Commun. Math. Phys. 168, 227 – 247 (1995)

Communications in Mathematical Physics

U (1) Gauge Theory on a Torus

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Received: 8 February 1993/in revised form: 26 April 1994

Abstract: $U(1)$ gauge theory with the Villain action on a cubic lattice approximation of three- and four-dimensional torus is considered. As the lattice spacing approaches zero, provided the coupling constant correspondingly approaches zero, the naturally chosen correlation functions converge to the correlation functions of the **R**-gauge electrodynamics on three- and four-dimensional torus. When the torus radius tends to infinity these correlation functions converge to the correlation functions of the **R**-gauge Euclidean electrodynamics.

1. Introduction

The compact lattice gauge field theory models introduced by K. Wilson [1] preserve the differential geometric structures of the continuum theory. This paper is concerned with the case where the gauge group is $U(1) = \mathbf{R}/2\pi\mathbf{Z}$. Let $h(\theta)$ be a real twice continuously differentiable even periodic function with period 2π. Any such function will be called an energy function. The main examples of interest are the Wilson [1] energy function $h(\theta) = 1 - \cos\theta$ and the Villain [2] energy function

$$\exp\,[-\beta h_\beta(\theta)] = c_\beta \sum_{n=-\infty}^{\infty} \exp\,[-\beta(\theta - 2\pi n)^2/2]\,, \tag{1.1}$$

where $\beta > 0$ and c_β is a constant chosen so that the right-hand side is one for $\theta = 0$.

Let e_i, $i = 1,\ldots,d$ be the standard unit vectors in $\mathbf{R}^d$, and p be a non-negative integer less than d. The p-cells based at $\mathbf{m} \in \mathbf{Z}^d$ are the formal symbols: ($\mathbf{m}$; $e_{i_1},\ldots,e_{i_p}$), where the unit vectors differ from each other.

Let G be one of three abelian groups: **Z**, **R** and $U(1) = \mathbf{R}/2\pi\mathbf{Z}$. A p-cochain with the coefficients in G is a G-valued function on p-cells $f(\mathbf{m}; e_{i_1},\ldots,e_{i_p}) \equiv f_{i_1\ldots i_p}(\mathbf{m})$ which is antisymmetric under the permutations of the indices $i_1,\ldots,i_p$.

★ Supported by the Russian Foundation of Fundamental Researches under Grant 93-011-147

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Let $\Lambda = \{\mathbf{m} \in \mathbf{Z}^d : N_1 \leqq m_i \leqq N_2, i = 1,\dots,d\}$ be a cube in $\mathbf{Z}^d$ for some integers N_1 and N_2. The free boundary conditions are equivalent to setting that $f_{i_1\dots i_p}(\mathbf{m})$ vanishes except for $\{\mathbf{m} \in \mathbf{Z}^d : N_1 \leqq m_i \leqq N_2, i \neq i_1,\dots,i_p; N_1 \leqq m_{i_k} \leqq N_2 - 1, k = 1,\dots,p\}$. Dirichlet boundary conditions correspond to setting that $f_{i_1\dots i_p}(\mathbf{m})$ vanishes except for $\{\mathbf{m} \in \mathbf{Z}^d : N_1 + 1 \leqq m_i \leqq N_2, i \neq i_1,\dots,i_p; N_1 \leqq m_{i_k} \leqq N_2, k = 1,\dots,p\}$. The conditions

$$f_{i_1\dots i_p}(m_1,\dots,m_j + N,\dots,m_d) = f_{i_1\dots i_p}(\mathbf{m}), \tag{1.2}$$

where $N = N_2 - N_1 + 1$, for every $j = 1,\dots,d$ and $\mathbf{m} \in \mathbf{Z}^d$ correspond to the choice of periodic boundary conditions. For the periodic boundary conditions we define the boundary operator

$$(\partial f)_{i_1\dots i_{p-1}}(\mathbf{m}) = \sum_{\varepsilon=0,1}\sum_{i_0=1}^{d}(-1)^{\varepsilon+1} f_{i_0 i_1\dots i_{p-1}}(\mathbf{m} - \varepsilon e_{i_0}) \tag{1.3}$$

and the coboundary operator

$$(\partial^* f)_{i_1\dots i_{p+1}}(\mathbf{m}) = \sum_{\varepsilon=0,1}\sum_{k=1}^{p+1}(-1)^{\varepsilon+k} f_{i_1\dots \widehat{i_k}\dots i_{p+1}}(\mathbf{m} + \varepsilon e_{i_k}). \tag{1.4}$$

For Dirichlet or free boundary conditions we need to modify the definitions (1.3) and (1.4), respectively, in an obvious way.

For the p-cochains with coefficients in $\mathbf{Z}$ and $\mathbf{R}$ the inner product is defined by

$$(f,g) = \sum_{i_1<\dots<i_p}\sum_{\mathbf{m}\in\Lambda} f_{i_1\dots i_p}(\mathbf{m}) g_{i_1\dots i_p}(\mathbf{m}). \tag{1.5}$$

For a smooth differential p-form $f(\mathbf{x}) = \sum_{i_1<\dots<i_p} f_{i_1\dots i_p}(\mathbf{x}) dx^{i_1} \wedge \dots \wedge dx^{i_p}$ on $\mathbf{R}^d$ we introduce two lattice approximations: $(f_a)_{i_1\dots i_p}(\mathbf{m}) = f_{i_1\dots i_p}(a\mathbf{m})$ and

$$(f^a)_{i_1\dots i_p}(\mathbf{m}) = \int_{[0,a]^{\times p}} f_{i_1\dots i_p}\left(a\mathbf{m} + \sum_{k=1}^{p} s_k e_{i_k}\right) d^p s. \tag{1.6}$$

The energy function h and 1-cochain θ on Λ provide two 2-cochains on Λ with real coefficients. These 2-cochains are defined for any indices $i_1 < i_2$ by the following relations $(h(\partial^*\theta))_{i_1 i_2}(\mathbf{m}) \equiv h((\partial^*\theta)_{i_1 i_2}(\mathbf{m}))$, $(h'(\partial^*\theta))_{i_1 i_2}(\mathbf{m}) \equiv h'((\partial^*\theta)_{i_1 i_2}(\mathbf{m}))$. By 1 we denote the 2-cochain $(1)_{i_1 i_2}(\mathbf{m}) = 1$ for any indices $i_1 < i_2$.

The finite volume Gibbs state in a cube $\Lambda \subset \mathbf{Z}^d$, at inverse temperature β and with energy function h is given by

$$\langle F \rangle_{\Lambda,\beta} = Z^{-1}\left[\prod_{\mathbf{m}\in\Lambda}\prod_{i=1,\dots,d}\int_{-\pi}^{\pi} d\theta_i(\mathbf{m})\right] F(\theta) \exp\left[-\beta(h(\partial^*\theta),1)\right]. \tag{1.7}$$

Here θ is a 1-cochain on Λ with coefficients in $U(1)$ and θ satisfies the periodic boundary conditions. For Dirichlet or free boundary conditions some $\theta_i(\mathbf{m})$ are equal to zero and the corresponding integrations in (1.7) are omitted. The measure $d\theta_i(\mathbf{m})$ is Lebesgue measure on $[-\pi,\pi]$. Z is the normalization constant and F is a function of the bond variables $\theta_i(\mathbf{m})$.

Let $\langle\rangle_\beta$ be any translation invariant infinite volume limit Gibbs state. L. Gross [3] proved that for Wilson and Villain energy functions h and for every smooth differential real 1-form j on $\mathbf{R}^3$ the following equality holds:

$$\lim_{a\to 0}\langle \exp\,[i(h'(\partial^*\theta),\partial^* j_a)]\rangle_{(ag^2)^{-1}} = \exp\,[-g^2(dj,dj)/2]\,, \tag{1.8}$$

where g is a strictly positive real number and d is a differential operator on the differential 1-forms on $\mathbf{R}^3$. The inner product of the differential 2-forms on $\mathbf{R}^d$ is similar to the inner product (1.5).

Let ψ be a smooth real differential 3-form on $\mathbf{R}^3$ and r be any number in $[1,\infty)$. L.Gross [3] proved also that for the Villain energy function

$$\lim_{a\to 0}\langle |(h'_{(ag^2)^{-1}}(\partial^*\theta),\partial\psi_a)|^r\rangle_{(ag^2)^{-1}} = 0\,. \tag{1.9}$$

In the four dimensional case B.Driver [4] proved that for the Wilson energy function and "for all but at most countable numbers of $g > 0$"

$$\lim_{a\to 0}\langle \exp\,[i(h'(\partial^*\theta),\partial^* j^a)]\rangle_{g^{-2}} = \exp\,[-\alpha g^2(dj,dj)/2]\,, \tag{1.10}$$

where $\langle\rangle_{g^{-2}}$ is any translation and 90°-rotation invariant infinite volume Gibbs state, j is any real smooth differential 1-form on $\mathbf{R}^4$, j^a is its lattice approximation (1.6) and the number $\alpha \geqq 0$ is independent of the particular choice $\langle\rangle_{g^{-2}}$.

This paper is concerned with the case of the Villain energy function and the periodic boundary conditions. We study the correlation functions: $\langle \exp\,[i(j,\theta)]\rangle_{\Lambda,\beta}$, where j is a 1-cochain on Λ with the integer coefficients. The inner product (j,θ) is not defined for a 1-cochain θ on Λ with coefficients in $U(1) = \mathbf{R}/2\pi\mathbf{Z}$, but exp $[i(j,\theta)]$ is well defined. It is easy to show that $\langle \exp\,[i(j,\theta)]\rangle_{\Lambda,\beta} = 0$ if $j \neq \partial\phi$ for some 2-cochain ϕ on Λ with the integer coefficients (see, for example, [5]). In view of the periodic boundary conditions we can identify the opposite vertices of the cube $[N_1, N_2+1]^{\times d}$ and obtain a lattice approximation $\mathbf{T}_N^d$ of the torus $\mathbf{T}^d$ of radius R.

Let $f_{i_1\ldots i_p}(\mathbf{x})$ be the coefficients of a real smooth differential p-form on the torus $\mathbf{T}^d$. We define the integer valued p-cochain on $\boldsymbol{T}_N^d$,

$$(f_{N,b})_{i_1\ldots i_p}(\mathbf{m}) = [N^b f_{i_1\ldots i_p}(2\pi R N^{-1}\mathbf{m})]\,, \tag{1.11}$$

where $N = N_2 - N_1 + 1$, b is a strictly positive integer and $[r]$ is the integer part of the real number r. In order to define the continuum limit we need to know how the constant β in the Villain energy function (1.1) depends on the lattice spacing parameter $a = 2\pi R N^{-1}$. B.K. Driver requires that being multiplied by β(a) the scalar product (1.5) of two lattice approximations (1.6) of the smooth differential 2-forms (electromagnetic field strength is the differential 2-form) tends as $a \to 0$ to the usual scalar product of these smooth differential 2-forms multiplied by the constant g^{-2}. This requirement implies that $\beta(a) = g^{-2}a^{d-4}$. For the dimensions $d = 3,4$ we get $\beta(a) = (ag^2)^{-1}$ and $\beta(a) = g^{-2}$, respectively. Due to Theorem 4.1 from [4] for the dimensions $d > 4$ this scaling implies that the continuum limit on the current sector of $U(1)$ gauge lattice models is degenerate. We choose the non-standard scaling, when $\beta(a) = g_1^{-2}a^{-d-2b}$, where b is a strictly positive integer introduced above. We require that being multiplied by $(\beta(N))^{-1}$ the scalar product (1.5) of two lattice approximations (1.11) of the smooth differential p-forms tends as $N \to \infty$

to the usual scalar product of these smooth differential p-forms multiplied by the constant g^2. Let the function $f(\mathbf{x})$ on the torus $\mathbf{T}^d$ be equal to one. By the definition (1.11) the 0-cochain $f_{N,b}(\mathbf{m}) \equiv N^b$. The definition (1.5) implies that $(f_{N,b}, f_{N,b}) = N^{d+2b}$. Choose $\beta(N)$ such that $\beta(N)^{-1}(f_{N,b}, f_{N,b}) = g^2(2\pi R)^d$, where $(2\pi R)^d$ is the volume of the torus $\mathbf{T}^d$ and $g > 0$, i.e. $\beta(N) = g^{-2}(2\pi R)^{-d}N^{d+2b}$. It seems that this geometrical definition of the continuum limit may be useful also for the $U(1)$ gauge models including the interaction with the fermions. In the next sections it will be proved that for any real smooth differential 2-form ϕ on the torus $\mathbf{T}^d$, $d = 3,4$,

$$\lim_{N\to\infty} \langle \exp\,[i(\partial\phi_{N,b}, \theta)]\rangle_{\mathbf{T}^d_N, \beta(N)} = \exp\,[-g^2(d^*\phi, G(d^*\phi))/2]\,, \tag{1.12}$$

where d^* is the adjoint operator of the differential operator d, the operator G is the Green operator for the Laplace–Beltrami operator on the differential 1-forms on the torus $\mathbf{T}^d$ and the inner product of the differential 1-forms on the torus is similar to the inner product (1.5).

