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The Markov Branching Random Walk and Systems of Reaction-Diffusion (Kolmogorov–Petrovskii–Piskunov) Equations★

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Abstract: A general model of a branching random walk in $\mathbb{R}^1$ is considered, with several types of particles, where the branching occurs with probabilities determined by the type of a parent particle. Each new particle starts moving from the place where it was born, independently of other particles. The distribution of the displacement of a particle, before it splits, depends on its type. A necessary and sufficient condition is given for the random variable

$$X^0 = \sup_{n \geqq 0} \max_{1 \leqq k \leqq N_n} X_{n,k}$$

to be finite. Here, $X_{n,k}$ is the position of the k^{th} particle in the n^{th} generation, N_n is the number of particles in the n^{th} generation (regardless of their type). It turns out that the distribution of X^0 gives a minimal solution to a natural system of stochastic equations which has a linearly ordered continuum of other solutions. The last fact is used for proving the existence of a monotone travelling-wave solution to systems of coupled non-linear parabolic PDE's.

1. Introduction and the Results

The purpose of this paper is twofold. First, we extend (and make more precise) results concerning the asymptotics of the single-type branching random walk obtained in [KKS 1–3] to the multi-type case. Secondly, and perhaps more importantly, we derive, from our results, a new theorem about the existence of monotone travelling waves, for a general system of coupled reaction-diffusion equations (otherwise known as Fisher or Kolmogorov–Petrovskii–Piskunov equation (see [F] and [KoPP])[1]. The connection between the reaction-diffusion

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[1] In their papers, Fisher and Kolmogorov, Petrovskii and Piskunov considered the case of a single equation only; by the present time the theory of a single reaction-diffusion equation is much more elaborated than the theory of reaction-diffusion systems, where many basic questions remain open

equations and branching random walk has been widely used since the papers by McKean [McK 1, 2]. Namely, it is the so-called branching diffusion process that figures in a formula for the solution of the Cauchy problem for reaction-diffusion equations. However, in this paper we use a *different* branching random walk model, called exponential, which is directly related to travelling waves. The literature devoted to the problem of existence (and uniqueness) of travelling wave solutions and of convergence to these solutions is vast; a detailed reference list may be found, e.g., in a recent paper [VV] and a forthcoming book [VVV].[2] Still, our method seems new: we believe that the opportunities provided by this method are not at all exhausted by the present result.

An alternative approach, also based on probabilistic ideas, was recently developed in [CHTWW]. [It is the case of two equations that was considered there.] The classes of reaction-diffusion systems considered in [CHTWW] and in the present paper are slightly different. Nevertheless, some comparison seems appropriate. In both papers, the conditions proposed are only sufficient, for the existence of a monotone travelling wave with a given velocity. [A necessary condition was established in [VV, VVV], but this condition is hard to check.] However, in [CHTWW] the authors are able to prove the uniqueness of such a solution. We believe that the uniqueness should also hold for our class of systems, but this question remains open. Another important question left open in this paper is that of the convergence of a solution of the Cauchy problem to a travelling wave, as time increases indefinitely. This question was partially answered by the authors in [CHTWW] for their class of systems.

1.1. A multi-type branching random walk: General results. Consider the following model of a discrete-time branching random walk in $\mathbb{R}^1$, with particles of several types labelled $1, \ldots, M$. A particle of type j created at time t moves randomly, from the point where it was created, to another point and splits, at time $t+1$ (the time of its death), into a random collection of offspring particles described by a vector $\mathbf{l} = (l_1, \ldots, l_M)$, where l_k is the number of type k offspring created. The probability of creating a sample $\mathbf{l} = (l_1, \ldots, l_M)$, is determined by the type j of the ancestor and denoted by $q(j; \mathbf{l})$. Each newly born particle moves independently of other particles, and the distribution $P_{j,k}$ of its random displacement, $\xi_{j,k}$, from the birthplace is determined by its type k and the type j of its ancestor.

Given that the initial particle is, at time zero, at the origin and has a prescribed type j, we can speak of the (absolute) positions of particles of the n^{th} generation at the time of their deaths. These particles are naturally labelled by the paths of distance n on a (random) oriented marked Cayley tree Γ, with a distinct initial vertex $\mathbf{O}$, which represents the structure of the descendants of the initial particle. By orientation we mean that each edge of the tree is directed out of vertex $\mathbf{O}$, and the "marks" assigned to the vertices are simply their types. The branching at each vertex occurs independently and is described by distribution $q(j; \cdot)$, where j is the type of the vertex. To each edge e of tree Γ we assign a random variable $\xi(e)$ that equals the displacement of the corresponding particle from its birthplace. Given a realization of the tree, the random variables $\xi(e)$ are (conditionally) independent and have distributions $P_{j,k}$. If $\mathbf{L}$ is a (finite) path on Γ initiated at $\mathbf{O}$ then the

[2] It is worth noting that these publications contain many results previously published in Russian and no widely available in the West

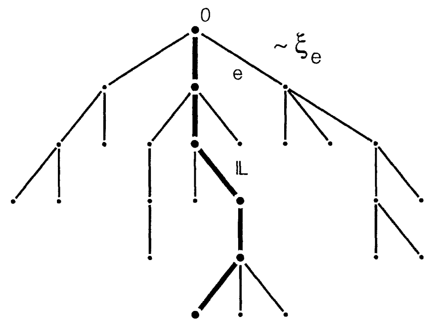

Fig. 1. A random process on a Cayley tree

positions $X_{\mathbf{L}}$ of particle $\mathbf{L}$ is given by the sum $\sum_{e \in \mathbf{L}} \xi(e)$. We are interested in the behaviour of the supremum

$$X^0 = \sup_{\mathbf{L}} X_{\mathbf{L}} \tag{1.1}$$

taken over all finite paths on the random tree which start at $\mathbf{O}$. See Fig. 1.

The first question we address is: under what condition is the distribution of X^0 proper (that is, $X^0 < \infty$ with probability one)? Denote by Π_j^0 the distribution of X^0 in the case where the initial particles has type j, $j = 1, \dots, M$. Vector $\boldsymbol{\Pi}^0$ is defined as $(\Pi_1^0, \dots, \Pi_M^0)$.

