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Fractal Drums and the *n*-Dimensional Modified Weyl–Berry Conjecture

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Abstract: In this paper, we study the spectrum of the Dirichlet Laplacian in a bounded (or, more generally, of finite volume) open set $\Omega \in \mathbf{R}^n$ $(n \geqq 1)$ with fractal boundary $\partial\Omega$ of interior Minkowski dimension $\delta \in (n-1, n]$. By means of the technique of tessellation of domains, we give the exact second term of the asymptotic expansion of the "counting function" $N(\lambda)$ (i.e. the number of positive eigenvalues less than λ) as $\lambda \to +\infty$, which is of the form $\lambda^{\delta/2}$ times a negative, bounded and left-continuous function of λ. This explains the reason why the modified Weyl–Berry conjecture does not hold generally for $n \geqq 2$. In addition, we also obtain explicit upper and lower bounds on the second term of $N(\lambda)$.

1. Introduction

Let Ω be an arbitrary non-empty bounded (or, more generally, of finite volume) open set in $\mathbf{R}^n$ $(n \geqq 1)$ with boundary $\partial\Omega$. We consider the following variational eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega\,, \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{P}$$

where Δ denotes the Dirichlet Laplacian in Ω and the problem (P) is to be interpreted in the following sense: we say that the scalar λ is an eigenvalue of (P) if there exists $u \neq 0$ in $H_0^1(\Omega)$ satisfying $-\Delta u = \lambda u$ in the distributional sense.

It is well-known that the spectrum of (P) is discrete and consists of an infinite sequence of positive eigenvalues with finite multiplicity, which may be ordered as

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \cdots \tag{1.1}$$

with $\lambda_k \to +\infty$, as $k \to +\infty$.

We introduce the counting function $N(\lambda)$, which is the number of eigenvalues of (P) less than λ, i.e.

$$N(\lambda) \equiv N(\lambda, -\Delta, \Omega) = \#\{k | \lambda_k < \lambda\}\,. \tag{1.2}$$

In this paper, we are interested in the asymptotic behaviour of the counting function $N(\lambda)$ as $\lambda \to +\infty$. It is well known that the following asymptotic estimate holds:

$$N(\lambda) \sim (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2}, \text{ as } \lambda \to +\infty\,, \tag{1.3}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$, $|\cdot|_n$ denotes the n-dimensional Lebesgue measure; and $f(\lambda) \sim g(\lambda)$ as $\lambda \to +\infty$ means $\lim_{\lambda\to+\infty}\dfrac{f(\lambda)}{g(\lambda)} = 1$.

The asymptotic estimate (1.3) was first proved in 1911 by H. Weyl [1, 2] for smooth boundaries and then extended more recently to irregular boundaries; see for example [3–6]. At present, we know (1.3) holds for very irregular boundaries, i.e. for fractal boundaries (cf. [7–11]).

Here we are more interested in the asymptotic behaviour of the second term of the counting function $N(\lambda)$. In [2], Weyl conjectured that

$$N(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} + O(\lambda^{\frac{n-1}{2}}), \text{ as } \lambda \to +\infty\,, \tag{1.4}$$

that is to say the correction is of the order $\lambda^{\frac{n-1}{2}}$.

In 1966, M. Kac [12] made a deep study of Weyl's conjecture (1.4) and proposed a number of approaches for obtaining further terms in (1.4). From then on, many interesting works, including Seeley [13, 14], Ivrii [15, 16], Melrose [17, 18], Hörmander [19], Vassiliev [20] and others, appeared. By using microlocal analysis and wave operator methods, their results showed, under a variety of geometrical and regularity conditions, that

$$N(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} - C'_n|\partial\Omega|_{n-1}\lambda^{\frac{n-1}{2}} + o(\lambda^{\frac{n-1}{2}})\,, \tag{1.5}$$

as $\lambda \to +\infty$, where $C'_n = \left[4(4\pi)^{\frac{n-1}{2}}\Gamma\left(1+\dfrac{n-1}{2}\right)\right]^{-1}$, is a universal constant. However their methods cannot be applied to the case of "rough" boundaries.

In 1979, M.V. Berry [21, 22], motivated in part by the study of scattering of light by random surfaces, extended Weyl's conjecture (1.5) to the case of fractal boundaries (i.e. Ω is a drum with fractal boundary). He made the following conjecture; if $\partial\Omega$ has Hausdorff dimension $H \in (n-1, n)$, then

$$N(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} - C_{n,H}H(\partial\Omega)\lambda^{H/2} + o(\lambda^{H/2})\,, \tag{1.6}$$

as $\lambda \to +\infty$, where $H(\partial\Omega)$ denotes the H-dimensional Hausdorff measure of $\partial\Omega$, and $C_{n,H}$ is a positive constant depending only on n and H. By analogy with the "smooth" case, Berry [21] even suggested an explicit value of $C_{n,H}$, i.e.

$$C_{n,H} = \left[4(4\pi)^{\frac{H}{2}}\Gamma\left(1+\frac{H}{2}\right)\right]^{-1}\,. \tag{1.7}$$

Brossard and Carmona [23] (1986) constructed a number of counter-examples to (1.6) which proved that Berry's conjecture could not be true. They suggested that the Hausdorff dimension H should be replaced by the interior Bouligand–Minkowski dimension δ (or the interior Minkowski dimension for simplicity) of the boundary by obtaining one-and two-sided estimates (expressed in terms of δ) for the asymptotic second term (as $t \to 0^+$) of the partition function $Z(t)$, the trace of the heat semigroup $e^{t\Delta}$. Indeed, in [7–11], a partial resolution of the modified conjecture (1.6) has been proved by showing that

$$N(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} + O(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty . \tag{1.8}$$

This means that the interior Minkowski dimension is more appropriate than the Hausdorff dimension as a measure of the "roughness" of the boundary $\partial\Omega$.

In this paper, we will study the modified Berry conjecture (1.6), which has also been called the "modified Weyl–Berry conjecture." The plan of the paper is as follows: In Sect. 2 we introduce the modified Weyl–Berry conjecture and its weaker form and state our main results. From these results, one can easily understand the reason why the modified Weyl–Berry conjecture does not hold generally for $n \geqq 2$. At the same time, our results also imply that the weaker form of the conjecture might be true. The proofs of the main results are given in Sects. 3, 4 and 5. Finally several examples, including the examples recently reported in the literature, are discussed in Sects. 7 and 8.

2. Concepts and Main Results

Let us first give the concepts of the interior (Bouligand-) Minkowski dimension and measure of $\partial\Omega$. Given $\varepsilon > 0$, define

$$\Omega^i_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) < \varepsilon\} , \tag{2.1}$$

where $d(x, \partial\Omega)$ denotes the Euclidean distance of x to the boundary $\partial\Omega$. The set Ω^i_ε is called the interior ε-neighborhood of $\partial\Omega$. For $l \geqq 0$, let

$$\mu^*(l, \partial\Omega) = \lim_{\varepsilon\to 0+} \sup\, \varepsilon^{-(n-l)}|\Omega^i_\varepsilon|_n , \tag{2.2}$$

$$\mu_*(l, \partial\Omega) = \lim_{\varepsilon\to 0+} \inf\, \varepsilon^{-(n-l)}|\Omega^i_\varepsilon|_n . \tag{2.3}$$

The interior Minkowski dimension of $\partial\Omega$ is defined as:

$$\begin{aligned} \delta &= \inf\{l \in \mathbf{R}_+ \mid \mu^*(l, \partial\Omega) = 0\} \\ &= \sup\{l \in \mathbf{R}_+ \mid \mu^*(l, \partial\Omega) = +\infty\} . \end{aligned} \tag{2.4}$$

Observe $\delta \in [n-1, n]$ and $\mu^*(\delta, \partial\Omega) \in [0, +\infty]$. On the other hand, if H is the Hausdorff dimension of $\partial\Omega$, then we know $H \leqq \delta$. Further, if δ is the interior Minkowski dimension of $\partial\Omega$ and

$$0 < \mu_*(\delta, \partial\Omega) = \mu^*(\delta, \partial\Omega) < +\infty ,$$

we say that the boundary $\partial\Omega$ is interior δ-Minkowski measurable, and denote by

$$\mu(\delta, \partial\Omega) = \lim_{\varepsilon\to 0+} \varepsilon^{-(n-\delta)}|\Omega^i_\varepsilon|_n , \tag{2.5}$$

the interior δ-Minkowski measure of $\partial\Omega$.

Now, the modified Weyl–Berry conjecture (see [23, 10, 24–26, 29]) can be stated in the form:

$$N(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} - C_{n,\delta}\mu(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}) , \tag{2.6}$$

as $\lambda \to +\infty$, where $\delta \in (n-1, n)$ and $C_{n,\delta}$ is a positive constant depending only on δ and n.

Lapidus–Pomerance [24, 25], under the so-called asymptotics condition, proved that the modified Weyl–Berry conjecture (2.6) is true for the case $n = 1$. Nevertheless several examples (see [26–29]) have suggested that the conjecture (2.6) might be false for $n \geqq 2$. Therefore it is natural to consider in a more general context the second term of $N(\lambda)$, namely under the condition of

$$0 < \mu_*(\delta, \partial\Omega) \leqq \mu^*(\delta, \partial\Omega) < +\infty . \tag{2.7}$$

Conjecture (2.6) is then modified as

$$\begin{aligned} C_{n,\delta}\mu_*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}) &\leqq (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{\delta/2} - N(\lambda, -\Delta, \Omega) \\ &\leqq C'_{n,\delta}\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}) , \end{aligned} \tag{2.8}$$

as $\lambda \to +\infty$, where $\mu^*(\delta, \partial\Omega)$ (μ_* $(\delta, \partial\Omega)$) is called the interior δ-Minkowski upper (lower) content of $\partial\Omega$ and $C_{n,\delta}$ and $C'_{n,\delta}$ are two positive constants depending only on δ and n.

Notice that inequality (2.7) implies that $\partial\Omega$ has interior Minkowski dimension δ. We call the conjecture (2.8) the "weaker form" of the modified Weyl–Berry conjecture (2.6), (cf. [10, 24, 25, 26]).

In order to prove the conjecture (2.8), we first follow the usual method of constructing a sequence of finer and finer tessellations of $\mathbf{R}^n$ (i.e. the Whitney covering) by cubes $\{Q_\xi^k\}_{\xi\in \mathrm{z}^n}$. That is, for each k, $\{Q_\xi^k\}_{\xi\in \mathrm{z}^n}$ is a tessellation of $\mathbf{R}^n$ into a countable family of congruent and non-overlapping open cubes with sides of length b_k, such that

$$\mathbf{R}^n = \bigcup_{\xi\in \mathrm{z}^n} \overline{\mathrm{Q}_\xi^\mathrm{k}} .$$

By finer and finer tessellations we mean that $b_{k+1} < b_k$ and b_k tends to zero as k tends to infinity. Note that the construction of the tessellations always starts at the origin. This method enables one to determine the influence of Ω and especially the influence of the irregular fractal nature of its boundary $\partial\Omega$.