When $\phi = dj$ the right-hand side of the equality (1.12) is a torus analogue of the right-hand sides of the equalities (1.8) and (1.10) for $\alpha = 1$. Due to $(d^*)^2 = 0$ the substitution $\phi = d^*\psi$ into the right-hand side of the equality (1.12) yields 1 and we obtain a torus generalization of the equalities (1.9). It is important to note that the right-hand side of the equality (1.12) coincides with the correlation function of $\mathbf{R}$-gauge electrodynamics on the torus [7]. As the torus radius R tends to infinity the limit of (1.12) gives the correlation function of $\mathbf{R}$-gauge Euclidean electrodynamics [7]. We studied the continuum limit (1.12) of the correlation functions of the free $U(1)$ gauge model. We believe that the limit (1.12) may be applied for the study of the correlation functions of the $U(1)$ gauge model which includes the interaction with the fermions.

The remaining sections are devoted to the proofs of the equality (1.12) for $d = 3,4$.

2. Three Dimensional Torus

The p-cochains with the coefficients in the abelian group $G = \mathbf{Z},\ \mathbf{R}, U(1) = \mathbf{R}/2\pi\mathbf{Z}$ satisfying the periodic boundary conditions (1.2) form the abelian group $C^p(\mathbf{T}^d_N, G)$, where $N = N_2 - N_1 + 1$. In order to simplify the situation we assume that $N_1 = 0, N_2 = N - 1$. Now the definition (1.7) for the correlation function may be rewritten in the form

$$\langle \exp\,[i(j,\theta)]\rangle_{\mathbf{T}^d_N,\beta} = Z^{-1}_{\mathbf{T}^d_N} \int_{C^1(\mathbf{T}^d_N, U(1))} \exp\,[i(j,\theta) - \beta(h_\beta(\partial^*\theta), 1)]d\theta\,, \tag{2.1}$$

where a 1-cochain $j \in C^1(\mathbf{T}^d_N, \mathbf{Z})$ and $\exp\,[i(j,\theta)]$ is a character of the compact group $C^1(\mathbf{T}^d_N, U(1))$. The Villain energy function $h_\beta(\theta)$ is given by (1.1). Here $d\theta$ is the normalized Haar measure on the compact group $C^1(\mathbf{T}^d_N, U(1))$ and $Z_{\mathbf{T}^d_N}$ is the normalization constant.

By [5, Lemma 1] the correlation function (2.1) isn't zero only for the boundaries $j = \partial\phi$, where $\phi \in C^2(\mathbf{T}^d_N, \mathbf{Z})$, and

$$\langle \exp\,[i(\partial\phi,\theta)]\rangle_{\mathbf{T}^d_N,\beta} = Z^{-1}_{\mathbf{T}^d_N} \int_{B^2(\mathbf{T}^d_N, U(1))} \exp\,[i(\phi,\psi) - \beta(h_\beta(\psi), 1)]d\psi\,, \tag{2.2}$$

where the group of coboundaries $B^2(\mathbf{T}_N^d, U(1))$ is the image of the homomorphism $\partial^* : C^1(\mathbf{T}_N^d, U(1)) \to C^2(\mathbf{T}_N^d, U(1))$ and $d\psi$ is the normalized Haar measure on the compact group $B^2(\mathbf{T}_N^d, U(1))$.

It is easy to compute the Fourier transform of the function (1.1),

$$1/2\pi \int_0^{2\pi} \exp\,[in\theta - \beta h_\beta(\theta)]d\theta = c_\beta(2\pi\beta)^{-1/2}\exp\,[-(2\beta)^{-1}n^2]\,. \tag{2.3}$$

By using the Fourier transform on the group $B^2(\mathbf{T}_N^d, U(1))$, due to the formula (2.3) and [5, Proposition 1] we obtain

$$\begin{aligned}\langle \exp\,[i(\partial\phi, \theta)]\rangle_{\mathbf{T}_N^d,\beta} &= Z_{\mathbf{T}_N^d}^{-1} c_\beta^g (2\pi\beta)^{-g/2} \\ &\quad \times \sum_{\mathbf{m}\in\mathbf{Z}^g} \exp\left[-1/2\beta(\phi + \sum_{i=1}^g m_i z_i, \phi + \sum_{i=1}^g m_i z_i)\right],\end{aligned} \tag{2.4}$$

where $z_1, \ldots, z_g$ form a basis of the group of 2-cycles $Z_2(\mathbf{T}_N^d, \mathbf{Z})$ which is the kernel of the homomorphism $\partial : C^2(\mathbf{T}_N^d, \mathbf{Z}) \to C^1(\mathbf{T}_N^d, \mathbf{Z})$. The symmetric $g \times g$ matrix $\Omega_{ij} = (z_i, z_j)$ is positively definite and invertible. Let us introduce the dual basis $\bar{z}_i = \sum_{j=1}^g \Omega_{ij}^{-1} z_j$. For every $i = 1, \ldots, g$ the 2-cochain $\bar{z}_i \in Z_2(\mathbf{T}_N^d, \mathbf{R})$ has the following properties: $(\bar{z}_i, z_j) = \delta_{ij}$ and $(\bar{z}_i, \bar{z}_j) = \Omega_{ij}^{-1}$. Let $\overline{Z}_2(\mathbf{T}_N^d, \mathbf{Z})$ be a free group with the basis $\bar{z}_1, \ldots, \bar{z}_g$. The group $\overline{Z}_2(\mathbf{T}_N^d, \mathbf{Z})$ may be defined also as the maximal subgroup of $Z_2(\mathbf{T}_N^d, \mathbf{R})$ so that for any elements $z \in Z_2(\mathbf{T}_N^d, \mathbf{Z})$ and $\bar{z} \in \overline{Z}_2(\mathbf{T}_N^d, \mathbf{Z})$ the inner product $(z, \bar{z})$ is an integer.

Applying the Poisson summation formula

$$\sum_n f(n) = \sum_n \int dx f(x) \exp\,[2\pi i n x]\,, \tag{2.5}$$

we can rewrite the relation (2.4) as

$$\langle \exp\,[i(\partial\phi, \theta)]\rangle_{\mathbf{T}_N^d,\beta} = Z_{\mathbf{T}_N^d}^{-1} c_\beta^g [\det\,\Omega]^{-1/2} W_{\mathbf{T}_N^d,\beta}(\partial\phi)\Theta((\phi, \bar{\mathbf{z}})|2\pi i\beta\Omega^{-1})\,, \tag{2.6}$$

where

$$W_{\mathbf{T}_N^d,\beta}(\partial\phi) = \exp\left[-1/2\beta\left[(\phi,\phi) - \sum_{i=1}^g (\phi, z_i)(\phi, \bar{z}_i)\right]\right] \tag{2.7}$$

is the correlation function calculated for $\mathbf{R}$-gauge electrodynamics on the lattice $\mathbf{T}_N^d$ in the paper [6] and the Riemann θ-function

$$\Theta(\mathbf{y}|\omega) = \sum_{\mathbf{m}\in\mathbf{Z}^g} \exp\left[i\pi \sum_{j,k=1}^g m_k \omega_k m_j + 2\pi i \sum_{j=1}^g m_j y_j\right] \tag{2.8}$$

depends on the vector $\mathbf{y} \in \mathbf{C}^g$ and on the symmetric $g \times g$ matrix ω with the positively definite imaginary part. In our case $\omega = 2\pi i\beta\Omega^{-1}$.

Taking the trivial 2-cochain $\phi = 0$ we obtain

$$Z_{\mathbf{T}_N^d} = c_\beta^g [\det\,\Omega]^{-1/2} \Theta(\mathbf{0}|2\pi i\beta\Omega^{-1})\,. \tag{2.9}$$

The substitution of the equality (2.9) into the right-hand side of the relation (2.6) gives

$$\langle \exp\, [i(\partial\phi,\theta)]\rangle_{\mathbf{T}_N^d,\beta} = W_{\mathbf{T}_N^d,\beta}(\partial\phi)\frac{\Theta((\phi,\overline{\mathbf{z}})|2\pi i\beta\Omega^{-1})}{\Theta(\mathbf{0}|2\pi i\beta\Omega^{-1})}\,. \tag{2.10}$$

This formula was obtained in the paper [6] for the Wilson energy function but only in the weak-coupling region when the inverse temperature approaches infinity.

The definition of the group $\overline{Z}_2(\mathbf{T}_N^d,\mathbf{Z})$ and the definition (2.8) imply that

$$\Theta((\phi,\overline{\mathbf{z}})|2\pi i\beta\Omega^{-1}) = \sum_{\overline{z}\in Z_2(\mathbf{T}_N^d,\mathbf{Z})} \exp\, [-2\pi^2\beta(\overline{z},\overline{z}) + 2\pi i(\phi,\overline{z})]\,. \tag{2.11}$$

To obtain the equality (2.10) we used the Fourier transform on the group $U(1)$. Our space $\mathbf{T}^d$ is a product of groups $U(1)$ and a lattice approximation $\mathbf{T}_N^d = \mathbf{Z}_N^d$, where a group $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$. In order to study the groups $Z_2(\mathbf{T}_N^d,\mathbf{Z})$ and $\overline{Z}_2(\mathbf{T}_N^d,\mathbf{Z})$ we use the Fourier transform on the group $\mathbf{Z}_N^d$. The Fourier transform of a p-cochain $f \in C^p(\mathbf{T}_N^d,\mathbf{R})$ is defined by

$$f^{\sim}_{i_1\dots i_p}(\mathbf{l}) = \sum_{\mathbf{m}\in\Lambda} \exp\left[2\pi i N^{-1}\sum_{k=1}^{d} l_k m_k\right] f_{i_1\dots i_p}(\mathbf{m})\,, \tag{2.12}$$

where $\mathbf{l}\in\Lambda$ and a cube $\Lambda = \{\mathbf{m}\in\mathbf{Z}^d : 0 \leqq m_i \leqq N-1, i=1,\dots,d\}$. We denote the group of functions (2.12) by $C^p(\mathbf{T}_N^d,\mathbf{R})^{\sim}$.

The following relation

$$N^{-1}\sum_{k=0}^{N-1} \exp\, [2\pi i N^{-1} km] = \delta_{m,0} \tag{2.13}$$

holds for any integer $-N+1 \leqq m \leqq N-1$. The relations (1.5) and (2.13) imply

$$(f,g) = N^{-d}\sum_{i_1<\dots<i_p}\sum_{\mathbf{l}\in\Lambda} \overline{f^{\sim}_{i_1\dots i_p}(\mathbf{l})}\, g^{\sim}_{i_1\dots i_p}(\mathbf{l})\,. \tag{2.14}$$

The right-hand side of the equality (2.14) we denote by $(f^{\sim},g^{\sim})$.

Applying the Fourier transform (2.12) we can rewrite the equalities (1.3) and (1.4) as

$$(\tilde{\partial} f^{\sim})_{i_1\dots i_{p-1}}(\mathbf{l}) = \sum_{i_0=1}^{d} (\exp\, [2\pi i N^{-1} l_{i_0}] - 1) f^{\sim}_{i_0 i_1\dots i_{p-1}}(\mathbf{l})\,, \tag{2.15}$$

$$(\tilde{\partial}^* f^{\sim})_{i_1\dots i_{p+1}}(\mathbf{l}) = \sum_{k=1}^{p+1} (-1)^{k+1}(\exp\, [-2\pi i N^{-1} l_{i_k}] - 1) f^{\sim}_{i_1\dots \widehat{i_k}\dots i_{p+1}}(\mathbf{l})\,. \tag{2.16}$$

Therefore a lattice Laplace–Beltrami operator is given by

$$((\tilde{\partial}^*\tilde{\partial} + \tilde{\partial}\tilde{\partial}^*) f^{\sim})_{i_1\dots i_p}(\mathbf{l}) = \sum_{k=1}^{d} |\exp\, [2\pi i N^{-1} l_k] - 1|^2 f^{\sim}_{i_1\dots i_p}(\mathbf{l})\,. \tag{2.17}$$

A p-cochain f is said to be harmonic if the expression (2.17) equals zero.