Denote by $K(j,k)$ the expected occupation numbers:

$$K(j,k) = \sum_{\mathbf{l} \geqq \mathbf{0}} q(j,\mathbf{l}) l_k, \quad 1 \leqq j,k \leqq M \ . \tag{1.2}$$

Throughout the paper we assume that $K(j,k) < \infty$ for any $j,k = 1, \dots, M$, and moreover that there exist finite exponential moments

$$\sum_{\mathbf{l} \geqq 0} q(j;\mathbf{l}) \exp\left(R_0 \sum_{k=1}^{M} l_k \right) < \infty, \quad 1 \leqq j \leqq M \ ,$$

where $R_0 > 0$. This means that the moment generating functions

$$\varphi_j(z_1, \dots, z_M) = \mathbf{E}_{q(j;\cdot)} z_1^{l_1} \dots z_M^{l_M}, \quad 1 \leqq j \leqq M \ , \tag{1.3}$$

are analytic in a poly-circle $|z_1|, \dots, |z_M| < \exp R_0$.

Another standard assumption is that all distributions $P_{j,k}$ possess exponential moments

$$\mathbf{E} e^{a \xi_{j,k}} = \int P_{j,k}(dx) e^{ax} < \infty, \quad a \in (0, a^0) \ , \tag{1.4}$$

for some a^0, $0 < a^0 \leqq \infty$. [We always refer to $a \in (0, a^0)$ while using these moments.]

Sometimes we impose various additional conditions, For example,

$$K(j,k) > 0, \quad 1 \leqq j,k \leqq M \ , \tag{1.5}$$

and

$$\min_j \sum_k K(j,k) > 1 \ . \tag{1.6}$$

Another condition: for any $j, k = 1, \ldots, M$ and any $x < 0$,

$$\int_{-\infty}^{x} [P_{j,k}](dy) > 0 \tag{1.7}$$

is also assumed at a certain point. Finally, when dealing with travelling wave solutions, we introduce further conditions on $q(j; \cdot)$ (see (1.28), (1.29)).

An important role is played by the family of matrices $\mathbf{A}(a)$, with non-negative entries

$$(\mathbf{A}(a))_{j,k} = K(j,k)\mathbf{E}e^{a\xi_{jk}}, \quad 1 \leqq j, k \leqq M . \tag{1.8}$$

According to the Perron–Frobenius Theorem (see [S]), $\mathbf{A}(a)$ has an eigenvector, $\mathbf{C} = (C_1, \ldots, C_M)$, with non-negative components C_j, such that the corresponding eigenvalue $\rho > 0$ is no less than any other eigenvalue of $\mathbf{A}(a)$. If condition (1.5) holds, all entries of $\mathbf{A}(a)$ are strictly positive, and so are the components C_j of eigenvector $\mathbf{C}$. Furthermore, eigenvalue ρ in this case is simple. [Matrix $\mathbf{A}(a)$ in general may have other (non-negative) eigenvalues with non-negative eigen-vectors.]

We sometimes write $\rho(a)$, $\mathbf{C}(a)$ and $C_j(a)$, $1 \leqq j \leqq M$, in order to stress the dependence on a.

Theorem 1. *The condition*

$$\text{there exists } a \text{ with } \rho(a) \leqq 1 \tag{1.9}$$

is sufficient for all distributions Π_j^0, $1 \leqq j \leqq M$, *to be proper* (*that is* $\Pi_j^0(\mathbb{R}^1) = 1$). *Under assumption* (1.5)–(1.7) *it is also necessary.*

The sufficiency assertion in Theorem 1 was proved in an earlier paper by Biggins [B].

It appears that the random variable X^0 satisfies a natural system of (stochastic) equations. More precisely, the equations involve vector $\boldsymbol{\Pi}^0$. If we denote by X_j the version of the random variable X^0 under the condition that the initial particle has type j, then the equations may be written in the form:

$$X_j \simeq \max\Big[0, \max_{1 \leqq k_1 \leqq l_1(j)} (X_1^{(k_1)} + \xi_{j,1}^{(k_1)}), \ldots,$$
$$\max_{1 \leqq k_M \leqq l_M(j)} (X_M^{(k_M)} + \xi_{j,M}^{(k_M)})\Big], \qquad 1 \leqq j \leqq M . \tag{1.10}$$

Here, symbol $\simeq$ means the equality in law. The random numbers $l_1(j), \ldots, l_M(j)$ indicate how many offspring of given type were produced by the initial j-vertex $\mathbf{O}$. The distribution of vector $\mathbf{l}(j) = (l_1(j), \ldots, l_M(j))$ is described by the transition probabilities $q(j; \cdot)$. Given a sample vector $\mathbf{l}(j) = (l_1(j), \ldots, l_M(j))$, the random variables $X_m^{(k_m)}$, $1 \leqq k_m \leqq l_m(j)$, $1 \leqq m \leqq M$, in the j^{th} equation (1.10) are independent; for a fixed m they have the same distribution as the random variable X_m in the l.h.s of the m^{th} equation. [These distributions are the unknowns in (1.10)]. Similarly, random variables $\xi_{j,m}^{(k_m)}$, $1 \leqq k_m \leqq l_m(j)$, $1 \leqq m \leqq M$, in the j^{th} equation (1.10) are (conditionally) independent and do not depend on the $X_m^{(k_m)}$'s; $\xi_{j,m}^{(k_m)}$ has distribution $P_{(j,m)}$. If some numbers among the $l_m(j)$'s are zeroes, the corresponding maximum is omitted.

Pictorially speaking, $X_m^{(k_m)}$ in the r.h.s of the j^{th} equation (1.10) represents a random variable of type (1.1) "viewed" from a type m vertex of the first generation on the tree: altogether there are $l_m(j)$ such vertices labeled by $k_m = 1, \ldots, l_m(j)$.

A solution to (1.10) is therefore a vector of (possibly improper) distributions, $\boldsymbol{\Pi} = (\Pi_1, \dots, \Pi_M)$. Speaking of an order between solutions, and in particular of a minimal solution, we mean the standard distribution ordering between the components of the corresponding vectors.

Theorem 2. *Vector $\boldsymbol{\Pi}^0 = (\Pi_1^0, \dots, \Pi_M^0)$ always gives a (unique) minimal solution to* (1.10). *Therefore, condition* (1.9) *is sufficient for* (1.10) *to have a solution where all distributions $\Pi_1, \dots, \Pi_M$ are proper. Under assumptions* (1.5)–(1.7) *it is also necessary.*

A surprising fact is that the solution to system (1.10), provided that it exists, is non-unique. We denote $b^+ = \max[0, b]$, $b \in \mathbb{R}^1$.

