}\,(y',z_i)\, d^2 y' \,. \tag{199}$$

The kernel $(\widehat{C}_\gamma S(h))(z_i, w)$ is then taken out of the respective tensor product which now depends on the parameter z_i. The integration over the z_i may be performed later using the exponential decay of $S(h)$. The second ζ-derivative $\frac{\delta}{\delta\zeta(w)}$ may apply again to det $(1+Q)$ or to any other $G_i\gamma$ a fact which admittedly makes it hard to present the expansion terms by explicit expressions. This does not alter the bounds, however, or change the mechanisms by which we control the expansion and which were described when treating the h-derivatives.

To resume we get from deriving n times det $(1+Q)$ an expression which may be bounded by

(i) a factor $\leqq O(1/m)N^{-2/5}$ per sfd.
(ii) a factor $e^{-\eta' m\, l(T)}, \eta' \leqq 1/2$, e.g. $\eta' = 1/4$.

(iii) a (large) constant $(O(m^{-2}))^n$ generated by the integrations over kernels and from the LFP bounds and which also contains the combinatoric and all other m-independent constants. (ζ-fields from the V-region generated by derivatives are also included in this bound once they have been integrated with the aid of (136).)
(iv) a factor of $1+\delta\,(\delta \ll 1)$ per sfd and a factor of $1+O(m^2|L_a|)$ per large field block L_a.

These factors come from bounding the terms generated by $(1+K_+ + K_- + \sum_a A_{L_a})^{-1}$ on iterated application of derivatives, taking into account support restrictions (see above (195)). The bound on the operator norm $\|(1+K_\pm)^{-1}\|$ follows from the small field condition, the other one from

Lemma 18. $0 \leqq (1+A_{L_a})^{-1} \leqq 1 + O(m^2|L_a|)$ *in the operator sense on* $\mathscr{L}^2(L_a)$.

Proof. We have

$$\begin{aligned} 1 + A_{L_a} &= 1 + g^2\,\sigma_{L_a}\,\frac{1}{p^2+m^2}\,\sigma_{L_a} - P_{L_a}\,\frac{m^2}{p^2+m^2}\,P_{L_a} \\ &\geqq 1 - P_{L_a}\,\frac{m^2}{p^2+m^2}\,P_{L_a}\,; \end{aligned}$$

now one verifies

$$p^2 + m^2 \geqq m^2 + O(1)\,|L_a|^{-1} \tag{200}$$

on E, where E is the space of smooth functions with support in $L_a \backslash \partial L_a$. E is dense in $\mathscr{L}^2(L_a)$ as is $\widetilde{E} := ((p^2+m^2)(E)) \subset E$, and on $\widetilde{E}$ (from (200))

$$\frac{1}{p^2+m^2} \leqq \frac{1}{m^2 + O(1)\,|L_a|^{-1}}\,,$$

which then holds by continuity on $\mathscr{L}^2(L_a)$.

Therefore

$$P_{L_a}\frac{m^2}{p^2+m^2}\,P_{L_a} \leqq \frac{m^2}{m^2+O(1)\,|L_a|^{-1}}$$

and

$$(1+A_{L_a})^{-1} \leqq \left(1 - \frac{m^2}{m^2+O(1)\,|L_a|^{-1}}\right)^{-1} = 1 + O(m^2\,|L_a|)\,.$$

For (very) large $|L_a|$ these factors are easily controlled by (135), (136). QED

$\det_3^{1/2}(1+K_\pm)$. The essential modification comes from the $\det_3$. Taking one h-derivative we obtain

$$\partial_h \det_3(1+K) = \mathrm{Tr}\left(\frac{1}{1+K}\,K^2\partial_h K\right)\det_3(1+K)\,.$$

Furthermore n derivatives produce

$$\left\{n!\,\mathrm{Tr}\left(\frac{1}{1+K}\,(K^2\partial_{h_1}K)\,\Lambda \ldots \Lambda\,\frac{1}{1+K}\,(K^2\partial_{h_n}K)\right) + \mathrm{rt}\right\}\det_3(1+K)\,, \tag{201}$$

to be compared to (192) (see also [4]).