For a sequence of given tessellations of $\mathbf{R}^n$, the tessellation of Ω is defined by induction on k as follows:

$$A_0 = \{\xi \in \mathbf{Z}^\mathrm{n} \mid Q_\xi^0 \subset \Omega\}, \Omega'_0 = \bigcup_{\xi\in A_0} Q_\xi^0, \text{ and } \Omega''_0 = \Omega\backslash\overline{\Omega'_0} ;$$

$$A_1 = \{\xi \in \mathbf{Z}^\mathrm{n} \mid Q_\xi^1 \subset \Omega''_0\}, \Omega'_1 = \Omega'_0 \cup \left(\bigcup_{\xi\in A_1} Q_\xi^1\right), \text{ and } \Omega''_1 = \Omega\backslash\overline{\Omega'_1} ;$$

$$\cdots\cdots$$

$$A_k = \{\xi \in \mathbf{Z}^\mathrm{n} \mid Q_\xi^k \subset \Omega''_{k-1}\}, \Omega'_k = \Omega'_{k-1} \cup \left(\bigcup_{\xi\in A_k} Q_\xi^k\right), \text{ and } \Omega''_k = \Omega\backslash\overline{\Omega'_k} .$$

$$\cdots\cdots$$

We denote by Ω'_∞ the "limit" of Ω'_k as $k \to +\infty$, and correspondingly $N(\lambda, -\Delta, \Omega'_\infty)$ and $N(\lambda, -\Delta, Q_\xi^k)$ the counting functions for Ω'_∞ and Q_ξ^k respectively.

If $\varphi(\lambda) \equiv (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2}$ (or $\varphi_\infty(\lambda) \equiv (2\pi)^{-n}\omega_n|\Omega'_\infty|_n\lambda^{n/2}$) is the first term (i.e. Weyl term) of $N(\lambda)$ (or $N(\lambda, -\Delta, \Omega'_\infty)$), we call

$$\psi(\lambda) \equiv \varphi(\lambda) - N(\lambda, -\Delta, \Omega)\,, \tag{2.9}$$

$$\psi_\infty(\lambda) \equiv \varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty)\,, \tag{2.10}$$

the remainder terms of $N(\lambda)$ and $N(\lambda, -\Delta, \Omega'_\infty)$. Furthermore if $\delta \in (n-1, n]$ and $\mu^*(\delta, \partial\Omega)$ is finite, then we know (cf. [10]) that $\psi(\lambda)$ (or $\psi_\infty(\lambda)) \in O(\lambda^{\delta/2})$. Observe that $|\Omega|_n = |\Omega'_k|_n + |\Omega''_k|_n$, and $|\Omega''_k|_n \to 0$ if $\delta \in (n-1, n)$ and $k \to +\infty$; this implies that, for $\delta \in (n-1, n)$, we have $|\Omega|_n = |\Omega'_\infty|_n$ and $\varphi(\lambda) = \varphi_\infty(\lambda)$.

Our first main result is concerned with the upper bound on the second term of the counting function $N(\lambda)$. Here we shall consider the case when the interior Minkowski dimension $\delta \in (n-1, n]$ and $\mu^*(\delta, \partial\Omega) < +\infty$. We have the following result:

Theorem 2.1. *Let Ω be an arbitrary non-empty bounded (or, more generally, of finite volume) open set in $\mathbf{R}^n (n \geqq 1)$ with boundary $\partial\Omega$. Suppose that the interior Minkowski dimension of $\partial\Omega$, $\delta \in (n-1, n]$ and $\mu^*(\delta, \partial\Omega) < +\infty$. Then for any $a_0 \in (0, 1)$ we have*

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \leqq \overline{C}_{n,\delta}(a_0)\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2})\,,$$

as $\lambda \to +\infty$, where

$$\overline{C}_{n,\delta}(a_0) = \left(1 + \frac{1}{\pi}\right)^{\frac{n-1}{2}} \left[\frac{\sqrt{n}}{a_0}\right]^{n-\delta} [1 - a_0^{\delta-n+1}]^{-1} + (2\pi)^{-n}\omega_n\, n^{\frac{n-\delta}{2}} \tag{2.11}$$

is a positive constant depending only on δ, n and a_0.

In the proof of Theorem 2.1, we shall let $b_k = a_0^k$ in the tessellation of Ω (see Sect. 3 below) and call a_0 the size of the tessellation. Observe that $\varphi(\lambda) - N(\lambda, -\Delta, \Omega)$ is independent of a_0, and $\overline{C}_{n,\delta}(a_0) \to +\infty$ if $\delta \in (n-1, n)$ and a_0 tends to 0 or 1. Sometimes it is convenient to choose $a_0 = \frac{1}{2}$ (i.e. the mid-point of (0,1)) (cf. [7–10, 27, 28]). However one can easily see that the optimal constant can be obtained by taking the minimum of the right-hand side of (2.11) over $a_0 \in (0, 1)$, i.e. $C'_{n,\delta} = \overline{C}_{n,\delta}(a_0^*) = inf_{a_0 \in (0,1)}\overline{C}_{n,\delta}(a_0)$, where $a_0^* = (n-\delta)^{\frac{1}{\delta-(n-1)}}$ is the unique root of the equation $\frac{d}{da_0}\overline{C}_{n,\delta}(a_0) = 0$ in (0,1). This implies that by choosing $a_0^* = (n-\delta)^{\frac{1}{\delta-(n-1)}}$ as the size of the tessellation of Ω, we should be able to obtain the optimal upper bound estimate for the second term of $N(\lambda)$. Actually we have the following obvious corollary:

Corollary 2.1. *Under the conditions of Theorem 2.1, if $\delta \in (n-1, n)$, we have*

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \leqq C'_{n,\delta}\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2})\,,$$

as $\lambda \to +\infty$, where

$$C'_{n,\delta} = \left(1 + \frac{1}{\pi}\right)^{\frac{n-1}{2}} n^{\frac{n-\delta}{2}}(n-\delta)^{\frac{\delta-n}{\delta-(n-1)}}(1 + \delta - n)^{-1} + (2\pi)^{-n}\omega_n\, n^{\frac{n-\delta}{2}} \tag{2.12}$$

is a positive constant depending only on δ and n.

Secondly, we give a necessary condition for the weaker form (2.8) of the modified Weyl–Berry conjecture to be valid. Actually we need the following asymptotic hypothesis:

(C1) For any fixed $a_0 \in (0,1)$, the size of a given tessellation of Ω, there exists a positive constant c, depending only on δ and n, such that $b_k^\delta(\# A_k) \geqq c$, as $k \to +\infty$, (where $b_k = a_0^k$).

We have

Theorem 2.2. *Let $n \geqq 1$, $\delta \in (n-1,n)$. If the conjecture* (2.8) (*i.e. the weaker form of the modified Weyl–Berry conjecture* (2.6)) *holds, then the asymptotic condition* ($C1$) *is certainly satisfied.*

The next question to ask is; how to estimate the lower bound on second term of $N(\lambda)$? As is well-known, this problem is rather complicated. In order to make the problem more tractable, we introduce a further asymptotic condition as follows:

(C2) There exists a suitable tessellation of Ω, such that

$$\psi(\lambda) - \psi_\infty(\lambda) = o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty .$$

To obtain an explicit lower bound for the second term of $N(\lambda)$, we still let $b_k = a_0^k$ (although it is not necessary in condition (C2)) in the tessellation of Ω. Our next main result is the following:

Theorem 2.3. *Let $n \geqq 1$, the interior Minkowski dimension of $\partial\Omega$, $\delta \in (n-1,n]$, $\mu^*(\delta, \partial\Omega) < +\infty$ and there exists a suitable tessellation of Ω (i.e. especially here, there exists a suitable size a_0 in* $(0,1)$), *such that the conditions* ($C2$) *and* ($C1$) *are satisfied. Then*

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \geqq C_{n,\delta}\lambda^{\delta/2} + o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty ,$$

where

$$\begin{aligned} &C_{1,\delta} > \frac{c}{\pi}[1 + (a_0^{\delta-1} - 1)^{-1}] \textit{ for } \delta \in (0,1) , \\ &\textit{and } C_{1,1} > \frac{c}{\pi} \textit{ in the case } n = 1 ; \\ &C_{n,\delta} = cc_n \left(\frac{1}{\pi}\right)^{n-1-\varepsilon_n} \left[1 - a_0^{\delta-n+1+\varepsilon_n}\right]^{-1} + (2\pi)^{-n}\omega_n c(a_0^{\delta-n} - 1)^{-1} \\ &\textit{for } \delta \in (n-1,n), n \geqq 2, \textit{ and } C_{n,n} = cc_n \left(\frac{1}{\pi}\right)^{n-1-\varepsilon_n} \left[1 - a_0^{1+\varepsilon_n}\right]^{-1} . \end{aligned} \tag{2.13}$$

Here $c > 0$ is the constant appearing in ($C1$) *and $\varepsilon_n \geqq 0$, $c_n > 0$ are two constants depending only on n (more precisely $\varepsilon_n \in [0, n-1]$).*

If there exist two positive constants c_1 and c_2, such that $c_1 f(\lambda) \leqq g(\lambda) \leqq c_2 f(\lambda)$ as $\lambda \to +\infty$, we denote $f(\lambda) \approx g(\lambda)$ as $\lambda \to +\infty$. From the results of Theorem 2.1 and Theorem 2.3 directly, we have the following corollary:

Corollary 2.2. *Under the conditions of Theorem* 2.3, *we have*

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \approx \lambda^{\delta/2}, \text{ as } \lambda \to +\infty , \tag{2.14}$$

where the constants c_1 and c_2, appearing in "$\approx$", depend only upon n and δ.

It is obvious that, under (C1) and (C2), the weaker form (2.8) of the n-dimensional modified Weyl–Berry conjecture (2.6) has been proved directly from Corollary 2.2. Indeed, we only need the weaker conditions (i.e. $\delta \in (n-1, n]$, $\mu^*(\delta, \partial\Omega) < +\infty$) in Corollary 2.2. At the same time we have also given explicit values for $C'_{n,\delta}$ (see (2.12)) and $C_{n,\delta}$ (see (2.13)). On the other hand, we know that the condition (C1) is necessary (cf. Theorem 2.2 above).

It is easy to see that the asymptotic hypothesis (C2) here is strictly weaker than the asymptotics hypothesis (H1) which appeared in [29]. On the other hand, (C2) is always satisfied in the case of $n = 1$ and $\psi(\lambda) \sim \psi_\infty(\lambda)$ as $\lambda \to +\infty$ for $n \geqq 2$. This is because a 1-dimensional open cube is just an open interval, so one can always choose the tessellation of Ω to be Ω itself (also see Sect. 6 below). However here, we are unable to give an easy way to check (C2) in the case $n \geqq 2$; this is a possible weakness of the paper. In Sects. 7 and 8, we give several interesting examples of $n \geqq 2$, in which (C2) is satisfied (in fact (C2) is always satisfied in nearly all the examples we know in the literature). Recently Fleckinger–Vassiliev [27, 28] constructed an interesting example in $\mathbf{R}^2$ (cf. Example 8.3 below) and gave the exact second term asymptotics of $N(\lambda)$ for their example, in which they actually disproved the modified Weyl–Berry conjecture in case $n = 2$. It is more interesting here, under the condition (C2), that by using the tessellation method and the main idea of [27, 28] we are able to deduce the exact second term in the asymptotic expansion of $N(\lambda)$ for rather general cases, which suggest a satisfactory explanation of why the modified Weyl–Berry conjecture (2.6) does not hold for $n \geqq 2$. Actually our next result (i.e. Theorem 2.4) is an extension of [27, 28].