Theorem 3. *Assume that conditions* (1.5) *and* (1.6) *are fulfilled and suppose that the eigenvalue $\rho(a) < 1$ for some $a \in (0, a^0)$. Denote by α the smallest root of the equation $\rho(\alpha) = 1$:*

$$\alpha = \inf[a > 0 : \rho(a) = 1] \; ; \tag{1.11}$$

then $\alpha > 0$. Denote by β the second smallest root of equation $\rho(a) = 1$, so that $\rho(a) < 1$ for $a \in (\alpha, \beta)$. Then, for any $\gamma > 0$, there exists a unique solution $(\Pi_1^{(\gamma)}, \dots, \Pi_M^{(\gamma)})$ to (1.10) *such that, as $x \to \infty$, the distribution function $F_j^{(\gamma)}(x) = \Pi_j^{(\gamma)}((-\infty, x])$ have the representations*[3]

$$F_j^{(\gamma)}(x) = (1 - \gamma C_j(\alpha) e^{-\alpha x})^+ + O(e^{-px}) \,, \quad 1 \leqq j \leqq M \,, \tag{1.12}$$

where p may be chosen arbitrarily close, from the left, to $\min[2\alpha, \beta]$. *Therefore, there exists a linearly ordered continuum of distinct solutions to* (1.10). *Furthermore, $\boldsymbol{\Pi}^0$ is precisely the minimal solution.*

The proofs of Theorem 1–3 are carried out in Sect. 2.

The question whether the set of solutions is exhausted by those listed in Theorem 3 is one of the important open questions in the theory of branching random walk. In some cases the set of solutions is much larger. For example, the structure of set of the solutions is known for the case where $\xi_{j,k}$ takes a finite number of values (see [KKS 1], Propositions 4.5 and 4.6). Here we focus on two models, where $\xi_{j,k}$ has continuous distributions. These models will later be used for analysis of travelling wave solutions for systems of reaction-diffusion equations,

1.2. Exponential branching random walk and branching diffusion. For the sake of simplicity, we assume, up to the end of this section, that the distribution $P_{j,k}$ of the random displacement $\xi_{j,k}$ does not depend on ancestor type j. The notation ξ_k and P_k is therefore used, instead of $\xi_{j,k}$ and $P_{j,k}$, and index j indicating the ancestor type is systematically omitted.

The first model under consideration is where the distribution P_k of the random variable ξ_k is the difference of two independent exponentially distributed variables, with means μ_k^{-1} and λ_k^{-1}, respectively. We call the corresponding model exponential. It has important applications in queueing network theory (see [KKS 1, DKS]).

[3] Symbol $O(e^{-px})$ is used below for a function that does not exceed, in absolute value, be^{-px}, $x > 0$, where $b > 0$ and $p > 0$ are constants

The second model is where ξ_k is given by the displacement of a Brownian particle at the end of its exponentially distributed lifetime. Here we denote by ϖ_k^{-1} the mean value of the lifetime distribution and by β_k and $\Sigma_k^2/2$ the drift and diffusion coefficient for type k particles. In this model, ξ_k is again distributed as the difference of two independent exponentially distributed random variables. The model may be easily set in continuous time (see below); we call it the branching diffusion model.

In the case of the exponential model, a key instrument of investigation is a non-linear dynamical system of ordinary differential equations. Distribution P_k in this model has the density

$$p_k(x) = \frac{\lambda_k \mu_k}{\lambda_k + \mu_k}\left(e^{\lambda_k x}(1 - \Theta(x)) + e^{-\mu_k x}\Theta(x)\right), \tag{1.13a}$$

and the Laplace transform $\mathbf{E}e^{a\xi_k}$ reads as

$$\mathbf{E}e^{a\xi_k} = \frac{\lambda_k \mu_k}{\lambda_k \mu_k + (\mu_k - \lambda_k)a - a^2}. \tag{1.13b}$$

In Sect. 3 we prove the following result.

Theorem 4. *Let* $\boldsymbol{\Pi} = (\Pi_1, \ldots, \Pi_M)$ *be an arbitrary solution to system* (1.10), *in the exponential model. Denote by* G_k *the distribution function of the convolution* $P_k * \Pi_k$, $1 \leqq k \leqq M$. *Then the pairs* (U_k, V_k), *where*

$$U_k(x) = G_k(x), \quad V_k(x) = \frac{d}{dx}G_k(x),$$

satisfy the following system of ordinary differential equations:

$$\begin{aligned} U_k'(x) &= V_k(x), \\ V_k'(x) &= -(\mu_k - \lambda_k)V_k(x) + \lambda_k\mu_k(U_k(x) - \varphi_k(U_1(x), \ldots, U_M(x))), \\ &\quad x > 0, \quad 1 \leqq k \leqq M, \end{aligned} \tag{1.14}$$

with the conditions

$$\frac{V_k(0)}{U_k(0)} = \lambda_k, \tag{1.15}$$

$$(U_k(x), V_k(x)) \in (0,1) \times (0,\infty), \quad x > 0, \tag{1.16}$$

$$\lim_{x \to \infty} U_k(x) = 1. \tag{1.17}$$

Here φ_j *is the moment-generating function* (1.4).

Furthermore, for $x < 0$,

$$G_k(x) = G_k(0)\exp(\lambda_k x). \tag{1.18}$$

The correspondence between the solutions to (1.10) *and the trajectories of problem* (1.14)–(1.18) *is one-to-one.*

Theorem 4 allows us to state the results concerning solutions to system (1.10) in terms of the phase portrait of system (1.14) in $\mathbb{R}^M \times \mathbb{R}^M$. This is carried out in Sect. 3 (see Lemma 3.1).

Passing to the branching diffusion model, we first note that the Laplace transform $\mathbf{E}e^{a\xi_k}$ equals here

$$\frac{\varpi_k}{\varpi_k + \beta_k a - (\Sigma_k^2/2)a^2}\,. \tag{1.19}$$

We can say no more about the solutions to system (1.10) for this model than was said in Theorems 1–3, in a general situation. [It is partly caused by the fact that system (1.10) leads, for the branching diffusion model, to a (non-linear) system of ordinary differential equations with delay, rather than to a system of type (1.14).] However, the information we gain from our analysis allows us to answer the question whether the particles in branching diffusion ultimately "reach" $+$ or $-\infty$, or both infinities, or neither. Another point is that the branching diffusion model has a remarkable relation to reaction-diffusion equations.