Lemma 2.1. *Any harmonic p-cochain $f \in C^p(\mathbf{T}_N^d,\mathbf{R})$ is constant.*

Proof. Taking the inner product (2.14) of the functions $f^{\sim}_{i_1\dots i_p}(\mathbf{l})$ and (2.17) we get that p-cochain $f_{i_1\dots i_p}(\mathbf{m})$ is harmonic iff for any $k = 1,\dots,d$,

$$(\exp\,[2\pi iN^{-1}l_k] - 1)f^{\sim}_{i_1\dots i_p}(\mathbf{l}) = 0\,. \tag{2.18}$$

Hence $f^{\sim}_{i_1\dots i_p}(\mathbf{l})$ isn't zero only for $l_k = 0, k = 1,\dots,d$. By using the inverse Fourier transform we have that the p-cochain $f_{i_1\dots i_p}(\mathbf{m})$ is constant.

Let P be an orthogonal projector on the subspace $Z_2(\mathbf{T}^d_N,\mathbf{R})$ of the linear space $C^2(\mathbf{T}^d_N,\mathbf{R})$. Lemma 2.1 and the equalities (2.15)–(2.17) imply that

$$\begin{aligned}(Pf)^{\sim}_{i_1i_2}(\mathbf{l}) &= f^{\sim}_{i_1i_2}(\mathbf{0})\prod_{k=1}^{d}\delta_{l_k,0}\\ &\quad+\left(\tilde{\partial}\left(\sum_{k=1}^{d}|\exp\,[2\pi iN^{-1}l_k]-1|^2\right)^{-1}\tilde{\partial}^{*}f^{\sim}\right)_{i_1i_2}(\mathbf{l})\,.\end{aligned} \tag{2.19}$$

Due to equalities (2.15) and (2.16) we may consider the second term in the right-hand side of (2.19) to be equal zero at $\mathbf{l} = \mathbf{0}$. We denote the right-hand side of the relation (2.19) by $(\tilde{P}f^{\sim})_{i_1i_2}(\mathbf{l})$.

Proposition 2.2. *The Fourier transform* (2.12) *of every 2-cycle $z \in Z_2(\mathbf{T}^3_N,\mathbf{Z})$ has the form: for any permutation i_1,i_2,i_3 of the numbers* 1, 2, 3

$$z^{\sim}_{i_1i_2}(\mathbf{l}) = a^{\sim}_{i_1i_2}(\mathbf{l}) + (\exp\,[2\pi iN^{-1}l_{i_3}]-1)[b^{\sim}_{i_2i_3}(\mathbf{l}) - b^{\sim}_{i_1i_3}(\mathbf{l}) + c^{\sim}_{i_1i_2i_3}(\mathbf{l})]\,, \tag{2.20}$$

where a 2-cochain $a_{j_1j_2}(\mathbf{m})$ is independent of the variables m_{j_1},m_{j_2}, a 2-cochain $b_{j_1j_2}(\mathbf{m})$ depends on the variables m_{j_1},m_{j_2} only and it is equal to zero if one of these variables equals $N-1$, a 3-cochain $c_{i_1i_2i_3}(\mathbf{m})$ equals zero if one of the variables m_1,m_2,m_3 is equal to $N-1$. The above cochains determine the 2-cycle given by (2.20) uniquely.

Proof. By using the formula for the sum of geometric progression we get

$$z^{\sim}_{i_1i_2}(\mathbf{l}) = z^{\sim}_{i_1i_2}(\mathbf{l})|_{l_{i_3}=0} + (\exp\,[2\pi iN^{-1}l_{i_3}]-1)(z^{\sim})'_{i_1i_2,i_3}(\mathbf{l})\,, \tag{2.21}$$

where

$$(z^{\sim})'_{i_1i_2;i_3}(\mathbf{l}) = \sum_{m_{i_1},m_{i_2}=0}^{N-1}\ \sum_{m_{i_3}=1}^{N-1}\ \sum_{m'_{i_3}=0}^{m_{i_3}-1}\exp\left[2\pi iN^{-1}\left[\sum_{k=1.2}l_{i_k}m_{i_k} + l_{i_3}m'_{i_3}\right]\right]z_{i_1i_2}(\mathbf{m})\,. \tag{2.22}$$

Since a function $z^{\sim}_{i_1i_2}(\mathbf{l})$ is antisymmetric under the permutation of the indices i_1,i_2 the substitution $l_{i_3} = 0$ into two equations $(\tilde{\partial}z^{\sim})_{i_1}(\mathbf{l}) = 0$ and $(\tilde{\partial}z^{\sim})_{i_2}(\mathbf{l}) = 0$ provides two equations $(\exp[2\pi iN^{-1}l_{i_k}]-1)z^{\sim}_{i_1i_2}(\mathbf{l})|_{l_{i_3}=0} = 0$, $k = 1,2$. Hence a function $z^{\sim}_{i_1i_2}(\mathbf{l})|_{l_{i_3}=0}$ is not equal to zero only at $l_{i_1} = 0, l_{i_2} = 0$ and by the relation (2.13) we have

$$z^{\sim}_{i_1i_2}(\mathbf{l})|_{l_{i_3}=0} = N^2\delta_{l_{i_1},0}\delta_{l_{i_2},0}a_{i_1i_2}\,, \tag{2.23}$$

where a constant $a_{i_1i_2}$ is antisymmetric under a permutation of the indices i_1,i_2.

The equation $(\tilde{\partial} z^{\sim})_{i_1}(\mathbf{l}) = 0$ and the equalities (2.21), (2.23) imply that

$$\left[\prod_{k=2,3} (\exp\,[2\pi i N^{-1} l_{i_k}] - 1)\right] [(z^{\sim})'_{i_1 i_2; i_3}(\mathbf{l}) + (z^{\sim})'_{i_1 i_3; i_2}(\mathbf{l})] = 0\,. \tag{2.24}$$

Let us introduce the function

$$\begin{aligned} b^{\sim}_{i_2 i_3}(\mathbf{l}) = {} & \delta_{l_{i_1},0} \sum_{l'_{i_1}=0}^{N-1} \exp\,[-2\pi i N^{-1}(N-1) l'_{i_1}] (z^{\sim})'_{i_1 i_2; i_3}(\mathbf{l})|_{l_{i_1} = l'_{i_1}} \\ & - \delta_{l_{i_1},0}\delta_{l_{i_2},0} \sum_{l'_{i_1}, l'_{i_2}=0}^{N-1} \exp\,[-2\pi i N^{-1}(N-1)(l'_{i_1} + l'_{i_2})] (z^{\sim})'_{i_1 i_2; i_3}(\mathbf{l}')|_{l'_{i_3} = l_{i_3}}\,. \end{aligned} \tag{2.25}$$

By definition the function (2.25) satisfies Eq. (2.24) and therefore it satisfies the equation $b^{\sim}_{i_2 i_3}(\mathbf{l}) + b^{\sim}_{i_3 i_2}(\mathbf{l}) = N^2 \delta_{l_{i_1},0}[\delta_{l_{i_2},0} f_3(l_{i_3}) + \delta_{l_{i_3},0} f_2(l_{i_2})]$. The definitions (2.22) and (2.25) imply that the Fourier expansion of the left-hand side of this equation does not contain the components with $m_{i_2} = N-1$ or $m_{i_3} = N-1$. It is easy to show now that $b^{\sim}_{i_2 i_3}(\mathbf{l}) + b^{\sim}_{i_3 i_2}(\mathbf{l}) = 0$. Hence the function $b^{\sim}_{i_2 i_3}(\mathbf{l})$ is antisymmetric under a permutation of indices i_2, i_3. By the definitions (2.22), (2.25) and the relation (2.13) all components in the Fourier expansion of the function $b^{\sim}_{i_2 i_3}(\mathbf{l})$ are integers. Then the function $b^{\sim}_{i_2 i_3}(\mathbf{l})$ is the Fourier transform (2.12) of some 2-cochain $b_{i_2 i_3}(\mathbf{m}) \in C^2(\mathbf{T}^3_N, \mathbf{Z})$. Due to equality (2.25) a cochain $b_{i_2 i_3}(\mathbf{m})$ depends on the variables m_{i_2}, m_{i_3} only and it is equal to zero if one of these variables equals $N-1$.

We define the function $c^{\sim}_{i_1 i_2 i_3}(\mathbf{l})$ by the following equality:

$$\begin{aligned} (z^{\sim})'_{i_1 i_2; i_3}(\mathbf{l}) = {} & c^{\sim}_{i_1 i_2 i_3}(\mathbf{l}) + b^{\sim}_{i_2 i_3}(\mathbf{l}) - b^{\sim}_{i_1 i_3}(\mathbf{l}) \\ & + \delta_{l_{i_1},0}\delta_{l_{i_2},0} \sum_{l'_{i_1}, l'_{i_2}=0}^{N-1} \exp\,[-2\pi i N^{-1}(N-1)(l'_{i_1} + l'_{i_2})] (z^{\sim})'_{i_1 i_2; i_3}(\mathbf{l}')|_{l'_{i_3} = l_{i_3}}\,. \end{aligned} \tag{2.26}$$

A function $c^{\sim}_{i_1 i_2 i_3}(\mathbf{l})$ is obviously antisymmetric under a permutation of indices i_1, i_2. By definitions (2.22), (2.25) and (2.26) its Fourier expansion does not contain the components with $m_k = N-1$, where k is one of the numbers 1,2,3 and the remaining components are integers. It is easy now to verify that a function $c^{\sim}_{i_1 i_2 i_3}(\mathbf{l})$ satisfies Eq. (2.24) and therefore it is antisymmetric under a permutation of indices i_2, i_3. Hence it is antisymmetric under a permutation of all indices i_1, i_2, i_3. Then a function $c^{\sim}_{i_1 i_2 i_3}(\mathbf{l})$ is the Fourier transform (2.12) of some 3-cochain $c_{i_1 i_2 i_3}(\mathbf{m}) \in C^3(\mathbf{T}^3_N, \mathbf{Z})$ which equals zero if one of the variables m_1, m_2, m_3 is equal to $N-1$.

Now the equalities (2.21), (2.23) and (2.26) imply the equality (2.20), where a function $a^{\sim}_{i_1 i_2}(\mathbf{l})$ is the Fourier transform (2.12) of some 2-cochain $a_{i_1 i_2}(\mathbf{m}) \in C^2(\mathbf{T}^3_N, \mathbf{Z})$ independent of the variables m_{i_1}, m_{i_2}. By definitions the cochains in the expansion (2.20) determine the 2-cycle defined by (2.20) uniquely.