Let us first extend the definitions of b_k and $\# A_k$ to negative integer k as follows:

$$b_k = b_{-k}^{-1}, \quad \# A_k = c b_k^{-\delta}, \ (c > 0 \text{ a constant}) \quad \text{for } k \leqq -1 \,. \tag{2.15}$$

Next we define

$$f_k^n(\lambda) = [(2\pi)^{-n}\omega_n(b_k\lambda^{1/2})^n - N(\lambda, -\Delta, Q_\xi^k)] \,. \tag{2.16}$$

Furthermore we assume that the rate of convergence of $b_k \to 0$ is fast enough so that

$$\sum_{k=1}^{+\infty} b_k^{\delta-(n-1)} < +\infty, \quad \delta \in (n-1, n] \,. \tag{2.17}$$

Then we have the following main result:

Theorem 2.4. *Under the assumptions of Theorem* 2.1; *if the conditions* (*C*2) *and* (2.17) *are satisified, then*

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = F_n(\lambda)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty \,, \tag{2.18}$$

where

$$F_n(\lambda) = \sum_{k=-\infty}^{+\infty} (\# A_k) f_k^n(\lambda)\lambda^{-\delta/2} \,, \quad \lambda > 0 \tag{2.19}$$

is a well-defined, positive, bounded and left-continuous function of λ; *furthermore its set of points of discontinuity is dense in* $\mathbf{R}_+$.

In the particular case of a tessellation of exponential size $b_k = a_0^k$ for some $a_0 \in (0,1)$ the result of Theorem 2.4 gives

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = \overline{F}_n(y)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty \,, \tag{2.20}$$

where $y = \dfrac{log\lambda - 2log\pi}{2log(1/a_0)}$, and

$$\overline{F}_n(y) = \left(\frac{1}{\pi}\right)^{\delta} \sum_{k=-\infty}^{+\infty} (\# A_k) a_0^{\delta y} f_k^n(\pi^2 a_0^{-2y}) \tag{2.21}$$

is a well-defined, positive, bounded and left-continuous functioin of y and the set of points of discontinuity for $\overline{F}_n(y)$ is dense in $\mathbf{R}$.

It is obvious that if the positive limit of $F_n(\lambda)$ (or $\overline{F}_n(y)$) does not exist as $\lambda \to +\infty$, then Theorem 2.4 tells us that the modified Weyl–Berry conjecture (2.6) is false. In the particular case, where $f_k^n(\lambda) = h_n(\pi^{-1} b_k \lambda^{1/2})$, with

$$h_n(r_k) = [2^{-n}\omega_n r_k^n - N(\pi^2 b_k^{-2} r_k^2, -\Delta, Q_\xi^k)], \quad r_k = \frac{b_k}{\pi}\lambda^{1/2}, \tag{2.22}$$

formula (2.21) becomes

$$\overline{F}_n(y) = \left(\frac{1}{\pi}\right)^{\delta} \sum_{k=-\infty}^{+\infty} (\# A_k) a_0^{\delta y} h_n(a_0^{k-y}). \tag{2.23}$$

Further, if there exists a constant $c > 0$, such that

$$\# A_k \sim c a_0^{-\delta k}, \quad as\ k \to +\infty, \tag{2.24}$$

then we can choose the same constant c in the definition of $\# A_k$ for $k \leqq -1$ (see (2.15)). We can then easily deduce that

$$\overline{F}_n(y) \sim \left(\frac{1}{\pi}\right)^{\delta} \sum_{k=-\infty}^{+\infty} c a_0^{-\delta(k-y)} h_n(a_0^{k-y}), \quad \text{as } y \to +\infty \tag{2.25}$$

Since $k - y = (k+1) - (y+1)$, $\overline{F}_n(y)$ is equivalent to a 1-periodic function. This means that the limit of $\overline{F}_n(y)$, as $y \to +\infty$, does not exist, which suggests that the modified Weyl–Berry conjecture might be false. In Sect. 8, we study several examples under the condition $\# A_k = c a_0^{-\delta k}$, which actually explains the reason why the modified Weyl–Berry conjecture (2.6) is not true in the case $n \geqq 2$.

Remark. 2.1. It is worth pointing out that there is not any restriction on $\mu_*(\delta, \partial\Omega)$ in our results here. We shall see in Sect. 6 that condition (C1) implies $\mu_*(\delta, \partial\Omega) > 0$.

Remark. 2.2. Lapidus–Pomerance [25, 26] proved the conjectures (2.6) and (2.8) in the case $n = 1$, $\delta \in (0, 1)$. Here our results give proofs for $n \geqq 2$. As is well known, there are essential differences in this matter between the cases of $n = 1$ and $n \geqq 2$. Furthermore our results include the particular case of $\delta = n$, which is a new one even for $n = 1$.

Remark. 2.3. It is worth stressing that we do not make any assumption of self-similarity (or, more generally, self-alikeness) in the sense of Mandelbrot [30] about $\partial\Omega$.

Remark. 2.4. The results of this paper also suggest an almost satisfactory answer to the main drawback of Proposition 3.3 in Brossard–Carmona [23, pp. 115], where they were unable to prove that the constant $C_{n,\delta}$ is finite.

Remark. 2.5. Our condition (C2) is essentially similar to the asymptotic condition (4.10) in Lapidus [34], so our results actually have given a partial resolution of conjecture 3 in [34].

3. Proofs of Theorem 2.1 and Theorem 2.2

Let Q_b be an n-dimensional cube with side length b; we know that the positive eigenvalues of the Dirichlet Laplacian problem for Q_b are given by

$$\pi^2 b^{-2}(q_1^2 + q_2^2 + \cdots + q_n^2), \text{ with } q_j \in \mathbf{N},\ 1 \leqq j \leqq n\,. \tag{3.1}$$

Then

$$N(\lambda, -\Delta, Q_b) = \#\left\{(q_1, q_2, \ldots, q_n) \in \mathbf{N}^n \mid \sum_{j=1}^{n} q_j^2 < \left(\frac{b}{\pi}\right)^2 \lambda\right\}. \tag{3.2}$$

This is the number of positive lattice-points in an n-dimensional ball with radius $r = \dfrac{b}{\pi}\lambda^{1/2}$. We define

$$p_n(r) = N(\pi^2 b^{-2} r^2,\ -\Delta, Q_b) = \#\left\{(q_1, q_2, \ldots, q_n) \in \mathbf{N}^n \Big| \sum_{j=1}^{n} q_j^2 < r^2\right\}. \tag{3.3}$$

It is well known (cf. Gauss [31]) that there exists a positive constant d_n, which does not depend on r, such that

$$0 < 2^{-n}\omega_n r^n - p_n(r) < d_n r^{n-1}, \text{ for } n \geqq 1,\ r > 0\,. \tag{3.4}$$

Let

$$c_n(r) = \frac{2^{-n}\omega_n r^n - p_n(r)}{r^{n-1}}, \tag{3.5}$$

then

$$0 < c_n(r) < d_n, \text{ for } n \geqq 1,\ r > 0\,. \tag{3.6}$$

More precisely, we know from [29, Lemma 3.1] that

$$0 < c_n(r) r^{n-1} < \pi^{\frac{n-1}{2}}(1+\pi)^{\frac{n-1}{2}} r^{n-1}, \text{ for } r \geqq \frac{1}{\pi}\,. \tag{3.7}$$

Furthermore, we have

Lemma 3.1. *$c_n(r)$ is a piecewise smooth, positive and bounded function in $\mathbf{R}_+$. If there exist n strictly positive integers $q_1, q_2, \cdots, q_n$, satisfying $\sum_{j=1}^n q_j^2 = r^2$, then r is a point of discontinuity of $c_n(r)$.*

Proof. From (3.6), $c_n(r)$ is positive and bounded. It is obvious that

$$p_n(r) = \begin{cases} 0 & \text{when } r \leqq \sqrt{n}\,, \\ 1 & \text{when } \sqrt{n} < r \leqq \sqrt{n+3}\,. \end{cases}$$

Hence $r = \sqrt{n}$ is a point of discontinuity of $p_n(r)$. Similarly whenever there exist n strictly positive integers $q_j (1 \leqq j \leqq n)$ such that $\sum_{j=1}^n q_j^2 = r^2$, r is a point of discontinuity of $p_n(r)$. Thus $p_n(r)$ is a piecewise constant function, which implies that $c_n(r)$ is a piecewise smooth function in $\mathbf{R}_+$ and Lemma 3.1 is proved.

Proof of Theorem 2.1. Observing $\Omega'_L \subseteq \Omega$ for any $L \geqq 0$, then by the monotonicity of the counting function, we have

$$N(\lambda, -\Delta, \Omega) \geqq N(\lambda, -\Delta, \Omega'_L), \quad \forall\, L \geqq 0 .$$

This means that for any $L \geqq 0$,

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \leqq \varphi(\lambda) - N(\lambda, -\Delta, \Omega'_L) = (2\pi)^{-n}\omega_n|\Omega'_L|_n\lambda^{n/2}$$
$$- N(\lambda, -\Delta, \Omega'_L) + (2\pi)^{-n}\omega_n[|\Omega|_n - |\Omega'_L|_n]\lambda^{n/2} .$$

Using the Dirichlet–Neumann bracketing method, or, more generally, by the results of [10, Lemma 4.2] and [32, XIII. 15, Prop. 3], we know that

$$(2\pi)^{-n}\omega_n|\Omega'_L|_n\lambda^{n/2} - N(\lambda, -\Delta, \Omega'_L) = \sum_{k=0}^{L}(\# A_k)[2^{-n}\omega_n r_k^n - p_n(r_k)] ,$$

where $r_k = \dfrac{b_k}{\pi}\lambda^{1/2}$ *and* $p_n(r_k)$ is defined by (3.3).

By using the estimate (3.7), we have

$$\sum_{k=0}^{L}(\# A_k)[2^{-n}\omega_n r_k^n - p_n(r_k)] \leqq \pi^{\frac{n-1}{2}}(1+\pi)^{\frac{n-1}{2}}\sum_{k=0}^{L}(\# A_k)r_k^{n-1} .$$

On the other hand, we know that

$$\varepsilon^{-(n-\delta)}|\Omega^i_\varepsilon|_n \leqq \mu^*(\delta, \partial\Omega) + o(1), \text{ as } \varepsilon \to 0+ .$$

Taking $\varepsilon_k = \sqrt{n}b_k$, $b_k = a_0^k$ $(a_0 \in (0,1))$, then $\Omega''_k \subset \Omega^i_{\varepsilon_k}$ and

$$b_k^n(\# A_k) \leqq |\Omega''_{k-1}|_n \leqq |\Omega^i_{\varepsilon_{k-1}}|_n \leqq [\mu^*(\delta, \partial\Omega) + o_k]n^{\frac{n-\delta}{2}}b_{k-1}^{n-\delta} ,$$

where $o_k \to 0$ as $k \to +\infty$. Hence we have

$$\# A_k \leqq \left[\frac{\sqrt{n}}{a_0}\right]^{n-\delta} b_k^{-\delta}[\mu^*(\delta, \partial\Omega) + o_k] . \tag{3.8}$$

Consequently

$$\sum_{k=0}^{L}(\# A_k)r_k^{n-1} \leqq \left[\frac{\sqrt{n}}{a_0}\right]^{n-\delta}\mu^*(\delta, \partial\Omega)\sum_{k=0}^{L}b_k^{-\delta}r_k^{n-1}$$
$$+ \left[\frac{\sqrt{n}}{a_0}\right]^{n-\delta}\sum_{k=0}^{L}o_k b_k^{-\delta}r_k^{n-1} .$$