So the essential modification is the replacement $\partial_h K \longrightarrow K^2 \partial_h K$. The small factor per sfd is now again $O(m^{-3}) N^{-1/5}$ (see 158), (192)) (ζ-derivatives produce an additional $N^{-1/10}$). Note that the small factor would not have been produced on expanding det or $\det_2$ but comes from the third power of K, each power contributing $N^{-2/5}$. The remaining terms 'rt' from (201) either stem from rederiving the $K^2 \partial_h K$ which is now possible at most once with respect to h (using support restrictions) or five times with respect to ζ. Each ζ-derivative produces a new $N^{-1/10}$ whereas a second ∂_{h_2} applied to $K^2 \partial_{h_1} K$ does not bring down a new small factor (unless it vanishes). Thus in the worst case we only get

$$O(m^{-3}) N^{-1/5} = o(N^{-1/10}) \tag{202}$$

per 2 h-derivatives. These terms, however, are not accompanied by large combinatoric factors and have additional distance decay due to the trace conditions.

Among the rt-terms we also count the correction terms appearing when we derive again $(1+K)^{-1}$ and then want to bring the result back into the form $(1+K)^{-1} K^2 \partial_h K$, e.g.:

$$\begin{aligned}
\partial_{h_2} \operatorname{Tr}\left(\frac{1}{1+K} K^2 (\partial_{h_1} K)\right) &= -\operatorname{Tr}\left(\frac{1}{1+K} \partial_{h_2} K \frac{1}{1+K} (K^2 \partial_{h_1} K)\right) + \ldots \\
&= -\operatorname{Tr}\left(\frac{1}{1+K} K^2 (\partial_{h_2} K) \frac{1}{1+K} K^2 \partial_{h_1} K\right) - \operatorname{Tr}\left((\partial_{h_2} K) \frac{1}{1+K} K^2 \partial_{h_1} K\right) \\
&\quad + \operatorname{Tr}\left(K (\partial_{h_2} K) \frac{1}{1+K} K^2 \partial_{h_1} K\right) + \ldots .
\end{aligned}$$

The first term is of appropriate shape to be grouped together with the term where the derivative applies to $\det_3 (1+K)$. The other terms are estimated separately and contain one or two supplementary $O(N^{-2/5})$.

We note that it is not really necessary to perform this regrouping of terms and to use the antisymmetric tensor product structure here, since (the kernel of) $(1+K)^{-1}$ is bounded (and exponentially decreasing) in the small field domain. So the LFP is sufficient to estimate the sum of all terms. The rest of the discussion is analogous as (but somewhat simpler than) that of $\det(1+Q)$. So we stop here.

$\det_2^{1/2}(1+A_{L_a})$. The treatment is as before, but the A_{L_a} are independent of h-parameters. ζ-derivatives produce antisymmetric tensor products of

$$\frac{1}{1+A_{L_a}} A_{L_a} \frac{\delta A_{L_a}}{\delta \zeta} \tag{203}$$

joined to $\partial_h C_\gamma^Y$'s, which always have to bridge a large gap, producing an N^{-2}, which we do not really need, however, in the large field domain. One $A_{L_a} \frac{\delta A_{L_a}}{\delta \zeta}$ may be rederived at most three times, any rederivation being accompanied again by a negative power of N through the distance gaps. Lemma 18 can be used to bound $(1+A_{L_a})^{-1}$ as before.

IV.4. Results. Now we come to the

Proof of Proposition 3. We have to bound a $(Y^{l,s})$ for given $Y^{l,s}$ consisting of n_s small field squares Δ_j and of n large field blocks Γ_ι with fixed assignments s and

$$\cup \Gamma_\iota = \Gamma \subset Y,\ \cup \Delta_j = S \subset Y,\ Y = S \cup \Gamma \tag{204}$$

(i.e. all regions are restricted to $Y = Y^{l,s}$). From the previous sections $a(Y^{l,s})$ is bounded by a product of factors:

(1) (i) a small factor $\leqq (m^{-3} N^{-1/5})^{n_s/2}$ (see (202)).
(ii) $\exp\,(O(1)N^{-1/10}\, n_s)$ (see Proposition 2, Corollary 1) from the small field region.
(2) $a(Y^{l,s})$ contains a sum over trees T. For given tree we have in the bound a factor

$$\exp\left(-\eta m\, l\,(T, Y)\right) \quad \text{with (e.g.)}\ \eta = 1/3$$

(see (196)).
(3) In the large field region we get a bound

$$\exp\left\{-\frac{b}{2}\,|\Gamma|\,N^{1/5}\right\}, \tag{205}$$

which may be deduced as follows:
From Prop. 2 and (136) we have a bound in the large field domain of

$$\begin{aligned}\exp\Bigg\{&-b\,|\Gamma|\,N^{1/5} + (\ln 2)\,|\gamma|\,N^{1/10} - \frac{1}{100}\,|V|\,N \\ &+ O\left(\frac{1}{N}\right)\left(|L|\,(\|\zeta_L\| + 1) + \|\zeta_L\|^2 - \frac{1}{60}\,\|\zeta_v\|^2\right)\Bigg\}\,.\end{aligned} \tag{206}$$

Here we used only a fraction of (136). The rest is used to integrate and then bound the contribution of the ζ-fields descended by derivatives in the V-region. ζ-fields descended in $l\backslash V$ region are bounded by (129). From those and from Lemma 18 we get factors $\sim O(|L_a|)$ per L_a. All those contributions are easily incorporated in (205) since we have replaced $b \longrightarrow \frac{b}{2}$.
(4) Finally we have a product of $R_1{}^{n_s} R_2{}^{n_l}$ with N-independent constants R_1, R_2. These terms take care of all combinatoric factors from the choices onto which $G_{i\gamma}$ to apply the h- and ζ-derivatives and how to apply them within each $G_{i\gamma}$. They take into account the negative powers of m^2, generated by applying the LFP in its possible forms. This also includes the integration of the descended ζ-fields in the (highly suppressed) V-region. Finally estimates of the type (194) and the bounds on the kernels in Lemmas 5 and 7 contribute to $R_{1,2}$. Just taking maximal values everywhere leads to tremendous constants R_1, R_2. For simplicity and laziness we do not optimize them. Note, however that any constant generated per lfd can be easily absorbed in the strong bound (205), contributions from V are really tiny. Finally putting $m^{3/2}N^{-1/10}$ per sfd in (1) corresponds to the worst case, appearing only in (202) for a single choice of applying the derivatives (and which for many trees is only allowed much less than n_s times). All other choices imply additional suppression factors of $N^{-1/20}, N^{-1/10}, N^{-1/2}, \ldots$ per derivative which then beat the contributions to $R_1^{n_s}$ completely.

To pass from $a(Y^{l,s})$ to $b(Y)$ we have to sum over the admitted assignments l, s for Y. By bold overestimation this leads to an additional $3^{|Y|}$.

In fact, however, picking any square out of Y and assuming that the assignment of all other squares is fixed – no matter how – there is at most(!) one assignment for this square such that it is in $Y \setminus \Gamma$. If it is in Γ we have the factor $e^{-bN^{1/5}}$ at our disposal and may assume that the change $b \to b/2$ in (205)–(206) also involves taking away $\delta \ll 1$ from $e^{-bN^{1/5}}$ (i.e. add ln $\delta|\Gamma|$ in the exponent of (206)). Then we may replace 3 by $1+2\delta$ and include this factor in R below (207).

Collecting everything we thus find

$$\begin{aligned}
&\left|\sum_{O\in Y\subset\Lambda} b(Y)\,e^{|Y|}\right| \\
&\leqq \left|\sum_{O\in Y^{l,s}\subset\Lambda, l,s}\ \sum_{T} e^{-1/3\, m\, l(T,Y^{l,s})}\,(eR)^{|Y|}\left(m^{-\frac{3}{2}}N^{-\frac{1}{10}}\right)^{n_s}\exp\left(-\frac{b}{2}|\Gamma|\,N^{1/5}\right)\right. \\
&\leqq \sum_{n\geqq 2}\left(R'N^{-\frac{1}{10}}m^{-\frac{7}{2}}\right)^{n} \leqq 1/2 \qquad (207)
\end{aligned}$$

for N sufficiently large, with the above remarks on $R(\sim R_1)$ and $R' = O(1)\,eR$. The sum over the trees T is performed as in [9], Lemma III.1.4, using Cayley's theorem. QED

As noted in Sect. IV.2, the bound of Prop. 3 is sufficient to prove the convergence of the series (184) for $S_2(f_1, f_2)$ in the infinite volume limit. Our aim is now to prove also that this series decays exponentially with the distance of the supports of f_1 and f_2.