The continuous-time setting of the branching diffusion model is as follows. At time zero we have a type j particle, $j = 1, \ldots, M$, that splits into a sample of offspring, according to distribution $q(j;\cdot)$. An offspring particle of type k moves along $\mathbb{R}^1$, according to the Wiener law, with drift β_k and the diffusion coefficient Σ_k, and independently of other particles. After an exponential lifetime, with mean ϖ_k^{-1}, it splits, according to $q(k;\cdot)$, and each new offspring then proceeds according to the same rule. Under condition (1.9) it is easy to check that the continuous-time supremum

$$X^* = \sup_{t \geqq 0} \sup_{\mathbf{L} \in \mathscr{L}(t)} X_{\mathbf{L}}(t) \tag{1.20}$$

(cf. (1.1)) is finite. Furthermore, X^* gives the (unique) minimal solution to a natural stochastic equation that again admits a continuum of other solutions. Here $\mathscr{L}(t)$ denotes the set of paths on the random Cayley tree $\Gamma(t)$ built by time $t \geqq 0$ ($\mathscr{L}(t)$ is a random set). Furthermore, $X_{\mathbf{L}}(t)$ stands for the position, at time t, of the particle labelled by a finite path $\mathbf{L} \in \mathscr{L}(t)$ (as before, we assume that after splitting the parent particle is frozen at its final position).

The relations with the reaction-diffusion equations are given by Theorem 5 below which is a straightforward generalization of a result of McKean [McK 1,2]. In this theorem, we follow a particular branch corresponding to a single particle produced at time zero. Correspondingly, $\mathbf{E}_k$ denotes the expectation value in the branching diffusion, following an initial offspring of type k, and $\mathscr{L}_k(t)$ stands for the subtree, up to time t, along this branch. Furthermore, for a given $\mathbf{L} \in \mathscr{L}_k(t)$, $n(\mathbf{L})$ denotes the type of a particle at the end of path $\mathbf{L}$.

Theorem 5. *In the branching diffusion model, let $f_k(t,x)$ denote the expectation value*

$$f_k(t,x) = \mathbf{E}_k\left[\prod_{\mathbf{L} \in \mathscr{L}_k(t)} f^0_{n(\mathbf{L})}(x + X_{\mathbf{L}}(t))\right],$$

where $f^0_k : \mathbb{R}^1 \to \mathbb{R}^1$ is a C^1-function, $1 \leqq k \leqq M$. Then functions $f_k(t,x)$ satisfy the following system of partial differential equations:

$$\begin{aligned}\frac{\partial}{\partial t} f_k(t,x) = &- \beta_k \frac{\partial}{\partial x} f_k(t,x) + \frac{1}{2}\Sigma_k^2 \frac{\partial^2}{\partial x^2} f_k(t,x)\,,\ t \geqq 0\,,\ x \in \mathbb{R}^1,\ 1 \leqq k \leqq M\,,\\ &+ \varpi_k(\varphi_k(f_1(t,x), \ldots, f_M(t,x)) - f_k(t,x))\,,\end{aligned} \tag{1.21}$$

with the Cauchy data

$$f_k(0,x) = f_k^0(x)\,, \quad x \in \mathbb{R}^1, \quad 1 \leqq k \leqq M\,. \tag{1.22}$$

1.3. Travelling waves for reaction-diffusion equations. System (1.21) is called a system of reaction-diffusion (or Fisher–Kolmogorov–Petrovskii–Piskunov) equations. See, e.g., [Bri, R, Sm]. One of the main problems in the theory of reaction-diffusion equations is that of the convergence, of the solution $f_k(t,x)$, as $t \to \infty$. One would expect that, if the initial date $f_k^0(x)$ is reasonably chosen, the solution converges to a (generalized) *travelling wave*. This means that, for some choice of (real) constants c_k, $1 \leqq k \leqq M$,

$$\lim_{t \to \infty} f_k(t, x - c_k t) = W_k(x)\,, \quad x \in \mathbb{R}^1, \quad 1 \leqq k \leqq M\,. \tag{1.23}$$

Here functions $W_k : \mathbb{R}^1 \to \mathbb{R}^1$, $1 \leqq k \leqq M$, determine the "profile" of the travelling wave; they should have the property that

$$(t,x) \mapsto W_k(x + c_k t)\,, \quad 1 \leqq k \leqq M\,,$$

is a solution to (1.21).

The last property means that $W_1, \ldots, W_M$ give a solution to a system of ordinary differential equations which differs from (1.14) only notationally:

$$\begin{aligned} W'_k(x) &= V_k(x)\,, \\ V'_k(x) &= (-2(\beta_k + c_k)/\Sigma_k^2)V_k(x) \\ &\quad + (2\varpi_k/\Sigma_k^2)(W_k(x) - \varphi_k(W_1(x), \ldots, W_M(x)))\,, \quad x \in \mathbb{R}^1, \quad 1 \leqq k \leqq M\,. \end{aligned} \tag{1.24}$$

We call system (1.24) (and equivalent system (1.14)) the travelling-wave system.

In the sequel, speaking of a travelling wave profile, for system (1.21), we refer to a vector-function $\mathbf{W} = (W_1, \ldots, W_M)$. Vector $\mathbf{c} = (c_1, \ldots, c_M)$ is called the (generalized) travelling wave velocity vector. The case $c_1 = \cdots = c_M = c$ corresponds to a "regular" travelling wave; the regular travelling waves are important in the situation where the drift coefficients vanish: $\beta_1 = \cdots = \beta_M = 0$.

Remark. A natural conjecture (motivated by results from [VV, VVV]) is that, under a certain non-decomposability condition on functions φ_k in (1.21) (and for $\beta_1 = \cdots = \beta_M = \beta$), the convergence in (1.23) is always to a proper travelling wave. An opposite example is where φ_k depends on the k^{th} argument only: in this case system (1.21) is decomposed into isolated reaction-diffusion equations, and the convergence in (1.23) is valid, in general, for different constants c_k. Our interest in this paper is focused on properties of system (1.24) (with general β_k), not on the convergence in (1.23), and we consider all possible vectors $\mathbf{c}$.

The simplest case where the convergence (1.23) is expected to hold is where the travelling wave profile $\mathbf{W}$ consists of probability distribution functions. That is, one must have

$$V_k(x) > 0\,, \quad \lim_{x \to -\infty} W_k(x)\,, \quad \lim_{x \to \infty} W_k(x) = 1, \quad 1 \leqq k \leqq M\,; \tag{1.25}$$

bound $V_k > 0$, or equivalently, the monotonicity of the travelling wave profiles W_k, is the most important property here. This is suggested by results obtained in the case of a single equation in (1.21).

In the case of several equations, under the condition that φ_k is the moment generating function of a probability distribution, and for $c_1 = \cdots = c_k = c$, it was established in [VV, VVV] that the set of values of c for which there exists a proper travelling wave is an interval $[c^0, \infty)$. In our Theorem 6 (see below), we give a simple characterization of the value c^0.

We adopt the point of view that properties (1.25) are included in the definition of a travelling wave solution. Thus the question we address in this paper is:

Given $\mathbf{c} \in \mathbb{R}^M$, does there exist a travelling wave profile $\mathbf{W}$ (obeying (1.25)), with velocity vector $\mathbf{c}$?

