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Mapping Class Group Actions on Quantum Doubles

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Abstract: We study representations of the mapping class group of the punctured torus on the double of a finite dimensional possibly non-semisimple Hopf algebra that arise in the construction of universal, extended topological field theories. We discuss how for doubles the degeneracy problem of TQFT's is circumvented. We find compact formulae for the $S^{\pm 1}$-matrices using the canonical, non-degenerate forms of Hopf algebras and the bicrossed structure of doubles rather than monodromy matrices. A rigorous proof of the modular relations and the computation of the projective phases is supplied using Radford's relations between the canonical forms and the moduli of integrals. We analyze the projective $SL(2, Z)$-action on the center of $U_q(sl_2)$ for q an $l = 2m + 1^{\text{st}}$ root of unity. It appears that the $3m + 1$-dimensional representation decomposes into an $m + 1$-dimensional finite representation and a $2m$-dimensional, irreducible representation. The latter is the tensor product of the two dimensional, standard representation of $SL(2, Z)$ and the finite, m-dimensional representation, obtained from the truncated TQFT of the semisimplified representation category of $U_q(sl_2)$.

1. Introduction

Since the seminal paper of Atiyah [A] on the abstract definition of a topological quantum field theory (TQFT) much progress has been made in finding non-trivial examples and extended structures. The most interesting developments took place in three dimensions where actual models of quantum field theory, like rational conformal field theories and Chern–Simons theory led to the discovery of new invariants. See [Cr] and [Wi].

In an attempt to counterpart these heuristic theories by mathematically rigorous constructions the field theoretical machinery had been replaced by quasitriangular Hopf algebras, or quantum groups. The resulting invariants are described in [TV] and [RT]. From here it is not hard to understand how to associate a TQFT to a rigid, abelian, monoidal category and an extended TQFT to a braided tensor category (BTC). In order for these theories to be well defined one has to make a few more assumptions. One is that the category shall obtain only a finite number of inequivalent, simple objects, i.e., it is rational. The other is a technical non-degeneracy condition, called "modularity" in [T], which is to assure that elementary cobordisms are associated to identifications rather than projections. Alternatively, if the modularity condition fails to hold, it is standard in the Atiyah [A] description to define a truncated TQFT by reducing the vectorspaces to the images of the projections.

All of the mentioned TQFT's are semisimple, i.e., they rely on the decompositions into simple objects. Clearly, semisimplicity cannot be an assumption of a fundamental but only of a technical nature. In seeking *universal* constructions of TQFT's, which do not refer to decompositions, one should thus not only generalize the existing ones to non-semisimple theories but also gain a deeper understanding of the structure underlying them.

A partial answer for the genus one case of how a universal TQFT should look like had been given by Lyubashenko in [Ly]. These representations of the mapping class group $\mathscr{D}$ of the punctured torus are constructed as a subgroup of the *End* set of a coend in a BTC with certain finiteness conditions. For the representation category of a finite dimensional Hopf algebra the coend turns out to be the algebra acting on itself by the adjoint action. A number of explicit formulae for the action of genus one mapping class groups on Hopf algebras had been derived from this by Lyubashenko and Majid [LyM].

One of the objectives of this paper is to give natural definitions of the modular operators and independent, rigorous proofs of the relations that rely mainly on the theory of integrals on Hopf algebras as developed by Larson, Sweedler and Radford. In doing so we will be able to give the precise relation of the projective phases of the representation to the basic invariant of a Hopf algebra obtained from the moduli.

Starting with nonsemisimple Hopf algebras it is a natural question to ask how the universal TQFT relates to the reduced TQFT defined by the semisimplified representation category of the same algebra. In the second part we give the precise connection for the mapping class group $SL(2,\mathbb{Z})$ of the closed torus and the quantum group $U_q(sl_2)$. In the universal picture the representation of $SL(2,\mathbb{Z})$ is found as the restriction of the action of $\mathscr{D}$ to the center. The usual modular representation will appear in a tensorproduct with the fundamental, algebraic representation besides an additional, inequivalent finite representation.

In order to give an idea where these results fit into the general framework of a TQFT we give here an outline of the construction of an extended three dimensional TQFT with BTC's. The axioms are essentially due to Kazhdan and Reshetikhin, [KR], and differ from other definitions in that they make no use of higher algebraic structures like 2-categories. We shall give the objects assigned to compact, oriented surfaces with boundaries both in the case of the TQFT constructed in [RT] for semisimple categories and for the universal TQFT associated to a Hopf algebra $\mathscr{A}$.

Extended Three Dimensional Topological Quantum Field Theories. As in [A] an extended TQFT is defined as a functor or, more precisely, a collection of functors from cobordism categories to abelian categories over an algebraically closed field k.

To a given one dimensional manifold S we can associate a cobordism category Cob_S as follows: The objects of the category are compact, oriented two-folds Σ with coordinate maps $S \overset{\sim}{\to} \partial\Sigma$. A morphism between Σ_1 and Σ_2 is a 3-fold M whose boundary is parametrized by $-\Sigma_1 \amalg_S \Sigma_2 \cong \partial M$. The composition of two morphisms is given by an identification along a common surface. An extended TQFT assigns to every surface S a category $\mathscr{C}_S$ and a functor

$$\Phi_S : Cob_S \to \mathscr{C}_S .$$

Assuming that $\mathscr{C}_\emptyset = Vect(k)$ this implies the original definition of [A]. We have a natural inclusion of categories $Cob_S \times Cob_{S'} \subsetneq Cob_{S \amalg S'}$. For the respective abelian categories we also assume a functor

$$\odot : \mathscr{C}_S \times \mathscr{C}_{S'} \to \mathscr{C}_{S \amalg S'} \tag{1.1}$$

compatible with Φ. We require this to be a tensorproduct of abelian categories in the sense of [D]. Note that this is consistent with $\mathscr{C}_\emptyset \odot \mathscr{C} = Vect(k) \odot \mathscr{C} \cong \mathscr{C}$.

A standard consequence of this are representations of mapping class groups. To see this we consider $M = \Sigma \times I \amalg_\sim \partial\Sigma$, where the relation $\sim$ is $(s, t) \sim s \, \forall s \in \partial\Sigma$, $t \in \mathrm{I}$ and I is the unit interval. For the boundary $\partial M = \Sigma \amalg_{\partial\Sigma} \Sigma$ we choose different coordinate maps for the two boundary pieces coinciding on $\partial\Sigma$. If we denote by $\mathscr{D}\!\mathit{iff}(\Sigma, \partial\Sigma)$ the group of homeomorphisms of Σ to itself which are identity on the boundary we obtain from these cobordisms a representation:

$$\pi_o(\mathscr{D}\!\mathit{iff}(\Sigma, \partial\Sigma)) \to End_{\mathscr{C}_{\partial\Sigma}}(X_\Sigma) . \tag{1.2}$$

Here we denoted by $X_\Sigma = \Phi_{\partial\Sigma}((\Sigma, \partial\Sigma))$.

Next we formulate the axiom that leads to lower dimensional cobordism functors. To this end suppose that $S = A \amalg B$ and $S' = B \amalg C$, then for tensor categories the contraction functor $\mathrm{Hom}(1, \_\otimes\_) : \mathscr{C}_B \times \mathscr{C}_B \to Vect(k)$ induces a bilinear, covariant functor

$$\mathscr{C}_{A \amalg B} \times \mathscr{C}_{B \amalg C} \to \mathscr{C}_{A \amalg C} . \tag{1.3}$$

On the side of the cobordism categories we consider two three manifolds M and M'that belong to $Cob_{A \amalg B}$ and $Cob_{B \amalg C}$ respectively. We can consider half tubular neighborhoods of the 1-folds B in the boundaries of M and M'. These define oriented ribbon graphs in the boundaries along which we can glue the two manifolods M and M'. The result is again a three manifold $M \amalg_B M'$. The boundary pieces are the boundary pieces of the individual 3-folds glued along B. This way we obtain a cobordism in $Cob_{A \amalg C}$ from $\Sigma_1 \amalg_B \Sigma'_2$. The assignment

$$Cob_{A \amalg B} \times Cob_{B \amalg C} \to Cob_{A \amalg C} \tag{1.4}$$

is easily seen to be a functor. The next axiom of an extended TQFT asserts that the functors Φ intertwines the two functors in (1.3) and (1.4).

This axiom allows us to define a functor from the category of 2-cobordisms between 1-folds and the category of abelian, tensor categories. The assignment of morphisms is given by the composition:

$$\mathscr{F}_\Sigma : \mathscr{C}_A \xrightarrow{1 \odot X_\Sigma} \mathscr{C}_A \odot \mathscr{C}_A \odot \mathscr{C}_B \xrightarrow{\mathrm{Hom}(1,\, \_\otimes\_) \odot id} \mathscr{C}_B . \tag{1.5}$$

Here Σ denotes a 2-manifold cobording the pieces A and B by some coordinate maps $-A \to \partial\Sigma \leftarrow B$.

In order to check functoriality of $A \to \mathscr{C}_A$ and $\Sigma \to \mathscr{F}_\Sigma$ we consider again the manifold $M = \Sigma \times \mathrm{I} \amalg_\sim \partial\Sigma$ as in (1.2) now with the same coordinate maps for the boundary pieces but two components for $\partial\Sigma$. Specializing to surfaces of the form

$\Sigma = S \times \mathrm{I}$, we get as in (1.2) a homomorphism

$$\pi_o(\mathscr{D}\!\mathit{iff}(S)) \to End_{Cat}(\mathscr{C}_S)\,. \tag{1.6}$$

For compact S and by (1.1) we easily identify (1.6) as the homomorphism from the permutation group of circles to the permutations of tensor factors.

The functors associated to the elementary cobordisms, given by spheres with one, two, and three punctures (denoted P_1, P_2 and P_3 respectively) have a specific meaning for the circle category. Since P_2, seen as a cobordism from S^1 to S^1 with the same coordinate maps is a unit in the cobordism category we want the associated $\mathscr{F}_{P_2}$ to be the identity functor in the basic category $\mathscr{C}_1$ of the circle. Regarding P_3 as a cobordism from $S^1 \amalg S^1$ to S^1 the associated functor defines a tensor product $\mathscr{F}_{P_3} = \otimes : \mathscr{C}_1 \odot \mathscr{C}_1 \to \mathscr{C}_1$, which we assume to be the same as the one used in (1.5). Finally, the functor of $P_1 : \emptyset \to S^1$ clearly gives the injection of an identity object with respect to $\otimes$ and $P_1 : S^1 \to \emptyset$ is assigned to the invariance functor $\mathrm{Hom}(1, \_)$.

In an extended TQFT we can also consider 3-cobordisms of 2-cobordisms, which yield natural transformations. More precisely, let M have boundary pieces Σ_i, $i = 1, 2$ and $\partial\Sigma_i = A \amalg B$. The functor $\Phi_{\partial\Sigma}$ associates to the surfaces Σ_i objects $X_i \in \mathscr{C}_A \odot \mathscr{C}_B$ and a morphism $f_M \in \mathrm{Hom}(X_1, X_2)$. For an object $Y \in \mathscr{C}_A$ we apply to the morphism $id \odot f_M : Y \odot X_1 \to Y \odot X_2$ the functor $\mathrm{Hom}(1, \_ \otimes \_) \odot id$ as in (1.5) to give us a morphism $\tilde{f}_M : \mathscr{F}_{\Sigma_1}(Y) \to \mathscr{F}_{\Sigma_2}(Y)$. It is easy to see that this defines a natural transformation $\tilde{f}_M : \mathscr{F}_{\Sigma_1} \to \mathscr{F}_{\Sigma_2}$ and thereby a functor $Cob_{A \amalg B} \to Funct(\mathscr{C}_A, \mathscr{C}_B)$.

A special type of natural transformations are generated by cobordisms of the form $M = S \amalg_\alpha (S \times \mathrm{I} \times \mathrm{I}) \amalg_\beta S$ with relations $\alpha : s \sim (s, 0, t)$ and $\beta : (s, 1, t) \sim F(s, t) \forall s \in S\; t \in \mathrm{I}$. Here F is a homotopy in the set of homeomorphisms $\mathscr{D}\!\mathit{iff}(S)$ of S to itself. Confining ourselves to loops, i.e., $F(s, 1) = F(s, 0) = s$, we obtain a homomorphism

$$\pi_1(\mathscr{D}\!\mathit{iff}(S)) \to Nat(id, id)\,. \tag{1.7}$$

Reconsidering the elementary cobordisms P_i, we can discuss some elementary natural transformations that identify the circle category $\mathscr{C}_1$ as a BTC. The 2π rotation of S^1 generating $\pi_1(\mathscr{D}\!\mathit{iff}(S^1))$ gives us by (1.7) a natural transformation, denoted $\theta \in Nat(id)$. We can also cobord the surface P_3 to P_3 with exchanged coordinate maps for the $S^1 \amalg S^1$ piece of the boundary by moving the circles around each other in one of two directions. The TQFT assigns a transformation $\varepsilon^{\pm} \in Nat(\otimes, P\otimes)$. The square of this cobordism is homeomorphic to the one where annuli around the punctures are twisted by 2π so we obtain the identity of natural transformations:

$$\varepsilon(Y, X)\varepsilon(X, Y) = \theta(X \otimes Y)\theta(X)^{-1} \otimes \theta(Y)^{-1}\,. \tag{1.8}$$

This means θ is a balancing of $\mathscr{C}_1$. The associativity constraint is obtained in a similar way.

Let us discuss for a surface Σ' whose boundary is the union on n circles and the corresponding closed surface Σ a connection between (1.2) and (1.6). We have fibrations

$$\mathscr{D}\!\mathit{iff}(\Sigma', \partial\Sigma') \hookrightarrow \mathscr{D}\!\mathit{iff}(\Sigma') \twoheadrightarrow \mathscr{D}\!\mathit{iff}(\partial\Sigma')$$

$$\text{and} \quad \mathscr{D}\!\mathit{iff}(\Sigma') \hookrightarrow \mathscr{D}\!\mathit{iff}(\Sigma) \twoheadrightarrow K_n\,,$$

where K_n is the symmetrized configuration space of n points in Σ. From the long exact sequence for the first fibration and the injection of the second we obtain the top row of the following commutative diagram:

$$\begin{tikzcd}
\pi_1(\mathscr{D}iff(\partial\Sigma')) \arrow[r] \arrow[d] & \pi_0(\mathscr{D}iff(\Sigma',\partial\Sigma')) \arrow[r] \arrow[d] & \pi_0(\mathscr{D}iff(\Sigma)) \arrow[d] \\
Nat_{\partial\Sigma'}(id, id) \arrow[r] & End_{\mathscr{C}\partial\Sigma'}(X_{\Sigma'}) \arrow[r] & End_k(V(\Sigma))
\end{tikzcd} \tag{1.9}$$

In the bottom row the left map is simply the evaluation of a natural transformation on an object. The second homomorphism is given by the invariance functor $\mathrm{Hom}(1,\_)^{\odot n}$ acting on $\mathscr{C}_{\partial\Sigma'} \cong \mathscr{C}_1^{\odot n}$ and $V(\Sigma) = \mathrm{Hom}(1,\_)^{\odot n}(X_{\Sigma'})$ is the vectorspace associated to the closed surface.

Examples and the Degeneracy Problem. The objects associated to punctured surfaces can be identified up to isomorphie for two types of categories. One is a semisimple, rational BTC $\mathscr{C}_o$ with simple objects $\mathscr{I}$, the other is the representation category $R(\mathscr{A})$ of a finite dimensional Hopf algebra $\mathscr{A}$. Quite generally it is possible to produce a semisimple, rational category from $R(\mathscr{A})$ by a generalized GNS construction with respect to a canonical categorial trace tr, see for example [K]. Thus in principle there are two ways of constructing TQFT's from a given Hopf algebra $\mathscr{A}$ which will lead to a different representations, e.g., of mapping class groups. The precise connection in one example will be discussed in an example in the last chapter.