Our main result is

Theorem. *The infinite volume two point function decays exponentially*:

$$|S_2(f_1, f_2)| \leqq O(1)\exp\{-m'\,\mathrm{dist}\,(f_1, f_2)\}\,, \qquad (208)$$

where

$$m' = m(1+|O(1)m| + o(N^{-1/10})) \qquad (209)$$

dist $(f_1, f_2) = \inf\{|x_1 - x_2|; x_{1,2} \in \mathrm{supp}\, f_{1,2}\}$. *$O(1)$ is an N-independent number. The estimate on m' is (of course) not optimal. The term proportional to m^2 is due to the UV-cutoff (Lemma 5).*

Proof. When treating external polymers we have to apply the cluster expansion also to

$$F_2(\zeta) = \left\langle f_1, \frac{1}{\not p + g\sigma} f_2\right\rangle .$$

There is a slight technical nuisance with this term, since the cluster-expansion as applied here requires the expanded kernels to be symmetric in (x, y).

Remembering that $1+K = \left(1 + g\tau\frac{1}{-\not p+m}\right)\left(1+\frac{1}{\not p+m}g\tau\right)$ (24), we therefore write:

$$\left\langle f_1, \frac{1}{\not p + g\sigma} f_2\right\rangle = \left\langle r, \frac{1}{1+K} f_2\right\rangle + \left\langle s(\tau), \frac{1}{1+K} f_2\right\rangle , \qquad (210)$$

where

$$r = \frac{1}{-\not p + m} f_1, \quad s(\tau) = \frac{1}{\not p + m} g\tau \frac{1}{-\not p + m} f_1 ,$$

so that (210) equals:

$$\sum_{\Delta'}\left\langle r_{\Delta'}, \frac{1}{1+K} f_2\right\rangle + \sum_{\Delta'\Delta''}\left\langle s_{\Delta''}(\tau_{\Delta'}), \frac{1}{1+K} f_2\right\rangle \tag{211}$$

with

$$r_{\Delta'} = \chi_{\Delta'}\frac{1}{-\not{p}+m} f_1, \quad s_{\Delta''}(\tau_{\Delta'}) = \chi_{\Delta''}\frac{1}{\not{p}+m} g\,\tau_{\Delta'}\frac{1}{-\not{p}+m} f_1\,,$$

and the sums are over all squares which form Λ.

From the exponential decay of $(\pm\not{p}+m)^{-1}$ (and the established estimates to bound τ'_{Δ}) it is then obvious that the Theorem holds for $F_2(\zeta)$ if it holds on replacing F_2 by

$$\widetilde{F}(\zeta) = \left\langle f_1, \frac{1}{1+K} f_2\right\rangle\,, \tag{212}$$

since the sums in (211) are easily estimated. So we replace $F_2 \to \widetilde{F}$ in the following. As we know from Sect. IV.3, $1+K$ may approach 0 for large volume. We therefore group together

$$\widetilde{F}(\zeta)\det{}_2^{1/N}(1+K) =: \widetilde{F}(\zeta)\det{}_2(1+k)\,, \tag{213}$$

where $\det_2^{1/N}(1+K)$ may be isolated from (53), thereby replacing $N \to N-2$ in the following. The power $1/N$ just means that we do not take a flavour trace. For an external polymer we now get also contributions from applying derivatives to (213). The treatment is analogous to that of the determinants in Sect. IV.3. But the contributions due to sfd are now much smaller since we do not have a factor N from the traces to beat. Bounding the expression analogous to (192) we find (see also [14, 15])