As in [29], we always take $L \sim \dfrac{ln\lambda}{2ln(1/a_0)}$. This is equivalent to $\lambda^{1/2} \sim b_L^{-1}$ or $\lambda \sim a_0^{-2L}$. It is obvious that $\lambda \to +\infty$ is equivalent to $L \to +\infty$. Hence

$$\sum_{k=0}^{L} b_k^{-\delta} r_k^{n-1} = \left(\frac{1}{\pi}\right)^{n-1} \sum_{k=0}^{L} (b_k^2 \lambda)^{\frac{n-1-\delta}{2}} \lambda^{\delta/2} \sim \left(\frac{1}{\pi}\right)^{n-1} \sum_{k=0}^{L} [a_0^{k-L}]^{n-1-\delta} \cdot \lambda^{\delta/2}$$
$$= \left(\frac{1}{\pi}\right)^{n-1} \lambda^{\delta/2} \cdot \sum_{k=0}^{L} [a_0^{\delta-n+1}]^{L-k} .$$

Taking $L - k = l$ and observing that $\delta > n - 1$, then

$$\sum_{k=0}^{L} [a_0^{\delta-n+1}]^{L-k} = \sum_{l=0}^{L} [a_0^{\delta-n+1}]^{l} \leqq \frac{1}{1 - a_0^{\delta-n+1}} . \tag{3.9}$$

On the other hand, we have

$$\sum_{k=0}^{L} o_k b_k^{-\delta} r_k^{n-1} \sim \left(\frac{1}{\pi}\right)^{n-1} \lambda^{\delta/2} \sum_{k=0}^{L} o_k [a_0^{\delta-(n-1)}]^{L-k} ,$$

where $\delta > n - 1$ and $L \sim \dfrac{ln\lambda}{2ln(1/a_0)}$. Furthermore we can prove that

$$\sum_{k=0}^{L} o_k [a_0^{\delta-(n-1)}]^{L-k} = \sum_{l=0}^{L} o_{L-l} [a_0^{\delta-(n-1)}]^{l} = o(1) \quad \text{as } \lambda \to +\infty .$$

Actually we have

Lemma 3.2. *Let* $x_0 \in (0, 1)$, *then*

$$\sum_{l=0}^{L} o_{L-l} x_0^l \to 0 \quad \text{as } L \to +\infty . \tag{3.10}$$

Proof. Since $o_k \to 0$ as $k \to +\infty$, then for any $\varepsilon > 0$, there exists $M > 0$ such that $|o_{L-l}| < \varepsilon$ for $L - l \geqq M$. Thus

$$\left|\sum_{l=0}^{L} o_{L-l} x_0^l\right| \leqq \varepsilon \left|\sum_{l=0}^{L-M} x_0^l\right| + |o_M| \sum_{l=L-M+1}^{L} x_0^l$$
$$\leqq \varepsilon + |o_M| x_0^{L-M+1} M .$$

Since $x_0^{L-M+1} \to 0$ as $L \to +\infty$; (3.10) is proved.

From (3.8), (3.9) and Lemma 3.2, we have

$$(2\pi)^{-n} \omega_n |\Omega'_L|_n \lambda^{n/2} - N(\lambda, -\Delta, \Omega'_L) \leqq \left(1 + \frac{1}{\pi}\right)^{\frac{n-1}{2}} \left[\frac{\sqrt{n}}{a_0}\right]^{n-\delta} \frac{1}{1 - a_0^{\delta-n+1}}$$
$$\cdot \mu^*(\delta, \partial\Omega) \lambda^{\delta/2} + o(\lambda^{\delta/2}) \quad \text{as } \lambda \to +\infty . \tag{3.11}$$

Next, we have $0 < |\Omega|_n - |\Omega'_L|_n = |\Omega''_L|_n \leqq |\Omega^i_{\varepsilon_L}|_n$, $\varepsilon_L = \sqrt{n} b_L$, thus

$$\varepsilon_L^{n-\delta} \cdot \varepsilon_L^{-(n-\delta)}(|\Omega|_n - |\Omega'_L|_n) \leqq \varepsilon_L^{n-\delta}[\mu^* \ (\delta, \partial\Omega) + o_L] \, . \tag{3.12}$$

Because $\lambda^{1/2} \sim b_L^{-1}$, we obtain

$$\begin{aligned}(2\pi)^{-n}\omega_n[|\Omega|_n - |\Omega'_L|_n]\lambda^{n/2} &\leqq (2\pi)^{-n}\omega_n n^{\frac{n-\delta}{2}} b_L^{n-\delta}\lambda^{n/2}[\mu^* \ (\delta, \partial\Omega) + o_L] \\ &\sim (2\pi)^{-n}\omega_n n^{\frac{n-\delta}{2}} \mu^* \ (\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty \, . \end{aligned} \tag{3.13}$$

Thus combining the estimates (3.11) and (3.13), we get

$$\varphi(\lambda) - N(\lambda) \leqq \overline{C}_{n,\delta}(a_0)\mu^* \ (\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty \, , \tag{3.14}$$

where the computable positive constant $\overline{C}_{n,\delta}(a_0)$ is given by (2.11), which depends only on n, δ and a_0. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. If there exists a size of the tessellation of $\Omega, a_0 \in (0,1)$ for which condition (C1) is not satisfied, we know that there exists a "subsequence" $b^{\delta}_{n_k}(\# A_{n_k}) \to 0$, as $k \to +\infty$ (where $b_{n_k} = a_0^{n_k}$). Because $\{\Omega'_{n_k}\}$ is also a tessellation of Ω, we can assume that

$$\# A_{n_k} = o_k \, b_{n_k}^{-\delta} \, , \tag{3.15}$$

where $o_k \to 0$ as $k \to +\infty$.

On the other hand, we know that

$$\varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) = \sum_{k=0}^{+\infty}(\# A_{n_k})[2^{-n}\omega_n r^n_{n_k} - p_n(r_{n_k})], \quad r_{n_k} = \frac{b_{n_k}\lambda^{1/2}}{\pi} \, ,$$

$N(\lambda, -\Delta, \Omega) \geqq N(\lambda, -\Delta, \Omega'_\infty)$, $\varphi(\lambda) = \varphi_\infty(\lambda)$ for $\delta \in (n-1, n)$ and $2^{-n}\omega_n r^n - p_n(r) > 0$ for any $r > 0$ (see (3.4)). Then

$$\begin{aligned}\varphi(\lambda) - N(\lambda, -\Delta, \Omega) &\leqq \sum_{k=0}^{+\infty}(\# A_{n_k})[2^{-n}\omega_n r^n_{n_k} - p_n(r_{n_k})] \\ &\leqq \sum_{l=0}^{+\infty}(\# A_l)[2^{-n}\omega_n r^n_l - p_n(r_l)] \, ,\end{aligned} \tag{3.16}$$

where we may take $\# A_l = o_l b_l^{-\delta}$ and $b_l = a_0^l$ for any $l \in \mathbf{Z}_+$ and $o_l \to 0$ as $l \to +\infty$. Now write

$$\begin{aligned}\sum_{l=0}^{+\infty}(\# A_l)[2^{-n}\omega_n r^n_l - p_n(r_l)] &= \sum_{l=0}^{L}(\# A_l)[2^{-n}\omega_n r^n_l - p_n(r_l)] \\ &\quad + \sum_{l=L+1}^{+\infty}(\# A_l)[2^{-n}\omega_n r^n_l - p_n(r_l)] \, ,\end{aligned}$$

where $L \sim \dfrac{ln\lambda}{2ln(1/a_0)}$. Thus we have

$$\sum_{l=0}^{L}(\# A_l)[2^{-n}\omega_n r_l^n - p_n(r_l)] \leqq \left(1+\frac{1}{\pi}\right)^{\frac{n-1}{2}} \sum_{l=0}^{L} o_l\, b_l^{n-1-\delta}\lambda^{\frac{n-1-\delta}{2}} \cdot \lambda^{\delta/2}$$

$$\sim \left(1+\frac{1}{\pi}\right)^{\frac{n-1}{2}} \lambda^{\delta/2} \cdot \sum_{l=0}^{L} o_l[a_0^{\delta-(n-1)}]^{L-l} .$$

By means of Lemma 3.2, we know that

$$\sum_{l=0}^{L}(\# A_l)[2^{-n}\omega_n r_l^n - p_n(r_l)] = o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty . \tag{3.17}$$

Secondly $p_n(r_l) = 0$ for $l \geqq L$, thus we obtain

$$\sum_{l=L+1}^{+\infty}(\# A_l)[2^{-n}\omega_n r_l^n - p_n(r_l)] = \sum_{l=L+1}^{+\infty} o_l(2\pi)^{-n}\omega_n b_l^{n-\delta}\lambda^{\frac{n-\delta}{2}}\lambda^{\delta/2}$$

$$\sim (2\pi)^{-n}\omega_n\lambda^{\delta/2}\sum_{l=L+1}^{+\infty} o_l(a_0^{n-\delta})^{l-L} ,$$

which implies that

$$\sum_{l=L+1}^{+\infty}(\# A_l)[2^{-n}\omega_n r_l^n - p_n(r_l)] = o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty . \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we obtain

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \leqq o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty , \tag{3.19}$$

which means the conjecture (2.8) does not hold and Theorem 2.2 is proved.

4. Proof of Theorem 2.3

Without loss of generality, we can assume that condition (C1) is of the form

$$b_k^{\delta}(\# A_k) \geqq c \quad \text{for all } k \geqq 0 . \tag{4.1}$$

Otherwise we can use another small constant instead of c.

Proof of Theorem 2.3. Let us first consider the case of $n = 1$. Now

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = \varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) + (\psi_\infty(\lambda) - \psi(\lambda))$$

$$= \sum_{k=0}^{+\infty}(\# A_k)[r_k - p_1(r_k)] + o(\lambda^{\delta/2}) .$$

From condition (C1) and noting that $p_1(r_k) = 0$ when $k \geqq L$, we have

$$\sum_{k=0}^{+\infty}(\# A_k)[r_k - p_1(r_k)] \geqq \sum_{k=0}^{L} c\, b_k^{-\delta}[r_k - p_1(r_k)] + \sum_{k=L+1}^{+\infty} c\, b_k^{-\delta} r_k ,$$

where $L \sim \dfrac{ln\lambda}{2ln(1/a_0)}$. Consequently

$$\sum_{k=0}^{L} b_k^{-\delta}[r_k - p_1(r_k)]\lambda^{-\frac{\delta}{2}}\lambda^{\delta/2} \sim \sum_{k=0}^{L}[r_k - p_1(r_k)](a_0^\delta)^{L-k} \cdot \lambda^{\delta/2} .$$

From the estimate (3.4), we have

$$0 < \sum_{k=0}^{L}[r_k - p_1(r_k)]a_0^{\delta(L-k)} \leqq \sum_{l=0}^{L} a_0^{\delta l} < +\infty , \quad \text{as } L \to +\infty ,$$

which implies that there exists a constant $c' > 0$ (where $c' > r_L \sim \frac{1}{\pi}$), such that

$$\sum_{k=0}^{L} c\, b_k^{-\delta}[r_k - p_1(r_k)] \geqq cc'\,\lambda^{\delta/2} , \quad \text{as } \lambda \to +\infty .$$