The assignment of objects for the two punctured sphere is easily inferred from $\mathscr{F}_{P_2} = id$ and formula (1.5). In $\mathscr{C}_o$ the answer is $X_{P_2} \cong \sum_{j\in\mathscr{I}} j \odot j^\vee$ and in $R(\mathscr{A})$ the module X_{P_2} is given by $\mathscr{A}$ with $\mathscr{A}^{\odot 2}$-action given by $a \odot b \cdot (x) = axS(b)$. Moreover, $\mathscr{F}_{P_3} = \otimes$ implies that $X_{P_3} \cong \sum_{ij\in\mathscr{I}} i \odot j \odot (i \otimes j)^\vee$ or $X_{P_3} = \mathscr{A} \otimes \mathscr{A}$ with $\mathscr{A}^{\odot 3}$-action $(a \odot b \odot c) \cdot (x \otimes y) = (ax \otimes by)\Delta(S(c))$. This allows us to identify the object associated to the punctured torus T' with $\partial T' = S^1$ by contracting the objects X_{P_3} and X_{P_2} along the category of the $S^1 \amalg S^1$-boundary pieces. In $\mathscr{C}_o$ the resulting object is $X_{T'} = \sum_{j\in\mathscr{I}} j \otimes j^\vee$ and in $R(\mathscr{A})$ by the module $\mathscr{A}$ with adjoint action. The objects of all other surfaces are now found easily by sewing along circles. For example the surface $\Sigma_{g,1}$ of genus g with one puncture is assigned to $X_{T'}^{\otimes g}$. The object of the $(n+1)$-punctured sphere P_{n+1} has object $X_{P_{n+1}} = \sum_{i_k} i_1 \odot \ldots \odot i_n \odot (i_1 \otimes \ldots \otimes i_n)^\vee$ in $\mathscr{C}_o$ and in $R(\mathscr{A})$ the module $X_{P_{n+1}} = \mathscr{A}^{\otimes n}$, where the $\mathscr{A}^{\odot(n+1)}$-action is given by the obvious generalization of the cases $n = 2, 3$. The object for a general compact, orientable surface is found by sewing $X_{P_{n+1}}$ and $X_{\Sigma_{g,1}}$. For $R(\mathscr{A})$ this gives for example the module $\mathrm{Hom}_{\mathscr{A}}(\mathscr{A}^{\otimes g}, \mathscr{A}^{\otimes n})$ of intertwiners for one of the $\mathscr{A}$-actions.

Let us discuss the case $g = 1$, $n = 1$ in some more detail. The mapping class group $\mathscr{D} = \pi_o(\mathscr{D}iff(T', \partial T'))$ maps by (1.2) into $\mathrm{End}(T')$, so that we obtain in $R(\mathscr{A})$ an action of $\mathscr{D}$ on $\mathscr{A}$ intertwining the adjoint action. Following (1.9) we obtain a representation of the modular group $\pi_o(\mathscr{D}iff(T))$ on $V(T) = \mathrm{Hom}(1, T)$, which for $R(\mathscr{A})$ is just the restriction of the $\mathscr{D}$ action to the center $Z(\mathscr{A}) = \mathrm{Hom}(1, \mathscr{A})$. In order to interpret the rest of (1.9) recall that for $R(\mathscr{A})$ the natural transformations of the identity functor are given by the action of central elements of $\mathscr{A}$. In particular the generator θ of $\pi_1(\mathscr{D}iff(S))$ acts on $\mathscr{A}$ as $ad(v)$, where $v = \theta(\mathscr{A})$ is the central "ribbon element," see [RT]. The Dehn twist along the boundary can also be given by $\mathcal{S}^4$, where $\mathcal{S}$ is the standard generator of $\mathcal{D}$. The restriction of $\mathcal{S}^4 = ad(v)$ to the center is clearly trivial. The second generator of $\mathcal{D}$, the Dehn twist at a handle, $\mathcal{T}$ is given by the action of θ on the constituent X_{P_2}, i.e., by multiplication of v on $\mathcal{A}$.

The definition of a TQFT we presented so far is not quite complete. Clearly, there are many ways of sewing up a surface Σ so we have many ways to construct the object X_Σ. For example instead of using the center of $\mathcal{A}$ as the vectorspace for the closed torus $V(T)$, we can also choose the space $\mathrm{Hom}(\mathcal{A}, 1)$ – which is isomorphic to the space of characters on $\mathcal{A}$ – or we could have chosen the endomorphism set $End(X_{P_2}) = End_{\mathcal{A}\odot 2}(\mathcal{A})$. These spaces are isomorphic to each other but there is no one canonical isomorphism identifying two of them. Instead the sewing procedure used to find the object defines a surface with a cut diagram of decoration. Thus we should take as objects of the cobordism categories surfaces Σ together with a Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$ which must be compatible with the cobording 3-manifolds. The functor of the TQFT is now allowed to have projective phases. This means for two cobordisms M_1 and M_2 with a common, decorated boundary component that

$$\Phi(M_1 M_2) = c^{\mu}\Phi(M_1)\Phi(M_2)\,, \tag{1.10}$$

where μ is the Maslov index of a triple of Lagrangian subspace defined by the cobordisms. It also measures the non-additivity of the signature of the 4-manifolds cobording the M_i to the corresponding union of handlebodies. If the M_i are invertible morphisms in the cobordism category we obtain projective representations of the modular groups. For details see [T]. The main result of the first chapter is the relation of the phase c to intrinsic invariants of the Hopf algebra $\mathcal{A}$.

In order to discuss the modularity condition we recall how the $\mathcal{S}$ matrix can be obtained from the [RT]-construction for standard TQFT's with $S = \emptyset$. The cobordism describing the action of $\mathcal{S}$ on T' is a 3-manifold whose boundary is $\partial M = T' \amalg_{S^1} T'$, the closed surface of genus two, and can thus be considered a cobordism $\Sigma_2 \to \emptyset$. In [RT] the vectorspace associated to Σ_2 is $\bigoplus_{ij} \mathrm{Hom}(i \otimes i^\vee \otimes j \otimes j^\vee, 1)$. The linear form assigned to $\Sigma_2 \to \emptyset$ is found by computing the invariant in S^3 of the ribbon graph embedded in the outside of Σ_2. On a vector f its value is

$$1 \xrightarrow{coev \otimes coev} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{1 \otimes \varepsilon^2 \otimes 1} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{f} 1\,.$$

In the description of an extended TQFT we have to consider this as the matrix element of $\mathcal{S} \in \mathrm{End}(T') \cong \oplus_{ij} \mathrm{Hom}(j \otimes j^\vee, i \otimes i^\vee)$. Thus on a summand we have

$$\mathcal{S}: j \otimes j^\vee \xrightarrow{coev \otimes 1} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{1 \otimes \varepsilon^2 \otimes 1} i \otimes i^\vee \otimes j \otimes j^\vee \xrightarrow{1 \otimes c_j} i \otimes i^\vee\,,$$

where c_j is proportional to $j \otimes j^\vee \xrightarrow{\varepsilon} j^\vee \otimes j \xrightarrow{ev} 1$. The generalization of this formula to non-semisimple categories is described by [Ly] and will be reviewed in the next chapter.

The matrix elements of the restriction of $\mathcal{S}$ to $V(T) = \mathrm{Hom}(1, X_{T'}) = \bigoplus_j \mathrm{Hom}(1, j \otimes j^\vee) \cong k^{\mathcal{I}}$ are given by $\mathcal{S}_{ij} = tr_{i \otimes j}(\varepsilon(i,j)\varepsilon(j,i))$, where tr is the usual trace of a balanced category.

A priori the operations $\mathcal{S}$ and $\mathcal{T}$ defined for a general semisimple $\mathcal{C}_o$ do not yield a projective representation of $SL(2, Z)$ unless we impose one further condition.

This is the rather specialized "modularity condition" introduced in [T] asserting that the $\mathcal{S}$-matrix is invertible. In case this condition is violated we may still apply Atiyah's prescription and reduce the space $k^{\mathcal{I}}$ by the projection $P = \mathcal{S}^{-}$, where the matrix $\mathcal{S}^{-}_{ij} = \mathcal{S}_{ij^\vee}$ is assigned by the [RT]-prescription to the inverse cobordism.

For example if $\mathscr{C}_o$ is a symmetric category, the $\mathcal{S}$ matrix is of rank one so the $SL(2, \mathbb{Z})$ representation is one dimensional. A degeneracy problem occurs quite generally if $\mathcal{I}$ contains a subset $\mathcal{I}_o$, of irreducible objects, which braid trivially, i.e., $\varepsilon(k, j)\varepsilon(j, k) = 1$ for all $k \in \mathcal{I}_o, j \in \mathcal{I}$. In case $\mathcal{I}_o$ is a subgroup of invertibles $\{\sigma\}$ we have for the natural action of its elements on $k^{\mathcal{I}}$ that $\mathcal{S}\sigma = \mathcal{S}$. Hence $\mathcal{S}$ and $\mathcal{T}$ can be defined on the orbit space $im(\sum_{\sigma \in \mathcal{I}_o} \sigma)$, where we can hope for the modularity condition to hold.

This situation occurs for the semisimplified representation categories of quantum groups at certain roots of unity. The example we will come back to in the last chapter is $U_q(sl_2)$, where $q^{1/2}$ is an $l = 2m + 1^{\text{th}}$ root of unity. We have $|\mathcal{T}| = 2m$ and the $2m^{\text{th}}$ representation braids trivially and is invertible of order two. The truncated theory yields an m-dimensional representation of $SL(2, \mathbb{Z})$.

The problem of degeneracy is resolved in a very natural way in the universal picture for $R(\mathscr{A})$ by choosing $\mathscr{A}$ to be a double constructed algebra. In this situation we find very simple formulae for $\mathcal{S}$ and its inverse.

Survey of Contents and Summary of Results

In Chapter 2 we define and study the action of operators generating the mapping class group $\mathscr{D} := \pi_o(\mathscr{D}i\!f\!f(T, D))$ on the double $D(\mathscr{A})$ of a finite dimensional Hopf algebra. We start in Sect. 1 with a review of the bicrossed structure of a double and properties of an isomorphism $D(\mathscr{A})^* \xrightarrow{\sim} D(\mathscr{A})$. These are in particular the relations between traces and characters on $D(\mathscr{A})$ and central and group like elements in $D(\mathscr{A})$. We also recall the definitions of canonical and balancing elements in quasitriangular Hopf algebras. For later application we derive a relation for the monodromy matrix of $D(\mathscr{A})$. The next section is a recollection from [Rd] of relations between several nondegenerate bilinear forms and moduli defined by the integrals of a finite dimensional Hopf algebra. This leads for $D(\mathscr{A})$ to the Drinfeld–Radford formula $S^4 = Ad(g)$. In Section 3 we determine the integral and cointegral of a double $D(\mathscr{A})$. In particular we find that the comodulus is trivial and that the modulus is the canonical element g. This allows us to show that a pair of non-degenerate, canonical traces on $D(\mathscr{A})$ can be defined very simply from the natural contraction on $D(\mathscr{A})$. The balancing of a double $D(\mathscr{A})$ is related in Section 4 to second order roots of the moduli and a fourth order root v of the ω- invariant of $\mathscr{A}$. Guided by categorical constructions in [Ly] we define in Section 5 the action of the generators of the mapping class group $\mathscr{D}$ on $D(\mathscr{A})$. We obtain an intriguingly simple expression for the actions of $\mathcal{S}$ and $\mathcal{S}^{-1}$ involving only non degenerate forms on $\mathscr{A}$ and $\mathscr{A}^*$ and the bicrossed isomorphism of $D(\mathscr{A})$. Similarly we have a formula for the braided antipode. The results from all previous sections are used in Section 6 to give a rigorous proof of the modular relations and determine the protective phase of the universal TQFT as v^{-3}.

In Chapter 3 we find the structure of the representation of $SL(2, \mathbb{Z})$ on the center of $U_q(sl_2)$ by restricting the action of $\mathscr{D}$. The non-degenerate forms and moduli of the double of B_q, the Borel algebra of $U_q(sl_2)$, are given in Section 1. In

Section 2 we determine the center of $D(B_q)$, which for the TQFT is the vectorspace of the torus. If q is an $l = 2m + 1$-st primitive root of unity it is given by $\mathbb{C}[\mathbb{Z}/l] \otimes \mathscr{V}$. Here $\mathscr{V}$ is a $3m + 1$-dimensional algebra with a basis of $m + 1$ idempotents and $2m$ nilpotents. The balancing element of $D(B_q)$ is expressed in terms of this basis, see Section 3. In doing so we propose a method to generate new partition identities. In Section 4 we compute the matrix elements of the $SL(2, \mathbb{Z})$-action on the center of $D(B_q)$. This requires us to find transformations from the PBW basis of $U_q(sl_2)$ to the algebra $\mathscr{V} \otimes \mathbb{C}[K]$, where K is a Cartan element. We analyze this representation in Section 6. We find evidence for the decomposition of the representation into two irreducibles. One of which is a finite, $m + 1$-dimensional representation, the other is the tensor product of the two dimensional standard representation of $SL(2, \mathbb{Z})$ and the finite, m-dimensional representation obtained from the semisimplified representation category of $U_q(sl_2)$. The conjecture is verified in the last section. Here we find the decomposition and the explicit finite representation for two non trivial roots of unity.

2. Mapping Class Group Action on Doubles

1. Double Algebras and Balancing. In this section we recall some basic facts and notions on Hopf algebras that we will use later. For a finite dimensional Hopf algebra $\mathscr{A}$ over a field k we denote by $\mathscr{A}^o$ the dual Hopf algebra with opposite comultiplication. We shall always assume that $\mathscr{A}$ has a counit ε and an invertible antipode S. The antipode of $\mathscr{A}^o$ is thus given by S^{-1*}.

For $\lambda, \mu \in \mathscr{A}^o$ and $a, b \in \mathscr{A}$ we have the following relations:

$$\langle \lambda\mu, a \rangle = \langle \lambda \otimes \mu, \Delta(a) \rangle \quad \langle \Delta(\lambda), a \otimes b \rangle = \langle \lambda, ba \rangle \,, \tag{2.11}$$

$$\Delta \otimes id(R) = R^{13}R^{23}, \quad id \otimes \Delta(R) = R^{13}R^{12} \,, \tag{2.12}$$

$$\langle S^{\mp 1}(l), a \rangle = \langle l, S^{\mp 1}(a) \rangle, \quad S \otimes id(R) = id \otimes S^{-1}(R) = R^{-1} \,. \tag{2.13}$$

Here $\langle \, , \rangle \colon \mathscr{A}^o \otimes \mathscr{A} \to k$ is the usual contraction and R is the canonical element $R \in \mathscr{A} \otimes \mathscr{A}^o$. Thus, if $\{e_i\}$ is a basis of $\mathscr{A}$ and $\{f_i\}$ the respective dual basis $\mathscr{A}^o$ we can write $R = \sum_i e_i \otimes f_i$. A bicrossed product of the two algebra is a Hopf algebra D which contains $\mathscr{A}$ and $\mathscr{A}^o$ as sub-Hopf algebras such that $\cdot \colon \mathscr{A} \otimes \mathscr{A}^o \xrightarrow{\sim} D$ is an isomorphism, where $\cdot$ is the multiplication in D. Clearly, a bicrossed structure is uniquely determined by an isomorphism $\bowtie \colon \mathscr{A} \otimes \mathscr{A}^o \xrightarrow{\sim} \mathscr{A}^o \otimes \mathscr{A}$ which by $\cdot \bowtie = \cdot$ defines an associative product, such that the coproduct on D defined by the coproducts on $\mathscr{A}$ and $\mathscr{A}^o$ extends to a homomorphism into $D^{\otimes 2}$. In [Dr0] it is shown that there is precisely one bicrossed product $D(\mathscr{A})$, the double, such that

$$R\Delta(y) = \Delta'(y)\mathrm{R} \quad \text{for all } y \in D(\mathscr{A}) \,. \tag{2.14}$$

Here $\Delta' = \tau \circ \Delta$ is the opposite comultiplication and τ is the flip $\tau(a \otimes b) = b \otimes a$. The bicrossed structure is given explicitly by

$$\begin{aligned} \bowtie \colon \mathscr{A} \otimes \mathscr{A}^o &\xrightarrow{\tau} \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{\Delta \otimes \Delta'} \mathscr{A}^o \otimes \mathscr{A}^o \otimes \mathscr{A} \otimes \mathscr{A} \\ &\xrightarrow{1 \otimes S \otimes 1 \otimes 1} \mathscr{A}^o \otimes \mathscr{A}^o \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{1 \otimes \langle , \rangle \otimes 1} \mathscr{A}^o \otimes \mathscr{A} \\ &\xrightarrow{\Delta' \otimes \Delta} \mathscr{A}^o \otimes \mathscr{A}^o \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{1 \otimes \langle , \rangle \otimes 1} \mathscr{A}^o \otimes \mathscr{A} \,. \end{aligned} \tag{2.15}$$

If we use the usual abbreviation $\Delta(a) = \sum_i a_i' \otimes a_i'' = a' \otimes a''$, $\Delta^2(a) = a' \otimes a'' \otimes a'''$ (2.15) can be summarized in the formula

$$\bowtie (y \otimes \lambda) = \lambda'' \otimes y'' \langle \lambda', y' \rangle \langle S(\lambda'''), y''' \rangle \,.$$

The inverse is given similarly by

$$\bowtie^{-1}: \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{\Delta' \otimes \Delta} \mathscr{A}^o \otimes \mathscr{A}^o \otimes \mathscr{A} \otimes \mathscr{A} \xrightarrow{1 \otimes S \otimes 1 \otimes 1} \mathscr{A}^o \otimes \mathscr{A}^o \otimes \mathscr{A} \otimes \mathscr{A}$$

$$\xrightarrow{1 \otimes \langle , \rangle \otimes 1} \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{\Delta \otimes \Delta'} \mathscr{A}^o \otimes \mathscr{A}^o \otimes \mathscr{A} \otimes \mathscr{A}$$

$$\xrightarrow{1 \otimes \langle , \rangle \otimes 1} \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{\tau} \mathscr{A} \otimes \mathscr{A}^o \,. \tag{2.16}$$