$$\begin{aligned}\left|\prod_{i=1}^{n}(1+\kappa_i)^{-1}\det{}_2(1+k)\right| &\leqq \prod_{i=n+1}^{\infty}(1+\kappa_i)e^{-\sum_{i=1}^{\infty}\kappa_i} \\ &= \exp\sum_{i=n+1}^{\infty}(\ln(1+\kappa_i)-\kappa_i)e^{-\sum_{i=1}^{n}\kappa_i} \leqq e^n\,,\end{aligned} \tag{214}$$

where κ_i are the eigenvalues of $\kappa(\zeta)$, ordered as $\kappa_1 \leqq \kappa_2 \leqq \ldots$, and we used $\kappa > -1$ (in finite volume).

To prove exponential decay we have to look at the external polymer Y contained in $b^T(M)$ (184). We have $\Delta(f_1), \Delta(f_2) \subset Y$, where $\Delta(f_1), \Delta(f_2)$ contain the supports of f_1, f_2. By definition of Y there is for any tree T a sequence of links joining the two squares which contain these supports (in any assignment $Y^{l,s}$ of Y). We first suppose Y to be assigned such that l is empty, which is the dominant contribution. The problem consists in putting aside the distance decay between $\Delta(f_1), \Delta(f_2)$ without invalidating the convergence proof. We proceed as follows: For any tree T we split up

$$T = T' \cup T''\,, \tag{215}$$

where T' is made of a *minimal* number of links in T joining $\Delta(f_1), \Delta(f_2)$. Thus the tree T' has at least one link, and all of its coordination numbers d'_i fulfill: $d'_i \leqq 2$. Corresponding to T' we put aside a factor

$$\varepsilon^{|T'|}\prod_{l'\in T'} P_{l'}(x_{l'}, y_{l'})\,, \tag{216}$$

where $|T'|$ is the number of links l' in T', and $\varepsilon = O(1)m^{-3}N^{-1/5} = o(N^{-1/10})$. (One realizes that the special choice of derivatives (202) does not have to be taken into account here. Otherwise we had to replace $\varepsilon \to \varepsilon^{1/2}$). $P_{l'}(x_{l'}, y_{l'})$ is a shorthand notation for the kernels associated to the links $l' \in T'$ and generated by the respective derivatives. All these kernels decay at least exponentially with constant m (even with $m + |O(1)m^2|$) according to Lemma 5. And so the product in (216) is bounded in modulus by

$$\varepsilon^{|T'|} \prod_{l \in T'} \left(\frac{1}{p^2 + m^2} \right) (\operatorname{dist}(\Delta_{a(l')}, \Delta_{l'})) \leqq (O(1)\varepsilon)^{|T'|} \prod_{l \in T'} e^{-m \operatorname{dist}(\Delta_{a(l')}, \Delta_{l'})}, \tag{217}$$

where $\Delta_{a(l')}, \Delta_{l'}$ are the squares linked by l', and we used the kernel of $(p^2 + m^2)^{-1} : (\frac{1}{p^2+m^2})(x - y)$. For shortness we call $a'(Y, T, T')$ the remainder of the amplitude of Y for given T.

The important point is that the LFP may still be used to bound the remainder of $a(Y, T)$ without using up (217): the LFP is only needed (in a small field polymer), when the tree T has large coordination numbers $d_j \gg 1$. But since $d'_i \leqq 2$, the coordination numbers of T'' fulfill $d''_j \geqq d_j - 2$, and the reduction by at most 2 is unimportant in (190). So the LFP works as before. We estimate

$$\left| \sum_T \sum_Y a(Y, T) \right| \leqq \sum_{T,Y} \varepsilon^{|T'|} \sup_{\substack{x \in \Delta(f_1) \\ y \in \Delta(f_2)}} \left(\frac{1}{p^2 + m^2} \right)^{|T'|} (x - y) \, |a'(Y, T, T')| \tag{218}$$

$a'(Y, T, T')$ is the rest of $a(Y, T)$ after taking out the P_l. It thus includes also all choices as to which the kernel P_l may be. All these kernels are bounded in modulus by the one of $(p^2 + m^2)^{-1}$ (Lemmas 5,7). Finally we used in (218) the pointwise positivity of the kernels of $(p^2 + m^2)^{-n}$ to estimate for any $O \subset \mathbb{R}^2$,

$$\int_O \left(\frac{1}{p^2 + m^2} \right) (x - y) \left[\left(\frac{1}{p^2 + m^2} \right)^n \right] (y - z) d^2 y \leqq \left[\left(\frac{1}{p^2 + m^2} \right)^{n+1} \right] (x - z), \tag{219}$$

etc.