In terms of system (1.24), the question is: does there exist a trajectory in the phase space $\mathbb{R}^M \times \mathbb{R}^M$ which goes, over infinite time, from point

$$S \sim w_1 = \cdots = w_M = v_1 = \cdots = v_M = 0 \quad \text{(the origin)} \tag{1.26a}$$

to point

$$T \sim w_1 = \cdots = w_M = 1\ , \quad v_1 = \cdots = v_M = 0\ , \tag{1.26b}$$

and is confined to the strip

$$0 \leqq w_k \leqq 1, \quad v_k \geqq 0, \quad 1 \leqq k \leqq M\ ? \tag{1.27}$$

We use here the notation w_k for the co-ordinate in $\mathbb{R}^M \times \mathbb{R}^M$ corresponding to W_k for the co-ordinate in $\mathbb{R}^M \times \mathbb{R}^M$ corresponding to W_k, and v_k for that corresponding to V_k.

To state our Theorem 6, we need a few basic definitions and facts from the theory of dynamical systems. The reader is referred for the detail to [H, Ar or AAr]. We also need two more conditions on probability distributions $q(j;\cdot)$: for each $j = 1, \ldots, M$,

$$q(j;0,\ldots,0) = 0\ , \quad \text{i.e. } \varphi_j(0,\ldots,0) = 0 \tag{1.28}$$

and

$$q(j;\{l_j = 1,\ l_k = 0 \text{ for } k \neq j\}) < 1\ . \tag{1.29}$$

Under condition (1.28), (1.29), the origin S is a saddle equilibrium point for system (1.24), with stable and unstable manifolds of dimension M. Point T is another equilibrium point for (1.24), but its status is related to the behaviour of functions φ_k near point $z_1 = \cdots = z_M = 1$, and it depends on the choice of vector $\mathbf{c}$. System (1.24) has no other equilibrium points in strip (1.27) (see Proposition 4.3). Geometrically, conditions (1.25) means that the corresponding trajectory must lie down on the intersection of three sets: 1) $\mathscr{U}(S)$, the unstable manifold of point S, 2) $\mathscr{S}(T)$, the stable manifold of point T, and 3) strip (1.27).

Given a vector $\mathbf{c} = (c_1, \ldots, c_M) \in \mathbb{R}^M$ and a value $a \in (0, a^1)$, consider an $M \times M$ matrix $\mathbf{B}(a)(=\mathbf{B}_{\mathbf{c}}(a))$ with the elements

$$(\mathbf{B}(a))_{j,k} = K(j,k)\frac{\varpi_k}{\varpi_k + (\beta_k + c_k)a - (\Sigma_k^2/2)a^2}\ . \tag{1.30}$$

Here, a_1 is the smallest positive root of the equation

$$\prod_{k=1}^{M} (\varpi_k + (\beta_k + c_k)a - (\Sigma_k^2/2)a^2) = 0\ .$$

[Matrix $\mathbf{B}(a)$ resembles $\mathbf{A}(a)$ from (1.8), in the case of an exponential model; in fact, both matrices coincide, modulo the change of variables that transforms system (1.24) into (1.14).] Under condition (1.5), $\mathbf{B}(a)$, $a \in (0, a_1)$, as a matrix with positive elements, possesses an eigenvector $\mathbf{L} = (L_1, \ldots, L_M)$ with positive components such that the corresponding eigenvalue κ is positive; the second eigenvalue is strictly less, in the absolute value, then κ. As before, we use the notation $\kappa(a)$, $\mathbf{L}(a)$ and $L_j(a)$, $j = 1, \ldots, M$, to stress the dependence on a.

Theorem 6. *Assume conditions* (1.5), (1.6), (1.28) *and* (1.29) *to hold. Also suppose that*

$$c_k > -\beta_k, \quad 1 \leqq k \leqq M . \tag{1.31}$$

Then the condition

$$\text{there exists } a > 0 \text{ with } \kappa(a) < 1 \tag{1.32}$$

is sufficient for the existence of a generalised travelling wave profile $\mathbf{W}$ *satisfying* (1.25), *with the velocity vector* $\mathbf{c}$.

Remarks. 1. It is interesting to compare the situation described in Theorem 6 with the case where function $\varphi_k(z_1, \ldots, z_M)$ depends only on z_k, $k = 1, \ldots, M$ (that is, system (1.21) is decomposed into independent single equations). This case is not covered by Theorem 6, since condition (1.5) is violated. In the case of a decoupled system, matrix $\mathbf{B}(a)$ is diagonal and has M eigenvectors with non-negative components. The corresponding eigenvalues $\kappa_j(=\kappa_j(a))$, $j = 1, \ldots, M$, are positive. The necessary and sufficient condition for the existence of a travelling wave profile $\mathbf{W}$ in this case is

$$\inf_{a>0} \max_{1 \leqq j \leqq M} \kappa_j(a) \leqq 1 ;$$

in an explicit form,

$$\inf_{a>0} \max_{1 \leqq k \leqq M} K(k,k)\mathbf{E}_{P_k} \exp(a\xi_k) \leqq 1 .$$

See [U, Br, VVV] and the references therein, and also [KKS 1]. Under this condition the travelling wave profile with velocity vector $\mathbf{c}$ is unique, and one can write down conditions on initial functions f_k^0, $k = 1, \ldots, M$, which are necessary and sufficient for convergence (1.23) (see again [U, Br and VVV]).

2. Theorem 6 gives only a sufficient condition for the existence of a travelling wave solution with property (1.25). A natural conjecture is that, in the situation of Theorem 6, possibly under additional mild "non-degeneracy" assumptions, condition (1.32), with the bound $\kappa(a) \leqq 1$ instead of $\kappa(a) < 1$, is necessary and sufficient for the existence of a travelling wave profile $\mathbf{W}$ satisfying (1.25). It is also expected that, under the condition (1.32), there exists a unique travelling wave profile $\mathbf{W}$ with velocity $\mathbf{c}$.

3. In the case of a proper travelling wave, with $\mathbf{c} = (c, \ldots, c)$, the values of c for which condition (1.32) holds form an interval (c^0, ∞). We believe that our c^0 coincides with the value indicated in [VV, VVV].

The proof of Theorem 6 is carried out in Sect. 4, together with some other results concerned with system (1.21).