For a Hopf algebra D, with $\dim(D) < \infty$ we denote by $G(D)$ the finite group of *group like* elements g, characterized by $\Delta(g) = g \otimes g$. Also we shall use the notation $Ch(D) \cong G(D^*)$ for the group of one dimensional representations of D. For doubles we have the following easy fact:

Lemma 1. *For a Hopf algebra $\mathscr{A}$ the multiplication map $\cdot : \mathscr{A} \otimes \mathscr{A}^o \xrightarrow{\sim} D(\mathscr{A})$ yields a group isomorphism*:

$$G(\mathscr{A}) \oplus Ch(\mathscr{A}) \xrightarrow{\sim} G(D(\mathscr{A})) \,.$$

Similarly the sum of restrictions yields:

$$Ch(D(\mathscr{A})) \xrightarrow{\sim} Ch(\mathscr{A}) \oplus G(\mathscr{A}) \,.$$

Proof. For $b \in D(\mathscr{A})$ let $\underline{b}$ be the corresponding element in $\mathrm{End}(\mathscr{A})$. For the coproduct this means $\underline{\Delta(b)}\,(x \otimes y) = \Delta(\underline{b}(yx))$, which for $b \in G(\mathscr{A})$ has to equal $\underline{b}(x) \otimes \underline{b}(y)$. Inserting $y = 1$ and applying $\varepsilon \otimes 1$ we find that $\underline{b}$ is of rank one and $\underline{b} = g \cdot \gamma$, where $g = \underline{b}(1)$ and $\gamma = \varepsilon \circ \underline{b}$. Inserting instead $x = 1$ and $y = 1$ (applying $\varepsilon \otimes \varepsilon$) shows that $g \in G(\mathscr{A})$ ($\gamma \in Ch(\mathscr{A})$). The adjoint action of $Ch(\mathscr{A})$ on the double $D(\mathscr{A})$ stabilizes $\mathscr{A}$ and, there, coincides with the coadjoint action, i.e., we have $\gamma \cdot y \cdot \gamma^{-1} = \gamma \rightharpoonup y \leftharpoonup \gamma^{-1}$ for all $y \in \mathscr{A}$. Since the coadjoint action on group-likes in $\mathscr{A}$ is trivial, the images of $G(\mathscr{A})$ and $Ch(\mathscr{A})$ centralize each other and the inclusion factors into the direct sum. Injectivity now follows from linear independence of group-likes, see [Ab], and injectivity of $\cdot$. □

Here we used the notation $\rightharpoonup$ ($\leftharpoonup$) as in [Ab] for the left (right) action of D^* on a Hopf algebra D. Similarly, we use $(a \triangleright \lambda)\,(y) := \lambda(ya)$ for the left action of D on D^* and $\triangleleft$ for the corresponding right action. We also use the adjoint actions of D on itself given by $ad(a)(y) = a' y S(a'')$ and on D^* given by $ad_*(a)\,(\lambda) = a'' \triangleright \lambda \triangleleft S(a')$. The invariance in D under the adjoint action is precisely the *center* $Z(D)$ and the invariance in D^* are the *q-characters* $\bar{C}(D) = \{\lambda \in D^* : \lambda(xy) = \lambda(S^2(y)x)\}$, which were introduced in [Dr]. In [Dr1] it is shown that these two spaces are related to each other by the map

$$\bar{f}: D^* \to D : \lambda \to \lambda \otimes 1(M) \,. \tag{2. 17}$$

Here $M \in D^{\otimes 2}$ is the element $M = \tau(R)R = \sum_{ij} f_j e_i \otimes e_j f_i = \Sigma_k m_k \otimes n_k$. In the case of a double $D = D(\mathscr{A})$ $\{m_k\}$ and $\{n_k\}$ are a different basis of $D(\mathscr{A})$ so that M is nondegenerate. The following is a slightly extended version of a lemma in [Dr1].

Lemma 2. 1. *The map $\bar{f}: D(\mathscr{A})^* \xrightarrow{\sim} D(\mathscr{A})$ is an isomorphism of $D(\mathscr{A})$-modules with respect to the adjoint actions.*

2. *$\bar{f}: \bar{C}(D(\mathscr{A})) \xrightarrow{\sim} Z(D(\mathscr{A}))$ is an isomorphism of algebras.*
3. *$\bar{f}: Ch(D(\mathscr{A})) \xrightarrow{\sim} G(D(\mathscr{A}))$ is the group isomorphism $(g, \gamma) \mapsto (\gamma, g)$*
4. *We have $\bar{f}^* \circ S^* = S^{-1} \circ \bar{f}$.*

Proof. The fact that $\bar{f}$ intertwines the actions of $D(\mathscr{A})$ follows from basic Hopf algebra relations, (2.14) and the identity $(S(y') \otimes 1)M(y'' \otimes 1) = (S(y') \otimes 1)M(y'' \otimes (y''' S(y'''')))= (S(y') \otimes 1)(y'' \otimes y''')M(1 \otimes S(y'''')) = (1 \otimes y')M(1 \otimes S(y''))$.

It is clear that $\bar{f}$ is an isomorphism. In particular we can write it as the composition:

$$D(\mathscr{A})^* \xrightarrow{*(\bowtie)*} (\mathscr{A}^* \otimes \mathscr{A})^* = \mathscr{A} \otimes \mathscr{A}^* \overset{*}{\xrightarrow{\sim}} D(\mathscr{A})\,.$$

Clearly, the invariances are mapped isomorphically to each other and a computation in [Dr] shows that this is a homomorphism. In fact we have $\bar{f}(\chi\lambda) = \bar{f}(\chi)\bar{f}(\lambda)$ for $\chi \in \bar{C}(D(\mathscr{A}))$ and any $\lambda \in D(\mathscr{A})^*$. 3) follows from Lemma 1 and the form of M. Finally, (2.13) implies $S \otimes S(M) = \tau(M)$ and thereby 4). $\square$

It follows from (2.12) that the R matrices satisfy the Yang Baxter equation $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$. For later computations of the modular relation we derive here an analogous equation for the M matrices.

Lemma 3. *For M and bases $\{e_i\}$, $\{f_i\}$, $\{n_k\}$, $\{m_k\}$ as above we have*

$$(\tau(M) \otimes 1)(1 \otimes M) = \sum_{kj} n'_k f_j \otimes m_k \otimes S^{-1}(e''_j) n''_k e'_j\,,$$

or equivalently

$$1 \otimes \lambda \otimes 1((\tau(M) \otimes 1)(1 \otimes M)) = \sum_{ij} \bar{f}(\lambda)' f_i f_j \otimes S^{-1}(e_j) \bar{f}(\lambda)'' e_i\,.$$

Proof. If we multiply R matrices from the left and right to the Yang Baxter equation and permute the first and third factor we obtain $(R^{-1})^{31}(R^{-1})^{32}R^{21}R^{31} = R^{21}(R^{-1})^{32}$. Applying $1 \otimes 1 \otimes S^{-1}$ to this equation and using (2.13) we find

$$\sum_{if} f_i \otimes e_i f_j \otimes e_j = \sum_{i,j,k,l} f_i f_j f_k \otimes f_l e_j \otimes S^{-1}(e_k) e_l e_i\,.$$

Multiplication with $R \otimes 1$ from the left and $1 \otimes R$ from the right yields

$$\begin{aligned}(\tau(M) \otimes 1)(1 \otimes M) &= \sum_{tsijkl} e_t f_i f_j f_k \otimes f_t f_l e_j e_s \otimes S^{-1}(e_k) e_l e_i f_s \\ {}^{by\ (2.12)} &= \sum_{ijkl} e'_l f_i f''_j f_k \otimes f_l e_j \otimes S^{-1}(e_k) e''_l e_i f'_j \\ {}^{by\ (2.14)} &= \sum_{ijkl} e'_l f'_j f_i f_k \otimes f_l e_j \otimes S^{-1}(e_k) e''_l f''_j\,.\end{aligned}$$

The formulas follow now from $\Delta(n_{(lj)}) = e'_l f'_j \otimes e''_l f''_j$ and again (2.12). $\square$

Let us also record here the canonical elements from [Dr1] and [Ly1] implementing the square of the antipode. They are defined by

$$u := \sum_i S(f_i) e_i \quad \text{and} \quad \hat{u} := \sum_i S^2(e_i) f_i \tag{2.18}$$

and satisfy the relations

$$S^2(y) = uyu^{-1} = \hat{u}y\hat{u}^{-1}, \quad \hat{u} = S(u)^{-1}$$
$$\text{and} \quad M = u \otimes u\Delta(u^{-1}) = \hat{u}^{-1} \otimes \hat{u}^{-1}\Delta(\hat{u}) . \tag{2.19}$$

From u and $\hat{u}$ one has two further elements of a quasitriangular Hopf algebra D with special properties:

$$g := u\hat{u}. \quad \text{with} \quad g \in G(D) \quad \text{and} \quad S^4(y) = gyg^{-1} , \tag{2.20}$$

$$z := u\hat{u}^{-1}, \quad \text{with} \quad z \in Z(D) \quad \text{and} \quad M^2 = z \otimes z\Delta(z^{-1}) . \tag{2.21}$$

2. Integrals, Moduli and Radford's Relations. We start this section with a review of basic facts from Hopf algebra theory and a summary of the formulae in [Rd] which we will use in this paper. The analysis of integrals of Hopf algebras in [LSw] is based on the fundamental theorem of Hopf modules. It asserts that a Hopf module M of a Hopf algebra D is free in the sense that $M^{cov} \otimes D \xrightarrow{\sim} M$ is an isomorphism of Hopf modules. Here D acts on itself by multiplication and comultiplication, M^{cov} is the coinvariance of the coaction and $\xrightarrow{\sim}$ is given by the left action on M. It is instructive to apply this to the situation where $M = D^*$ with actions $h \cdot \lambda := \lambda \triangleleft S(h)$ and coaction $\delta(\lambda) = \lambda \otimes 1\Delta \in \mathrm{End}(D) = D \otimes D^*$. The isomorphism $J \otimes D \xrightarrow{\sim} D^*$ then implies that $J = \{\lambda : \lambda \otimes 1\Delta(y) = 1\lambda(y)\}$ – the space of *right integrals* – is one dimensional and every nonzero element induces a nondegenerate bilinear form. Analogous statements are found if we use left actions or consider Hopf bilinear form. Let us fix once and for all a left integral μ and a left cointegral x with the properties.

$$1 \otimes \mu\Delta(h) = 1\mu(h), \quad hx = \varepsilon(h)x, \quad \text{and} \quad \mu(x) = 1 . \tag{2.22}$$

As in [Rd] we use notations for the following isomorphisms:

$$\beta_l, \beta_r : D \xrightarrow{\sim} D^* \quad \text{with} \quad \beta_l(h) = \mu \triangleleft h \quad \text{and} \quad \beta_r(h) = h \triangleright \mu , \tag{2.23}$$

$$\bar{\beta}_l, \bar{\beta}_r : D^* \xrightarrow{\sim} D \quad \text{with} \quad \bar{\beta}_l(\lambda) = x \leftharpoondown \lambda \quad \text{and} \quad \bar{\beta}_r(\lambda) = \lambda \rightharpoonup x . \tag{2.24}$$

They intertwine the right and left actions as in

$$\beta_l(kh) = \beta_l(k) \triangleleft h \quad \text{and} \quad \beta_r(kh) = k \triangleright \beta_r(h) . \tag{2.25}$$

It is obvious that xh is again a left cointegral for any $h \in D$. Hence by uniqueness of x, we find $\alpha \in Ch(D)$ and for the dual situation $a \in G(D)$ such that

$$\alpha(h)x = xh \quad \text{and} \quad \mu \otimes 1\Delta(h) = a\mu(h) . \tag{2.26}$$

Since D is finite dimensional, both the *modulus* α and the *comodulus* a are of finite order and

$$\omega := \alpha(a) \tag{2.27}$$

is a root of unity. Note, that in the following we use the opposite comultiplication for D^* so that, e.g., $S^{-1} = S^*(= \gamma$ in [Rd]). The antipode acts on the integrals as follows:

$$S^{-1}(\mu) = a \triangleright \mu = \omega S(\mu) = \omega\mu \triangleleft a , \tag{2.28}$$

$$S(x) = \alpha \rightharpoonup x = \omega S^{-1}(x) = \omega x \leftharpoondown \alpha . \tag{2.29}$$

The compositions of isomorphisms in (2.23) and (2.24) are given by the following formulae. Each one can be given on D^* or the adjoint one on D using $\beta_l^* = \beta_r$:

$$\beta_r\bar{\beta}_l(\lambda) = S^{-1}(\lambda)\,, \qquad \bar{\beta}_r\beta_l(h) = S(h)\,, \tag{2.30}$$

$$\beta_r\bar{\beta}_r(\lambda) = S(a(\lambda)\,, \qquad \bar{\beta}_l\beta_l(h) = S^{-1}(h)\cdot a\,, \tag{2.31}$$

$$\beta_l\bar{\beta}_l(\lambda) = S(\alpha^{-1})\,, \qquad \bar{\beta}_r\beta_r(h) = S^{-1}(\alpha \rightharpoonup h)\,, \tag{2.32}$$

$$\beta_l\bar{\beta}_r(\lambda) = \alpha\cdot S^{-1}(\lambda \triangleleft a)\,, \qquad \bar{\beta}_l\beta_r(h) = a\cdot S(h \leftharpoonup \alpha)\,. \tag{2.33}$$

From (2.28)–(2.30) we can derive further useful relations between adjoints:

$$\beta_l(S(h)) = \omega a^{-1} \triangleright S(\beta_r(h)) \quad \text{and} \quad \beta_l(S^{-1}(h)) = a^{-1} \triangleright S^{-1}(\beta_r(h))\,, \tag{2.34}$$

$$\beta_l(a \rightharpoonup h) = \beta_r(S^2(h)) \quad \text{resp.} \quad \mu((\alpha \rightharpoonup k)h) = \mu(hS^2(k))\,, \tag{2.35}$$

$$\bar{\beta}_l(S(\lambda)) = \alpha^{-1} \rightharpoonup S(\bar{\beta}_r(\lambda)) \quad \text{and} \quad \bar{\beta}_l(S^{-1}(\lambda)) = \omega\alpha^{-1} \rightharpoonup S^{-1}(\lambda)\,, \tag{2.36}$$

$$\bar{\beta}_r(S^{-2}(\lambda)) = \bar{\beta}_l(a \triangleright \lambda) \quad \text{resp.} \quad S^2 \otimes 1\Delta'(x) = \Delta(x)(a \otimes 1)\,. \tag{2.37}$$

Combining these identities we find Radfords formula for S^4:

$$S^4 = ad^*(\alpha) \circ ad(a^{-1})\,. \tag{2.38}$$

Since in a double $D(\mathscr{A})$ the adjoint action of $G(\mathscr{A}^o)$ coincides with the coadjoint action of $Ch(\mathscr{A})$ on $\mathscr{A}$ we find from that (2.38) and the corresponding equation on $\mathscr{A}^o$ that the group-like element $\alpha \otimes a^{-1}$ implements S^4 on $D(\mathscr{A})$. The same is true for the element defined in (2.17). In fact we have the following result of Drinfeld for doubles:

Proposition 4 [Dr1].

$$g = \alpha\cdot a^{-1}\,.$$

3. Integrals and Canonical Traces of Doubles. In the construction of representations of the modular group the integrals of the defining algebra play an important role. The integral μ_D and the cointegral x_D of a double $D(\mathscr{A})$ clearly have to be related to the integrals and cointegrals of $\mathscr{A}$. In this section we are also interested in finding the moduli α_D and a_D. Comparing Proposition 4 to (2.38) we are led to expect that $\alpha_D = 1$ and a_D is the same as g. We shall prove triviality of the comodulus first:

Proposition 5. *For left integrals μ and x as in (2.22) define the canonical element in $D(\mathscr{A})$ by $p = \mu\cdot S^{-1}(x)$. Then*

1. $S(p) = p$.
2. *p is both a right and left cointegral in $D(\mathscr{A})$ and $\alpha_D = 1$.*
3. *Let $P \in End(\mathscr{A})$ be the image of p under the map $\cdot^{-1} : D(\mathscr{A}) \xrightarrow{\sim} \mathscr{A} \otimes \mathscr{A}^* =$ End$(\mathscr{A})$. It is the projector onto the space of left cointegrals and is given by*:

$$\mathscr{A} \xrightarrow{R\otimes 1} \mathscr{A} \otimes \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{1\otimes\tau} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}^o \xrightarrow{\bullet\otimes 1} \mathscr{A} \otimes \mathscr{A}^o \xrightarrow{\Delta\otimes 1} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}^o$$

$$\xrightarrow{1\otimes\tau} \mathscr{A} \otimes \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{1\otimes S^2} \mathscr{A} \otimes \mathscr{A}^o \otimes \mathscr{A} \xrightarrow{1\otimes\langle,\rangle} \mathscr{A}\,.$$