Let S_n be a symmetric group, i.e. a group of all permutations of the numbers $1,\dots,n$. For any permutation $\sigma \in S_3$ and for any point $\mathbf{m} \in \mathbf{T}_N^3$ we introduce three 2-cochains from $C^2(\mathbf{T}_N^3, \mathbf{Z})$ by defining their Fourier transforms

$$(\hat{a}[\sigma(1),\sigma(2);m_{\sigma(3)}])^{\sim}_{i_1 i_2}(\mathbf{l}) = (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)}) \times \exp\,[2\pi i N^{-1}((N-1)(l_{\sigma(1)} + l_{\sigma(2)}) + m_{\sigma(3)} l_{\sigma(3)})]\,, \tag{2.27}$$

$$\begin{aligned}(\hat{b}[\sigma(2),\sigma(3);m_{\sigma(2)},m_{\sigma(3)}])^{\sim}_{i_1 i_2}(\mathbf{l}) &= (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)}) \Bigg[\prod_{k=2,3} (\exp\,[2\pi i N^{-1} m_{\sigma(k)} l_{\sigma(k)}] \\ &\quad - \exp\,[2\pi i N^{-1}(N-1) l_{\sigma(k)}])\Bigg] (\exp\,[-2\pi i N^{-1} l_{\sigma(3)}] - 1)^{-1} \\ &\quad \times \exp\,[2\pi i N^{-1}((N-1) l_{\sigma(1)}]\,, \end{aligned} \tag{2.28}$$

$$\begin{aligned}(\hat{c}[\sigma(1),\sigma(2),\sigma(3);\mathbf{m}])^{\sim}_{i_1 i_2}(\mathbf{l}) &= (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)}) \Bigg[\prod_{k=1,2,3} (\exp\,[2\pi i N^{-1} m_{\sigma(k)} l_{\sigma(k)}] \\ &\quad - \exp\,[2\pi i N^{-1}(N-1) l_{\sigma(k)}])\Bigg] (\exp\,[-2\pi i N^{-1} l_{\sigma(3)}] - 1)^{-1}\,. \end{aligned} \tag{2.29}$$

The inner product (2.14) of the 2-cochains given by their Fourier transforms (2.20) and (2.27) is equal to $a_{\sigma(1)\sigma(2)}(m_{\sigma(3)})$. It is antisymmetric under a permutation of the indices $\sigma(1), \sigma(2)$. The function (2.27) has the same property. Hence the independent functions (2.27) are related to three permutations $\sigma \in S_3$ satisfying the condition $\sigma(1) < \sigma(2)$. The inner product (2.14) of the 2-cochains given by their Fourier transforms (2.20) and (2.28) is equal to $b_{\sigma(2)\sigma(3)}(\mathbf{m})$. Since it is antisymmetric under a permutation of the indices $\sigma(2), \sigma(3)$ the independent projections on the subspace $Z_2(\mathbf{T}_N^3, \mathbf{R})$ of the 2-cochains given by the Fourier transforms (2.28) correspond to three permutations $\sigma \in S_3$ satisfying the condition $\sigma(2) < \sigma(3)$. The inner product (2.14) of the 2-cochains given by their Fourier transforms (2.20) and (2.29) is equal to $c_{\sigma(1)\sigma(2)\sigma(3)}(\mathbf{m})$. Since it is antisymmetric under a permutation of all indices $\sigma(1), \sigma(2), \sigma(3)$ the only projection on the subspace $Z_2(\mathbf{T}_N^3, \mathbf{R})$ of the 2-cochain given by the Fourier transform (2.29) corresponding to the identity permutation $\sigma \in S_3$ is independent. Thus we have proved the following

Proposition 2.3. *Every element $\bar{z} \in \bar{Z}_2(\mathbf{T}_N^3, \mathbf{Z})$ has the following form:*

$$\bar{z}_{i_1 i_2}(\mathbf{m}) = (P\hat{z})_{i_1 i_2}(\mathbf{m})\,, \tag{2.30}$$

$$\begin{aligned}\hat{z}_{i_1 i_2}(\mathbf{m}) &= \sum_{\sigma \in S_3;\sigma(1)<\sigma(2)} \sum_{k_{\sigma(3)}=0}^{N-1} \bar{a}_{\sigma(1)\sigma(2)}(k_{\sigma(3)})(\hat{a}[\sigma(1),\sigma(2);k_{\sigma(3)}])_{i_1 i_2}(\mathbf{m}) \\ &\quad + \sum_{\sigma \in S_3;\sigma(2)<\sigma(3)} \sum_{k_{\sigma(2)},k_{\sigma(3)}=0}^{N-2} \bar{b}_{\sigma(2)\sigma(3)}(k_{\sigma(2)},k_{\sigma(3)}) \\ &\quad \times (\hat{b}[\sigma(2),\sigma(3);k_{\sigma(2)},k_{\sigma(3)}])_{i_1 i_2}(\mathbf{m}) \\ &\quad + \sum_{k_1,k_2,k_3=0}^{N-2} \bar{c}_{123}(\mathbf{k})(\hat{c}[1,2,3;\mathbf{k}])_{i_1 i_2}(\mathbf{m})\,, \end{aligned} \tag{2.31}$$

where P is the projector (2.19)*, the 2-cochains* $(\hat{a}[\sigma(1),\sigma(2);k_{\sigma(3)}])_{i_1 i_2}(\mathbf{m}), (\hat{b}[\sigma(2),\sigma(3);k_{\sigma(2)},k_{\sigma(3)}])_{i_1 i_2}(\mathbf{m})$ *and* $(\hat{c}[1,2,3;\mathbf{k}])_{i_1 i_2}(\mathbf{m})$ *are defined by their Fourier transforms* (2.27), (2.28) *and* (2.29)*. The integer valued functions* $\bar{a}_{\sigma(1)\sigma(2)}(k_{\sigma(3)})$, $\bar{b}_{\sigma(2)\sigma(3)}(k_{\sigma(2)},k_{\sigma(3)})$ *and* $\bar{c}_{123}(\mathbf{k})$ *in the equalities* (2.30), (2.31) *are independent and they determine the element* (2.30) *uniquely.*

As explained above for the continuum limit $\mathbf{T}_N^3 \to \mathbf{T}^3$ of the correlation function (2.10) we need to choose the special sequence (1.11) of the 2-cochains $\phi_N \in C^2(\mathbf{T}_N^3,\mathbf{Z})$ and the inverse temperature $\beta = \beta_0 N^{3+2b}$, where $\beta_0^{-1} = g^2(2\pi R)^3 > 0$ and b is a strictly positive integer.

Proposition 2.4. *Let a θ-function $\Theta((\phi,\bar{\mathbf{z}})|2\pi i\beta\Omega^{-1})$ be given by the equality* (2.11)*. Then for any sequence $\phi_N \in C^2(\mathbf{T}_N^3,\mathbf{Z})$ and for any numbers $\beta_0 > 0, \gamma > 3$,*

$$\lim_{N\to\infty} \Theta((\phi_N,\bar{\mathbf{z}})|2\pi i\beta_0 N^\gamma \Omega^{-1}) = 1 . \tag{2.32}$$

Proof. It follows from the equalities (2.16), (2.27)–(2.29) and (2.31) that: for $0 \leqq m_1,m_2,m_3 \leqq N-2$,

$$(\partial^*\hat{z})_{123}(\mathbf{m}) = \bar{c}_{123}(\mathbf{m}) , \tag{2.33}$$

for $0 \leqq m_1,\widehat{m_j},m_3 \leqq N-2, m_j = N-1, j = 1,2,3,$

$$(\partial^*\hat{z})_{123}(\mathbf{m}) = (-1)^{j+1}\bar{b}_{\widehat{1j3}}(m_1,\widehat{m_j},m_3) - \sum_{m_j'=0}^{N-2} \bar{c}_{123}(\mathbf{m})|_{m_j=m_j'} , \tag{2.34}$$

for $0 \leqq m_j \leqq N-2, m_1,\widehat{m_j},m_3 = N-1, j = 1,2,3,$

$$\begin{aligned}(\partial^*\hat{z})_{123}(\mathbf{m}) = (-1)^{j+1}&(\bar{a}_{\widehat{1j3}}(m_j+1) - \bar{a}_{\widehat{1j3}}(m_j)) \\ &- \sum_{\sigma\in S_3; j=\sigma(2)<\sigma(3)} \operatorname{sgn}\sigma \sum_{m'_{\sigma(3)}=0}^{N-2} \bar{b}_{j\sigma(3)}(m_j,m'_{\sigma(3)}) \\ &- \sum_{\sigma\in S_3;\sigma(2)<\sigma(3)=j} \operatorname{sgn}\sigma \sum_{m'_{\sigma(2)}=0}^{N-2} \bar{b}_{\sigma(2)j}(m'_{\sigma(2)},m_j) \\ &+ \sum_{m_k'=0; k\neq j}^{N-2} \bar{c}_{123}(\mathbf{m}')|_{m_j'=m_j} ,\end{aligned} \tag{2.35}$$

where $\operatorname{sgn}\sigma$ is a parity of permutation.

Due to equalities (2.27)–(2.29) for $1 \leqq i_1,i_2 \leqq 3$ we get

$$(\hat{z})^{\sim}_{i_1,i_2}(\mathbf{0}) = \sum_{k=0}^{N-1} \bar{a}_{i_1 i_2}(k) . \tag{2.36}$$

By definition the terms in the right-hand side of the relation (2.19) are orthogonal to each other. Now the equalities (2.15), (2.16), (2.36) and the obvious estimate for any integers $l_k, k = 1,2,3,$

$$\left(\sum_{k=1}^{3}|1 - \exp\,[2\pi i N^{-1}l_k]|^2\right)^{-1} \geqq 1/6 \tag{2.37}$$

imply the following estimate

$$|\Theta((\phi,\mathbf{z})|2\pi i\beta\Omega^{-1})-1| \leqq \sum_{P\hat{z}\in Z_2(\mathbf{T}_N^3,\mathbf{Z})} \exp\Bigg[-2\pi^2\beta N^{-3} \times \sum_{\sigma\in S_3;\sigma(1)<\sigma(2)} \left(\sum_{m=0}^{N-1} \bar{a}_{\sigma(1)\sigma(2)}(m)\right)^2 - \pi^2\beta/3 \sum_{m_1,m_2,m_3=0}^{N-1} ((\partial^*\hat{z})_{123}(\mathbf{m}))^2\Bigg] - 1 . \tag{2.38}$$

Since

$$\bar{a}_{\sigma(1)\sigma(2)}(m) = \bar{a}_{\sigma(1)\sigma(2)}(0) + \sum_{k=0}^{m-1} (\bar{a}_{\sigma(1)\sigma(2)}(k+1) - \bar{a}_{\sigma(1)\sigma(2)}(k)) , \tag{2.39}$$

it is possible to consider $\bar{a}_{\sigma(1)\sigma(2)}(0)$ and the right-hand sides of the equalities (2.33)–(2.35) as the summation variables in the sum (2.38). It follows from the equality (2.39) that

$$\sum_{m=0}^{N-1} \bar{a}_{\sigma(1)\sigma(2)}(m) = N\bar{a}_{\sigma(1)\sigma(2)}(0) + \sum_{m=0}^{N-2} (N-m-1)(\bar{a}_{\sigma(1)\sigma(2)}(m+1) - \bar{a}_{\sigma(1)\sigma(2)}(m)) . \tag{2.40}$$

Extending the summation over the integer variables $\bar{a}_{\sigma(1)\sigma(2)}(0)$ in the sum (2.38) into the summation over $\bar{a}_{\sigma(1)\sigma(2)}(0) \in N^{-1}\mathbf{Z}$ in view of (2.40) we get the extended sum (2.38), where the independent summation variables are the right-hand sides of the equalities (2.33)–(2.36). Now if we leave in the second exponent (2.38) the components (2.33)–(2.35) only we obtain the obvious estimate for this extended sum and therefore for the left-hand side of the inequality (2.38),

$$|\Theta((\phi,\mathbf{z})|2\pi i\beta\Omega^{-1})-1| \leqq (\Theta(0|2\pi i\beta N^{-3}))^3(\Theta(0|i\pi\beta/3))^{N^3-1} - 1 , \tag{2.41}$$

where N^3-1 is the total number of the component (2.33)–(2.35), i.e. the total number of the generators of the group of 2-boundaries on $\mathbf{T}_N^3$.

Since for any strictly positive integer n the following estimate $n^2 > n$ holds, the definition (2.8) of the one dimensional θ-function implies that for any $t > 0$,

$$1 < \Theta(0|it) < 1 + 2(e^{\pi t} - 1)^{-1} . \tag{2.42}$$

By using this estimate we have

$$0 < (\Theta(0|i\pi\beta_0 N^\gamma/3))^{N^3-1} - 1 < \sum_{k=1}^{N^3-1} \frac{(N^3-1)\cdots(N^3-k)}{k!} 2^k(\exp[\pi^2\beta_0 N^\gamma/3] - 1)^{-k} . \tag{2.43}$$

The estimates (2.41)–(2.43) imply the relation (2.32).