On the other hand, if $\delta \in (0,1)$, then

$$\sum_{k=L+1}^{+\infty} cb_k^{-\delta} r_k \sim \frac{c}{\pi}\sum_{k=L+1}^{+\infty}(a_0^{1-\delta})^{k-L} \cdot \lambda^{\delta/2} = \frac{c}{\pi}(a_0^{\delta-1} - 1)^{-1}\lambda^{\delta/2} .$$

From this we obtain

$$\varphi(\lambda) - N(\lambda) \geqq \left[cc' + \frac{c}{\pi}(a_0^{\delta-1} - 1)^{-1}\right]\lambda^{\delta/2} + o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty . \tag{4.2}$$

Next, if $n \geqq 2$, from (3.5) and (3.6) it is obvious that the positive bounded function $c_n(r)$ is at most polynomial decreasing as $r \to +\infty$ (cf. [31, 33]). Thus we can choose two suitable constants $\varepsilon_n \geqq 0$ (more precisely $\varepsilon_n \in [0, n-1]$) and $c_n > 0$, depending only on n, satisfying

$$\inf_{r \geqq \frac{1}{\pi}}(c_n(r)r^{\varepsilon_n}) \geqq c_n > 0 . \tag{4.3}$$

This implies that

$$c_n(r)r^{n-1} = 2^{-n}\omega_n r^n - p_n(r) \geqq c_n\, r^{n-1-\varepsilon_n}, \quad \text{for } n \geqq 2,\ r \geqq \frac{1}{\pi} . \tag{4.4}$$

From (C2) we know that

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = \varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) + o(\lambda^{\delta/2}) . \tag{4.5}$$

Using the same method as in Sect. 3, we have

$$\begin{aligned}\varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) &= \sum_{k=0}^{L}(\# A_k)[2^{-n}\omega_n r_k^n - p_n(r_k)] \\ &\quad + \sum_{k=L+1}^{+\infty}(\# A_k)(2^{-n}\omega_n)r_k^n ,\end{aligned} \tag{4.6}$$

where $r_k = \dfrac{b_k \lambda^{1/2}}{\pi}$, $L \sim \dfrac{ln\lambda}{2ln(1/a_0)}$, and $p_n(r_k) = 0$ for $k \geqq L$. Using (4.4), we obtain

$$\sum_{k=0}^{L} (\# A_k)[2^{-n}\omega_n r_k^n - p_n(r_k)] \geqq c c_n \sum_{k=0}^{L} b_k^{-\delta} r_k^{n-1-\varepsilon_n}\,, \quad \varepsilon_n \geqq 0,\ c_n > 0$$

$$\sim c c_n \left(\frac{1}{\pi}\right)^{n-1-\varepsilon_n} \lambda^{\delta/2} \cdot \sum_{k=0}^{L} [a_0^{\delta-(n-1-\varepsilon_n)}]^{L-k}$$

$$= c c_n \left(\frac{1}{\pi}\right)^{n-1-\varepsilon_n} [1 - a_0^{\delta-n+1+\varepsilon_n}]^{-1} \lambda^{\delta/2} + o(\lambda^{\delta/2})\,, \quad \text{as } \lambda \to +\infty\,.$$

On the other hand, if $\delta \in (n-1, n)$, then

$$\begin{aligned} \sum_{k=L+1}^{+\infty} (\# A_k)(2^{-n}\omega_n) r_k^n &\geqq (2\pi)^{-n}\omega_n c \sum_{k=L+1}^{+\infty} (a_0^{n-\delta})^{k-L} \lambda^{\delta/2} \\ &= (2\pi)^{-n}\omega_n c [a_0^{\delta-n} - 1]^{-1} \lambda^{\delta/2}\,. \end{aligned} \tag{4.7}$$

Hence Theorem 2.3 is proved.

5. Proof of Theorem 2.4

From (C2), we have

$$\begin{aligned} \varphi(\lambda) - N(\lambda) &= \varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) + \psi_\infty(\lambda) - \psi(\lambda) \\ &= \varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) + o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty\,. \end{aligned}$$

By using the Dirichlet–Neumann bracketing method, we know that

$$\varphi_\infty(\lambda) - N(\lambda, -\Delta, \Omega'_\infty) = \sum_{k=-\infty}^{+\infty} (\# A_k) f_k^n(\lambda) - \sum_{k=-\infty}^{-1} (\# A_k) f_k^n(\lambda)\,. \tag{5.1}$$

Here

$$0 < \sum_{k=-\infty}^{-1} (\# A_k) f_k^n(\lambda) < d_n \sum_{k=-\infty}^{-1} c\, b_{-k}^{\delta} \left(\frac{b_{-k}^{-1}}{\pi}\right)^{n-1} \lambda^{\frac{n-1}{2}}\,. \tag{5.2}$$

By taking $k = -j$, we have

$$\sum_{k=-\infty}^{-1} b_{-k}^{\delta} \left(\frac{1}{b_{-k}\pi}\right)^{n-1} = \left(\frac{1}{\pi}\right)^{n-1} \sum_{j=1}^{+\infty} b_j^{\delta-(n-1)}\,. \tag{5.3}$$

From the condition (2.17), it follows that

$$\sum_{k=-\infty}^{-1} (\# A_k) f_k^n(\lambda) = O(\lambda^{\frac{n-1}{2}}) = o(\lambda^{\delta/2}), \text{ as } \lambda \to +\infty\,, \tag{5.4}$$

which proves (2.18).

Secondly, from (2.16), (2.19), (3.4) and the fact that

$$\varphi(\lambda) - N(\lambda) \in O(\lambda^{\delta/2}), \quad \delta \in (n-1, n]\,,$$

it follows that $F_n(\lambda)$ is well defined, positive and bounded. Next, from Lemma 3.1, we know that the function $f_k^n(\lambda)$ (and so $F_n(\lambda)$) is left-continuous with discontinuity in $\lambda \in \mathbf{R}_+$, satisfying

$$\left(\frac{b_k}{\pi}\lambda^{1/2}\right)^2 = \sum_{j=1}^{n} q_j^2, \quad k \in \mathbf{Z}, \quad q_j \in \mathbf{N}, \quad 1 \leqq j \leqq n, \tag{5.5}$$

or equivalently

$$ln\lambda = -2ln b_k + 2ln\pi + ln\left(\sum_{j=1}^{n} q_j^2\right) . \tag{5.6}$$

Taking

$$y = ln\lambda \in \mathbf{R}, \quad q_1 = q_2 = \cdots = q_n = q\,, \tag{5.7}$$

$f_k^n(e^y)$ is discontinuous at those $y \in \mathbf{R}$, where

$$y = -2ln b_k + 2ln\pi + ln n + 2ln q, \quad k \in \mathbf{Z}, \quad q \in \mathbf{N}\,. \tag{5.8}$$

For any given $\lambda_0 \in \mathbf{R}_+$, define $x_0 = ln\lambda_0$. Observe that

$$ln b_{-k} \to +\infty \quad as\, k \to +\infty\,,$$

and so we can choose $k \in \mathbf{N}$, large enough, so that

$$-2ln b_{-k} + 2ln\pi + ln n < x_0\,.$$

By choosing q_k, the largest positive integer, for which

$$y_k = -2ln b_{-k} + 2ln\pi + ln n + 2ln q_k \leqq x_0\,,$$

then

$$0 \leqq x_0 - y_k < 2ln(q_k + 1) - 2ln q_k = 2ln(1 + q_k^{-1})\,. \tag{5.9}$$

Observe that $q_k \to +\infty$ as $k \to +\infty$ implies

$$y_k \to x_0 \text{ as } k \to +\infty\,, \tag{5.10}$$

or equivalently

$$\lambda_k = e^{y_k} \to e^{x_0} = \lambda_0, \text{ as } k \to +\infty\,. \tag{5.11}$$

Hence the set of points of discontinuity for $F_n(\lambda)$ is dense in $\mathbf{R}_+$. Theorem 2.4 is thus proved.

6. Remarks on the Condition (C1)

Let $n = 1$, Ω be an (non-empty) open subset of $\mathbf{R}$ with finite content $|\Omega|_1$ and fractal boundary $\partial\Omega$ of interior Minkowski dimension $\delta \in (0,1)$. We write Ω as the union of its connected components:

$$\Omega = \bigcup_{j=0}^{+\infty} I_j\,, \tag{6.1}$$

where the open intervals I_j are pairwise disjoint and of length l_j.

Without loss of generality, we assume that

$$l_0 \geqq l_1 \geqq l_2 \geqq \cdots \geqq l_j \geqq \cdots > 0\,, \quad \text{and} \quad \lim_{j\to+\infty} l_j = 0\,,$$

where the "lengths" l_j are repeated according to their "multiplicity, denoted by $\# A_j$." Hence we can rewrite Ω as

$$\Omega = \bigcup_{k=0}^{+\infty} (\# A_k)\tilde{I}_k\,, \tag{6.2}$$

where $\{\tilde{I}_k\}_{k\in\mathbf{Z}_+}$ is a "subsequence" of $\{I_j\}_{j\in\mathbf{Z}_+}$, with "lengths" $\{\tilde{l}_k\}_{k\in\mathbf{Z}_+}$ $(\subset \{l_j\}_{j\in\mathbf{Z}_+})$ and

$$\tilde{l}_0 > \tilde{l}_1 > \cdots > \tilde{l}_k > \cdots > 0\,, \quad \lim_{k\to+\infty} \tilde{l}_k = 0\,.$$

Here we still assume there exists a constant $a_0 \in (0,1)$ (the size of the tessellation) such that $\tilde{l}_k = a_0^k$.

Since $\tilde{I}_k$ are one-dimensional open cubes with multiplicity $\# A_k$, condition (C2), as claimed in Sect. 2, is certainly satisfied. Consider condition (C1); we have the following result:

Proposition 6.1. *When* $n = 1$, $\delta \in (0,1)$, *the following conditions are equivalent*:
(i) $\# A_k \approx \tilde{l}_k^{-\delta}$, *as* $k \to +\infty$.
(ii) *Condition* (C1) *holds and* $\mu^*(\delta, \partial\Omega) < +\infty$.
(iii) $l_j \approx j^{-\frac{1}{\delta}}$, *as* $j \to +\infty$.
(iv) $0 < \mu_*(\delta, \partial\Omega) \leqq \mu^*(\delta, \partial\Omega) < +\infty$.
(v) $\varphi(\lambda) - N(\lambda) \approx \lambda^{\delta/2}$, *as* $\lambda \to +\infty$.

Proof. From [25, Theorem 3. 13], we know that the conditions (iii), (iv) and (v) are equivalent. On the other hand, by Theorems 2.1, 2.2, 2.3 and the estimate (3.8), we can easily see that (v) implies (i) and (ii), and (ii) implies (i) and (v). It only remains to prove that (i) implies the other conditions. Here we prove that (i) implies (iii).

Since

$$(a_0^{-\delta})^{k+1} - 1 = (a_0^{-\delta} - 1)(1 + a_0^{-\delta} + a_0^{-2\delta} + \cdots + a_0^{-k\delta})\,,$$

we obtain

$$1 + a_0^{-\delta} + a_0^{-2\delta} + \cdots + a_0^{-k\delta} \approx a_0^{-k\delta}, \quad \text{as } k \to +\infty\,. \tag{6.3}$$

Without loss of generality, on taking $j = j(k) = (\# A_0) + (\# A_1) + \cdots + (\# A_k) \to +\infty$, as $k \to +\infty$, we know that $l_j = \tilde{l}_k$, and $l_{j+1} = \tilde{l}_{k+1}$.