Proof. Since $S(p) = x\cdot S(\mu)$ and $S^{-1}(x)$ and $S(\mu)$ are right integrals it is clear that 1) implies that p is an invariant with respect to right and left multiplication of $\mathscr{A}$ and

$\mathscr{A}^o$. This shows 2). Assuming that 1) is true we show 3):

$$\bowtie^{-1}(\mu \otimes S^{-1}(x)) = \langle S(\mu'), S^{-1}(x)' \rangle \langle \mu'', S^{-1}(x)''' \rangle S^{-1}(x)'' \otimes \mu'' \tag{2.39}$$

$$\begin{aligned}
&\overset{by(2.28)}{=} \omega^{-1} \langle S(\mu'), S(x''') \rangle \langle \mu''', S(x') \rangle S(x'') \otimes \mu'' \\
&= \omega^{-1} \langle \mu', x'' \rangle \langle \mu''', S(x')'' \rangle \, S(x')' \otimes \mu'' \\
&= \omega^{-1} \sum_{ij} \langle \mu', x'' \rangle \, \langle \mu''', e_j'' \rangle \, \langle \mu'', e_i \rangle \, \langle f_j, S(x') \rangle e_j' \otimes f_i \\
&= \omega^{-1} \sum_{ij} \langle \mu'', e_j'' e_i \rangle \, \langle f_j, S\bar{\beta}_l(\mu') \rangle e_j' \otimes f_i \\
&= \omega^{-1} \sum_{ij} \langle f_j, S\bar{\beta}_l \beta_r(e_j'' e_i) \rangle e_j' \otimes f_i \\
&\overset{by(2.30)}{=} \omega^{-1} \sum_{ij} \langle f_j, S^2(e_j'' e_i) \rangle e_j' \otimes f_i \overset{by(1.))}{=} x \otimes S(m) \,.
\end{aligned} \tag{2.40}$$

If we apply $1 \otimes S^2$ to both sides of the last equation and use (2.28) we find

$$\begin{aligned}
P &= \sum_{ij} \langle f_j, S^2(e_j'') e_i \rangle e_j' \otimes f_i = \sum_{i} \langle f_j'', S^2(e_j'') \rangle e_j' \otimes f_j' \\
&\overset{by(2.12)}{=} \sum_{ij} \langle f_j, S^2(e_j'' e_i'') \rangle e_j' e_i' \otimes f_i \,,
\end{aligned}$$

which is precisely the equation given in 3). The same formula has been proven in [Dr1] using the theory of Hopf modules directly. From (2.28) we find $S(\mu)(x) = \mu(ax) = 1$ so that P as in (2.40) is a rank one projection. It remains to show the first part of the proposition. For this purpose we need two identities for the integrals, namely

$$\begin{aligned}
S^2((S(\mu)'' \lhd a^{-1})\alpha) \otimes S(\mu)' &= S^2((S(\mu) \lhd a^{-1})''\alpha) \otimes (S(\mu) \lhd a^{-1})' \\
&\overset{by(2.28)}{=} S^2(\mu''\alpha) \otimes \mu' \\
&\overset{by(2.35)}{=} \mu' \otimes \mu'' \quad = \Delta(\mu) \,,
\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}
((S^2(x'')a^{-1}) \leftharpoondown \alpha) \otimes x' &\overset{by(2.37)}{=} ((x') \leftharpoondown \alpha) \otimes x'' \\
&= (x \leftharpoondown \alpha)' \otimes (x \leftharpoondown \alpha)'' \\
&\overset{by(2.28)}{=} S^{-1}(x)' \otimes S^{-1}(x)'' = \Delta(S(x)^{-1}) \,.
\end{aligned} \tag{2.42}$$

Inserting (2.41) and (2.42) into the expression for the bicrossed product of p in (2.29) we find:

$$\begin{aligned}
\bowtie^{-1}(\mu \otimes S^{-1}(x)) &= \langle S(\mu'), S^{-1}(x)' \rangle \langle \mu''', S^{-1}(x)''' \rangle \, S^{-1}(x)'' \otimes \mu'' \\
&= \langle S^3((S(\mu))''' \lhd a^{-1}), S^2((x''a^{-1}) \leftharpoondown \alpha) \rangle \langle S(\mu)'', x'' \rangle x' \otimes S(\mu)' \\
&= \langle \alpha^{-1}(a \rhd S(S(\mu)''')), (x'''a^{-1}) \leftharpoondown \alpha \rangle \, \langle S(\mu)'', x'' \rangle \, x' \otimes S(\mu)' \\
&= \langle S(S(\mu)'''), x''' \rangle \, \langle S(\mu)'', x'' \rangle x' \otimes S(\mu)' \\
&= \langle S(\mu)'' S(S(\mu)'''), x'' \rangle x' \otimes S(\mu)' = x \otimes S(\mu) \,.
\end{aligned} \tag{2.43}$$

Hence $\mu \cdot S^{-1}(x) = S(\mu \cdot S^{-1}(x)) = x \cdot S(\mu)$ and we have shown 1) of the proposition. $\square$

Thanks to the simple comultiplicative structure of $D(\mathscr{A})$ it is much easier to find the integral. Since $S^{-1}(\mu)$ is a right integral of $\mathscr{A}$ and since we have opposite comultiplication on $\mathscr{A}^o$ a *right* integral of $D(\mathscr{A})$ is given by

$$\mu_D(\lambda\cdot y):=\lambda(x)\mu(S(y))\quad \text{for all } \lambda\in\mathscr{A}^o,\text{ and } y\in\mathscr{A}. \tag{2.44}$$

Its properties are described next:

Proposition 6. *Let x and μ be the left integrals of $\mathscr{A}$ as in* (2.22). *Then*

1. *μ_D is a right integral of $D(\mathscr{A})$ with $\mu_D(p)=1$.*
2. *The modulus (as defined in 2.26) of $D(\mathscr{A})$ is $a_D=g^{-1}$.*
3. *$\mu_D\in\bar{C}(D(\mathscr{A}))$, in particular $\mu_D(\lambda\cdot y)=\omega\mu_D(y\cdot\lambda)\ \forall y\in\mathscr{A},\ \lambda\in\mathscr{A}^0$.*

Proof. Part 1) is clear. Also, it follows directly from the definitions of the moduli of $\mathscr{A}$ that $1\otimes\mu_D\Delta(h)=\alpha\cdot a^{-1}\mu_D(h)$. 2) follows if we apply the antipode and use that $S(\mu_D)$ is a left integral of $D(\mathscr{A})$. In order to show 3) we observe that the equation for right integrals analogous to (2.35) is $\mu_D(S^2(k)h)=\mu_D(h(k\leftharpoondown\alpha_D^{-1}))$. Together with Proposition 5.2 this shows $\mu_D\in\bar{C}(D(\mathscr{A}))$. In the case where k and h are in the special subalgebras we use (2.28) to show $\mu_D(S^2(\lambda)y)=\omega^{-1}\mu_D(\lambda y)$ which yields the last equation in part 3. □

The fact that the right integral of a double algebra is invariant under the coadjoint action allows us to identify as an object in the representation category of $D(\mathscr{A})$, namely with the integral of the "braided algebra" [Ly] of the category. Before we explain this aspect in more detail in the next section let us discuss a few more consequences of Proposition 6 for doubles.

It is easy to see that an element of a Hopf algebra $w\in D$ with $S^2(y)=wyw^{-1}$ provides us with an isomorphism $\bar{C}(D)\xrightarrow{\sim}C_0(D):\lambda\mapsto\lambda\lhd w$. Here $C_0(D)$ denotes as in [Dr1] the space of traces on D. Given the two canonical elements in (2.19) we wish to compute the respective traces for μ_D. To this end define the following linear forms on $D(\mathscr{A})$:

$$\chi:D(\mathscr{A})\xrightarrow{\bullet^{-1}}\mathscr{A}^0\otimes\mathscr{A}\xrightarrow{\langle,\rangle}k\,,$$

$$\hat{\chi}:D(\mathscr{A})\xrightarrow{\bullet^{-1}}\mathscr{A}^0\otimes\mathscr{A}\xrightarrow{1\otimes S^{-1}}\mathscr{A}^0\otimes\mathscr{A}\xrightarrow{\langle,\rangle}k\,. \tag{2.45}$$

The forms on $D(\mathscr{A})$ and the canonical elements are now related as follows.

Proposition 7.

$$\chi=\mu_D\lhd u\,,\quad \hat{\chi}=\omega^{-1}\mu_D\lhd\hat{u}\,, \tag{2.46}$$

and both χ and $\hat{\chi}$ are <u>*nondegenerate traces*</u> *on $D(\mathscr{A})$.*

Proof. From previous considerations it is clear that $\mu_D\lhd u$ and $\mu_D\lhd\hat{u}$ are traces. The rest of the proof are straightforward computations:

$$\begin{aligned}(\mu_D\lhd u)(\lambda\cdot y)&=(\mu_D\lhd u)(y\cdot\lambda)=\sum_i\mu_D(S(f_i)\cdot e_i\cdot y\cdot\lambda)\\ &=\sum_i\mu_D(S^2(\lambda)\cdot S(f_i)\cdot e_i\cdot y)=\sum_i(S^2(\lambda)S(f_i))(x)\,\mu(S(y)S(e_i))\\ &=\sum_i f_i(x\leftharpoondown S^2(\lambda))\mu(S(y)e_i)=\mu(S(y)(x\leftharpoondown S^2(\lambda)))\\ &=\mu(S(y)\bar{\beta}_l(S^2(\lambda)))=\beta_r\bar{\beta}_l(S^2(\lambda))(S(y))\\ {\scriptstyle by(2.30)}&=S(\lambda)(S(y))=\lambda(y)=\chi(\lambda\cdot y)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}
(\mu_D \triangleleft \hat{u})(\lambda \cdot y) &= \sum_i \mu_D(f_i \cdot \lambda \cdot y \cdot e_i) = \sum_i f_i \cdot \lambda(x)\mu(S(ye_i)) \\
&= \mu(S(y(\lambda \rightharpoonup x)))\overset{by(2.28)}{=} \omega\mu(ay(\lambda \rightharpoonup x)) \\
&= \omega\mu(ay\bar{\beta}_r(\lambda)) = \omega\beta_r\bar{\beta}(\lambda)(ay) \\
&\overset{by(2.31)}{=} \omega S(a \triangleright \lambda)(ay) = \omega S(\lambda)(y) = \omega\hat{\chi}(\lambda \cdot y) .
\end{aligned}$$

Nondegeneracy of χ and $\hat{\chi}$ follow directly from nondegeneracy of μ_D. □

4. Balancing in Doubles. In a rigid BTC any object X is isomorphic to its double conjugate $X^{\vee\vee}$. Yet the only isomorphism that is a priori canonical is between X and $X^{\vee\vee\vee\vee}$. Thus in addition to the usual axioms defining a BTC one often requires the existence of a $\otimes-$ natural isomorphism of the functor $X \to X^{\vee\vee}$ to the identity, which squares to the canonical one from $X \to X^{\vee\vee\vee\vee}$ to the identity. For the representation category of a quasitriangular Hopf algebra D this is equivalent to the existence of a group like element k with:

$$k \in G(D), \quad g = k^2, \quad \text{and} \quad S^2(y) = kyk^{-1} . \tag{2.47}$$

It is clear that a balancing does not exist since often g is not a square in $G(D)$. If it does it is unique up to multiplication with central, group-like elements of order two, i.e., elements in $\Sigma(D) := {}_2G(D) \cap Z(D)$.

Equivalently, we can consider the corresponding element $v := u \cdot k^{-1} = \hat{u}^{-1} \cdot k$. Inspecting (2.21) it is easily verified that v defines a balancing of elements iff

$$v \in Z(D), \quad S(v) = v, \quad \text{and} \quad M = v \otimes v\Delta(v^{-1}) . \tag{2.48}$$

From these conditions $\varepsilon(v) = 1$ and $v^2 = z$ follow. This point of view has been introduced in [RT0] where v is called a *ribbon element.* In their context the eigenvalue of v in an irreducible representation yields the framing anomalies of colored link.

For a double $D(\mathscr{A})$ the existence of a balancing can be phrased as a property of the moduli of $\mathscr{A}$.

Proposition 8.

1. *k is a balancing of $D(\mathscr{A})$ if and only if*

$$k = \sqrt{\alpha} \cdot (\sqrt{a})^{-1} , \tag{2.49}$$

where $\sqrt{\alpha} \in G(\mathscr{A}^0)$, $\sqrt{a} \in G(\mathscr{A})$ square to α and a respectively, and

$$S^2 = ad^*(\sqrt{\alpha}) \circ ad(\sqrt{a}^{-1}) \ \text{on} \ \mathscr{A} . \tag{2.50}$$

2. *To a given balancing we associate the number v defined by*

$$v^{-1} = \chi(k) = \sqrt{\alpha}(\sqrt{a}) . \tag{2.51}$$

If $D(\mathscr{A})$ admits a balancing v is a root of unity, $v^4 = \omega$ and v^2 does not depend on the choice of balancing.

Proof. From Lemma 1 and (2.47) we infer that k has to be a product of group-likes of the special subalgebras. By definition of doubles (2.50) is the same as $S^2(y) = kyk^{-1} \forall y \in \mathscr{A}$. The inverse adjoint of (2.50) yields the same equation on $\mathscr{A}^0$ and thereby (2.47). For part 2 we remark that two balancings k and k' are related by

$k' = k \cdot R$, where $R = \rho \cdot r \in \Sigma(D(\mathscr{A})) \cong \Sigma(\mathscr{A}^0) \oplus \Sigma(\mathscr{A})$. Then $\chi(k') \cdot \chi(k)^{-1} = \sqrt{\alpha}(r)\,\rho(\sqrt{a})\rho(r)$ which is of order two since ρ and r are. □

In particular, the last statement implies that once a balancing exists the intrinsic quantity ω has a canonical square root.

5. Representation of Mapping Class Groups on Doubles. In several papers [Ly] Lyubashenko has developed the notion of a Hopf algebra F in a braided tensor category $\mathscr{C}$. It is an analogue to the notion of a braided group, as defined by Majid [M2]. As an object F in a category with all limits the algebra is the constant functor of the coend $\langle \underline{\mathrm{Hom}};\, h: \underline{\mathrm{Hom}} \xrightarrow{\cdot\cdot} F \rangle$ of the functor $\underline{\mathrm{Hom}}: \mathscr{C}^{opp} \times \mathscr{C} \to \mathscr{C}: (X, Y) \mapsto X^\vee \otimes Y$. For definitions see [Mc]. The multiplication and comultiplication of F are induced by certain compositions of dinatural transformations using universality of the coend. As opposed to symmetric categories the definition of the multiplication of F depends on the choice of a commutativity isomorphism. The same is true for the axiom replacing cohomomorphie of the multiplication. An analogous statement of the fundamental theorem of Hopf modules holds for the braided algebras so that under certain finiteness conditions the algebra has an integral $\mu \in Hom(1, F)$. The algebra also possesses a braided antipode $\Gamma \in End(F)$. Lyubashenko constructs, in analogy to the definitions for semisimple categories, modular operators $\mathscr{T}, \mathscr{S} \in End(F)$. They are determined by the coend properties of F and the following commutative diagrams:

$$\begin{array}{ccccccc} X^\vee \otimes X & \xrightarrow{1 \otimes u_X} & X^\vee \otimes X & \quad & X^\vee \otimes X & \xrightarrow{\gamma_X} & X^{\vee\vee} \otimes X^\vee \\ \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_X} & & \downarrow{\scriptstyle h_{X^\vee}} \\ F & \xrightarrow{\mathscr{T}} & F & & F & \xrightarrow{\Gamma} & F \end{array} \; . \tag{2.52}$$

Here $v_- \in Nat(id)$ is the balancing and $\gamma_X := q_X \otimes 1\ \varepsilon(X^\vee, X)$, where ε is the commutativity constraint and $q_X: X \to X^\vee \otimes X^{\vee\vee} \otimes X \xrightarrow{1 \otimes \varepsilon} X^\vee \otimes X \otimes X^{\vee\vee} \to X$. Furthermore,

$$\begin{array}{ccccc} Y^\vee \otimes Y & \xrightarrow{\mu \otimes 1} & F \otimes Y^\vee \otimes Y & \xleftarrow{h_X \otimes 1} & X^\vee \otimes X \otimes Y^\vee \otimes Y \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle (1 \otimes 1 \otimes ev)(1 \otimes \varepsilon^2 \otimes 1)} \\ F & \xrightarrow{\mathscr{S}} & F & \xleftarrow{h_X} & X^\vee \otimes X \end{array} \; . \tag{2.53}$$

The coend and integral exist if $\mathscr{C}$ is the representation category of a finite dimensional Hopf algebra D. Specifically, we have that $F = D^*$, which is a D-module by ad_*-action. The comultiplication is just the multiplication on D. However, the multiplication in F stems from a distorted coproduct Δ_B on D as the usual one is not ad_*-covariant. In one convention we have, e.g., $\Delta_{Br}(y) = e_i'' y' S^{-1}(e_i') \otimes f_i y''$. As remarked in [LyM] the right integral for the braided multiplication coincides with the ordinary right integral. This is seen easily, e.g., from the fact that μ_D for a double is ad_*-invariant. An antipode $\Gamma \in End(D)$ of the braided multiplication consistent with Δ_B is

$$\Gamma(A) := \sum_i S(e_i) S(A) \hat{u} f_i \,. \tag{2.54}$$

The triple $(D, \cdot, \Delta_{Br})$ is the prototype of a braided group. For a thorough treatment of this structure which inspired the algebraic construction in [Ly] we refer to [M2]. However, the construction of the $\mathcal{S}$ and $\mathcal{T}$ given in [Ly] can also be translated into the context of ordinary Hopf algebras. The action of $\mathcal{T}$ is clearly given by multiplication of a ribbon element v. The identity of integrals allows us to derive from (2.53) a formula for $\mathcal{S}$ acting on a quasitriangular algebra D,

$$\mathcal{S}(A) = S(\bar{f}(\mu_D \triangleleft A)) = \sum_{i,j} \mu_D(A f_j e_i) S(e_j) \,. \tag{2.55}$$

This formula (with slightly different conventions) has been given in [LyM]. Using the form of the right integral given in (6) and applying the bicross formula (2.15) to order $A f_j e_i$ this formula can be worked out further. The formula $\mathcal{S}(\lambda \otimes h) = \sum_i f_i \otimes (x'') \langle \lambda, e_i''' x' S^{-1}(e_i') \rangle$ resulting from this has been in [M1].