Using (219) the kernel in (218) has already been freed from the positions of the intermediate squares between $\Delta(f_1), \Delta(f_2)$.

When summing over $|T'|$ we obtain for (218) the estimate

$$\left(\sum_{T,Y} \sup_{T'(T)} |a'(Y, T, T')| \right) \sum_{n \geqq 1} \sup_{\substack{x \in \Delta(f_1), \\ y \in \Delta(f_2)}} \left(\frac{\varepsilon}{p^2 + m^2} \right)^n (x - y), \tag{220}$$

and the right sum satisfies by geometric expansion and pointwise positivity

$$\begin{aligned} &\sup_{\substack{x \in \Delta(f_1), \\ y \in \Delta(f_2)}} \left(\frac{1}{p^2 + m^2 - \varepsilon} - \frac{1}{p^2 + m^2} \right) (x - y) \\ &\leqq O(\varepsilon) \sup_{\substack{x \in \Delta(f_1), \\ y \in \Delta(f_2)}} \exp\left\{ -m \left(1 - \frac{\varepsilon}{m} \right) (x - y) \right\}. \end{aligned} \tag{221}$$

We note at this stage that if some of the links of T' come from the third cluster expansion the small factor associated to them is taken from the exponential decay (see 187)). Since the position of the respective squares is fixed, however, we do not need their small factors ε and take out the distance factor directly without integrating as in (219). n in (220) is then the number of second kind links in T', and the result remains unaltered.

(When taking into account the effect of the UV regularization we may replace $m \to m(1 + |O(1)m^2|)$. The bound on the first sum in (220) is then achieved as that of Proposition 3, by our previous discussion – with one more explanation: if any of the $\Delta_{a(l')}, \Delta_{l'}$, linked by $l' \in T'$, appears only in links of T', but not in T'' – so that no decay is available to sum over its position in the first sum in (220), then we do *not* sum over its position in (220): this sum has already been taken account of by the bound (218). Instead of summing we only take the sup over all possible squares. (If there are additional links we may or may not sum.)

It is of course not true that S_2 vanishes for $\varepsilon \to 0\,(N \to \infty)$. The dominant contribution comes from (211)

$$\sum_{\Delta'} \left\langle r_{\Delta'}, \frac{1}{1+K} f_2 \right\rangle \text{ for } Y = \Delta(f_2) = \Delta' \,.$$

This contributes

$$\left\langle f_1, \frac{1}{\not{p}+m} \left(\frac{1}{1+K}\right)_Y f_2 \right\rangle = \left\langle f_1, \frac{1}{\not{p}+m} f_2 \right\rangle (1 + O(N^{-2/5}))\,. \tag{222}$$

Finally we consider the case when Y contains also large field squares. $\Delta(f_1), \Delta(f_2)$ may now be contained in two blocks $\Gamma_1 \neq \Gamma_2$, where at least one is large field, or in a single field block Γ. For large field blocks

$$\sup_{\substack{x \in \Gamma_1, \\ y \in \Gamma_2}} \left(\frac{1}{p^2+m^2}\right)^n (x-y)$$

or

$$\sup_{x,y \in \Gamma} \left(\frac{1}{p^2+m^2}\right)(x-y)$$

need not decrease. But in this case we may take the missing decay (and much more) from

$$\exp\left(-\frac{b}{2}|\Gamma_i|N^{1/5}\right), \exp\left(-\frac{b}{2}|\Gamma|N^{1/5}\right),$$

i.e. from the large fields suppression. The large field suppression also is (more than) sufficient to make the LFP work in the situations which only appear in large field polymers (Sect. IV.3). The essential reason that large field polymers do not pose any problem, is in the fact that the large field suppression exponential has not only a much larger coefficient than m, but is also proportional to $|\Gamma|$, and $|\Gamma| \gg \sup_{x,y \in \Gamma} \mathrm{dist}(x-y)$. So we stop here.