If (i) holds, then

$$j \approx 1 + a_0^{-\delta} + a_0^{-2\delta} + \cdots + a_0^{-k\delta} \approx a_0^{-k\delta}, \quad \text{as } k \to +\infty\,, \tag{6.4}$$

and so

$$l_j j^{\frac{1}{\delta}} \approx \tilde{l}_k a_0^{-k} = 1, \quad l_{j+1}(j+1)^{\frac{1}{\delta}} \approx a_0 \tilde{l}_k a_0^{-k} = a_0\,, \tag{6.5}$$

and so condition (iii) is satisfied.

From Proposition 6.1, we have the following obvious corollary.

Corollary 6.1. *Let* $n = 1$, $\delta \in (0,1)$ *and* $\mu^*(\delta, \partial\Omega) < +\infty$. *Then condition* (C1) *is equivalent to* $\mu_*(\delta, \partial\Omega) > 0$.

Secondly, consider $n \geqq 1$. First we introduce the following condition:

(C3) $$|\Omega|_n - |\Omega'_L|_n \approx b_L^{n-\delta}, \quad \text{as } L \to +\infty ,$$

where $b_L = a_0^L$ and a_0 is the size of the tessellation of Ω. Then we have

Proposition 6.2. *Let $n \geqq 1$, $\delta \in [n-1, n]$. If (C1) holds, then there exists a positive constant c_1, depending only on δ and n, such that*

$$|\Omega|_n - |\Omega'_L|_n \geqq c_1 b_L^{n-\delta}, \quad \text{as } L \to +\infty . \tag{6.6}$$

Furthermore the inequality (6.6) *implies that $\mu_*(\delta, \partial\Omega) > 0$.*

Proof. Since $b_L^{-(n-\delta)}[|\Omega|_n - |\Omega'_L|_n] = b_L^{-(n-\delta)}|\Omega''_L|_n$, we know that

$$b_L^{-(n-\delta)}|\Omega''_L|_n \geqq b_L^{-(n-\delta)} b_{L+1}^n (\# A_{L+1}) = b_1^{n-\delta} b_{L+1}^{\delta} (\# A_{L+1}) . \tag{6.7}$$

If (C1) holds, then $b_L^{-(n-\delta)}|\Omega''_L|_n \geqq b_1^{n-\delta} c$ as $L \to +\infty$, as required.

On the other hand, we know that $|\Omega|_n - |\Omega'_L|_n \leqq |\Omega^i_{\varepsilon_L}|_n$, $\varepsilon_L = \sqrt{n} b_L$; which means that

$$n^{-\frac{n-\delta}{2}} \cdot b_L^{-(n-\delta)}|\Omega''_L|_n \leqq \varepsilon_L^{-(n-\delta)}|\Omega^i_{\varepsilon_L}| .$$

By the condition (6.6), we have

$$\varepsilon_L^{-(n-\delta)}|\Omega^i_{\varepsilon_L}|_n \geqq c_1 n^{-\frac{n-\delta}{2}} \quad \text{as } L \to +\infty , \tag{6.8}$$

which implies that $\mu_*(\delta, \partial\Omega) > 0$, as claimed.

Next, we have

Proposition 6.3. *Let $n \geqq 1$, $\delta \in [n-1, n]$. If $\mu^*(\delta, \partial\Omega) < +\infty$, then there exists a positive constant c_2, depending only on δ and n, such that*

$$|\Omega|_n - |\Omega'_L|_n \leqq c_2 b_L^{n-\delta}, \quad \text{as } L \to +\infty . \tag{6.9}$$

Proof. Since $|\Omega''_L|_n \leqq |\Omega^i_{\varepsilon_L}|_n$, $\varepsilon_L = \sqrt{n}\, b_L$, then

$$b_L^{-(n-\delta)}[|\Omega|_n - |\Omega'_L|_n] \leqq n^{\frac{n-\delta}{2}} \varepsilon_L^{-(n-\delta)}|\Omega^i_{\varepsilon_L}|_n .$$

On the other hand, we know that

$$\varepsilon_L^{-(n-\delta)}|\Omega^i_{\varepsilon_L}|_n \leqq \mu^*(\delta, \partial\Omega) + o(1), \quad \text{as } L \to +\infty ,$$

which means that $\mu^*(\delta, \partial\Omega) < +\infty$ implies the estimate (6.9) holds, as required.

From Proposition 6.2 and Proposition 6.3, we have an obvious corollary as follows:

Corollary 6.2. *Let $n \geqq 1$, $\delta \in [n-1, n]$, if* (C1) *holds and $\mu^*(\delta, \partial\Omega) < +\infty$, then the condition* (C3) *holds.*

7. An Example

In order to obtain an explicit upper (or lower) bound on the second term of $N(\lambda)$, we often let $b_k = a_0^k$ for some size $a_0 \in (0,1)$ in the tessellation of Ω. On the other

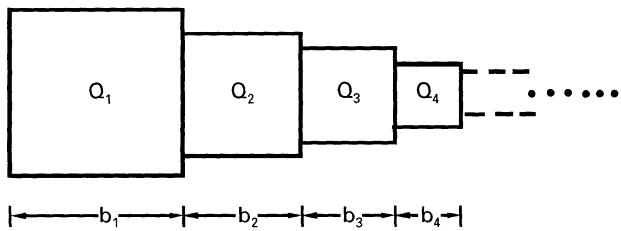

Fig. 7.1.

hand, as shown above, in order to obtain a positive lower bound on the second term of $N(\lambda)$, we have introduced the condition (C2), which makes the problem more tractable. Sometimes we can choose a special tessellation of Ω, so that the condition (C2) can be easily satisfied. However we can not ensure, in the meantime, that the tessellation size is exponential (i.e. $b_k = a_0^k$).

In the following, we study an example, which was mentioned in [26] and for which condition (C2) can be easily satisfied and the tessellation size is polynomial decreasing. However the weaker form (2.8) of the Weyl–Berry conjecture (2.6) remains valid.

Example 7.1. Let $\Omega = \bigcup_{j=1}^{+\infty} Q_j$ be the disjoint union of open cubes Q_j in $\mathbf{R}^n$ $(n \geqq 1)$ with sides of length b_j satisfying (see Fig. 7.1 above)

$$b_j = Kj^{-\frac{1}{\delta}}, \ \delta \in (n-1, n), \quad j \geqq 1, \ K > 0 \text{ a constant}. \tag{7.1}$$

From the figure above, we can see that although $\Omega \subset \mathbf{R}^n (n \geqq 2)$ is unbounded, it has finite volume $|\Omega|_n = K^n \sum_{j=1}^{+\infty} j^{-\frac{n}{\delta}} < +\infty$. As shown in [26], $\partial\Omega$ is δ-Minkowski measurable and $\mu(\delta, \partial\Omega)$ can be computed by means of δ and K.

By Corollary 2.1, we have

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \leqq C'_{n,\delta}\,\mu(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \tag{7.2}$$

as $\lambda \to +\infty$, where

$$C'_{n,\delta} = \left(1 + \frac{1}{\pi}\right)^{\frac{n-1}{2}} n^{\frac{n-\delta}{2}} (n-\delta)^{\frac{\delta-n}{\delta-(n-1)}} (1 + \delta - n)^{-1} + (2\pi)^{-n} \omega_n n^{\frac{n-\delta}{2}}.$$

In order to estimate the lower bound on the second term of $N(\lambda)$, we let the tessellation of Ω be Ω itself, i.e. $\Omega'_\infty = \Omega$ (actually the construction of the tessellation here is different to the tessellation as described in Sect. 2). Here condition (C2) is certainly satisfied, with $b_k = Kk^{-\frac{1}{\delta}}$ (for $k \geqq 1$), and is polynomial decreasing.

By using the same method as in Sects. 3 and 4, we have

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = \sum_{k=1}^{+\infty} [2^{-n}\omega_n r_k^n - p_n(r_k)], \quad r_k = \frac{b_k \lambda^{1/2}}{\pi}.$$

If $n = 1$ and taking $L \sim \pi^{-\delta} K^\delta \lambda^{\frac{\delta}{2}}$, then $r_k \leqq 1$ implies $k \geqq L$. We write

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = \sum_{k=1}^{L-1} [r_k - p_1(r_k)] + \sum_{k=L}^{+\infty} [r_k - p_1(r_k)].$$

Since $p_1(r_k) = 0$ for $k \geqq L$, we have

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \geqq \sum_{k=L}^{+\infty} r_k = \frac{K}{\pi} \sum_{k=L}^{+\infty} k^{-\frac{1}{\delta}} \lambda^{1/2} .$$

Observe that $\lambda^{\frac{1-\delta}{2}} \sim \left(\frac{\pi}{K}\right)^{1-\delta} L^{\frac{1-\delta}{\delta}}$, and so

$$\frac{K}{\pi} \sum_{k=L}^{+\infty} k^{-\frac{1}{\delta}} \lambda^{1/2} \sim \left(\frac{K}{\pi}\right)^{\delta} \sum_{k=L}^{+\infty} k^{-\frac{1}{\delta}} L^{\frac{1-\delta}{\delta}} \lambda^{\delta/2} = \left(\frac{K}{\pi}\right)^{\delta} \sum_{k=L}^{+\infty} \frac{1}{k} \left(\frac{k}{L}\right)^{\frac{\delta-1}{\delta}} \cdot \lambda^{\delta/2} .$$

Let $\dfrac{\delta - 1}{\delta} = \delta_1$, $B_k = \dfrac{k}{L}$ for $k \geqq L$, then we know that

$$\int_{B_{k-1}}^{B_k} x^{\delta_1 - 1} dx = \theta_k^{\delta_1 - 1} \frac{1}{L}, \quad \theta_k \in (B_{k-1}, B_k) .$$

Since $\delta_1 - 1 < 0$, we have

$$\frac{1}{k} B_k^{\delta_1} \left(1 + \frac{1}{k-1}\right)^{1-\delta_1} \geqq \int_{B_{k-1}}^{B_k} x^{\delta_1 - 1} dx \geqq \frac{1}{k} B_k^{\delta_1} . \tag{7.3}$$

Hence we obtain

$$\sum_{k=L}^{+\infty} \frac{1}{k} \left(\frac{k}{L}\right)^{\delta_1} \geqq \left(1 + \frac{1}{L-1}\right)^{\delta_1 - 1} \int_1^{+\infty} x^{\delta_1 - 1} dx = \frac{\delta}{1-\delta} + o(1), \quad \text{as } L \to +\infty .$$

This implies that

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) \geqq \left(\frac{K}{\pi}\right)^{\delta} \frac{\delta}{1-\delta} \lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty . \tag{7.4}$$

Further, let $n \geqq 2$ and $L \sim n^{\frac{-\delta}{2}} \left(\frac{K}{\pi}\right)^{\delta} \lambda^{\delta/2}$. Then $r_k \leqq \sqrt{n}$ implies that $k \geqq L$ and $p_n(r_k) = 0$. We have

$$\varphi(\lambda) - N(\lambda, -\Delta, \Omega) = \sum_{k=1}^{L-1} [2^{-n} \omega_n r_k^n - p_n(r_k)] + \sum_{k=L}^{+\infty} 2^{-n} \omega_n r_k^n . \tag{7.5}$$