Let us now use the properties of integrals given in the previous section and the identities for the canonical isomorphisms to derive an intriguingly, compact formula for $\mathcal{S}$. From this form the invertibility of $\mathcal{S}$ for doubles is obvious and the inverse readily computed from the identities (2.31) and following.

Proposition 9. *For a double $D(\mathcal{A})$ over a finite dimensional Hopf algebra $\mathcal{A}$ let μ_D be as in* (2.44) *and $\mathcal{S} \in End(D(\mathcal{A}))$ be defined as in* (2.55). *Then the following diagrams of isomorphisms commute*:

$$\begin{array}{ccccccc}
\mathcal{A} \otimes \mathcal{A}^0 & \xrightarrow{\beta_l \circ S^{-1} \otimes S \circ \bar{\beta}_l} & \mathcal{A}^0 \otimes \mathcal{A} & & \mathcal{A}^0 \otimes \mathcal{A} & \xrightarrow{\bar{\beta}_r \otimes \beta_l \circ L_a} & \mathcal{A} \otimes \mathcal{A}^0 \\
\downarrow \cdot & & \downarrow \cdot & & \downarrow \cdot & & \downarrow \cdot \\
D(\mathcal{A}) & \xrightarrow{\quad \mathcal{S} \quad} & D(\mathcal{A}) & & D(\mathcal{A}) & \xrightarrow{\quad \mathcal{S}^{-1} \quad} & D(\mathcal{A})
\end{array} \,. \tag{2.56}$$

Here L_a denotes left multiplication with a.

Proof. The first diagram is verified by direct computation:

$$\begin{aligned}
\mathcal{S}(y \cdot \lambda) &= \sum_{ij} \mu_D(y \cdot \lambda f_j e_i) \otimes S(f_i) S(e_j) \quad {}^{by(Pp6.3.)} = \sum_{ij} \mu_D(\lambda f_j \cdot e_i S^{-2}(y)) \\
{}^{by(2.44)} &= \sum_{ij} \lambda f_j(x) \mu(S^{-1}(y) e_i) f_i \cdot S(e_j) \qquad\qquad = \mu \triangleleft S^{-1}(y) \cdot S(x \leftharpoonup \lambda) \\
{}^{by(2.23)} &= \beta_l(S^{-1}(y)) \cdot S(\bar{\beta}_l(\lambda)) \,.
\end{aligned} \tag{2.57}$$

The second diagram follows immediately from relations (2.30) and (2.31), which allow us to invert β_l and $\bar{\beta}_l$. □

Let us also give a more convenient form for the braided antipode:

Lemma 10. *Let $\hat{\Gamma} : \mathcal{A}^0 \otimes \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes \mathcal{A}^0$ be given by*

$$\begin{aligned}
&\mathcal{A}^0 \otimes \mathcal{A} \xrightarrow{R \otimes 1^2 \otimes R} \mathcal{A} \otimes \mathcal{A}^0 \otimes \mathcal{A}^0 \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^0 \\
&\xrightarrow{S \otimes 1 \otimes S^{-1} \otimes 1^3} \mathcal{A} \otimes \mathcal{A}^0 \otimes \mathcal{A}^0 \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^0 \xrightarrow{1 \otimes \cdot \otimes \cdot \otimes 1} \\
&\mathcal{A} \otimes \mathcal{A}^0 \otimes \mathcal{A} \otimes \mathcal{A}^0 \xrightarrow{1 \otimes r \otimes 1} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^0 \otimes \mathcal{A}^0 \\
&\xrightarrow{\cdot \otimes \cdot} \mathcal{A} \otimes \mathcal{A}^0 \xrightarrow{S \otimes 1} \mathcal{A} \otimes \mathcal{A}^0
\end{aligned}$$

and Γ as in (2.54). *Then the following diagram commutes*:

$$\begin{array}{ccc} \mathscr{A}^0 \otimes \mathscr{A} & \xrightarrow{\tilde{\Gamma}} & \mathscr{A} \otimes \mathscr{A}^0 \\ \downarrow \cdot & & \downarrow \cdot \\ D(\mathscr{A}) & \xrightarrow{\Gamma} & D(\mathscr{A}) \end{array} \quad . \tag{2.58}$$

Proof. Straightforward computation:

$$\Gamma(\lambda \cdot y) = \sum_i S(e_i)S(y) \cdot S(\lambda)\hat{u} f_i = \sum_i S(e_i)S(y)\hat{u}S^{-1}(\lambda) f_i$$

$$= \sum_{ij} S(S(f_j) y e_i) \cdot e_j S^{-1}(\lambda) f_i\,, \tag{2.59}$$

which is precisely the above composition. □

6. Proof of Modular Relations and the Projective Phases. For the square of the braided antipode we easily verify

$$\Gamma^2 = ad^-(v^{-1})\,, \tag{2.60}$$

where v is any ribbon element and $ad^-(y) = S^{-1} \circ ad(y) \circ S$. Proposition 9 and Lemma 10 put us in a position to prove the next lemma. From this we will infer one of the modular relations and the correct projective phase.

Lemma 11. *We have the following relation for maps $\mathscr{A}^0 \otimes \mathscr{A} \xrightarrow{\sim} \mathscr{A} \otimes \mathscr{A}^0 \otimes \mathscr{A}$*:

$$\beta_l \circ S^{-1} \otimes S \circ \bar{\beta}_l \tilde{\Gamma} = \omega \bowtie \bar{\beta}_r \otimes (\beta_l L_a)\,. \tag{2.61}$$

Proof. We shall prove (2.61) by evaluating both sides on $\lambda \otimes y \in \mathscr{A}^0 \otimes \mathscr{A}$ individually and comparing results. For the right-hand side we have

$$\begin{aligned}
\bowtie \bar{\beta}_r \otimes (\beta_l L_a)(\lambda \otimes y) &= \bowtie(x'\lambda(x'') \otimes \mu \triangleleft (ay)) \\
&= \langle (\mu \triangleleft (ay))', x' \rangle \langle S((\mu \triangleleft (ay))'''), x''' \rangle \lambda(x'''') (\mu \triangleleft (ay))'' \otimes x'' \\
&= \langle \mu', x' \rangle \langle \mu''', ayS^{-1}(x''') \rangle \lambda(x'''') \mu'' \otimes x'' \\
&= \sum_i \langle \mu', x' \rangle \langle f_i, x'' \rangle \langle \mu''', ayS^{-1}(e_i'') \rangle \lambda(e_i''') \mu'' \otimes e_i' \\
&= \sum_i \langle \mu', \bar{\beta}_r(f_i) \rangle \langle \mu''', ayS^{-1}(e_i'') \rangle \lambda(e_i''') \mu'' \otimes e_i' \\
&= \sum_i \langle \beta_r \bar{\beta}_r(f_i)'', ayS^{-1}(e_i'') \rangle \lambda(e_i''') \beta_r \bar{\beta}_r(f_i)' \otimes e_i' \\
{}^{by (2.31)} &= \sum_i \langle (S(f_i) \triangleleft a^{-1})'', ayS^{-1}(e_i'') \rangle \lambda(e_i''') (S(f_i) \triangleleft a^{-1})' \otimes e_i' \\
&= \sum_i \langle S(f_i)'', yS^{-1}(e_i'') \rangle \lambda(e_i''') S(f_i)' \otimes e_i'\,.
\end{aligned} \tag{2.62}$$

The evaluation of the left-hand side gives:

$$\begin{aligned}
\beta_l\circ S^{-1}\otimes S\circ\bar\beta_l\tilde\Gamma(\lambda\otimes y) &= \sum_{ij}\beta_l\circ S^{-1}\otimes S\circ\bar\beta_l(S(S(e_i)ye_j)\otimes(f_iS^{-1}(\lambda)f_j))\\
&= \sum_{ij}\beta_l(S(e_i)ye_j)\otimes S\circ\bar\beta_l(f_iS^{-1}(\lambda)f_j)\\
&= \sum_{ij}\langle f_i,x'\rangle\langle S^{-1}(\lambda)f_j,x''\rangle\mu\triangleleft(S(e_i)ye_j)\otimes S(x''')\\
&= \sum_{j}\langle S^{-1}(\lambda)f_j,x''\rangle S((S^{-1}(ye_j)x')\triangleright S^{-1}(\mu))\otimes S(x''')\\
{}^{by(2.28)} &= \omega\sum_{ij}\langle f_i,x''\rangle\langle S^{-1}(\lambda)f_j,e_i'\rangle S((S^{-1}(ye_j)x')\triangleright\mu\triangleleft a)\\
&\quad\otimes S(e_i'')\\
&= \omega\sum_{ij}\langle S^{-1}(\lambda)f_j,e_i'\rangle S((S^{-1}(ye_j)\bar\beta_r(f_i))\triangleright\mu\triangleleft a)\otimes S(e_i'')\\
&= \omega\sum_{ij}\langle S^{-1}(\lambda)f_j,e_i'\rangle S((S^{-1}(ye_j))\triangleright(\beta_r\bar\beta_r(f_i))\triangleleft a)\otimes S(e_i'')\\
{}^{by(2.31)} &= \omega\sum_{ij}\langle S^{-1}(\lambda)f_j,e_i'\rangle S((S^{-1}(ye_j))\triangleright S(a\triangleright f_i)\triangleleft a)\otimes S(e_i'')\\
&= \omega\sum_{ij}\langle S^{-1}(\lambda)f_j,e_i'\rangle S^2(f_i)\triangleleft(ye_j)\otimes S(e_i'')\\
&= \omega\sum_{ij}\langle S(f_j)\lambda,e_i''\rangle S(f_i)\triangleleft(ye_j)\otimes e_i'\\
&= \omega\sum_{ij}\langle S(f_i)'',ye_j\rangle\langle S(f_j),e_i''\rangle\lambda(e_i''')S(f_i)'\otimes e_i'\\
&= \omega\sum_{i}\langle S(f_i)'',yS^{-1}(e_i'')\rangle\lambda(e_i''')S(f_i)'\otimes e_i'\,.
\end{aligned}\tag{2.63}$$

Comparison of (2.62) to (2.63) proves the assertion. □

The projective phases of the second modular relation arise in the computation of the value of $\mathscr{S}$ on the ribbon element.

Lemma 12. *Suppose v is a ribbon element of a double $D(\mathscr{A})$ and ν is the associated fourth root of ω (see Prop. 8.). Then we have for $\mathscr{S}$ as defined in* (2.55):

$$\mathscr{S}(v)=\nu^{-1}v^{-1}\,,\quad \mathscr{S}(v^{-1})=\nu^5 v\,.\tag{2.64}$$

Proof. Straightforward computation: Using $v=uk^{-1}$ we have

$$\begin{aligned}
\mathscr{S}(v) &= \sum_{ij}\mu_D(uk^{-1}f_je_i)S(f_i)\cdot S(e_j) &\quad {}^{by\,Prop.\,7.} &= \sum_{ij}\chi(e_ik^{-1}f_j)S(f_i)\cdot S(e_j)\\
&= \sum_{ij}\langle\sqrt{\alpha}^{-1}f_j,e_i\sqrt{a}\rangle S(f_i)\cdot S(e_j) & &= \sum_i S(f_i)\cdot S((e_i\sqrt{a})\leftharpoonup\sqrt{\alpha}^{-1})\\
&= \nu^{-1}\sum_i S(f_i)\cdot\sqrt{a}^{-1}S(e_i\leftharpoonup\sqrt{\alpha}^{-1}) & &= \nu^{-1}\sum_i S(\sqrt{\alpha}^{-1}f_i)\cdot\sqrt{a}^{-1}S(e_i)\\
&= \nu^{-1}\sum_i S(f_i)kS(e_i)=\nu^{-1}\sum_i f_iS^2(e_i)k & &= \nu^{-1}u^{-1}k=\nu^{-1}v^{-1}\,.
\end{aligned}\tag{2.65}$$

The second relation follows with $v^{-1} = \hat{u}k^{-1}$ from:

$$\begin{aligned}\mathcal{S}(v^{-1}) &= \sum_{ij} \mu_D(\hat{u}k^{-1} f_j e_i) S(f_i) \cdot S(e_j) = \omega \sum_{ij} \hat{\chi}(f_j \sqrt{\alpha}^{-1} \sqrt{a} S^2(e_i)) S(f_i) \cdot S(e_j) \\ &= \omega \sum_{ij} \langle f_j \sqrt{a}^{-1}, S(e_i) \sqrt{a}^{-1} \rangle S(f_i) \cdot S(e_j) \\ &= \omega \sum_i f_i \cdot S(\sqrt{\alpha}^{-1} \rightharpoonup (e_i \sqrt{a}^{-1})) \\ &= \omega v \sum_i f_i \cdot \sqrt{a} S(\sqrt{\alpha}^{-1} \rightharpoonup e_i) = \omega v \sum_i f_i \sqrt{\alpha}^{-1} \sqrt{a} S(e_i) \\ &= v^5 k^{-1} u = v^5 v\,. \end{aligned} \tag{2.66}$$

Let us now prove the second modular relation:

Proposition 13. *For a double $D(\mathcal{A})$ with balancing, let $\mathcal{S}$ be defined as in* (2.55) *and $\mathcal{T}$ by multiplication with v. Then*

$$\mathcal{S}\mathcal{T}^{-1}\mathcal{S} = v^5 \mathcal{T}\mathcal{S}\mathcal{T}\,. \tag{2.67}$$

Proof. If we apply $\eta \circ S^{-1} \otimes 1 \otimes S$ for some $\eta \in D(\mathcal{A})^*$ to both sides of the equation in Lemma 3 we find with $\tau(M) = S \otimes S(M)$ that

$$\begin{aligned} S \circ \bar{f}(\lambda \triangleleft S \circ \bar{f}(\eta)) &= (\lambda \otimes S)((S(\bar{f}(\eta)) \otimes 1)M) \\ &= \eta \circ S^{-1} \otimes \lambda \otimes S((\tau(M) \otimes 1)(1 \otimes M)) \\ &= \sum_{ij} \eta(S^{-1}(\bar{f}(\lambda)' f_i f_j)) \otimes S(e_i) S(\bar{f}(\lambda)'') e_j\,. \end{aligned} \tag{2.68}$$

Inserting into (2.68) the forms $\lambda = \mu_D \triangleleft \rho$ and $\eta = \mu_D \triangleleft A$ for some $A, \rho \in D(\mathcal{A})$ and by using the definition (2.55) we find:

$$\begin{aligned} \mathcal{S} \circ L_\rho \circ \mathcal{S}(A) &= \sum_{ij} \mu_D(AS^{-1}((S^{-1}(\mathcal{S}(\rho)))' f_i f_j)) S(e_i) S(S^{-1}(\mathcal{S}(\rho))'') e_j \\ &= \sum_{ij} \mu_D(AS^{-1}(f_j) f_i S^{-2}(\mathcal{S}(\rho)'')) S^2(e_i) \mathcal{S}(\rho)' e_j\,. \end{aligned} \tag{2.69}$$

Here L_ρ is the left multiplication with ρ. The left-hand side the assertion (2.67) is now found by specializing $\rho = v^{-1}$, where $L_v = \mathcal{T}$. In order to evaluate the right-hand side of (2.69) we notice that Lemma 12 implies the following identities:

$$\begin{aligned} \Delta(\mathcal{S}(v^{-1})) &= v^5 \Delta(v) = v^5 v \otimes v M^{-1} = v^5 v \otimes v \otimes v R^{-1} \tau(R^{-1}) \\ &= v^5 v \otimes v S^3 \otimes S^2(R) 1 \otimes S(\tau(R)) = v^5 \sum_{kl} v S^3(e_k) f_l \otimes v S^2(f_k) S(e_l)\,. \end{aligned}$$

Replacing $\mathcal{S}(v^{-1})' \otimes \mathcal{S}(v^{-1})''$ in (2.69) by this expression yields the assertion:

$$\begin{aligned} \mathcal{S}\mathcal{T}^{-1}\mathcal{S} &= v^5 \sum_{ijkl} \mu_D(AS^{-1}(f_j) f_i v f_k S^{-1}(e_l)) v S^2(e_i) S^3(e_k) f_l e_j \\ &= v^5 \sum_{jl} \sum_{ik} \mu_D(vAS^{-1}(f_j)(f_i f_k) S^{-1}(e_l)) v S^2(e_i S(e_k)) f_l e_j \\ &\overset{by(2.13)}{=} v^5 \sum_{jl} \mu_D(vAS^{-1}(f_j) S^{-1}(e_l)) v f_l e_j\,. \end{aligned} \tag{2.70}$$

We readily identify the last equation with the right-hand side of (2.67). This completes the proof.