In order to compute the continuum limit of the correlation function (2.10) it is necessary to calculate the continuum limit of the correlation function (2.7). Let us consider again the d-dimensional torus, $d > 2$. Due to [8, Sect. 22, Proposition 1] for any 2-cochain $\phi \in C^2(\mathbf{T}_{N^d}, \mathbf{Z})$,

$$(\phi,\phi) - \sum_{i=1}^{g}(\phi, z_i)(\phi, \bar{z}_i) = (\phi, Q\phi)\,, \tag{2.44}$$

where Q is the orthogonal projector on the subspace of the 2-coboundaries $B^2(\mathbf{T}_N^d, \mathbf{R})$. Lemma 2.1, the relations (2.14)–(2.17) and the relation $\exp\,[2\pi i N^{-1}(N - l)m] = \exp\,[-2\pi i N^{-1} lm]$ for any integers l, m imply that

$$\begin{aligned}(\phi, Q\phi) = \sum_{\mu=1}^{d} N^{-d} \sum_{l_1,\ldots,l_d=-(N-1)/2;\, l_1^2+\cdots+l_d^2 \neq 0}^{(N-1)/2} \left(\sum_{k=1}^{d} |\exp\,[2\pi i N^{-1} l_k] - 1|^2\right)^{-1} \\ \times \,|\sum_{\lambda=1}^{d}(\exp\,[2\pi i N^{-1} l_\lambda] - 1)\phi^{\sim}_{\lambda\mu}(\mathbf{l})|^2\,. \end{aligned} \tag{2.45}$$

Here we assume N to be odd. Let a 2-cochain $\phi_{N,b} \in C^2(\mathbf{T}_N^d, \mathbf{Z})$ be constructed from the coefficients $\phi_{i_1 i_2}(\theta)$ of a smooth differential 2-form on the torus $\mathbf{T}^d$ by means of the definition (1.11) for some strictly positive integer b. From the definitions (1.11) and (2.12) we get

$$\begin{aligned}\lim_{N\to\infty} N^{-d-b}(\phi_{N,b})^{\sim}_{i_1 i_2}(\mathbf{l}) &= \phi^{\sim}_{i_1 i_2}(\mathbf{l}) \\ &= (2\pi R)^{-d}\int_0^{2\pi R} d\theta_1 \cdots \int_0^{2\pi R} d\theta_d \exp\,[iR^{-1}\sum_{k=1}^{d} l_k\theta_k]\phi_{i_1 i_2}(\theta)\,. \end{aligned} \tag{2.46}$$

Hence the limit (2.46) is a square summable function of the variable $\mathbf{l} \in \mathbf{Z}^d$. Now the relations (2.45) and (2.46) imply that

$$\begin{aligned}&\lim_{N\to\infty} g^2(2\pi R)^d N^{-d-2b}(\phi_{N,b}, Q\phi_{N,b}) \\ &= g^2(2\pi R)^{-d}\sum_{\mu=1}^{d}\sum_{l_1,\ldots,l_d=-\infty;\, l_1^2+\cdots+l_d^2\neq 0}^{\infty} R^2(l_1^2 + \cdots \\ &\quad + l_d^2)^{-1}|(d^*\phi)^{\sim}_{\mu}(\mathbf{l})|^2\,, \end{aligned} \tag{2.47}$$

where d^* is the adjoint operator of the differential operator d

$$(d^*\phi)_\mu(\theta) = -\sum_{\lambda=1}^{d}\frac{\partial}{\partial\theta_\lambda}\phi_{\lambda\mu}(\theta)\,. \tag{2.48}$$

Now it follows from the relations (2.7), (2.44) and (2.47) that

$$\begin{aligned}&\lim_{N\to\infty} W_{\mathbf{T}_N^d, g^{-2}(2\pi R)^{-d}N^{d+2b}}(\partial\phi_{N,b}) \\ &= \exp\left[-(g^2/2)(2\pi R)^{-d}\sum_{\mu=1}^{d}\sum_{l_1,\ldots,l_d=-\infty;\, l_1^2+\cdots+l_d^2\neq 0}^{\infty} R^2(l_1^2 + \cdots \right. \\ &\quad \left. + l_d^2)^{-1}|(d^*\phi)^{\sim}_{\mu}(\mathbf{l})|^2\right]\,. \end{aligned} \tag{2.49}$$

It is interesting to note that the right-hand side of this relation is a correlation function of the $\mathbf{R}$-gauge electrodynamics on a torus [7]. The equalities (2.10), (2.32) and (2.49) imply the equality (1.12) for $d = 3$.

If the differential 2-form ϕ has a compact support independent of the radius R of a torus then by using the equalities (2.46) and (2.49) it is easy to prove that

$$\lim_{R\to\infty}\lim_{N\to\infty} W_{\mathbf{T}_N^d, g^{-2}(2\pi R)^{-d}N^{d+2b}}(\partial\phi_{N,b})$$
$$= \exp\left[-(g^2/2)(2\pi)^{-d}\int_{\mathbf{R}^d} d^d p(p_1^2+\cdots+p_d^2)^{-1}|(d^*\phi)_\mu^\sim(\mathbf{p})|^2\right], \tag{2.50}$$

where the operator d^* is defined by the same equality (2.48) and a function $f_\mu^\sim(\mathbf{p})$ is an usual Fourier transform of a function $f_\mu(\mathbf{x})$ on the Euclidean space $\mathbf{R}^d$. The right-hand side of the equality (2.50) is a correlation function of the $\mathbf{R}$-gauge Euclidean electrodynamics [7].

3. Four Dimensional Torus

In order to prove the relation (1.12) for $d=4$ we have to obtain the four dimensional versions of Propositions 2.2, 2.3, and 2.4.

Proposition 3.1. *The Fourier transform* (2.12) *of every 2-cycle* $z\in Z_2(\mathbf{T}_N^4,\mathbf{Z})$ *has the form: for any permutation* $\lambda\in S_4$

$$z_{\lambda(1)\lambda(2)}^\sim(\mathbf{l}) = a_{\lambda(1)\lambda(2)}^\sim(\mathbf{l}) + \sum_{\sigma,\tau\in S_4;\sigma(k)=k,k=1,2;\tau(k)=k,k=3,4} \operatorname{sgn}\tau$$
$$\times\Big[(\exp[2\pi i N^{-1}l_{\lambda\sigma(3)}]-1)((b^{(\lambda\sigma(4))})_{\lambda\tau(2)\lambda\sigma(3)}^\sim(\mathbf{l})$$
$$+1/2(c^{(\lambda\sigma(4))})_{\lambda\tau(1)\lambda\tau(2)\lambda\sigma(3)}^\sim(\mathbf{l}))+1/2(\prod_{k=3,4}(\exp[2\pi i N^{-1}l_{\lambda(k)}]-1))$$
$$\times(b_{[\lambda\tau(2);\{\lambda\sigma(3),\lambda\sigma(4)\}]}^\sim(\mathbf{l})+1/2c_{[\lambda\tau(1)\lambda\tau(2);\{\lambda\sigma(3),\lambda\sigma(4)\}]}^\sim(\mathbf{l}))\Big], \tag{3.1}$$

where a 2-cochain $a_{\lambda(1)\lambda(2)}(\mathbf{m})$ *is independent of the variables* $m_{\lambda(1)}, m_{\lambda(2)}$*; a 2-cochain* $b_{\lambda(2)\lambda(3)}^{(\lambda(4))}(\mathbf{m})$ *depends only on the variables* $m_{\lambda(2)}, m_{\lambda(3)}, m_{\lambda(4)}$ *and it is zero except for* $0\leqq m_{\lambda(2)}, m_{\lambda(3)}\leqq N-2, m_{\lambda(4)}=0$*; a 3-cochain* $(c^{(\lambda(4))})_{\lambda(1)\lambda(2)\lambda(3)}(\mathbf{m})$ *is zero except for* $0\leqq m_{\lambda(1)}, m_{\lambda(2)}, m_{\lambda(3)}\leqq N-2, m_{\lambda(4)}=0$*; a 1-cochain* $b_{[\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{m})$ *is symmetric under a permutation of the extra indices* $\lambda(3),\lambda(4)$ *and it satisfies the symmetry equation*

$$\sum_{\sigma\in S_4;\sigma(1)=1,\operatorname{sgn}\sigma=1} b_{[\lambda\sigma(2);\{\lambda\sigma(3),\lambda\sigma(4)\}]}(\mathbf{m})=0, \tag{3.2}$$

a 1-cochain $b_{[\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{m})$ *depends only on the variables* $m_{\lambda(2)}, m_{\lambda(3)}, m_{\lambda(4)}$ *and it is equal to zero if one of these variables equals* $N-1$*; a 2-cochain* $c_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{m})$ *is symmetric under a permutation of the extra indices* $\lambda(3),\lambda(4)$ *and it satisfies the symmetry equation*

$$\sum_{\sigma\in S_4;\sigma(1)=1,\operatorname{sgn}\sigma=1} c_{[\lambda(1)\lambda\sigma(2);\{\lambda\sigma(3),\lambda\sigma(4)\}]}(\mathbf{m})=0 \tag{3.3}$$

a 2-cochain $c_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{m})$ *equals zero if one of the variables* $m_1,\ldots,m_4$ *equals* $N-1$*. The above cochains determine the 2-cycle given by* (3.1) *uniquely.*

Proof. Let $\lambda \in S_4$ be a permutation of the numbers 1,2,3,4. Applying the formula for a sum of a geometrical progression we get a expansion for a Fourier transform (2.12) of a 2-cycle from $Z_2(\mathbf{T}_N^4, \mathbf{Z})$,

$$\begin{aligned} z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l}) &= -z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(3)}, l_{\lambda(4)}=0} \\ &\quad + z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{\lambda(3)=0} + z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(4)}=0} \\ &\quad + \left[\prod_{k=3,4} \left(\exp\left[2\pi i N^{-1} l_{\lambda(k)}\right] - 1\right)\right] (z^{\sim})''_{\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}}(\mathbf{l}) \,, \end{aligned} \tag{3.4}$$

$$\begin{aligned} (z^{\sim})''_{\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}}(\mathbf{l}) = & \sum_{m_{\lambda(1)}, m_{\lambda(2)}=0}^{N-1} \sum_{m_{\lambda(3)}, m_{\lambda(4)}=1}^{N-1} \sum_{k_{\lambda(3)}=0}^{m_{\lambda(3)}-1} \sum_{k_{\lambda(4)}=0}^{m_{\lambda(4)}-1} \\ & \times \exp\left[2\pi i N^{-1}(m_{\lambda(1)} l_{\lambda(1)} + m_{\lambda(2)} l_{\lambda(2)} \right. \\ & \left. + k_{\lambda(3)} l_{\lambda(3)} + k_{\lambda(4)} l_{\lambda(4)})\right] z_{\lambda(1)\lambda(2)}(\mathbf{m}) \,. \end{aligned} \tag{3.5}$$

It is obvious that the function (3.5) is symmetric under a permutation of the extra indices $\lambda(3), \lambda(4)$. By an argument analogous to the one given in Proposition 2.2 the function $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(3)}, l_{\lambda(4)}=0}$ has the form (2.23). The function $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(3)}=0}$ may be considered as a Fourier transform (2.12) of a 2-cochain which is not zero only for $m_{\lambda(3)} = 0$. In view of the equality (2.15) it is evident that the functions $z^{\sim}_{i_1 i_2}(\mathbf{l})|_{l_{\lambda(3)}=0}, i_1, i_2 = 1, \ldots, \widehat{\lambda(3)}, \ldots, 4$, of the variables $l_1, \ldots, \widehat{l_{\lambda(3)}}, \ldots, l_4$ form a function from the group $Z_2(\mathbf{T}_N^3, \mathbf{Z})^{\sim}$. Now by Proposition 2.2 for the function $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(3)}=0}$ the expansion (2.20) holds. Since a function $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})$ satisfies the equation $(\tilde{\partial} z^{\sim})_{\lambda(1)}(\mathbf{l}) = 0$, it follows from the relation (3.4), the explicit form (2.23) of the function $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(3)}, l_{\lambda(4)}=0}$ and the expansions of the type (2.20) for the functions $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(3)}=0}$, $z^{\sim}_{\lambda(1)\lambda(2)}(\mathbf{l})|_{l_{\lambda(4)}=0}$ that the function (3.5) satisfies the equation

$$\left[\prod_{k=2,3,4} \left(\exp\left[2\pi i N^{-1} l_{\lambda(k)}\right] - 1\right)\right] \sum_{\sigma \in S_4; \sigma(1)=1, \mathrm{sgn}\sigma=1} (z^{\sim})''_{\lambda(1)\lambda\sigma(2);\{\lambda\sigma(3),\lambda\sigma(4)\}}(\mathbf{l}) = 0 \,, \tag{3.6}$$

where the identity permutation and two cyclic permutations of the numbers 2,3,4 leaving invariant the number 1 are represented as the set $\{\sigma \in S_4 : \sigma(1) = 1, \mathrm{sgn}\sigma = 1\}$.