After extracting the exponential decay from a (Y, T) the convergence proof for (184) follows from Proposition 3 as in [9], ch. III.1. QED

We finish with the promised statement on the expectation value of σ.

Proposition 4. $\langle\sigma_\Delta\rangle = \frac{m}{g}\left(1 + o(N^{-1/20})\right)$.

Proof. We proceed as before. An external polymer is now one containing the (fixed) square Δ. If Δ is in S_+ and Y consists of Δ only, we have for the amplitude (see (160))

$$a(\Delta) = \frac{m}{g}\left(1 + o(N^{-1/10})\right),$$

since $\sigma_\Delta = \left(\frac{m}{g}\right)_\Delta + \zeta_\Delta$, and $\langle\zeta_\Delta\rangle \sim O(1)$ in s_+.

For larger polymers we may gain as before $o(N^{-1/20})$ per sfd, large field polymers are exponentially suppressed. The crucial point is the following: If $\Delta \in s_-$, the boundary conditions imply $\Gamma \neq \emptyset$. So this assignment, which potentially would give $\sim -\frac{m}{g}$, is always exponentially suppressed. QED

Conclusions

The Gross–Neveu-Model plays a prominent role in the program of constructive field theory. The reason is that it shares two important features with four dimensional gauge theories, though being technically much simpler. These are UV asymptotic freedom and a nonperturbative mechanism of mass generation. Gauge theories could so far only be studied in the UV regime by constructive methods (and this only near the edge of the region where technicalities become prohibitive) [16]. This paper starts the analysis of the Gross–Neveu-Model with discrete chiral symmetry from the IR-viewpoint. Constructive methods presently require a small expansion parameter which is here $\frac{1}{N}$ and which has to be unfortunately unrealistically small, but nevertheless finite! We prove the exponential decay of the two-point function and the existence of (at least) two pure phases.

As regards the UV part of the problem, the massive Gross–Neveu-Model (on introducing a bare mass by hand-also called Mitter–Weisz-Model [17]) has not only been constructed for an $N > 2$ and small renormalized coupling [2, 3], but also analyzed in more detail. Thus the Wilson–Zimmermann short-distance expansion and asymptotic completeness in the two-particle region have been established [18]. For (again very) large N the UV analysis even includes the perturbatively nonrenormalizable three-dimensional model [4]. As regards the future, it remains to close the gap between the UV and IR constructions by taking away the UV cutoff from the latter.

We also intend to analyze the model with continuous chiral symmetry whose infrared structure is certainly richer and more complicated. To go even further one would hope to get a complete picture of the phase structure of the large N models and to make contact with the results obtained using the so-called complete integrability of the Gross–Neveu Model. These results, while based on a certain number of unproven assumptions, provide a rich amount of information [19]. From the technical point of view the methods are completely separate from ours.

Acknowledgement. Christoph Kopper would like to thank the Theory Group at Ecole Polytechnique, especially the coauthors, for kind hospitality, and the CNRS for financial support during his stay in Palaiseau in 1991. We thank the referee for his very careful reading of this work and his comments, which led us to rewrite Sects. IV.1 and IV.2.