The $\mathcal{S}$ matrix was originally defined as an element in the *End*-set of the coend of the representation category. As a map on $D(\mathcal{A})$ it therefore intertwines the ad^--action (see (2.60)) of the algebra on itself. (This property can also be inferred directly from Lemma 2, 1.) The same is true for multiplications with central elements as for example for $\mathcal{T}$. Hence the center $Z(D(\mathcal{A}))$ – which is the invariance of the ad^--action – is an invariant subspace of both operators. It follows immediately from (2.59) that the restriction of Γ to the center is the usual antipode S and thus involutive.

We summarize these observations and the relations found in (2.60), (2.61), and (2.67) in the following theorem:

Theorem 1. *Suppose $D(\mathcal{A})$ is the double of a finite dimensional Hopf algebra. Assume that $D(\mathcal{A})$ admits a balancing and let v and ω be as in Proposition 8. Furthermore, let $\mathcal{T}$ be the multiplication with v, and $\mathcal{S}$ and Γ be defined as in (2.55) and (2.59), respectively. Then*

1. *The generators define a projective representation of the mapping class group $\mathcal{D} := \pi_0(\mathcal{D}\mathit{iff}(T,D))$ of torus maps fixing a disk with the following relations:*

$$\mathcal{S}^2 = \omega\Gamma^{-1}\,, \quad \mathcal{T}\Gamma = \Gamma\mathcal{T}\,, \tag{2.71}$$

$$(\mathcal{S}\mathcal{T})^3 = v^3\Gamma^{-2} = v^2 ad^-(v)\,. \tag{2.72}$$

2. *The maps $\mathcal{S}$ and $\mathcal{T}$ stabilize the center $Z(D(\mathcal{A}))$. The restrictions $\bar{\mathcal{S}}$ and $\bar{\mathcal{T}}$ satisfy*

$$(\bar{\mathcal{S}}\bar{\mathcal{T}})^3 = v^3\,, \quad \bar{\mathcal{S}}^2 = \omega S^{\pm 1}\,, \quad \mathcal{T}S = S\mathcal{T}\,, \tag{2.73}$$

where S is the involutive map given by the restriction of the antipode to the center.

The relations in (2.73) show that $\bar{\mathcal{S}}$ and $\bar{\mathcal{T}}$ define a projective representation of $SL(2,\mathbb{Z})$. The normalization of the $\mathcal{S}$-operation was defined by the canonical normalization of μ_D. For the computation of topological invariants it is often more convenient to have a normalization for which the operators are inverted if we invert the braided structure. For a given balancing k let $\bar{\mathcal{S}}'$ and $\bar{\mathcal{T}}'$ be the analogous operators defined with respect to $R' = \tau(R^{-1})$. Then as $u' = \hat{u}$ we have that $\bar{\mathcal{T}}' = \bar{\mathcal{T}}^{-1}$ is already correctly normalized. A computation similar to the one in Proposition 9 yields

$$\bar{\mathcal{S}}' = \omega\bar{\mathcal{S}}^{-1}\,.$$

Thus it is the matrix $\mathcal{S}_{\bullet} := v^2\bar{\mathcal{S}}^{-1}$ which inverts under inversion of the braided structure. For these generators we have the relations:

$$\mathcal{S}_{\bullet}^4 = 1\,, \quad (\mathcal{S}_{\bullet}\bar{\mathcal{T}})^3 = v^{-3}\mathcal{S}_{\bullet}^2\,. \tag{2.74}$$

Comparing (2.74) to relations in [T] and [RT] we find that the projective phase c of the functor Φ in (1.10) for a universal TQFT over a double $D(\mathcal{A})$ is given by:

$$c = v^{-3}\,.$$

3. The Relation of Universal and Semisimple TQFT's: An Example

In this section we shall analyze the proposed representation of the mapping class group $\mathcal{D}$ of the punctured torus explicitly in the example of the double of the quantum-sl_2-Borel algebra B_q.

1. The Algebra $D(B_q)$. Let q be a primitive l^{th} root of unity where $l = 2m + 1$, $m \in \mathbb{Z}_{\geqq 1}$. We denote by B_q the Hopf algebra with generators e, $k^{\pm 1}$ and relations:

$$\begin{aligned} &kek^{-1} = qe\,, && k^l = 1\,, && e^l = 0\,, \\ &\Delta(k) = k \otimes k\,, && \Delta(e) = e \otimes 1 + k^2 \otimes e\,, && \\ &S(e) = -k^{-2}e\,, && S(k) = k^{-1}\,, && \varepsilon(e) = 0 \;\; \varepsilon(k) = 1\,. \end{aligned} \tag{3.75}$$

As a PBW-basis for B_q we choose $e^n k^j$ with $n = 0, \ldots, l-1$ and $j \in \mathbb{Z}/l$. The left *cointegral* of B_q is given by

$$x = \left(\sum_{j=0}^{l-1} k^j\right) e^{l-1}\,, \tag{3.76}$$

and the left *integral* with normalization $m(x) = 1$ is

$$m(e^n k^j) = q^2 \delta_{j,2} \delta_{n,(l-1)}\,. \tag{3.77}$$

The *moduli* of these integrals are easily found to be

$$a = k^2 \quad \text{and} \quad \alpha(k) = q\,, \quad \alpha(e) = 0\,, \tag{3.78}$$

$$\text{so that} \qquad \omega = q^2\,. \tag{3.79}$$

Since we assumed l to be odd we can choose as generators of the dual algebra B_q^* the modulus α and the linear form f defined by $\langle f, e^n k^j \rangle = \delta_{n,1}$. The following relations together with those in (3.75) can be used as a definition for the double $D(B_q)$ containing B_q and B_q^* with opposite comultiplication:

$$\begin{aligned} &\alpha f \alpha^{-1} = q^2 f\,, && \alpha^l = 1\,, && f^l = 0\,, \\ &\alpha e \alpha^{-1} = q^{-2} e\,, && k f k^{-1} = q^{-1} f\,, && \\ &ef - fe = \alpha - k^2\,, && && \\ &\Delta(\alpha) = \alpha \otimes \alpha\,, && \Delta(f) = f \otimes \alpha + 1 \otimes f\,. && \end{aligned} \tag{3.80}$$

We shall sometimes refer to the *$\mathbb{Z}$-gradation* of $D(B_q)$ which is defined on the generators by $gr(e) = +1$, $gr(k) = 0$, $gr(f) = -1$, and $gr(\alpha) = 0$. The universal $\mathcal{R}$-matrix of this algebra is

$$\mathcal{R} = \left(\sum_{n=0}^{l-1} \frac{q^{-\frac{n(n-1)}{2}}}{[n]!} e^n \otimes f^n\right)\left(\frac{1}{l} \sum_{i,j \in \mathbb{Z}/l} q^{-ij} k^j \otimes \alpha^i\right). \tag{3.81}$$

Here $[n]! = [n][n-1] \ldots [1]$ with $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$. If we compute the expressions in (2.23) and (2.24) for the integrals in (3.76) and (3.77) we obtain the following isomorphisms between B_q and B_q^*:

$$\beta_l(k^j e^n) = \frac{q^{-\frac{(l-1-n)(l-2-n)}{2}}}{[l-1-n]!} f^{l-1-n} \frac{1}{l} \sum_{i \in \mathbb{Z}/l} q^{i(j-2)} \alpha^{i+1}\,, \tag{3.82}$$

and

$$\bar{\beta}_l(\alpha^i f^n) = (-1)^n [n]!\, q^{-\frac{n(n+3)}{2} - 2i} e^{l-1-n} \sum_{j \in \mathbb{Z}/l} q^{j(i-1)} k^j\,. \tag{3.83}$$

As an associative algebra $D(B_q)$ is isomorphic to the product $\mathbb{C}[\mathbb{Z}/l] \otimes U_q(sl_2)$, where the central group algebra $\mathbb{C}[\mathbb{Z}/l]$ is generated by

$$z := \alpha^{-m} k \, . \tag{3.84}$$

The generators of the $U_q(sl_2)$ factor are defined by

$$E := z^{-1} e, \quad F := -f, \quad K := \alpha^m k \tag{3.85}$$

and obey the relations

$$KEK^{-1} = q^2 E \, , \quad KFK^{-1} = q^{-2} F \, , \tag{3.86}$$

$$EF - FE = K - K^{-1} \, . \tag{3.87}$$

2. The Center of $D(B_q)$. Thanks to the above decomposition the center of $D(B_q)$ is given by $\mathbb{C}[\mathbb{Z}/l] \otimes \mathscr{V}$, where $\mathscr{V}$ is the center of $U_q(sl_2)$. In order to give a description of $\mathscr{V}$ it is convenient to introduce the projections

$$\pi_j(K) = \frac{1}{l} \sum_{i \in \mathbb{Z}/l} q^{2ij} K^i \quad j \in \mathbb{Z}/l \tag{3.88}$$

on the eigenspaces of K with eigenvalue q^{-2j}. Furthermore we introduce the projections

$$T_j = \sum_{s=j+1}^{l-1-j} \pi_s(K) \quad j = 0, \ldots, m-1 \, . \tag{3.89}$$

The standard quadratic Casimir of $U_q(sl_2)$ is given by:

$$X = EF + \frac{qK^{-1} + q^{-1}K}{q - q^{-1}} \, . \tag{3.90}$$

The trivially graded part U^0 of $U_q(sl_2)$ ($\mathrm{gr(E)} = 1, \mathrm{gr(F)} = -1, \mathrm{gr(K)} = 0$) is a free module over the ring $\mathbb{C}[K]$ with basis $\{X^j\}_{j=0,\ldots,l-1}$ and the minimal equation for X is:

$$\prod_{j=0}^{l-1} (X - b(j)) = 0 \, , \tag{3.91}$$

where the roots

$$b(j) = b(l-1-j) := \frac{q^{(2j+1)} + q^{-(2j+1)}}{q - q^{-1}} \tag{3.92}$$

are of order two for $j = 0, \ldots, (m-1)$ and of order one for $j = m$.

Using the polynomials

$$\phi_j(X) = \prod_{0 \leqq s \leqq (l-1) : b(s) \neq b(j)} (X - b(s)) \quad j = 0, \ldots, m \tag{3.93}$$

of order $(l-2)$ for $j < m$ and of order $(l-1)$ for $j = m$ we can define the idempotents and nilpotents associated to X:

$$\begin{aligned} P_j &= \frac{1}{\phi_j(b(j))} \phi_j(X) - \frac{\phi_j'(b(j))}{\phi_j(b(j))^2} (X - b(j)) \phi_j(X) \, , \quad && j = 0, \ldots, m \, , \\ N_j &= \frac{1}{\phi_j(b(j))} (X - b(j)) \phi_j(X) \, , && j = 0, \ldots, m-1 \, , \\ N_j^+ &= T_j N_j \, , \quad N_j^- = (1 - T_j) N_j \, . \end{aligned} \tag{3.94}$$

For example a general polynomial $\Psi(X)$ in X is expressed in terms of P_j and N_j by the formula:

$$\Psi(X) = \sum_{j=0}^{m} \Psi(b(j))P_j + \sum_{j=0}^{m-1} \Psi'(b(j))N_j \,. \tag{3.95}$$

The normalizations in (3.94) can be evaluated explicitly using

$$\begin{aligned} \phi_j(b(j)) &= \frac{l^2}{(q-q^{-1})^l} \frac{1}{[d_j^{\pm}]^2} \,, \\ \phi_j'(b(j)) &= -\frac{l^2}{(q-q^{-1})^{(l+1)}} \frac{[2d_j^{\pm}]}{[d_j^{\pm}]^5} \quad \text{for } j = 0, \ldots, (m-1) \,, \\ \phi_m(b(m)) &= \frac{l^2}{(q-q^{-1})^{(l-1)}} \,. \end{aligned} \tag{3.96}$$

The center of the quantum algebra does not only contain the subalgebra generated by X but also the above combinations of nilpotents with the weight-projectors T_j. More precisely, we have the following lemma:

Lemma 14. *The center, denoted by $\mathscr{V}$ of $U_q(sl_2)$ is the* $(3m+1)$*-dimensional algebra with basis* $\{P_i, N_j^{\pm} : i = 0, \ldots, m; j = 0, \ldots, m-1\}$ *and products*:

$$\begin{aligned} P_i P_j &= \delta_{ij} P_j \,, \\ P_i N_j^{\pm} &= \delta_{ij} N_j^{\pm} \,, \\ N_i^{\pm} N_j^{\pm} &= N_i^{\pm} N_j^{\mp} = 0 \,. \end{aligned} \tag{3.97}$$

Proof. We use the fact that every element y in the trivially graded part U_o has a unique presentation:

$$y = \sum_{s \in \mathbb{Z}/l} \pi_s(K) p_s(X) \,,$$

where the p_s are polynomials of order smaller than l. The condition that y commutes with E is then:

$$\sum_{s \in \mathbb{Z}/l} \pi_s(K)(p_s(X) - p_{s-1}(X)) \in \mathscr{I} \,.$$

Here we denote the ideal $\mathscr{I} = \{y \in U^o : Ey = 0\}$. It is clear that $\mathscr{I}$ is generated by

$$E^{l-1}F^{l-1} = \sum_{s \in \mathbb{Z}/l} \pi_{s+1}(K) \prod_{j \in \mathbb{Z}/l,\, j \neq s} (X - b(j)) \,.$$

The polynomials in X occurring in this sum are proportional to the nilpotents and idempotents defined in (3.94). The ideal $\mathscr{I}$ is therefore spanned by the elements

$$\pi_{s+1}(K) N_s \,, \quad s = 0, \ldots, m-1 \text{ and } \quad \pi_{m+1} P_m \,.$$

Solving the recursion for the p_j's we find that the center is generated by the elements in (3.94). The commutation with F yields exactly the same conditions. Linear independence of these generators can be shown by choosing special representation of X. □

3. Canonical Elements and Balancing. The canonical, group-like element, g, from (2.20) implementing the fourth order of the antipode is obtained from the equations for the moduli:

$$g = \alpha k^{-2} = K^{-2} \,. \tag{3.98}$$

For odd l this element has precisely one square root in the group-like elements,

$$\sqrt{g} = K^{-1} \,, \tag{3.99}$$

so that we have uniqueness of balancing. The fourth root of one associated to this by (2.51) is

$$v = q^{-m} \,. \tag{3.100}$$

The canonical element $u = S(R^{(2)})R^{(1)}$ can be expressed as the following product of commuting elements

$$u = u_Z u_K u_o \,, \tag{3.101}$$

where

$$u_Z = \frac{\gamma_q}{\sqrt{l}} \sum_{i \in \mathbb{Z}/l} q^{-mi^2} z^i \,, \quad u_K = \frac{\gamma_q^{-1}}{\sqrt{l}} \sum_{i \in \mathbb{Z}/l} q^{mi^2} K^i \,,$$

and

$$u_o = \sum_{n=0}^{l-1} \frac{q^{\frac{n(n+3)}{2}}}{[n]!} K^n F^n E^n \,.$$

Here we denote the Gauss sum $\gamma_q := \frac{1}{\sqrt{l}} \sum_{j=0}^{l-1} q^{mj^2}$, which is a phase for odd l and can be evaluated explicitly (see e.g. [L]). The unique ribbon element v can be written as a product of an element in the $\mathbb{C}[\mathbb{Z}/l]$ factor and an element in the $U_q(sl_2)$-factor of the algebra:

$$v = u_Z v_o \,,$$

where

$$v_o = K u_K u_o \,. \tag{3.102}$$

If we denote by $\mathcal{T}_Z$, $\mathcal{T}_o$ and $\mathcal{T}$ the linear operators on $\mathbb{C}[\mathbb{Z}/l]$, $U_q(sl_2)$ and $D(B_q)$ defined by multiplication with u_Z, v_o and v respectively this implies

$$\mathcal{T} = \mathcal{T}_Z \otimes \mathcal{T}_o \,. \tag{3.103}$$

We have the following expression for the central element v_o in terms of the basis given in Lemma 14:

Lemma 15. *The central ribbon element $v_o \in U_q(sl_2)$ has the Jordan decomposition*

$$v_o = q^m P_m + \sum_{j=0}^{m-1} q^{2j(j+1)} \left(P_j + \frac{d_j^+}{[d_j^+]} N_j^+ + \frac{d_j^-}{[d_j^-]} N_j^- \right) . \tag{3.104}$$

Here the basis elements of $\mathcal{V}$ are the same as in (3.94) *and the numbers $d_j^\pm = 1, \ldots, l-1$, are defined for $j = 0, \ldots, m-1$ by*

$$d_j^+ := 2j + 1 \quad d_j^- := l - (2j + 1) \,.$$

Proof. The computation of these coefficients is most conveniently done by multiplying the expression for v_o obtained from (3.101) by a weight projector $\pi_s(K)$. The result can be expressed in terms of a polynomial Ψ_s of the quadratic Casimir X:

$$\pi_s(K) v_o = \pi_s(K) \Psi_s(X) \,, \tag{3.105}$$

where

$$\Psi_s(X) = \sum_{n=0}^{l-1} \frac{q^{\frac{n(n+3)}{2}}}{[n]!} q^{2a(a-n-1)} \prod_{i=l-n}^{l-1} (X - b(i+s)) .$$

From the general expansion (3.95) we see that the coefficient of P_j is given by $\Psi_s(b(j))$ for any s and the coefficients of N_j^+ and N_j^- are given by $\Psi_s'(b(j))$, where $s = j+1, \ldots, l-1-j$ and $s = -j, \ldots, j$ respectively. For a choice of s with $b(s-1) = b(j)$, we can avoid one summation in the expressions for Ψ_s and Ψ_s'. In order to evaluate the remaining sum for Ψ' we invoke the partition identity for t with $t^i \neq 1$ for $i = 1, \ldots, d$:

$$\frac{d}{1-t^d} = \sum_{n \geqq 1} \frac{1}{1-t^n} \prod_{i=1}^{n-1} (1 - t^{(d-i)}) . \tag{3.106}$$

Remark. From the observation that the coefficients should be independent of the choice of the weight s we are led to new partition identities. For example in the computation of Ψ_s we find the formula:

$$t^{AB} = \sum_{n=0}^{min(A,B)} \prod_{i=0}^{n-1} \frac{(t^A - t^i)(t^B - t^i)}{t^i(t^{(i+1)} - 1)} .$$

4. The SL(2,$\mathbb{Z}$)-Action on the Center of $D(B_q)$. We use the formula obtained in (9) to give the explicit action of $\mathcal{S}$ on $D(B_q)$. Together with $\mathcal{T}$ defined by multiplication with the ribbon element this yields a representation of the mapping class group $\mathcal{D}$ on $D(B_q)$. If we insert the expressions for the integrals from (3.82) and (3.83) the action of $\mathcal{S}$ can be immediately written if we use both PBW bases $k^j e^n f^p \alpha^s$ and $\alpha^s f^n e^p k^j$ as:

$$\mathcal{S}(k^j e^n f^p \alpha^s) = \frac{(-1)^n [p]!}{[l-1-n]!} q^{\left(\frac{(n+1)(n+2)}{2} + (n+1)j + \frac{p(p-1)}{2}\right)}$$
$$\times \left(\frac{1}{l} \sum_{i \in \mathbb{Z}/l} q^{-ij} \alpha^i\right) f^{(l-1-n)} e^{(l-1-p)} \left(\sum_{i \in \mathbb{Z}/l} q^{i(s+p)} k^{-i}\right) . \tag{3.107}$$

A similar formula was obtained in [LyM]. It is immediate from the above form that the $\mathcal{S}$ matrix preserves the gradation $n - p \in \mathbb{Z}$ of a basis element. Given that the balancing element is trivially graded and acts by multiplication it follows that the $\mathcal{D}$-representation on $D(B_q)$ decomposes into a direct sum of the $2l-1$ spaces corresponding each gradation.

Clearly, the category from which $\mathcal{S}$ is obtained is the tensor product of the representation category of $U_q(sl_2)$ and $\mathbb{C}[\mathbb{Z}/l]$ as an abelian category. Also, since the balancing element and hence the monodromy can be factorized into a product of invertible elements from either algebra, the $\mathcal{S}$-matrix has to factorize too. More precisely we define the following isomorphisms on $C[\mathbb{Z}/l(z)]$:

$$\mathcal{S}_Z(z^n) := \frac{1}{\sqrt{l}} \sum_{j \in \mathbb{Z}/l} q^{-jn} z^j , \tag{3.108}$$

and on $U_q(sl_2)$

$$\mathcal{S}_0(K^j E^n F^p) := \frac{(-1)^p [p]!}{[l-1-n]!} q^{\left(\frac{(n-p)(n-p+1)}{2} + j(2n+1-p) + 1\right)}$$
$$\times \left(\frac{1}{\sqrt{l}} \sum_{k \in \mathbb{Z}/l} q^{k(j-n)} K^k\right) F^{(l-1-n)} E^{(l-1-p)} . \tag{3.109}$$

Using the isomorphism $D(B_q) \cong C[\mathbb{Z}/l] \otimes U_q(sl_2)$ defined by the change of basis in (3.84) and (3.85) we can now write the $\mathscr{S}$-matrix in the form:

$$\mathscr{S} = \mathscr{S}_Z \otimes \mathscr{S}_o \,. \tag{3.110}$$

Together with (3.103) this shows that the representation of $\mathscr{D}$ on $D(B_q)$ is given by the tensorproduct of two projective representations of $\mathscr{D}$. Since $C[\mathbb{Z}/l]$ is central in $D(B_q)$ we expect the representation generated by $\mathscr{T}_Z$ and $\mathscr{S}_Z$ to factor through a projective representation of $SL(2, \mathbb{Z})$. In fact we easily verify the following relations:

$$\mathscr{S}_Z^2 \mathscr{T}_Z = \mathscr{T}_Z \mathscr{S}_Z^2 \,, \quad \mathscr{S}_Z^2(z^n) = z^{-n} \,, \quad (\mathscr{S}_Z \mathscr{T}_Z)^3 = \gamma_q \mathbb{1} \,. \tag{3.111}$$

It is clear that the action of $\mathscr{S}_o$ on $U_q(sl_2)$ preserves the gradation in the same way as the action of $\mathscr{S}$ on $D(B_q)$. For example the restriction on the highest $l-1$ graded subspace defines for each $g_0 \in \mathscr{D}$ by

$$g_o(aKE^{l-1}) = \hat{g}(a)KE^{l-1}$$

an action $\hat{g}$ on an element a in the group algebra $\mathbb{C}[\mathbb{Z}/l]$ generated by K. It factors into an $SL(2, \mathbb{Z})$ representation and is equivalent to the one defined previously by $\mathscr{T}_Z$ and $\mathscr{S}_Z$ with q replaced by q^{-1}.

In the following we shall focus on the 0-graded part U_o of $U_q(sl_2)$ from where we wish to compute the restrictions to the center. We determine explicitly the $(3m+1)$-dimensional representation matrices of $SL(2, \mathbb{Z})$ which we obtain by restricting the action of $\mathscr{D}$ onto the center $\mathscr{V}$ of $U_q(sl_2)$. We choose the basis as in Lemma 14 in the order $P_0, N_0^+, N_0^-, P_1, \ldots, N_{(m-1)}^-, P_m$. On the subspace spanned by P_j, N_j^+, N_j^- we define the Jordan block:

$$\tau_j := q^{2j(j+1)} \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{d_j^+}{[d_j^+]} & 1 & 0 \\ \dfrac{d_j^-}{[d_j^-]} & 0 & 1 \end{bmatrix} \qquad \text{for } j = 0, \ldots, (m-1) \,.$$

Then it is obvious from the formula in Lemma 15 that the $\mathscr{T}_o$ matrix defined by multiplication of v_o is given by the direct sum:

$$\mathscr{T}_o = \tau_0 \oplus \tau_1 \oplus \ldots \oplus \tau_{(m-1)} \oplus q^m \,. \tag{3.112}$$

The restriction of $\mathscr{S}_o$ to the center is much more complicated and will be dealt with in the rest of this section. It involves finding transformations from the idempotents and nilpotents given in Lemma 14 to polynomials in X and K, to the standard PBW basis of $U_q(sl_2)$ and backwards. The transformations between the center and expressions in X and K can be obtained from the relations given in (3.94) and (3.95). In order to reexpress polynomials in X and K in terms of the basis $K^j E^n F^n$ and conversely, we need the following two technical lemmas. A special case of the first lemma we used already in the computation of the center. The proof is straightforward.

Lemma 16. *Let X be the quadratic Casimir defined in* (3.90) *and set*

$$Q_j = \frac{q^{(2j+1)}K^{-1} + q^{-(2j+1)}K}{q - q^{-1}}, \tag{3.113}$$

then the following relations hold:

$$E^j F^j = \prod_{s=0}^{j-1} (X - Q_s), \tag{3.114}$$

$$F^j E^j = \prod_{s=l-j}^{l-1} (X - Q_s). \tag{3.115}$$

Before we give the converse transformations let us state the following identity for general polynomials

Lemma 17. *Suppose $\lambda_0, \ldots, \lambda_N$ is an ordered set of roots and $0 \leqq a_1 < a_2 < \cdots < a_k \leqq N$ are an ordered set of k indices then we have*

$$\prod_{j=0, j \notin \{a_j\}}^{N} (X - \lambda_j) = \sum_{0 \leqq s_1 < s_2 < \ldots < s_k \leqq N} \prod_{i=0}^{s_1 - 1} (X - \lambda_i) \times \prod_{i=s_1+1}^{s_2-1} (\lambda_{a_1} - \lambda_i) \ldots \prod_{i=s_k+1}^{N} (\lambda_{a_k} - \lambda_i), \tag{3.116}$$

reexpressing a polynomial with omitted roots in terms of polynomials with consecutive roots.

Here an empty product is meant to be 1. The proof is a straightforward induction which is most conveniently done by assuming the statement for all (k, N') with $k' < k$ or $k' = k$ and $N' \leqq N$ and proving it for $k' = k$ and $N' = N + 1$ thus for all pairs with $k' = k$. This is followed by an induction in k. If we combine Lemma 16 and Lemma 17 we arrive at the following formula for the polynomials defined in (3.93)

Lemma 18. *Let $\phi_k(X)$ be the polynomials in the quadratic Casimir X as defined in* (3.93), *$\pi_s(K)$ the projector from* (3.88) *and $b(j)$ as in* (3.92). *Then*

$$\pi_t(K)(X - b(k))\phi_k(X) = \sum_{j=0}^{l-1} \prod_{i=j+1}^{l-1} (b(k) - b(i+t))\pi_t(K)E^j F^j, \tag{3.117}$$

$$\pi_t(K)\phi_m(X) = \sum_{j=0}^{l-1} \prod_{i=j+1}^{l-1} (b(m) - b(i+t))\pi_t(K)E^j F^j, \tag{3.118}$$

$$\pi_t(K)\phi_k(X) = \sum_{j=0}^{l-2} \prod_{s=j+1}^{l-1} \prod_{\substack{i=j+1 \\ i \neq s}}^{l-1} (b(j) - b(i+t))\pi_t(K)E^j F^j. \tag{3.119}$$

Proof. We apply Lemma 17 to the situation where $N = l - 1$, X is the quadratic Casimir and the roots λ_j are replaced by the elements Q_j defined in (3.113). The polynomials with consecutive roots on the right-hand side (3.116) are precisely those in (3.114). Thus for $k = 1, 2$ we obtain the specializations:

$$\prod_{\substack{j=0 \\ j \neq a}}^{l-1} (X - Q_j) = \sum_{j=0}^{l-1} \prod_{s=j+1}^{l-1} (Q_a - Q_s)E^j F^j, \tag{3.120}$$

and

$$\prod_{\substack{j=0\\ j\neq a,b}}^{l-1}(X-Q_j)=\sum_{j=0}^{l-1}\sum_{s=j+1}^{l}\prod_{i=j+1}^{s-1}(Q_p-Q_i)\prod_{i=s+1}^{l}(Q_b-Q_i)E^jF^j\,. \tag{3.121}$$

Notice that the polynomials $\phi_k(X)$ and $(X-b(k))\phi_k(X)$ are obtained from (3.91) by omitting one or two roots. If we multiply Eq. (3.120) and (3.121) with the projector $\pi_t(K)$ for suitable choices of a and b we obtain these polynomials on the left-hand side. The identities (3.117)–(3.119) follow using $\pi_t(K)Q_j = \pi_t(K)b(j+t)$. □

Lemma 18 puts us now in the position to determine the action of $\mathscr{S}_o$ on the polynomials $\phi_k(X)$ and $(X-b(k))\phi_k(X)$.

Insertion into (3.109) yields for $k=0,\ldots,(m-1)$:

$$\mathscr{S}_o(\pi_t(K)(X-b(k))\phi_k(X))=\sum_{b\in\mathbb{Z}/l}\pi_b(K)\Delta_b^{tk}(X)\,, \tag{3.122}$$

$$\mathscr{S}_o(\pi_t(K)\phi_k(X))=\sum_{b\in\mathbb{Z}/l}\pi_b(K)\Gamma_b^{tk}(X)\,, \tag{3.123}$$

$$\mathscr{S}_o(\pi_t(K)\phi_m(X))=\sum_{b\in\mathbb{Z}/l}\pi_b(K)\Delta_b^{tm}(X)\,. \tag{3.124}$$

The polynomials $\Delta_b^{tk}(X)$ and $\Gamma_b^{tk}(X)$ are defined by

$$\Delta_b^{tk}(X):=\frac{q}{\sqrt{l}}\sum_{j=0}^{l-1}\frac{(-1)^j[j]!}{[l-1-j]!}q^{(j+2b)(j+2t+1)}\prod_{i=j+1}^{l-1}(b(k)-b(i+t))$$
$$\times\prod_{i=j+1}^{l-1}(X-b(i+b))\,, \tag{3.125}$$

and

$$\Gamma_b^{tk}(X):=\frac{q}{\sqrt{l}}\sum_{j=0}^{l-2}\sum_{s=j+1}^{l-2}\frac{(-1)^j[j]!}{[l-1-j]!}q^{(j+2b)(j+2t+1)}$$
$$\times\prod_{\substack{i=j+1\\ i\neq s}}^{l-1}(b(k)-b(i+t))\prod_{i=j+1}^{l-1}(X-b(i+b))\,. \tag{3.126}$$

The action of $\mathscr{S}_o$ on $\mathscr{V}$ is now easily obtained by summing Eqs. (3.122) to (3.124) over an appropriate range of t and combining them in (3.94). The result contains products of projections $\pi_b(X)$ with polynomials of the form $\sum_t\Delta_b^{kt}(X)$ and $\sum_t\Gamma_b^{kt}(X)$. The polynomials can be expanded for every weight b into the basis $\mathscr{V}$ defined in Lemma 14; the coefficients of P_p and N_p are obtained by the values of the derivatives of these polynomials at $X=b(p)$.

By general construction the $\mathscr{S}_o$-matrix has to map $\mathscr{V}$ to itself. As in the remark following Lemma 15 this can be used to produce new families of partition identities.