Now $k \leqq L$ implies $r_k \geqq \dfrac{1}{\pi}$, and so by (4.3) and (4.4), we know that there exist two constants $\varepsilon_n \in [0, n-1]$ and $c_n > 0$, depending only on n, such that

$$\sum_{k=1}^{L-1} [2^{-n} \omega_n r_k^n - p_n(r_k)] \geqq c_n \left(\frac{K}{\pi}\right)^{n-1-\varepsilon_n} \sum_{k=1}^{L-1} k^{-\frac{n-1-\varepsilon_n}{\delta}} \lambda^{\frac{n-1-\varepsilon_n-\delta}{2}} \lambda^{\frac{\delta}{2}} . \tag{7.6}$$

Note that

$$\sum_{k=1}^{L-1} k^{-\frac{n-1-\varepsilon_n}{\delta}} \lambda^{\frac{n-1-\varepsilon_n-\delta}{2}} \sim n^{\frac{n-1-\varepsilon_n-\delta}{2}} \left(\frac{K}{\pi}\right)^{-(n-1-\varepsilon_n-\delta)} \sum_{k=1}^{L-1} \frac{1}{k} \left(\frac{k}{L}\right)^{\frac{\delta-(n-1)+\varepsilon_n}{\delta}} . \tag{7.7}$$

Let $\frac{\delta-(n-1)+\varepsilon_n}{\delta}=\delta_2$, $B_k=\frac{k}{L}$ for $0\leqq k\leqq L-1$, then

$$\int_{B_{k-1}}^{B_k} x^{\delta_2-1}dx=\theta_k^{\delta_2-1}\frac{1}{L},\quad \text{where } \theta_k\in(B_{k-1},B_k)\,. \tag{7.8}$$

Since $\delta_1-1\leqq 0$, we obtain

$$\frac{1}{k}\left(\frac{k}{L}\right)^{\delta_2}=B_k^{\delta_2-1}\frac{1}{L}\geqq\left(\frac{k-1}{k}\right)^{1-\delta_2}\int_{B_{k-1}}^{B_k}x^{\delta_2-1}dx\,,$$

which means that

$$\sum_{k=1}^{L-1}\frac{1}{k}\left(\frac{k}{L}\right)^{\delta_2}\geqq 2^{\delta_2-1}\int_{1/L}^{1-\frac{1}{L}}x^{\delta_2-1}dx\sim 2^{\frac{\varepsilon_n-(n-1)}{\delta}}\cdot\frac{\delta}{\delta-(n-1)+\varepsilon_n}+o(1),\quad \text{as } L\to+\infty\,.$$

Combining (7.6) and (7.7), we obtain

$$\sum_{k=1}^{L-1}[2^{-n}\omega_n r_k^n-p_n(r_k)]\geqq c_n\left(\frac{K}{\pi}\right)^{\delta}2^{\frac{\varepsilon_n-(n-1)}{\delta}}\cdot n^{\frac{n-1-\varepsilon_n-\delta}{2}}\cdot\frac{\delta}{\delta-(n-1)+\varepsilon_n}\lambda^{\delta/2}+o(\lambda^{\delta/2}),\quad \text{as } \lambda\to+\infty\,. \tag{7.9}$$

Next, we consider

$$\sum_{k=L}^{+\infty}2^{-n}\omega_n r_k^n=(2\pi)^{-n}\omega_n K^n\sum_{k=L}^{+\infty}k^{-\frac{n}{\delta}}\lambda^{\frac{n}{2}}\sim\left[\frac{\sqrt{n}}{2}\right]^n\omega_n\sum_{k=L}^{+\infty}\left(\frac{k}{L}\right)^{-\frac{n}{\delta}}\,.$$

Let $k=L+l$, $l\geqq 0$, then $\sum_{k=L}^{+\infty}\left(\frac{k}{L}\right)^{-\frac{n}{\delta}}=\sum_{l=0}^{+\infty}\left(1+\frac{l}{L}\right)^{-\frac{n}{\delta}}$. Write $B_l=1+\frac{l}{L}$ for $l\geqq 0$, $\delta_3=\frac{n}{\delta}$, then

$$\int_{B_{l-1}}^{B_l}x^{-\delta_3}dx=\theta_l^{-\delta_3}\cdot\frac{1}{L},\quad \theta_l\in(B_{l-1},B_l),\, l\geqq 1\,. \tag{7.10}$$

This implies that

$$B_l^{-\delta_3}\leqq L\int_{B_{l-1}}^{B_l}x^{-\delta_3}dx\leqq B_{l-1}^{-\delta_3},\quad l\geqq 1\,. \tag{7.11}$$

Hence we have

$$\sum_{k=L}^{+\infty}\left(\frac{k}{L}\right)^{-\frac{n}{\delta}}\geqq L\int_1^{+\infty}x^{-\delta_3}dx=\frac{L}{\delta_3-1}\sim n^{-\frac{\delta}{2}}\left(\frac{K}{\pi}\right)^{\delta}\cdot\frac{\delta}{n-\delta}\lambda^{\delta/2}\,. \tag{7.12}$$

Combining (7.9) and (7.12), we obtain

$$\varphi(\lambda)-N(\lambda)\geqq c_n\left(\frac{K}{\pi}\right)^{\delta}2^{\frac{\varepsilon_n-(n-1)}{\delta}}n^{\frac{n-1-\varepsilon_n-\delta}{2}}\frac{\delta}{\delta-(n-1)+\varepsilon_n}\lambda^{\delta/2}+2^{-n}\omega_n n^{\frac{n-\delta}{2}}\left(\frac{K}{\pi}\right)^{\delta}\frac{\delta}{n-\delta}\lambda^{\delta/2}+o(\lambda^{\delta/2}),\quad \text{as } \lambda\to+\infty\,. \tag{7.13}$$

Thus, for Example 7.1, we have proved that conjecture (2.8) is true from the estimates (7.2), (7.4) and (7.13).

Remark. 7.1 We cannot apply the result of Theorem 2.4 to Example 7.1 directly. This is because although condition (C2), under the assumption of the tessellation of Ω to be Ω itself, is satisfied, condition (2.17) does not hold.

8. Further Examples

Example 8.1. (*Cantor set*). Let Ω be the complement in [0, 1] of the triadic Cantor set Γ. Then $\partial\Omega = \Gamma$ and we have (cf. [24, 25])

$$\delta = H = log2/log3, \quad \mu_*(\delta, \partial\Omega) = \frac{log9}{log(3/2)}\left(\frac{log(3/2)}{log4}\right)^\delta, \quad \mu^*(\delta, \partial\Omega) = 2^{2-\delta}. \tag{8.1}$$

If we choose the size of the tessellation of Ω is $a_0 = \frac{1}{3}$, then condition (C2) holds and

$$b_k = 3^{-k}, \quad \# A_k = 2^{k-1}, \quad b_k^\delta(\# A_k) = \frac{1}{2}. \tag{8.2}$$

From Corollary 2.1 and Theorem 2.3, we have

$$\begin{aligned}
&\frac{1}{2\pi}[1 + (3^{1-\delta} - 1)^{-1}]\frac{log(3/2)}{log9}\left(\frac{log4}{log(3/2)}\right)^\delta \mu_*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2})\\
&\leqq \frac{|\Omega|_1}{\pi}\lambda^{1/2} - N(\lambda)\\
&\leqq \left[(1-\delta)^{1-\frac{1}{\delta}}\delta^{-1} + \frac{1}{\pi}\right]\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty.
\end{aligned} \tag{8.3}$$

Thus conjecture (2.8) holds. On the other hand, we see from Theorem 2.4 that

$$\frac{|\Omega|_1}{\pi}\lambda^{1/2} - N(\lambda) = \overline{F}_1\left(\frac{log\lambda - 2\,log\pi}{2log3}\right)\lambda^{\delta/2} + O(1), \quad \lambda \to +\infty, \tag{8.4}$$

where

$$\overline{F}_1(y) = \frac{1}{2}\left(\frac{1}{\pi}\right)^\delta \sum_{k=-\infty}^{+\infty} 3^{\delta(k-y)}h_1(3^{y-k}), \quad \delta = log2/log3, \tag{8.5}$$

is a well-defined, positive, bounded, 1-periodic and left-continuous function; its point set of discontinuity being dense in **R**. This implies that the function $\lambda^{-\frac{\delta}{2}}\left[\frac{|\Omega|_1}{\pi}\lambda^{1/2} - N(\lambda)\right]$ is oscillating, i.e. the limit of $\lambda^{-\delta/2}\left[\frac{|\Omega|_1}{\pi}\lambda^{1/2} - N(\lambda)\right]$, as $\lambda \to +\infty$, does not exist. Thus the modified Weyl–Berry conjecture (2.6) is not true for Ω, which in particular (cf. [24, 25]) means that the boundary $\partial\Omega$ is not δ-Minkowski measurable. Actually we have

$$0 < \mu_*(\delta, \partial\Omega) < \mu^*(\delta, \partial\Omega) < +\infty.$$

Example 8.2. Let $n = 1$, $\Omega = \bigcup_{k=1}^{+\infty}(\# A_k)I_k$ be the union of disjoint open intervals I_k with finite length and fractal boundary $\partial\Omega$, where

$$\# A_k = 3^{k-1}, \quad |I_k|_1 = 5^{-k}. \tag{8.6}$$

Then

$$\delta = log3/log5, \quad 0 < \mu_*(\delta, \partial\Omega) < \mu^*(\delta, \partial\Omega) < +\infty .$$

If choosing $a_0 = \frac{1}{5}$, then conditions (C1) and (C2) are satisfied, and $a_0^{k\delta}(\# A_k) = \frac{1}{3}5^{-k\delta}3^k = \frac{1}{3}$. Thus from Corollary 2.1 and Theorem 2.3 we have

$$\begin{aligned}\frac{1}{3\pi}[1 + (5^{1-\delta} - 1)^{-1}]\lambda^{\delta/2} + o(\lambda^{\delta/2}) &\leqq \frac{|\Omega|_1}{\pi}\lambda^{1/2} - N(\lambda, -\Delta, \Omega) \\ &\leqq \left((1-\delta)^{1-\frac{1}{\delta}}\delta^{-1} + \frac{1}{\pi}\right)\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty . \qquad (8.7)\end{aligned}$$

Consequently conjecture (2.8) is true. Secondly, by Theorem 2.4, we know that

$$\frac{|\Omega|_1}{\pi}\lambda^{1/2} - N(\lambda) = \overline{F}_1\left(\frac{log\lambda - 2log\pi}{2log5}\right)\lambda^{\delta/2} + O(1), \quad \text{as } \lambda \to +\infty , \qquad (8.8)$$

where

$$\overline{F}_1(y) = \frac{1}{3}\left(\frac{1}{\pi}\right)^{\delta}\sum_{k=-\infty}^{+\infty} 5^{\delta(k-y)}h_1(5^{y-k}) \qquad (8.9)$$

is a well-defined, positive, bounded, 1-periodic, left-continuous function whose set of points of discontinuity is dense in **R**. Hence the modified Weyl–Berry conjecture (2.6) is not true for Ω.

Next, we shall consider the case of $n \geqq 2$. We give three examples with disconnected domains. Observe, in these examples, that only in the case of calculating the exact or the lower bound of the second term of the counting function, we choose a special tessellation for Ω, i.e. let Ω'_∞ be Ω itself, so that condition (C2) can be easily satisfied. In this case, the construction of the tessellation is actually different from that of the tessellation as mentioned in Sect. 2. However we know that the counting function $N(\lambda)$ is actually independent of the tessellation, so there is no change essentially for the problem itself if we choose a special tessellation for Ω.