References

1. Gross, D.J., Neveu, A.: Dynamical symmetry breaking in asymptotically free field theories. Phys. Rev.D **10**, 3235 (1974)
2. Gawedzki, K., Kupiainen, A.: Gross–Neveu model through convergent perturbation expansions. Commun. Math. Phys. **102**, 1 (1986)
3. Feldman, J., Magnen, J., Rivasseau, V., Sénéor, R.: A Renormalizable Field Theory: The Massive Gross–Neveu Model in two Dimensions. Commun. Math. Phys. **103**, 67 (1986)
4. Faria da Veiga, P.A.: Construction de modèles non renormalisables en Théorie quantique des champs. Thesis, Ecole Polytechnique, (1991)
5. Glimm, J., Jaffe, A.: Quantum Physics, Berlin, Heidelberg, New York: Springer, 1987
6. Reed, M.C.: In: Constructive Field Theory. Proc. Erice 1973, Lecture Notes in Physics **25**, (1973)
7. Simon, B.: Trace Ideals and their Applications. London Mathematical Society Lecture Note Series **35**, Cambridge: Cambridge Univ. Press, 1979
8. Ruelle, D.: Statistical Mechanics – Rigorous Results. New York: Benjamin, 1969
9. Rivasseau, V.: From Perturbative to Constructive Renormalization. Princeton, NJ: Princeton University Press, 1991
10. Brydges, D.: In: Critical Phenomena, Random Systems, Gauge Theories. Proc. Les Houches 1984. Amsterdam: North Holland, 1986
11. Dunlop, F., Magnen, J., Rivasseau, V.: Mass generation for an interface in the mean field regime. Ann. de l'I.H.P. **58**, 333 (1993)
12. Battle, G., Federbush, P.: A phase cell cluster expansion for Euclidean field theories. Ann. Phys. **142**, 95 (1982)
13. Reed, M.C., Simon, B.: Methods of Modern Mathematical Physics IV. New York: Academic Press, 1979
14. Seiler, E.: Schwinger functions for the Yukawa Model in two dimensions with space time cutoff. Commun. Math. Phys. **42**, 163 1975
15. Seiler, E., Simon, B.: Bounds on the Yukawa_2 Quantum Field Theory: Upper bound on the pressure, Hamiltonian bound and linear lower bound. Commun. Math. Phys. **45**, 99 (1975)
16. Balaban, T.: Large Field Renormalization II: Localization, Exponentiation and bounds for the R-operation. Commun. Math. Phys. **122**, 355 (1989), and references cited there; Magnen, J., Rivasseau, V., Sénéor. Constructions of YM_4 with an infrared cutoff. Commun. Math. Phys. **155**, 325 (1993)
17. Mitter, P., Weisz, P.: Asymptotic scale invariance in a massive Thirring model with $U(n)$-symmetry. Phys. Rev. **D8**, 4410 (1973)
18. Iagolnitzer, D., Magnen, J.: Asymptotic completeness and multiparticle structure in field theories II: Theories with renormalization. Commun. Math. Phys. **111**, 81 (1987); Iagolnitzer, D., Magnen, J.: Large momentum properties and Wilson short distance expansion in non-perturbative field theory. Commun. Math. Phys. **119**, 609 (1988)
19. Zamolodchikov, A.B., Zamolodchikov, Al.B.: Relativistic Factorized S-Matrix in two dimensions having $O(N)$ isotopic symmetry. Nucl. Phys. **B133**, 525 (1978); Karowski, M., Thun H.J.: Complete S-Matrix of the $O(2N)$ Gross–Neveu model. Nucl. Phys. **B190**, 61

(1981); Forgacs, P., Niedermayer, F., Weisz, P.: The exact mass gap of the Gross–Neveu model I. The thermodynamic Bethe ansatz. Nucl. Phys. **B367**, 123 (91)
20. Glimm, J., Jaffe, A., Spencer, T.: Phase transitions for ϕ_2^4 quantum fields. Commun. Math. Phys. **45**, 203–216 (1975)
21. Balaban, T., Gawedzki, K.: A low temperature expansion for the pseudo-scalar Yukawa model of quantum fields in two space-time dimensions. Ann. Inst. Henri Poincaré, Sect. A **36**, 271–400 (1982)
22. Brydges, D., Kennedy, T.: Mayer expansions and the Hamilton–Jacobi Equation. Stat. Phys. **48**, 19 (1987)
23. Brydges, D., Yau, H.T.: Grad Φ Perturbations of massless Gaussian Fields. Commun. Math. Phys. **129**, 351 (1990)
24. Abdesselam, A., Rivasseau, V.: Trees, forests, jungles: A botanical garden for cluster expansions. Proc. of Int. Workshop on Constructive Theory, Berlin, Heidelberg, New York: Springer 1995

Communicated by G. Felder