Let us introduce the function

$$\begin{aligned} b^{\sim}_{[\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{l}) = & N^{-1} \sum_{l'_{\lambda(1)}=0}^{N-1} \exp\left[-2\pi i N^{-1}(N-1) l'_{\lambda(1)}\right] \\ & \times (z^{\sim})''_{\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}}(\mathbf{l})|_{l_{\lambda(1)}=l'_{\lambda(1)}} - \delta_{l_{\lambda(2)},0} N^{-1} \\ & \times \sum_{l'_{\lambda(1)}, l'_{\lambda(2)}=0}^{N-1} \exp\left[-2\pi i N^{-1}(N-1)(l'_{\lambda(1)} \right. \\ & \left. + l'_{\lambda(2)})\right] (z^{\sim})''_{\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}}(\mathbf{l})|_{l_{\lambda(k)}=l'_{\lambda(k)}; k=1,2} \,. \end{aligned} \tag{3.7}$$

By using the inverse transformation for the Fourier transformation (2.12) we obtain from the function (3.7) the 1-cochain $b_{[\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{m})$ which is symmetric under a permutation of the extra indices $\lambda(3), \lambda(4)$. Due to (3.5), (3.7) it depends on the variables $m_{\lambda(2)}, m_{\lambda(3)}, m_{\lambda(4)}$ only and it is equal to zero if one of these variables equals $N-1$. By definition the functions of the (3.7) satisfy Eq. (3.6). Now the arguments similar to those given in Proposition 2.2 lead to the statement that the inverse Fourier transform of the functions of the type (3.7) satisfy Eq. (3.2).

Let us define the function $c^{\sim}_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{l})$ by the following equality:

$$\begin{aligned}(z^{\sim})''_{\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}}(\mathbf{l}) &= c^{\sim}_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{l}) \\ &+ b^{\sim}_{[\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{l}) - b^{\sim}_{[\lambda(1),\{\lambda(3),\lambda(4)\}]}(\mathbf{l}) \\ &- \delta_{l_{\lambda(1)},0}\delta_{l_{\lambda(2)},0} \sum_{l'_{\lambda(1)},l'_{\lambda(2)}=0}^{N-1} \exp\,[-2\pi i N^{-1}(N-1)(l'_{\lambda(1)}+l'_{\lambda(2)})] \\ &\times (z^{\sim})''_{\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}}(\mathbf{l})|_{l_{\lambda(k)}=l'_{\lambda(k)};k=1,2}\,. \end{aligned} \tag{3.8}$$

The definitions (3.5), (3.7), (3.8) and the relation (2.13) imply that the function $c^{\sim}_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{l})$ is the Fourier transform (2.12) of the cochain $c_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}$ $(\mathbf{m}) \in C^2(\mathbf{T}^4_N, \mathbf{Z})$ which equals zero if one of the variables $m_1, \ldots, m_4$ is equal to $N-1$. It follows from the definitions (3.7), (3.8) and Eqs. (3.2), (3.6) that the functions $c^{\sim}_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{l})$ are symmetric under permutations of the extra indices $\lambda(3), \lambda(4)$ and satisfy Eq. (3.6). Now since the 2-cochains $c_{[\lambda(1)\lambda(2);\{\lambda(3),\lambda(4)\}]}(\mathbf{m})$ equal zero if one of the variables $m_1, \ldots, m_4$ is equal to $N-1$, applying the arguments of Proposition 2.2 we get that these 2-cochains satisfy Eq. (3.3).

The expansions (2.20), (3.4), (3.8) and the equality of the type (2.23) imply the expansion (3.1). Due to the construction the cochains contained in the expansion (3.1) define the 2-cycle given by (3.1) uniquely.

For any permutation $\sigma \in S_4$ and any point $\mathbf{m} \in \mathbf{T}^4_N$ we introduce five 2-cochains from $C^2(\mathbf{T}^4_N, \mathbf{Z})$ by defining their Fourier transforms

$$\begin{aligned}(\hat{a}[\sigma(1),\sigma(2);\mathbf{m}])^{\sim}_{i_1 i_2}(\mathbf{l}) &= (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)}) \\ &\times \exp\,[2\pi i N^{-1}((N-1)(l_{o(1)}+l_{o(2)}) \\ &+ m_{o(3)}l_{o(3)} + m_{o(4)}l_{o(4)})]\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned}&(\hat{b}[\sigma(2),\sigma(3);(\sigma(4));\mathbf{m}])^{\sim}_{i_1 i_2}(\mathbf{l}) \\ &= (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)}) \Bigg[\prod_{k=2,3} (\exp\,[2\pi i N^{-1} m_{\sigma(k)} l_{\sigma(k)}] \\ &-\exp\,[2\pi i N^{-1}(N-1)l_{\sigma(k)}])\Bigg] (\exp\,[-2\pi i N^{-1} l_{\sigma(3)}] - 1)^{-1} \\ &\times N\delta_{l_{\sigma(4)},0} \exp\,[2\pi i N^{-1}(N-1)l_{\sigma(1)}]\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned}&(\hat{b}[[\sigma(2);\{\sigma(3),\sigma(4)\}];\mathbf{m}])^{\sim}_{i_1 i_2}(\mathbf{l}) \\ &= (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)}) \exp\,[2\pi i N^{-1}(N-1)l_{\sigma(1)}] \\ &\times (\exp\,[2\pi i N^{-1} m_{\sigma(2)} l_{\sigma(2)}] - \exp\,[2\pi i N^{-1}(N-1)l_{\sigma(2)}]) \end{aligned}$$

$$\times\left[\prod_{k=3,4}((\exp\,[2\pi iN^{-1}m_{\sigma(k)}l_{\sigma(k)}] - \exp\,[2\pi iN^{-1}(N-1)l_{\sigma(k)}])\right.$$
$$\times(\exp\,[-2\pi iN^{-1}l_{\sigma(k)}]-1)^{-1}) + \sum_{\tau\in S_4;\tau(k)=k,k=1,2}(\exp\,[2\pi iN^{-1}m_{\sigma\tau(4)}l_{\sigma\tau(4)}]$$
$$\left.-\exp\,[2\pi iN^{-1}(N-1)l_{\sigma\tau(4)}])(\exp\,[-2\pi iN^{-1}l_{\sigma\tau(4)}]-1)^{-1}N\delta_{l_{\sigma\tau(3)},0}\right], \quad (3.11)$$

$$(\hat{c}[\sigma(1),\sigma(2),\sigma(3);(\sigma(4));\mathbf{m}])^{\sim}_{i_1 i_2}(\mathbf{l}) = (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)})N\delta_{l_{\sigma(4)},0}$$
$$\times\left[\prod_{k=1,2,3}(\exp[2\pi iN^{-1}m_{\sigma(k)}l_{\sigma(k)}] - \exp\,[2\pi iN^{-1}(N-1)l_{\sigma(k)}])\right]$$
$$\times(\,\exp\,[-2\pi iN^{-1}l_{\sigma(3)}]-1)^{-1}\,, \quad (3.12)$$

$$(\hat{c}[[\sigma(1),\sigma(2);\{\sigma(3),\sigma(4)\}];\mathbf{m}])^{\sim}_{i_1 i_2}(\mathbf{l})$$
$$= (\delta_{i_1,\sigma(1)}\delta_{i_2,\sigma(2)} - \delta_{i_1,\sigma(2)}\delta_{i_2,\sigma(1)})\left(\prod_{k=1,2}(\exp\,[2\pi iN^{-1}m_{\sigma(k)}l_{\sigma(k)}]\right.$$
$$\left.-\,\exp\,[2\pi iN^{-1}(N-1)l_{\sigma(k)}])\right)\left[\prod_{k=3,4}((\exp\,[2\pi iN^{-1}m_{\sigma(k)}l_{\sigma(k)}]\right.$$
$$-\,\exp\,[2\pi iN^{-1}(N-1)l_{\sigma(k)}])(\exp\,[-2\pi iN^{-1}l_{\sigma(k)}]-1)^{-1})$$
$$+\sum_{\tau\in S_4;\tau(k)=k,k=1,2}(\exp\,[2\pi iN^{-1}m_{\sigma\tau(4)}l_{\sigma\tau(4)}] - \exp\,[2\pi iN^{-1}(N-1)l_{\sigma\tau(4)}])$$
$$\left.\times(\exp\,[-2\pi iN^{-1}l_{\sigma\tau(4)}]-1)^{-1}N\delta_{l_{\sigma\tau(3)},0}\right]. \quad (3.13)$$

The inner product (2.14) of the 2-cochains given by the Fourier transforms (3.1) and (3.9) is equal to $a_{\sigma(1)\sigma(2)}(\mathbf{m})$. The function (3.9) is antisymmetric under a permutation of the indices $\sigma(1),\sigma(2)$ and it is symmetric under a permutation of the indices $\sigma(3),\sigma(4)$. Hence the independent functions (3.9) are related to six permutations $\sigma\in S_4$ satisfying the conditions $\sigma(1)<\sigma(2)$, $\sigma(4)<\sigma(3)$. The inner product (2.14) of the 2-cochains given by the Fourier transforms (3.1) and (3.10) is equal to $b^{(\sigma(4))}_{\sigma(2)\sigma(3)}(\mathbf{m})$. Since it is antisymmetric under a permutation of the indices $\sigma(2),\sigma(3)$ the independent projections on $Z_2(\mathbf{T}^4_N,\mathbf{R})$ of the 2-cochains given by the Fourier transforms (3.10) correspond to twelve permutations $\sigma\in S_4$ satisfying the condition $\sigma(2)<\sigma(3)$. The inner product (2.14) of the 2-cochains given by the Fourier transforms (3.1) and (3.12) is equal to $c^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}(\mathbf{m})$. Since it is antisymmetric under a permutation of the indices $\sigma(1),\sigma(2),\sigma(3)$ the independent projections on $Z_2(\mathbf{T}^4_N,\mathbf{R})$ of the 2-cochains given by the Fourier transforms (3.12) are related to four permutations $\sigma\in S_4$ satisfying the condition $\sigma(1)<\sigma(2)<\sigma(3)$. The inner product (2.14) of the 2-cochains given by the Fourier transforms (3.1) and (3.11) is equal to $b_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$. These functions are symmetric under a permutation of the indices $\sigma(3),\sigma(4)$ and satisfy the symmetry equation (3.2). This equation allows to represent $b_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$, where $\sigma(2)>\sigma(3),\sigma(4)$, as a sum of the functions $b_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$, where $\sigma\in S_4$ and $\sigma(3)>\sigma(2)$ or $\sigma(4)>\sigma(2)$. By using the symmetry under the indices $\sigma(3),\sigma(4)$ we can add the condition $\sigma(3)>\sigma(4)$. Thus the independent functions $b_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$ are related to eight permutations

$\sigma \in S_4$ satisfying the condition $\sigma(2), \sigma(4) < \sigma(3)$. Therefore the independent projections on $Z_2(\mathbf{T}_N^4, \mathbf{R})$ of the 2-cochains given by the Fourier transforms (3.11) correspond to eight permutations $\sigma \in S_4$ satisfying the condition $\sigma(2), \sigma(4) < \sigma(3)$. The inner product (2.14) of the 2-cochains given by the Fourier transforms (3.1) and (3.13) is equal to $c_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$. These functions are antisymmetric under a permutation of the indices $\sigma(1), \sigma(2)$ and they are symmetric under a permutation of the indices $\sigma(3), \sigma(4)$. They satisfy the symmetry equation (3.3). Applying the arguments given above it is possible to assume that for the independent functions $c_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$ the condition $\sigma(2), \sigma(4) < \sigma(3)$ holds. The antisymmetry under a permutation of the indices $\sigma(1), \sigma(2)$ enables us to add the condition $\sigma(1) < \sigma(2)$. Summing up we have $\sigma(1) < \sigma(2) < \sigma(3), \sigma(4) < \sigma(3)$. These conditions are equivalent to the conditions $\sigma(1) < \sigma(2) < \sigma(3), \sigma(4) \mp 4$. Thus the independent projections on $Z_2(\mathbf{T}_N^4, \mathbf{R})$ of the 2-cochains given by the Fourier transforms (3.13) correspond with three permutations $\sigma \in S_4$ satisfying the conditions $\sigma(1) < \sigma(2) < \sigma(3), \sigma(4) \mp 4$. Hence we have proved the following:

Proposition 3.2. *Every element $\bar{z} \in \bar{Z}_2(\mathbf{T}_N^4, \mathbf{R})$ has the following form*:

$$\bar{z}_{i_1 i_2}(\mathbf{m}) = (P\hat{z})_{i_1 i_2}(\mathbf{m}), \tag{3.14}$$

$$\begin{aligned}
\hat{z}_{i_1 i_2}(\mathbf{m}) = & \sum_{\sigma \in S_4; \sigma(1) < \sigma(2); \sigma(4) < \sigma(3)} \sum_{k_{\sigma(3)}, k_{\sigma(4)}=0}^{N-1} \bar{a}_{\sigma(1)\sigma(2)}(\mathbf{k})(\hat{a}[\sigma(1), \sigma(2); \mathbf{k}])_{i_1 i_2}(\mathbf{m}) \\
& + \sum_{\sigma \in S_4; \sigma(2) < \sigma(3)} \sum_{k_{\sigma(2)}, k_{\sigma(3)}=0}^{N-2} \bar{b}^{(\sigma(4))}_{\sigma(2)\sigma(3)}(\mathbf{k})(\hat{b}[\sigma(2), \sigma(3); (\sigma(4)); \mathbf{k}])_{i_1 i_2}(\mathbf{m}) \\
& + \sum_{\sigma \in S_4; \sigma(2), \sigma(4) < \sigma(3)} \sum_{k_{\sigma(2)}, k_{\sigma(3)}, k_{\sigma(4)}=0}^{N-2} \bar{b}_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{k}) \\
& \quad \times (\hat{b}[[\sigma(2); \{\sigma(3), \sigma(4)\}]; \mathbf{k}])_{i_1 i_2}(\mathbf{m}) \\
& + \sum_{\sigma \in S_4; \sigma(1) < \sigma(2) < \sigma(3)} \sum_{k_{\sigma(1)}, k_{\sigma(2)} k_{\sigma(3)}=0}^{N-2} \bar{c}^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}(\mathbf{k}) \\
& \quad \times (\hat{c}[\sigma(1), \sigma(2), \sigma(3); (\sigma(4)); \mathbf{k}]])_{i_1 i_2}(\mathbf{m}) \\
& + \sum_{\sigma \in S_4; \sigma(1) < \sigma(2) < \sigma(3), \sigma(4) \mp 4} \sum_{k_1, \cdots, k_4=0}^{N-2} \bar{c}_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{k}) \\
& \quad \times (\hat{c}[[\sigma(1), \sigma(2); \{\sigma(3), \sigma(4)\}]; \mathbf{k}])_{i_1 i_2}(\mathbf{m})
\end{aligned} \tag{3.15}$$

where P is the projector (2.19), *the* 2-*cochains* $(\hat{a}[\sigma(1), \sigma(2); \mathbf{k}])_{i_1 i_2}(\mathbf{m}), (\hat{b}[\sigma(2), \sigma(3); (\sigma(4)); \mathbf{k}])_{i_1 i_2}(\mathbf{m}), (\hat{b}[[\sigma(2); \{\sigma(3), \sigma(4)\}]; \mathbf{k}])_{i_1 i_2}(\mathbf{m}), (\hat{c}[\sigma(1), \sigma(2), \sigma(3); (\sigma(4)); \mathbf{k}])_{i_1 i_2}(\mathbf{m})$ *and* $(\hat{c}[[\sigma(1), \sigma(2); \{\sigma(3), \sigma(4)\}]; \mathbf{k}])_{i_1 i_2}(\mathbf{m})$ *are defined by the equalities* (3.9), (3.10), (3.11), (3.12) *and* (3.13) *respectively. The integer valued functions* $\bar{a}_{\sigma(1)\sigma(2)}(k_{\sigma(3)}, k_{\sigma(4)}), \bar{b}^{(\sigma(4))}_{\sigma(2)\sigma(3)}(k_{\sigma(2)}, k_{\sigma(3)}), \bar{b}_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(k_{\sigma(2)}, k_{\sigma(3)}, k_{\sigma(4)}), \bar{c}^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}$ $(k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)})$ *and* $\bar{c}_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{k})$ *are independent and they determine the element* (3.14) *uniquely.*

In order to obtain the four dimensional version of Proposition 2.4 it is necessary to know the explicit expressions similar to (2.33)–(2.36) of the independent functions contained in (3.15) through the 2-cochain (3.15). The definitions (3.9)–(3.13) and (3.15) imply that for $i_1 < i_2$,

$$(\hat{z})^{\sim}_{i_1 i_2}(\mathbf{0}) = \sum_{k_{i_3}, k_{i_4}=0}^{N-1} \bar{a}_{i_1 i_2}(k_{i_3}, k_{i_4}), \tag{3.16}$$

where $i_3 < i_4$ and $i_1, \ldots, i_4$ is a permutation of the numbers $1, \ldots, 4$.

It follows from the definitions (2.16) and (3.9)–(3.13) that

$$\begin{aligned}(\partial^* \hat{a}[\sigma(1), \sigma(2); \mathbf{k}])^{\sim}_{i_1 i_2 i_3}(\mathbf{l}) = & \sum_{\tau \in S_4; \tau(p)=p, p=1,2} \sum_{\rho \in S_3} \operatorname{sgn} \rho \delta_{i_{\rho(1)}, \sigma(1)} \delta_{i_{\rho(2)}, \sigma(2)} \delta_{i_{\rho(3)}, \sigma\tau(3)} \\ & \times \exp [2\pi i N^{-1}((N-1)(l_{\sigma(1)} + l_{\sigma(2)}) \\ & + k_{\sigma\tau(4)} l_{\sigma\tau(4)})] \\ & \times (\exp [2\pi i N^{-1}(k_{\sigma\tau(3)} - 1) l_{\sigma\tau(3)}] \\ & - \exp [2\pi i N^{-1} k_{\sigma\tau(3)} l_{\sigma\tau(3)}]),\end{aligned} \tag{3.17}$$

$$\begin{aligned}&(\partial^* \hat{b}[\sigma(2), \sigma(3); (\sigma(4)); \mathbf{k}])^{\sim}_{i_1 i_2 i_3}(\mathbf{l}) \\ &= \sum_{\rho \in S_3} \operatorname{sgn} \rho \delta_{i_{\rho(1)}, \sigma(1)} \delta_{i_{\rho(2)}, \sigma(2)} \delta_{i_{\rho(3)}, \sigma(3)} N \delta_{l_{\sigma(4)}, 0} \exp [2\pi i N^{-1}(N-1) l_{\sigma(1)}] \\ &\times \left[\prod_{p=2,3} (\exp [2\pi i N^{-1} k_{\sigma(p)} l_{\sigma(p)}] - \exp [2\pi i N^{-1}(N-1) l_{\sigma(p)}]) \right],\end{aligned} \tag{3.18}$$

$$\begin{aligned}&(\partial^* \hat{b}[[\sigma(2); \{\sigma(3), \sigma(4)\}]; \mathbf{k}])^{\sim}_{i_1 i_2 i_3}(\mathbf{l}) \\ &= \sum_{\tau \in S_4; \tau(p)=p, p=1,2} \sum_{\rho \in S_3} \operatorname{sgn} \rho \delta_{i_{\rho(1)}, \sigma(1)} \delta_{i_{\rho(2)}, \sigma(2)} \delta_{i_{\rho(3)}, \sigma\tau(3)} \\ &\times \exp [2\pi i N^{-1}(N-1) l_{\sigma(1)}] \\ &\times \left[\prod_{p=2,3} (\exp [2\pi i N^{-1} k_{\sigma\tau(p)} l_{\sigma\tau(p)}] - \exp [2\pi i N^{-1}(N-1) l_{\sigma\tau(p)}]) \right] \\ &\times \sum_{k'_{\sigma\tau(4)} = k_{\sigma\tau(4)}+1}^{N-1} \exp [2\pi i N^{-1} k'_{\sigma\tau(4)} l_{\sigma\tau(4)}],\end{aligned} \tag{3.19}$$

$$\begin{aligned}&(\partial^* \hat{c}[\sigma(1), \sigma(2), \sigma(3); (\sigma(4)); \mathbf{k}])^{\sim}_{i_1 i_2 i_3}(\mathbf{l}) \\ &= \sum_{\rho \in S_3} \operatorname{sgn} \rho \delta_{i_{\rho(1)}, \sigma(1)} \delta_{i_{\rho(2)}, \sigma(2)} \delta_{i_{\rho(3)}, \sigma(3)} \\ &\times \left[\prod_{p=1,2,3} (\exp [2\pi i N^{-1} k_{\sigma(p)} l_{\sigma(p)}] - \exp [2\pi i N^{-1}(N-1) l_{\sigma(p)}]) \right] \\ &\times N \delta_{l_{\sigma(4)}, 0},\end{aligned} \tag{3.20}$$

$$\begin{aligned}&(\partial^* \hat{c}[[\sigma(1), \sigma(2); \{\sigma(3), \sigma(4)\}]; \mathbf{k}])^{\sim}_{i_1 i_2 i_3}(\mathbf{l}) \\ &= \sum_{\tau \in S_4; \tau(p)=p, p=1,2} \sum_{\rho \in S_3} \operatorname{sgn} \rho \delta_{i_{\rho(1)}, \sigma(1)} \delta_{i_{\rho(2)}, \sigma(2)} \delta_{i_{\rho(3)}, \sigma\tau(3)} \\ &\times \left[\prod_{p=1,2,3} (\exp [2\pi i N^{-1} k_{\sigma\tau(p)} l_{\sigma\tau(p)}] - \exp [2\pi i N^{-1}(N-1) l_{\sigma\tau(p)}]) \right] \\ &\times \sum_{k'_{\sigma\tau(4)} = k_{\sigma\tau(4)}+1}^{N-1} \exp [2\pi i N^{-1} k'_{\sigma\tau(4)} l_{\sigma\tau(4)}].\end{aligned} \tag{3.21}$$

We notice that the functions (3.17)–(3.21) have the same properties with respect to $\sigma(1),\ldots,\sigma(4),\mathbf{k}$ as the cochains contained in the expansion (3.1) have. In particular, the functions (3.19) and (3.21) satisfy Eqs. (3.2) and (3.3), respectively.

Let us consider four permutations $\sigma_q \in S_4$ satisfying the conditions $\sigma_q(1) < \sigma_q(2) < \sigma_q(3), \sigma_q(4) = q, q = 1,2,3,4$. In the relations (3.17)–(3.21) we substitute the indices $i_p = \sigma_q(p), p = 1,2,3$. Now it is easy to show that the inverse Fourier transforms of the functions (3.17)–(3.19) and (3.21) are equal to zero for $0 \leqq m_{\sigma_q(1)}, m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2, m_{\sigma_q(4)} = 0$ and the inverse Fourier transform of the function (3.20) isn't zero if the sets of numbers $\{\sigma_q(1),\sigma_q(2),\sigma_q(3)\}$ and $\{\sigma(1),\sigma(2),\sigma(3)\}$ coincide. If $\sigma(1) < \sigma(2) < \sigma(3)$ this condition implies $\sigma_q = \sigma$. Therefore we have proved that for $0 \leqq m_{\sigma_q(1)}, m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2, m_{\sigma_q(4)} = 0, q = 1,2,3,4$

$$(\partial^* \hat{z})_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m}) = \bar{c}^{(\sigma_q(4))}_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m})\,. \tag{3.22}$$

The right-hand sides of the relations (3.22) provide all four independent functions $\bar{c}^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}(\mathbf{m})$ contained in the expansion (3.15).