Example 8.3. (*Fleckinger–Vassiliev's example*). Let $n = 2$ and consider the union Ω of disjoint open squares, where the central square Q_0 has side 1. The side of each consecutive square is s times smaller with $1 + \sqrt{2} < s < 3$. At the k^{th} step we have $\# A_k = 4 \times 3^{k-1}$ squares Q_k with sides $b_k = s^{-k}$ (See [27, 28, Figure 1]).

We know (cf. [27, 28]) that $\delta = log3/logs$, and $0 < \mu_*(\delta, \partial\Omega) < \mu^*(\delta, \partial\Omega) < +\infty$. If we choose the tessellation of Ω to be Ω itself, then $a_0 = \frac{1}{s}$ and $b_k^{\delta}(\# A_k) = \frac{4}{3}$ and so conditions (C1) and (C2) hold. By the results of this paper, we have

$$\begin{aligned}C_{2,\delta}\lambda^{\delta.2} + o(\lambda^{\delta/2}) &\leqq \frac{1}{4\pi}|\Omega|_2\lambda - N(\lambda, -\Delta, \Omega) \\ &\leqq C'_{2,\delta}\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty , \qquad (8.10)\end{aligned}$$

and

$$\frac{|\Omega|_2}{4\pi}\lambda - N(\lambda) = \overline{F}_2\left(\frac{log\lambda - 2log\pi}{2logs}\right)\lambda^{\delta/2} + O(\sqrt{\lambda}), \text{as } \lambda \to +\infty , \qquad (8.11)$$

where the constants

$$C_{2,\delta} = \frac{4}{3}c_2\left(\frac{1}{\pi}\right)^{1-\varepsilon_2}[1 - s^{1-\delta-\varepsilon_2}]^{-1} + \frac{1}{3\pi}(s^{2-\delta} - 1)^{-1}, \quad \varepsilon_2 \in [0,1], c_2 > 0\,,$$

$$C'_{2,\delta} = \left(1 + \frac{1}{\pi}\right)^{\frac{1}{2}} 2^{\frac{2-\delta}{2}}(2-\delta)^{\frac{\delta-2}{\delta-1}}(\delta - 1)^{-1} + \frac{1}{4\pi}2^{\frac{2-\delta}{2}}\,;$$

and

$$\overline{F}_2(y) = \frac{4}{3}\left(\frac{1}{\pi}\right)^{\delta}\sum_{k=-\infty}^{+\infty} s^{\delta(k-y)}h_2(s^{y-k})$$

is a well-defined, positive, bounded, 1-periodic and left-continuous function; its point set of discontinuity being dense in $\mathbf{R}$.

The above result implies that the modified Weyl–Berry conjecture (2.6) is not true for Ω (which is similar to the main result of [27, 28]), however here, the weaker form (2.8) of the conjecture (2.6) is true.

Our next example illustrates a similar phenomenon.

Example 8.4. (*Brossard–Carmona's example*). Let $\Omega \subset \mathbf{R}^2$ be a countable disjoint union of all the small open squares belonging to the successive "generations" defined below.

Let $\{P_j,\ j \geqq 1\}$ be a nondecreasing sequence of positive integers. The 0$^{\text{th}}$ generation contains one square of side 1; the 1$^{\text{st}}$ generation contains four large squares, each of which has side 1/3 and is divided into $(P_1)^2$ congruent smaller squares, etc. Similarly, the j$^{\text{th}}$ generation contains $4 \times 5^{j-1}$ large squares, each of which has side 3^{-j} and is divided into $(P_j)^2$ congruent smaller squares; and so on. (See [23, Fig. 1].)

As is shown in [23], irrespective of the sequence $\{P_j,\ j \geqq 1\}$, $H = log5/log3$ and $H(\partial\Omega) \in (0, \frac{36}{5}]$. Now, given a real number $a \geqq 1$ fixed, we let $P_j = [a^j]$ for any $j \in \mathbf{N}$. Then we know (see [23])

$$\delta = log(5a^2)/log(3a), \quad 0 < \mu_*(\delta, \partial\Omega) < \mu^*(\delta, \partial\Omega) < +\infty\,. \tag{8.12}$$

Observe that $\delta \in [H, 2)$, and $\delta = H$ if and only if $a = 1$. Without loss of generality, we can assume that $a \in \mathbf{N}$, and let the tessellation of Ω be Ω itself. Then $a_0 = \frac{1}{3a}$, $a_0^{k\delta}(\# A_k) = \frac{4}{5}$ and conditions (C1) and (C2) are satisfied. From Corollary 2.1 and Theorem 2.3, we have

$$\begin{aligned} C_{2,\delta}\lambda^{\delta/2} + o(\lambda^{\delta/2}) &\leqq \frac{1}{4\pi}|\Omega|_2\lambda - N(\lambda, -\Delta, \Omega) \\ &\leqq C'_{2,\delta}\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty\,, \end{aligned} \tag{8.13}$$

where

$$C_{2,\delta} = \frac{4}{5}c_2\left(\frac{1}{\pi}\right)^{1-\varepsilon_2}[1 - (3a)^{1-\delta-\varepsilon_2}]^{-1} + \frac{1}{5\pi}[(3a)^{2-\delta} - 1]^{-1}, \quad \varepsilon_2 \in [0,1], c_2 > 0,$$

$$C'_{2,\delta} = \left(1 + \frac{1}{\pi}\right)^{\frac{1}{2}} 2^{\frac{2-\delta}{2}}(2-\delta)^{\frac{\delta-2}{\delta-1}}(\delta - 1)^{-1} + \frac{1}{4\pi}2^{\frac{2-\delta}{2}}\,.$$

This means that conjecture (2.8) holds. On the other hand, Theorem 2.4 gives

$$\frac{|\Omega|_2}{4\pi}\lambda - N(\lambda) = \overline{F}_2\left(\frac{log\lambda - 2log\pi}{2log(3a)}\right)\lambda^{\delta/2} + O(\sqrt{\lambda}), \text{as } \lambda \to +\infty\,, \tag{8.14}$$

where

$$\overline{F}_2(y) = \frac{4}{5}\left(\frac{1}{\pi}\right)^{\delta}\sum_{k=-\infty}^{+\infty}(3a)^{\delta(k-y)}h_2((3a)^{y-k})$$

is a well-defined, positive, bounded, 1-periodic and left-continuous function; its set of points of discontinuity being dense in **R**. Therefore the modified Weyl–Berry conjecture (2.6) does not hold for Ω here. This explains the reason why the authors in [23] could not find the same upper and lower bounds for the second asymptotic term of the "partition function" $Z(t) = \int_0^{+\infty} exp(-\lambda t)dN(\lambda)$, as $t \to 0+$.

Finally we construct an n-dimensional example.

Example 8.5. Let $\Omega = \bigcup_{k=0}^{+\infty}(\# A_k)Q_k$ be the union of countably disjoint open cubes in $\mathbf{R}^n$ $(n \geqq 2)$ with finite volume and fractal boundary $\partial\Omega$. More precisely, the central cube Q_0 has side length 1. The side of each of the 2n consecutive cubes Q_1 is s_n times smaller. These cubes are "glued" to the middles of the sides of Q_0. Similarly, at the k$^{\text{th}}$ step, we have $\# A_k = 2n \times (2n-1)^{k-1}$ cubes Q_k of side $b_k = s_n^{-k}$, where $s_n = (2\alpha_n n)^{\frac{1}{n}} > 1$ and α_n a suitable constant depending only on n, satisfying

$$1 < \alpha_n < \frac{2n-1}{2n}(2n-1)^{\frac{1}{n-1}}, \quad (n \geqq 2)\,.$$

This ensures that Ω is the union of disjoint open cubes. Then we can deduce

$$\delta = \frac{log(2n-1)}{logs_n} = n\frac{log(2n-1)}{log(2\alpha_n n)} \in (n-1, n)\,,$$

$$0 < \mu_*(\delta, \partial\Omega) < \mu^*(\delta, \partial\Omega) < +\infty\,.$$

If we choose the tessellation of Ω to be Ω itself, then $a_0 = \frac{1}{s_n} \in (0,1)$, $a_0^{\delta k}(\# A_k) = \frac{2n}{2n-1}$ and conditions (C1) and (C2) are satisfied. Thus Corollary 2.1 and Theorem 2.3 tell us that

$$\begin{aligned} C_{n,\delta}\lambda^{\delta/2} + o(\lambda^{\delta/2}) &\leqq (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} - N(\lambda, -\Delta, \Omega) \\ &\leqq C'_{n,\delta}\mu^*(\delta, \partial\Omega)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad \text{as } \lambda \to +\infty\,, \end{aligned}$$

where

$$\begin{aligned} C_{n,\delta} = &\frac{2n}{2n-1}c_n\left(\frac{1}{\pi}\right)^{n-1-\varepsilon_n}\left[1 - (2\alpha_n n)^{\frac{1}{n}(n-\delta-1-\varepsilon_n)}\right]^{-1} \\ &+ \frac{2n}{2n-1}(2\pi)^{-n}\omega_n\left[(2\alpha_n n)^{\frac{n-\delta}{n}} - 1\right]^{-1}, \quad \varepsilon_n \in [0, n-1], c_n > 0\,; \end{aligned}$$

$$C'_{n,\delta} = \left(1 + \frac{1}{\pi}\right)^{\frac{n-1}{2}} n^{\frac{n-\delta}{2}}(n-\delta)^{\frac{\delta-n}{\delta-(n-1)}}(1+\delta-n)^{-1} + (2\pi)^{-n}\omega_n n^{\frac{n-\delta}{2}}\,.$$

Thus the weaker form (2.8) of the conjecture (2.6) holds. Next, from Theorem 2.4 we have

$$(2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2} - N(\lambda) = \overline{F}_n\left(\frac{n(log\lambda - 2log\pi)}{2log(2\alpha_n n)}\right)\lambda^{\delta/2} + O(\lambda^{\frac{n-1}{2}}),$$

as $\lambda \to +\infty$, where

$$\overline{F}_n(y) = \frac{2n}{2n-1}\left(\frac{1}{\pi}\right)^{\delta}\sum_{k=-\infty}^{+\infty}(2\alpha_n n)^{\frac{\delta}{n}(k-y)}h_n\left((2\alpha_n n)^{\frac{y-k}{n}}\right)$$

is a well-defined, positive, bounded, 1-periodic and left-continuous function; its set of points of discontinuity being dense in **R**. This means that the modified Weyl–Berry conjecture (2.6) is not true for Ω.

From the preceding Examples 8.3, 8.4 and 8.5, we know that $N(\lambda)$ does not admit an asymptotic second term proportional to $\lambda^{\delta/2}$. Although we have not shown that the boundaries (appearing in these examples) are δ-Minkowski measurable, it is apparent that the modified Weyl–Berry conjecture (2.6) does not hold for $n \geqq 2$. This is because although we have, in our examples, a strict inequality $0 < \mu_*(\delta, \partial\Omega) < \mu^*(\delta, \partial\Omega) < +\infty$, an equality $\mu_*(\delta, \partial\Omega) = \mu^*(\delta, \partial\Omega)$ can always be easily obtained by extracting a specially chosen infinite sequence of isolated points from Ω. This will not change the spectrum for $n \geqq 2$, but will change $\mu_*(\delta, \partial\Omega)$ and $\mu^*(\delta, \partial\Omega)$ (also see [28]).

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