2. An Analysis of Branching Random Walk: The Proof of Theorems 1–3

Proof of Theorems 1 *and* 2. By using the distribution functions $F_j(x) = \Pi_j((-\infty, x])$ and setting

$$[F_k * P_{j,k}](x) = \int [P_{j,k}](dy) F_k(x-y)\,, \tag{2.1}$$

we can re-write Eq. (1.10) as

$$F_j(x) = \Theta(x)\varphi_j([F_1 * P_{j,1}](x), \ldots, [F_M * P_{j,M}](x))\,, \quad x \in \mathbb{R}^1, \quad 1 \leqq j \leqq M\,. \tag{2.2}$$

Here, and below, Θ is the indicator function of the non-negative half-axis $\mathbb{R}^1_+ = [0, \infty)$. Denote by Λ_j the non-linear operator representing the r.h.s of (2.2):

$$\Lambda_j : \mathbf{u} = (u_1, \ldots, u_M) \mapsto \Theta\varphi_j(u_1 * P_{j,1}, \ldots, u_M * P_{j,M}), \quad 1 \leqq j \leqq M\,. \tag{2.3}$$

Operator Λ_j acts on vector-functions $(u_1, \ldots, u_M)$, where each u_k is a non-decreasing, left-continuous function $\mathbb{R}^1 \to [0,1]$ vanishing on $(-\infty, 0)$. The image $\Lambda_j \mathbf{u}$ is again a function of this type. Equations (2.2) take the form

$$\mathbf{F} = \Lambda \mathbf{F} \tag{2.4}$$

where $\mathbf{F} = (F_1, \ldots, F_M)$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_M)$.

First, note that operators Λ_j are monotonic, in the sense that if $u_k > u'_k$, $1 \leqq k \leqq M$, then $\Lambda_j \mathbf{u} \geqq \Lambda_j \mathbf{u}'$, $1 \leqq j \leqq M$. Therefore, the sequence $\Lambda^n \boldsymbol{\Theta}$, $n = 1, 2, \ldots$, where $\boldsymbol{\Theta} = (\Theta, \ldots, \Theta)$, converges pointwise to a limit $\mathbf{u}^0 = (u_1^0, \ldots, u_M)$ which is a fixed element for Λ. Any other fixed point $\mathbf{u} = (u_1, \ldots, u_M)$ is majorized by $\mathbf{u}^0$ in the sense that $u_j^0(x) \geqq u_j(x)$, $x \in \mathbb{R}^1$, $1 \leqq j \leqq M$ (which is equivalent to the inverse relation between the corresponding probability distributions).

Another obvious remark is that $u_k^0(x)$ is nothing but the distribution function of random variable X^0 from (1.1), under the condition that at time zero we have a particle of type k. That is, $u_k^0(x) = \Pi_k^0((-\infty, x))$, $x \in \mathbb{R}^1$. This distribution function may still be improper ($\lim_{x \to \infty} u_k^o(x)$ may be < 1).

To check if $\mathbf{u}^0$ is proper, we assume that condition (1.9) holds. We take a special initial vector

$$\mathbf{v} = (v_1, \ldots, v_M)\,, \quad v_j(x) = \Theta(x)(1 - C_j e^{-ax})^+, \quad 1 \leqq j \leqq M\,, \tag{2.5}$$

where value $a > 0$ is chosen so that $\rho(a) \leqq 1$ and the eigenvector $\mathbf{C}(a)$ is normalized so that $C_k < 1$, $1 \leqq k \leqq M$. Denote $b_k = a^{-1} \log C_k$ and write

$$\begin{aligned}
[v_k * P_{j,k}](x) &= \int [P_{j,k}](dy)\Theta(x-y)(1 - C_k e^{-a(x-y)})^+ \\
&= \int_{-\infty}^{x-b_k} [P_{j,k}](dy)(1 - C_k e^{-a(x-y)}) \\
&= 1 - \int_{x-b_k}^{\infty} [P_{j,k}](dy) - C_k \exp(-ax) \\
&\quad \times \left(\int [P_{j,k}](dy) e^{ay} - \int_{x-b_k}^{\infty} [P_{j,k}](dy) e^{ay} \right).
\end{aligned} \tag{2.6}$$

Since

$$C_k \exp(-ax) \int_{x-b_k}^{\infty} [P_{j,k}](dy) e^{ay} \geqq \int_{x-b_k}^{\infty} [P_{j,k}](dy)\,,$$

we have

$$[v_k * P_{j,k}](x) \geqq 1 - C_k \exp(-ax) \mathbf{E} e^{a\xi_{j,k}}\,. \tag{2.7}$$

Furthermore,

$$(\Lambda_j \mathbf{v})(x) = \Theta(x)[1 + (\varphi([v_1 * P_{j,1}](x), \ldots, [v_M * P_{j,M}](x)) - 1)]$$

$$= \Theta(x)\left[1 + \sum_{k=1}^{M} \left(\frac{\partial \varphi_j}{\partial z_k}([v_1 * P_{j,1}](x), \ldots, z_k, \ldots, [v_M * P_{j,M}](x))\bigg|_{z_k = \vartheta_k}\right) \times ([v_k * P_{j,k}](x) - 1)\right], \tag{2.8}$$

where ϑ_k is an intermediate point between $[v_k * P_{j,k}](x)$ and 1. The r.h.s. of (2.8) is

$$\geqq \Theta(x)\left[1 - \exp(-ax) \sum_{k=1}^{M} K(j,k) \mathbf{E} e^{a\xi_{j,k}} C_k\right]$$

$$\geqq \Theta(x)[1 - C_j \exp(-ax)] ;$$

in the last inequality we used the fact that $\mathbf{C}$ is an eigenvector of $\mathbf{A}(a)$ with the eigenvalue $\rho(a)$ that is $\leqq 1$.

We conclude that for any $j = 1, \ldots, M$, $(\Lambda_j \mathbf{v})(x) \geqq v_j(x)$. Therefore, there exists a pointwise limit $\mathbf{v}^0 = \lim_{n \to \infty} \Lambda^n \mathbf{v}$ giving a fixed point for Λ. Furthermore, for the components $v_1^0, \ldots, v_M^0$ of vector $\mathbf{v}^0$ we have bounds

$$v_j^0(x) \geqq \Theta(x)(1 - C_j e^{-ax})^+, \quad 1 \leqq j \leqq M . \tag{2.9}$$

This proves Theorem 1 and the sufficiency of condition (1.9) in Theorem 2.