Let us substitute in the relations (3.17)–(3.21) the indices $i_p = \sigma_q(p), p,q = 1,2,3$. The inverse Fourier transforms of the functions (3.17)–(3.19) are equal to zero for $0 \leqq m_{\sigma_q(1)}, m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2, 1 \leqq m_{\sigma_q(4)} \leqq N-1, q = 1,2,3$, and by the above arguments the inverse Fourier transform of the function (3.20) is equal to $\delta_{\sigma_q,\sigma}\prod_{p=1.2.3}\delta_{k_{\sigma(p)},m_{\sigma(p)}}$, if $\sigma \in S_4, \sigma(1) < \sigma(2) < \sigma(3)$. The inverse Fourier transform of the function (3.21) isn't zero if the set of numbers $\{\sigma_q(1),\sigma_q(2),\sigma_q(3)\}$ coincides with one of the sets $\{\sigma(1),\sigma(2),\sigma(3)\}$ or $\{\sigma(1),\sigma(2),\sigma(4)\}$. The set of the permutations $\sigma \in S_4$ satisfying the conditions $\sigma(1) < \sigma(2) < \sigma(3), \sigma(4) \neq 4$ consists of three permutations $\sigma_q, q = 1,2,3$. Since $\sigma_q(3) = 4$ the sets of numbers $\{\sigma_{q_1}(1),\sigma_{q_1}(2),\sigma_{q_1}(3)\}$ and $\sigma_{q_2}(1),\sigma_{q_2}(2),\sigma_{q_2}(4)\}$ do not coincide. The set of numbers $\{\sigma_{q_1}(1),\sigma_{q_1}(2),\sigma_{q_1}(3)\}$ and $\{\sigma_{q_2}(1),\sigma_{q_2}(2),\sigma_{q_2}(3)\}$ coincide only for $\sigma_{q_1} = \sigma_{q_2}$. Now it is clear that for $0 \leqq m_{\sigma_q(1)}, m_{\sigma_q(2)} m_{\sigma_q(3)} \leqq N-2, 1 \leqq m_{\sigma_q(4)} \leqq N-1, q = 1,2,3$,

$$\begin{aligned}(\partial^* \hat{z})_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m}) &= \bar{c}^{(\sigma_q(4))}_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}\mathbf{m}) \\ &\quad + \sum_{m'_{\sigma_q(4)}=0}^{m_{\sigma_q}(4)-1} \bar{c}_{[\sigma_q(1)\sigma_q(2);\{\sigma_q(3),\sigma_q(4)\}]}(\mathbf{m})|_{m_{\sigma_q(4)}=m'_{\sigma_q(4)}}\,.\end{aligned} \tag{3.23}$$

The linear combinations with integer coefficients of the right-hand sides of the relations (3.22), (3.23) enable us to obtain all the three independent functions $\bar{c}_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{m})$ contained in the expansion (3.15).

Let us substitute in the relations (3.17)–(3.21) the indices $i_p = \sigma_q(p), p = 1,2,3, q = 1,2,3,4$. The inverse Fourier transforms of the functions (3.17), (3.19) and (3.21) are equal to zero for $m_{\sigma_q(1)} = N-1, 0 \leqq m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2, m_{\sigma_q(4)} = 0$. Let us consider the inverse Fourier transform of the function (3.18). It is not zero if two sets of numbers $\{\sigma_q(1),\sigma_q(2),\sigma_q(3)\}$ and $\{\sigma(1),\sigma(2),\sigma(3)\}$ coincide. Due to conditions $m_{\sigma_q(1)} = N-1, 0 \leqq m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2$ this implies $\sigma(1) = \sigma_q(1)$. The additional condition $\sigma(2) < \sigma(3)$ gives $\sigma = \sigma_q$. Applying similar arguments and the arguments given for the proof of the relation (3.22) now we can prove that for $m_{\sigma_q(j)} = N-1, 0 \leqq m_{\sigma_q(2)}, \widehat{m_{\sigma_q(j)}}, m_{\sigma_q(3)} \leqq N-2, m_{\sigma_q(4)} = 0, j = 1,2,3, q = 1,2,3,4,$

$$(\partial^* \hat{z})_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m}) = (-1)^{j+1} \bar{b}^{(\sigma_q(4))}_{\sigma_q(1)\widehat{\sigma_q(j)}\sigma_q(3)}(\mathbf{m}) - \sum_{m'_{\sigma_q(j)}=0}^{N-1} \bar{c}^{(\sigma_q(4))}_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m})|_{m_{\sigma_q(j)}=m'_{\sigma_q(j)}} . \tag{3.24}$$

The linear combinations with integer coefficients of the right-hand sides of the relations (3.22) and (3.24) allow us to compute all the twelve independent functions $\bar{b}^{(\sigma(4))}_{\sigma(2)\sigma(3)}(\mathbf{m})$ contained in the expansion (3.15).

Let us substitute in the relations (3.17)–(3.21) the indices $i_p = \sigma_q(p), p,q = 1,2,3$. The inverse Fourier transform of the function (3.17) is equal to zero for $m_{\sigma_q(1)} = N-1, 0 \leqq m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2$. Let us consider the inverse Fourier transform of the function (3.19). It is not zero if the set of numbers $\{\sigma_q(1), \sigma_q(2), \sigma_q(3)\}$ coincides with one of the sets $\{\sigma(1),\sigma(2),\sigma(3)\}$ or $\{\sigma(1),\sigma(2),\sigma(4)\}$. The additional conditions $m_{\sigma_q(1)} = N-1, 0 \leqq m_{\sigma_q(2)}, m_{\sigma_q(3)} \leqq N-2$ imply that $\sigma_q(1) = \sigma(1)$. If we suppose that $\sigma(2),\sigma(4) < \sigma(3)$, the sets $\{\sigma_q(1),\sigma_q(2),\sigma_q(3)\}$ and $\{\sigma(1),\sigma(2),\sigma(4)\}$ do not coincide because of $\sigma_q(3) = 4 \neq \sigma(2), \sigma(4) < \sigma(3)$. There remains one possibility that the sets of numbers $\{\sigma_q(1),\sigma_q(2),\sigma_q(3)\}$ and $\{\sigma(1),\sigma(2),\sigma(3)\}$ coincide. Due to the condition $\sigma(2) < \sigma(3)$ this implies $\sigma = \sigma_q$. These and analogous arguments yield for $m_{\sigma_q(j)} = N-1, 0 \leqq m_{\sigma_q(1)}, \widehat{m_{\sigma_q(j)}}, m_{\sigma_q(3)} \leqq N-2, 1 \leqq m_{\sigma_q(4)} \leqq N-1, j = 1,2, q = 1,2,3,$

$$(\partial^* \hat{z})_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m}) = (-1)^{j+1} \sum_{m'_{\sigma_q(4)}=0}^{m_{\sigma_q}(4)-1} \bar{b}_{[\sigma_q(\tau_{12}(j));\{\sigma_q(3),\sigma_q(4)\}]}(\mathbf{m})|_{m_{\sigma_q(4)}=m'_{\sigma_q(4)}} + L_j(\bar{b}^{(\sigma(4))}_{\sigma(2)\sigma(3)}, \bar{c}^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}, \bar{c}_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}) , \tag{3.25}$$

where τ_{12} is the unique non-trivial permutation of the numbers 1,2 and L_j is a linear combination with integer coefficients of its variables.

The case $m_{\sigma_q(3)} = N-1, 0 \leqq m_{\sigma_q(1)}, m_{\sigma_q(2)} \leqq N-2$ needs a special consideration. Let us choose two permutations $\sigma_q \in S_4, q = 1,2$. They have the following property : $\sigma_q(2) = 3, \sigma_q(3) = 4$. If $\sigma(1) = \sigma_q(3) = 4$ and $\sigma(2),\sigma(4) < \sigma(3)$ the sets of numbers $\{\sigma(1),\sigma(2),\sigma(4)\}$ and $\{\sigma_q(1),\sigma_q(2),\sigma_q(3)\}$ cannot coincide because of $\sigma_q(2) = 3 \neq \sigma(2), \sigma(4) < \sigma(3)$. The remaining arguments are similar to the above used ones and for $m_{\sigma_q(3)} = N-1, 0 \leqq m_{\sigma_q(1)}, m_{\sigma_q(2)} \leqq N-2, 1 \leqq m_{\sigma_q(4)} \leqq N-1, q = 1,2$ we get

$$(\partial^* \hat{z})_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m}) = \sum_{m'_{\sigma_q(4)}=0}^{m_{\sigma_q}(4)-1} \bar{b}_{[\sigma_q(1);\{\sigma_q(2),\sigma_q(4)\}]}(\mathbf{m})|_{m_{\sigma_q(4)}=m'_{\sigma_q(4)}} + L_3(\bar{b}^{(\sigma(4))}_{\sigma(2)\sigma(3)}, \bar{c}^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}, \bar{c}_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}) . \tag{3.26}$$

The linear combinations with integer coefficients of the relations (3.22)–(3.26) give all the eight independent functions $\bar{b}_{[\sigma(1);\{\sigma(2),\sigma(4)\}]}(\mathbf{m})$ contained in the expansion (3.15).

Let us substitute in the relations (3.17)–(3.21) the indices $i_p = \sigma_q(p), p = 1,2,3, q = 1,2,3,4$. The inverse Fourier transform of the function (3.17) for $m_{\sigma_q(1)} =$

$m_{\widehat{\sigma_q(j)}} = m_{\sigma_q(3)} = N-1; 0 \leqq m_{\sigma_q(j)} \leqq N-2; j = 1,2,3; 0 \leqq m_{\sigma_q(4)} \leqq N-1; q = 1,$ $2,3,4$; isn't zero if two sets of numbers $\{\sigma_q(1), \widehat{\sigma_q(j)}, \sigma_q(3)\}$, $\{\sigma(1), \sigma(2)\}$ coincide and if the set of numbers $\{\sigma_q(1), \sigma_q(2), \sigma_q(3)\}$ coincides with one of the sets $\{\sigma(1), \sigma(2), \sigma(3)\}$ or $\{\sigma(1), \sigma(2), \sigma(4)\}$. Let us suppose that $\sigma(1) < \sigma(2)$, $\sigma(4) < \sigma(3)$. Then the above mentioned conditions imply that for $\sigma_q(j) > \sigma_q(4)$ we get $\sigma_q(j) = \sigma(3), \sigma_q(4) = \sigma(4)$ and for $\sigma_q(j) < \sigma_q(4)$ we get $\sigma_q(j) = \sigma(4), \sigma_q(4) = \sigma(3)$. Now it is simple to prove that for $m_{\sigma_q(1)} = m_{\widehat{\sigma_q(j)}} = m_{\sigma_q(3)} = N-1; 0 \leqq m_{\sigma_q(j)} \leqq N-2; 0 \leqq m_{\sigma_q(4)} \leqq N-1; j = 1,2,3; q = 1,2,3,4$; the following relation holds:

$$(\partial^* \hat{z})_{\sigma_q(1)\sigma_q(2)\sigma_q(3)}(\mathbf{m}) = (-1)^{j+1}(\bar{a}_{\sigma_q(1)\widehat{\sigma_q(j)}\sigma_q(3)}(m_{\sigma_q(j)} + 1, m_{\sigma_q(4)}) \\ - \bar{a}_{\sigma_q(1)\widehat{\sigma_q(j)}\sigma_q(3)}(m_{\sigma_q(j)}, m_{\sigma_q(4)})) + L_j(b,c)\,, \quad (3.27)$$

where $L_j(b,c)$ is a linear combination with integer coefficients of the independent functions $\bar{b}^{(\sigma(4))}_{\sigma(2)\sigma(3)}(\mathbf{k}), \bar{b}_{[\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{k}), \bar{c}^{(\sigma(4))}_{\sigma(1)\sigma(2)\sigma(3)}(\mathbf{k})$ and $\bar{c}_{[\sigma(1)\sigma(2);\{\sigma(3),\sigma(4)\}]}(\mathbf{k})$. The linear combinations with integer coefficients of the right-hand sides of the relations (3.22)–(3.27) give all the twelve functions $\bar{a}_{\sigma(1)\sigma(2)}(m_{\sigma(3)} + 1, m_{\sigma(4)}) - \bar{a}_{\sigma(1)\sigma(2)}(m_{\sigma(3)}, m_{\sigma(4)})$, $\bar{a}_{\sigma(1)\sigma(2)}(m_{\sigma(3)}, m_{\sigma(4)} + 1) - \bar{a}_{\sigma(1)\sigma(2)}(m_{\sigma(3)}, m_{\sigma(4)})$, where permutations $\sigma \in S_4, \sigma(1) < \sigma(2)$, $\sigma(4) < \sigma(3)$.

Applying the relations (3.16) and (3.22)–(3.27) it is possible to modify the proof of Proposition 2.4 for the four dimensional case.

Proposition 3.3. *Let a θ-function $\Theta((\phi, \bar{\mathbf{z}})|2\pi i\beta\Omega^{-1})$ be given by the equality (2.11). Then for any sequence $\phi_N \in C^2(\mathbf{T}^4_N, \mathbf{Z})$ and for any numbers $\beta_0 > 0$, $\gamma > 4$,*

$$\lim_{N\to\infty} \Theta((\phi_N, \bar{\mathbf{z}})|2\pi i\beta_0 N^\gamma \Omega^{-1}) = 1\,. \quad (3.28)$$

The relations (2.10), (2.49) and (3.28) imply the relation (1.12) for $d = 4$. For $d = 2$ this relation is also fulfilled but now the right-hand side of (1.12) has the continuum form of (2.7) for $g = 1$. When the torus radius tends to infinity this correlation function of type (2.7) converges to the trivial correlation function of the **R**-gauge Euclidean two dimensional electrodynamics [7].

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Communicated by A. Jaffe