In order to find the matrix coefficients of the $SL(2,\mathbb{Z})$ representation we need the following quantities:

$$\eta(d_A,d_B):=$$
$$[d_A]^2\frac{(q-q^{-1})^l}{l^2}\sum_{j=1}^{min(d_B,d_A-1)}\sum_{s=j+1}^{d_A}[l-1-j]!\frac{(-1)^j}{[j]}q^{(j-d_B)(j+1+d_A-2s)}$$
$$\times\prod_{i=1}^{j}(q-q^{-1})[s-i][d_A-s+i]\prod_{i=1}^{j-1}(q^{(d_B-i)}-q^{-(d_B-i)})\,, \tag{3.127}$$

$$\mu(d_A,d_B):=$$
$$[d_A]^2\frac{(q-q^{-1})^l}{l^2}\sum_{s\in\mathbb{Z}/l}\sum_{j=1}^{d_B}\sum_{r=1}^{j}[l-1-j]!\frac{(-1)^j}{[j]}q^{(j-d_B)(j+1+d_A-2s)}$$
$$\times\prod_{\substack{i=1\\ i\neq r}}^{j}(q-q^{-1})[s-i][d_A-s+i]\prod_{i=1}^{j-1}(q^{(d_B-i)}-q^{-(d_B-i)})\,, \tag{3.128}$$

and

$$\rho(d_B):=\frac{(q-q^{-1})^{l-1}}{l^2}\sum_{j=1}^{d_B}\sum_{s=j+1}^{l}[l-1-j]!\frac{q^{(j-d_B)(j+1-2s)}}{[j]}$$
$$\times\prod_{i=1}^{j}(q-q^{-1})[s-i]^2\prod_{i=1}^{j-1}(q^{(d_B-i)}-q^{-(d_B-i)})\,. \tag{3.129}$$

The main result of the previous calculation – the $SL(2,\mathbb{Z})$ representation on $\mathscr{V}$ – is described in the next theorem:

Theorem 2. *Let $P_0, N_0^+, N_0^-, P_1, \ldots, N_{(m-1)}, P_m$ be the ordered basis of $\mathscr{V}$ as defined in Lemma* 14. *Then the following matrices define a projective $SL(2,\mathbb{Z})$ representation.*
The $\mathscr{T}_o$ matrix is given by (3.112).
The $\mathscr{S}_o$ matrix is given by the following formulae:

1. *For $k=0,\ldots,m-1$*

$$\mathscr{S}_o(N_k^{\mp})=\frac{q}{\sqrt{l}}\frac{q-q^{-1}}{l}[d_k^{\pm}]^2\sum_{p=0}^{m-1}\frac{[d_k^{\pm}d_p^*]}{[d_p^*]}P_p$$
$$+\frac{q}{\sqrt{l}}[d_k^{\pm}]^2d_k^{\pm}P_m+\frac{q}{\sqrt{l}}\sum_{p=0}^{m-1}\sum_{\varepsilon=\pm}\eta(d_k^{\pm},d_p^{\varepsilon})N_p^{\varepsilon}\,.$$

2. *For $k=0,\ldots,m-1$*

$$\mathscr{S}_o(P_k)=\frac{q}{\sqrt{l}}\frac{[2d_k^*]}{[d_k^*]}P_m$$
$$+\frac{q}{\sqrt{l}}\sum_{p=0}^{m-1}\sum_{\varepsilon=\pm}\left(\mu(d_k^*,d_p^{\varepsilon})+\frac{[2d_k^*]}{[d_k^*]^3}\frac{\eta(d_k^+,d_p^{\varepsilon})+\eta(d_k^-,d_p^{\varepsilon})}{(q-q^{-1})}\right)N_p^{\varepsilon}\,.$$

3.

$$\mathscr{S}_o(P_m)=\frac{q}{\sqrt{l}}P_m+\frac{q}{\sqrt{l}}\sum_{p=0}^{m-1}\sum_{\varepsilon=\pm}\rho(d_p^{\varepsilon})N_p^{\varepsilon}\,.$$

These matrices satisfy the relations

$$(\mathcal{S}_o\mathcal{T}_o)^3 = \gamma_q^{-1} q^{(1-m)} \mathbb{1} \, ,$$
$$(\mathcal{S}_o)^2 = q^2 \mathbb{1} \, .$$

Here the superscript $*$ means that either $+$ or $-$ can be inserted yielding the same result.

5. The structure of the $SL(2,\mathbb{Z})$-Representation on $\mathcal{V}$. For small values of l the following polynomial identities hold true:

$$\eta(d_k^+, d_p^+) + \eta(d_k^-, d_p^+) = \eta(d_k^+, d_p^-) + \eta(d_k^-, d_p^-) \text{ and } \rho(d_p^+) = \rho(d_p^-) \, . \qquad (3.130)$$

In this case it is easy to see that the representation contains an $m+1$-dimensional subrepresentation spanned by the N_j's and P_m.

On this subspace the $\mathcal{T}$-matrix is diagonal and has eigenvalues $\{q^{2j(j+1)}, j = 0, \ldots, m\}$, i.e., one more than the finite m-dimensional representation obtained from the semisimplified representation category. For prime l it is not hard to see that the representation is irreducible. Also, for small l we find that it is finite.

It is clear by inspection of the $\mathcal{T}$-matrix that a complement to this representation has to contain the linearly independent nilpotents $\tilde{N}_j = \sum_{\varepsilon=\pm} \frac{d_j^\varepsilon}{[d_j^\varepsilon]} N_j^\varepsilon$. Thus it also contains the vectors $\mathcal{S}_o(\tilde{N}_j)_k$, where the subindex $k = 0, \ldots, m-1$ means that we take the component in the k^{th} eigenspace of $\mathcal{T}$. Since the elements $\mathcal{S}_0(\tilde{N}_0)_k$ are linearly independent from the nilpotents, a $2m$-dimensional component exists only if

$$\mathcal{S}_o(\tilde{N}_j)_k = c_{jk}\mathcal{S}_o(\tilde{N}_o)_k + b_{jk}\tilde{N}_k \, , \qquad (3.131)$$

for some coefficients c_{jk} and b_{jk}. Comparison of the coefficients of the idempotents shows that we need $c_{jk} = \frac{d_j}{[d_k]}[d_j d_k]$, i.e., the $m \times m$-matrix c defined by these coefficients is equivalent to the $\mathcal{S}$-matrix of the semisimple TQFT. We can write polynomial identities similar to (3.130) which are equivalent to (3.131) with $b_{jk} = 0$. Again, for small values of l we know that they hold true. Using that $\mathcal{S}^2$ is proportional to the identity they also imply that $\mathcal{S}$ decomposes into a tensor product $b \otimes c$, where b is a two by two matrix with vanishing diagonal elements. The $\mathcal{T}$-matrix on the second summand has eigenvalues $\{q^{2j(j+1)}, j = 0, \ldots, (m-1)\}$ all of which are doubly degenerate, with non-trivial Jordan-block. For a suitable normalization we thus expect the second summand to be the tensorproduct of the two dimensional standard representation and the known m-dimensional finite representation.

In a TQFT $\widetilde{SL(2,\mathbb{Z})}$ extends to representations of modular groups at higher genus. If these factor through their actions on the homology of the surface the projective $SL(2,\mathbb{Z})$-representation extends to representations of a higher symplectic group. It is a fact that for congruence groups at the higher symplectic groups over $\mathbb{Z}$ any irreducible representation is the tensorproduct of a finite and an algebraic representation, see [Kz]. Thus it is likely that the tensorproduct presentation described in the previous paragraph can also be inferred from rather general arguments.

We summarize our observations in the following conjecture. In the next section we show that it holds true for the five and seven dimensional representation.

Conjecture 1. *The projective, $3m+1$-dimensional $SL(2,\mathbb{Z})$ representation defined in Theorem 2 decomposes as*

$$\mathcal{V} = \mathcal{V}_N \oplus \mathcal{V}_{stan} \otimes \mathcal{V}_{semis} \, ,$$

where

1. $\mathscr{V}_N$ *is an* $(m+1)$*-dimensional, irreducible, finite representation spanned by* $N_j = N_j^+ + N_j^-$ *and* P_m, *see* (3.134) *or* (3.138).
2. $\mathscr{V}_S$ *is the 2m-dimensional subrepresentation spanned by*

$$\tilde{N}_j = \sum_{\varepsilon = \pm} \frac{d_j^\varepsilon}{[d_j^\varepsilon]} N_j^\varepsilon ,$$

and the j^{th} $\mathscr{T}$*-eigenspace components*

$$(\mathscr{S}_o(\tilde{N}_0))_j .$$

This representation is the tensorproduct $\mathscr{V}_S = \mathscr{V}_{stan} \otimes \mathscr{V}_{semis}$ *of*

(a) *the two dimensional, algebraic standard representation* $\mathscr{V}_{stan}$ *as in* (3.135) *or* (3.140) *and*

(b) *an m-dimensional finite representation* $\mathscr{V}_{semis}$ *which is isomorphic–up to a projective phase-to the* $SL(2,\mathbb{Z})$ *representation obtained from the semisimple subquotient category, see for example* (3.141).

6. The Examples $l = 3, 5$. In this section we verify the conjecture of the previous section for $l = 3$ and $l = 5$. We compute the explicit representation matrices of the various finite representation:

For $l = 3$ the matrices of the $SL(2,\mathbb{Z})$ are given in the basis P_0, N_0^+, N_0^-, P_1 by:

$$\mathscr{T}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \tag{3.132}$$

$$\mathscr{S}_0 = \frac{q}{\sqrt{3}} \begin{bmatrix} 0 & -\frac{1}{3}(q-q^{-1}) & \frac{1}{3}(q-q^{-1}) & 0 \\ \frac{2}{3}(q-q^{-1}) & -1 & 0 & -\frac{2}{3}(q-q^{-1}) \\ -\frac{7}{3}(q-q^{-1}) & -1 & 0 & -\frac{2}{3}(q-q^{-1}) \\ -1 & \frac{2}{3}(q-q^{-1}) & \frac{1}{3}(q-q^{-1}) & 1 \end{bmatrix}, \tag{3.133}$$

This representation decomposes into the sum of two irreducible, two-dimensional subrepresentations

$$\mathscr{V} = \mathscr{V}_N \oplus \mathscr{V}_S .$$

Here the subspace $\mathscr{V}_N$ is spanned by $N_0 = N_0^+ + N_0^-$ and P_1 with $\mathscr{S}$ and $\mathscr{T}$ acting as:

$$\mathscr{T}_N = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}, \quad \mathscr{S}_N = \frac{q}{\sqrt{3}} \begin{bmatrix} -1 & -\frac{2}{3}(q-q^{-1}) \\ (q-q^{-1}) & 1 \end{bmatrix}. \tag{3.134}$$

This subrepresentation $\mathscr{V}_S$ has basis vectors $\tilde{P}_o = P_o + \dfrac{1}{(q-q^{-1})} N_o$ and $\widetilde{N_o} := N_o^+ - 2N_o^-$ for which the $\mathscr{S}$ and $\mathscr{T}$ matrix have the form of the standard representation:

$$\mathscr{T}_S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathscr{S}_S = q\gamma_q \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.135}$$

For $l = 5$ the matrices are given for the basis $P_0, N_0^+, N_0^-, P_1\, N_1^+, N_1^-, P_2$:

$$\mathscr{T}_0 = \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 \\ & & & \vdots & & & & \vdots & 0 \\ 1 & 1 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 \\ -4 & 0 & 1 & \vdots & 0 & 0 & 0 & \vdots & 0 \\ \cdots\ \cdots & \cdots & \cdots\ \cdots & \vdots & \cdots\ \cdots & \cdots\ \cdots & \cdots & \vdots & \cdots\ \cdots \\ 0 & 0 & 0 & \vdots & q^{-1} & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & -q^{-1}\frac{3}{[2]} & q^{-1} & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & q^{-1}\frac{2}{[2]} & 0 & q^{-1} & \vdots & 0 \\ \cdots\ \cdots & \cdots & \cdots\ \cdots & \vdots & \cdots\ \cdots & \cdots\ \cdots & \cdots & \vdots & \cdots\ \cdots \\ 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}, \quad (3.136)$$

$$\mathscr{S}_0 = \frac{q}{\sqrt{5}} \begin{bmatrix} 0 & -\xi & \xi & \vdots & 0 & \xi(-1+2[2]) & \xi(1-2[2]) & \vdots & 0 \\ \xi(2+4[2]) & [2] & 0 & \vdots & \xi(2-6[2]) & \frac{1}{5}(-2+[2]) & \frac{1}{5}(2-6[2]) & \vdots & \xi(-4+2[2]) \\ \xi(4[2]-23) & [2] & 0 & \vdots & \xi(-23+19[2]) & \frac{1}{5}(-2+[2]) & \frac{1}{5}(2-6[2]) & \vdots & \xi(-4+2[2]) \\ \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \vdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \vdots & \cdots\ \cdots \\ 0 & -\xi & \xi & \vdots & 0 & -\xi[2] & \xi[2] & \vdots & 0 \\ -\xi(12+14[2]) & -3-2[2] & 0 & \vdots & \xi(18+16[2]) & \frac{1}{5}(-2+[2]) & \frac{1}{5}(2+4[2]) & \vdots & -\xi(6+2[2]) \\ \xi(13+11[2]) & -3-2[2] & 0 & \vdots & -\xi(7+9[2]) & \frac{1}{5}(-2+[2]) & \frac{1}{5}(2+4[2]) & \vdots & -\xi(6+2[2]) \\ \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \vdots & \cdots\ \cdots & \cdots\ \cdots & \cdots\ \cdots & \vdots & \cdots\ \cdots \\ [2] & 4\xi & \xi & \vdots & -1-[2] & \xi(2-2[2]) & \xi(3-3[2]) & \vdots & 1 \end{bmatrix} \quad (3.137)$$

where

$$\xi = \frac{q - q^{-1}}{5}\,.$$

This representation decomposes into two irreducible representations $\mathscr{V}_N$ and $\mathscr{V}_S$, with $dim(\mathscr{V}_N) = 3$ and $dim(\mathscr{V}_S) = 4$. The three dimensional representation is spanned by the vectors

$$N_0 = \frac{1}{q - q^{-1}}(N_0^+ + N_0^-) \quad N_1 = \frac{1}{q - q^{-1}}(N_1^+ + N_1^-) \text{ and } P_2\,.$$

The representation matrices are

$$\mathscr{T}_N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & q^2 \end{bmatrix}, \quad \mathscr{S}_N = \frac{q}{\sqrt{5}} \begin{bmatrix} [2] & -[2] & 2 \\ -(3 + 2[2]) & [2] & (4 + 2[2]) \\ 1 & (1 - [2]) & 1 \end{bmatrix}. \tag{3.138}$$

The four dimensional representation is spanned by

$$\tilde{P}_0 := P_0 - [2]N_0 \quad \tilde{P}_1 := \frac{1}{[2]}(P_1 + (3 + 2[2])N_1)$$

and

$$\tilde{N}_0 := (N_0^+ - 4N_0^-) \quad \tilde{N}_1 := \frac{1}{[2]^2}(-3N_1^+ + 2N_1^-)\,.$$

With ordering $\tilde{P}_0, \tilde{N}_0, \tilde{P}_1, \tilde{N}_1$ we find the matrices:

$$\mathscr{T}_S = \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & q^{-1} & q^{-1} \end{array}\right],$$

$$\mathscr{S}_S = \frac{q(q - q^{-1})}{\sqrt{5}} \left[\begin{array}{cc:cc} 0 & -1 & 0 & -[2] \\ 1 & 0 & [2] & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & -[2] & 0 & 1 \\ [2] & 0 & -1 & 0 \end{array}\right]. \tag{3.139}$$

Now it is easy to see that this can be written as a tensorproduct of $SL(2, \mathbb{Z})$ representations:

$$\mathscr{V}_S \cong \mathscr{V}_{stan} \otimes \mathscr{V}_{semis} \,.$$

In order to denote the isomorphism

$$\tilde{P}_i \to v_P \otimes w_i \,, \quad \tilde{N}_i \to v_N \otimes w_i \,,$$

we introduce bases $\{v_P, v_N\}$ and $\{w_0, w_1\}$ of $\mathscr{V}_{stan}$ and $\mathscr{V}_{semis}$ respectively. For these bases we can write

$$\mathscr{T}_N = \mathscr{T}_{stan} \otimes \mathscr{T}_{semis} \,, \quad \mathscr{S}_N = q \mathscr{S}_{stan} \otimes \mathscr{S}_{semis}$$

with

$$\mathscr{T}_{stan} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathscr{S}_{stan} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3.140}$$

and

$$\mathscr{T}_{semis} = \begin{bmatrix} 1 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad \mathscr{S}_{semis} = \frac{(q - q^{-1})}{\sqrt{5}} \begin{bmatrix} 1 & [2] \\ [2] & -1 \end{bmatrix}. \tag{3.141}$$

Conclusion

The results of Chapter 2 show that the construction of a universal TQFT should include two features. One is to avoid degeneracies by considering only doubles. The fact that the projective phases and the proofs of modular relations are most conveniently given in terms of the bilinear forms and moduli defined from the integrals is an indication that this is the correct language also for constructions at higher genus. In view of the gluing operations described in the introduction, the genus one case can in fact be thought of as a basic building block. It should be possible to understand more conceptually the appearance of the finite representation we know from the semisimple theory as a tensorproduct with the standard representation rather than a subrepresentation. In particular it should be interesting to see how the representation on general $D(\mathscr{A})$ is modified if we pass the semisimple quotient of the representation category of $D(\mathscr{A})$ and a possible truncation of the resulting TQFT.

Also, the appearance of algebraic representations is a novel feature of these theories. We expect to find higher dimensional algebraic representations of $SL(2, Z)$ if we start from higher rank quantum groups for which the orders of nilpotencies of central elements will be higher.

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Note added in proof: It appears that for prime l $\mathscr{V}_N \oplus \mathscr{V}_{\mathrm{semis}}$ is just the decomposition of the metaplectic representation of $SL(2, Z)$ on $\mathbb{C}[Z/l]$. I owe this remark to Vanghan Jones.