Let us prove the necessity. We now assume that conditions (1.5)–(1.7) are valid. Suppose that the minimal solution $\boldsymbol{\Pi}^0$ to (1.10) is composed of proper probability distributions, (or, equivalently, vector $\mathbf{u}^0$ is composed of proper distribution functions), but condition (1.9) fails, i.e. $\rho(a) > 1$ for all $a > 0$. The first remarks is that, without loss of generality, we can assume that distributions $P_{j,k}$ are supported on finite sets with some specific properties. More precisely, we shall assume that there exist a positive δ, a positive integer N, and a one-to-one map $\eta\colon j \mapsto \eta(j)$, $1 \leqq j$, $\eta(j) \leqq M$, such that (a) for any $j, k = 1, \ldots, M$, with $j \neq k$, the distribution $P_{\eta(j),k}$ is supported on a set

$$\mathbf{S} = \{-\infty, (-N+1)\delta, \ldots, N\delta\} \tag{2.10}$$

in the sense that $[P_{\eta(j),k}](\xi \in \mathbf{S}) = 1$ and $[P_{\eta(j),k}](\xi = s\delta) > 0$ for any $s = -N+1, \ldots, N$, and (b) for any $j = 1, \ldots, M$, the distribution $P_{\eta(j),j}$ is supported on

$$\bar{\mathbf{S}} = \{-\infty, -N\delta, \ldots, N\delta\} \tag{2.11}$$

in the same sense as before. In fact, choosing $\delta > 0$ small enough and N large enough, we can always diminish the distribution $P_{j,k}$, in the sense of stochastic ordering, so that the new distributions possess the property indicated, and condition (1.9) still fails (assumption (1.7) is important here). Obviously, for the new $P_{j,k}$'s, the distributions $\Pi_1, \ldots, \Pi_M$ are also proper.

Note that, under our assumptions about $P_{j,k}$, distributions Π_j are concentrated on subsets of the set $\{n\delta, n \in \mathbf{Z}_+^1\}$.

Denoting

$$[P_{j,k}](\xi_{j,k} = s\delta) = p_{j,k}(s), \quad s \in \mathbf{Z}^1 \text{ or } s = -\infty, \tag{2.12}$$

and

$$1 - u_j^0(n\delta) = y_j(n), \quad n \in \mathbf{Z}^1, \tag{2.13}$$

we can write

$$[u_k^0 * P_{j,k}](n\delta) = p_{j,k}(-\infty) + \sum_{s=-N+1}^{N} p_{j,k}(s)(1 - y_k(n-s))$$
$$= 1 - \sum_{s=-N+1}^{N} p_{j,k}(s) y_k(n-s), \quad n \in \mathbf{Z}^1,\ j,k = 1, \ldots, M,\ k \neq \eta(j), \tag{2.14a}$$

and

$$[u_k^0 * P_{j,k}](n\delta) = p_{j,k}(-\infty) + \sum_{s=-N}^{N} p_{j,k}(s)(1 - y_k(n-s))$$
$$= 1 - \sum_{s=-N}^{N} p_{j,k}(s) y_k(n-s), \quad n \in \mathbf{Z}^1, k = \eta(j), j = 1, \ldots, M. \tag{2.14b}$$

Note that, by the assumption, $\lim_{n\to\infty} y_j(n) = 0$, $1 \leqq j \leqq M$.

Equation $\mathbf{u}^0 = \Lambda \mathbf{u}^0$ now takes the form

$$y_j(n) = 1 - \Theta(n)\varphi_j\left(1 - \sum_{s_1} p_{j,1}(s_1) y_1(n - s_1), \ldots, 1 - \sum_{s_M} p_{j,M}(s_M) y_M(n - s_M)\right),$$
$$n \in \mathbf{Z}^1, \quad 1 \leqq j \leqq M; \tag{2.15}$$

the sum in the r.h.s. of (2.15) is over set $\bar{\mathbf{S}}$ for $s_{\eta(j)}$ and over set $\mathbf{S}$ for all other s_k's. Pictorially speaking, variables $y_{\eta(j)}(n+N)$ form a "future propagation front" in Eqs. (2.15).

Denoting $\tilde{\varphi}_j = \varphi_{\eta(j)}$, we now observe that functions $\tilde{\varphi}_j$, $1 \leqq j \leqq M$, admit the following representations, near points $z_j = 1$:

$$\tilde{\varphi}_j(z_1, \ldots, z_M) = 1 + \sum_{k=1}^{M} (z_k - 1) K(\eta(j), k) + \psi_j(z_1, \ldots, z_M), \tag{2.16}$$

where $|\psi_j(z_1, \ldots, z_M)| = O(\max_k |z_k - 1|^2)$. More precisely, there exist $c, \gamma > 0$ such that, for any $j = 1, \ldots, M$, the following holds. For any values $z_1, \ldots, z_M$ with $\max_k |z_k - 1| < \zeta$, function $\psi_j(z_1, \ldots, z_M)$ admits a bound

$$|\psi_j(z_1, \ldots, z_M)| \leqq c\zeta^2. \tag{2.17}$$

Since $K(\eta(j), k) > 0$ for any j, k, it means that functions $\tilde{\varphi}_j$ are invertible, in any of the variables z_k, in a circle about $z_k = 1$. Moreover, the inverse functions, as functions of the whole collection of the complex variables, are analytic in a poly-circle in $\mathbb{C}^M$ about point $z_1 = \cdots = z_M = 1$.

We want to re-write (2.15), by expressing the "front" variables in terms of other variables taking part in the equation, for which we need to invert the corresponding function φ_j. By the above remark, we can do so while all values $y_k(\cdot)$ taking part in (2.15) are confined to a small interval $(0, \tilde{\zeta})$ ($\tilde{\zeta} \leqq \zeta$), which may be achieved, for n large enough: $n \geqq n_0$, since $y_k(n)$ monotonically decrease to zero when $n \to \infty$.

Hence, for $n \geqq n_0$, Eqs. (2.15) may be written in the equivalent form

$$\begin{aligned} y_j(n) = f_j(&y_1(n-2N), \dots, y_1(n-1), \\ &y_2(n-2N), \dots, y_2(n-1), \\ &\dots, \\ &y_M(n-2N), \dots, y_M(n-1)), \quad 1 \leqq j \leqq M. \end{aligned} \tag{2.18}$$

Note that functions f_j are analytic in an open poly-circle about the origin and have their non-zero first derivatives.

We are interested in linearizing (2.18), by taking the linear parts of functions f_j near the origin. In fact, we can do so by linearizing (2.15), i.e., by taking the linear parts of functions $\varphi_j(z_1, \cdots, z_M)$, near point $z_1 = \cdots = z_M = 1$. The linearized equations (2.15) read as

$$y_j^*(n) = \sum_{1 \leqq k \leqq M} \sum_{1 \leqq s \leqq 2N} J_{j,k}(s) y_k^*(n-s), \quad 1 \leqq j \leqq M, \tag{2.19}$$

where

$$J_{j,k}(s) = K(j,k) p_{j,k}(s). \tag{2.20}$$

As to the linearization of (2.18), it takes, in the matrix notation, the following form:

$$Y^*(n) = \mathbf{Q} Y^*(n-1), \quad n \in \mathbf{Z}_+^1, \tag{2.21}$$

where $Y^*(n)$ denotes a real (column) vector, of dimension $M(2N-1)$, namely:

$$\begin{aligned} (Y^*(n))^T = (&y_1^*(n-2N+1), \dots, y_1^*(n), \\ &y_2^*(n-2N+1), \dots, y_2^*(n), \\ &\dots, \\ &y_M^*(n-2N+1), \dots, y_M^*(n)), \end{aligned}$$

and $\mathbf{Q}$ is a real $(M(2N-1)) \times (M(2N-1))$-matrix.

Matrix $\mathbf{Q}$ has a specific "block" structure: it contains M blocks $\mathrm{B}_1, \dots, \mathrm{B}_M$ of size $(2N-2) \times (2N-1)$, intermitted with M "full-length" rows $\mathbf{b}_1, \dots, \mathbf{b}_M$. All entries of $\mathbf{b}_1, \dots, \mathbf{b}_M$ are strictly positive; the total number of these entries is $M^2(2N-1)$. See Fig. 2a.

Each block B_i contains $2(N-1)$ unit elements, just above the main diagonal; all other entries of matrix $\mathbf{Q}$ are zeros. See Fig. 2b.

It is worth noting that the $M^2(2N-1)$ entries of rows $\mathbf{b}_1, \dots, \mathbf{b}_M$ are the only elements of matrix $\mathbf{Q}$ which vary when distributions $P_{j,k}$ vary (with N remained fixed). Pictorially speaking, matrix $\mathbf{Q}$ may be considered as a point in $\mathbb{R}^{M^2(2N-1)}$.

At this point we use a version of Siegel's Fundamental Lemma, for real-analytic transformations of a Euclidean space (see [Ar]). In our situation it guarantees that, unless the entries of matrix $\mathbf{Q}$ belong to a set $\mathscr{B}$ of the Lebesgue measure zero (in $\mathbb{R}^{M^2(2N-1)}$), the following holds true. The fact that the system of equations (2.18) (or equivalently, (2.15)) possesses a bounded non-negative solution $y_j(n)$, $n \geqq n_0$, $1 \leqq j \leqq M$, implies that the linearized system (2.19) (or equivalently, Eq. (2.21)) also possesses a bounded non-negative solution $y_j^*(n)$, $n \geqq n_0$, $1 \leqq j \leqq M$. Another remark is that, unless the entries of $\mathbf{Q}$ belong to another set, $\mathscr{D}$, again of the Lebesgue measure zero, matrix $\mathbf{Q}$ has $M(2N-1)$ linearly independent eigenvectors, and its

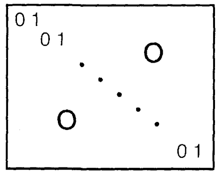

Fig. 2a. The block structure of matrix $\mathbf{Q}$

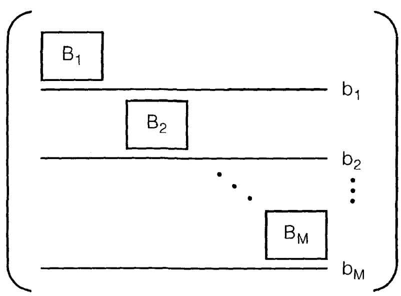

Fig. 2b. A single block B_i

eigenvalues satisfy the following conditions: (i) none of the eigenvalues has the absolute value one, (ii) for any $r > 0$, there are at most two (complex-conjugate) eigenvalues with the absolute value r, (iii) any pair of the complex-conjugate eigenvalues has an irrational arugment (modulo 2π).

It is clear that we can avoid sets $\mathscr{B}$ and $\mathscr{D}$, again by diminishing the distributions $P_{j,k}$, without violating our original assumptions (and actually without changing sets $\mathbf{S}$ and $\bar{\mathbf{S}}$).

But once we are out of these sets, the fact that there exists a bounded non-negative solution to (2.18) means that there exists an eigenvalue $\lambda_0 \in (0, 1)$, and the corresponding eigenvector $\mathbf{e} = (e_1, \ldots, e_{M(2N-1)})$ has non-negative components.

In terms of Eq. (2.19) it means that there exists a non-negative solution $y_j^{*(0)}(n)$, $n \in \mathbf{Z}$, $1 \leqq j \leqq M$, of the form

$$y_j^*(n) = \lambda_0^n y_j^{(0)} , \tag{2.22}$$

where $y^{(0)} = (y_1^{(0)}, \ldots, y_M^{(0)})$ is a fixed non-negative vector. Substituting (2.22) into (2.19) yields

$$\lambda_0^n y_j^{(0)} = \sum_{1 \leqq k \leqq M} \sum_{1 \leqq s \leqq 2N} Q_{j,k}(s) \lambda_0^{n-s} y_k^{(0)}, \quad 1 \leqq j \leqq M ,$$

or equivalently,

$$y_j^{(0)} = \sum_{1 \leqq k \leqq M} \sum_{1 \leqq s \leqq 2N} Q_{j,k}(s) \lambda_0^{-s} y_k^{(0)} . \tag{2.23}$$

Now set $a_0 = -\delta^{-1} \log \lambda_0$ and recall (2.20). We obtain that $\underline{y}^{*(0)}$ is an eigenvector of matrix $\mathbf{A}(a_0)$ with eigenvalue one. This completes the proof of the necessity.

The above analysis shows that a condition weaker than (1.9) guarantees the existence of a solution $\boldsymbol{\Pi}$ to (1.10) with at least one proper component Π_j.

Theorem 2.1. *System* (1.10) *has a solution, with at least one component* Π_j *being a proper distribution, if matrix* $\mathbf{A}(a)$ *has, for some* a, *an eigenvector* $\mathbf{C}$, *with non-negative components and with an eigenvalue* $\bar{\rho} \leqq 1$.

The meaning of Theorem 2.1 is that $\rho(a)$ may be > 1; eigenvectors $\mathbf{C}$ and $\bar{\mathbf{C}}$ do not need to coincide. We omit the proof of Theorem 2.1: it simply repeats the above argument.

Proof of Theorem 3. It is convenient to introduce, for any $\gamma > 0$, a class $\mathscr{F}(\gamma)$ consisting of vectors $\mathbf{F} = (F_1, \ldots, F_M)$, where each F_j is a distribution function admitting the asymptotics

$$F_j(x) = 1 - \gamma C_j(\alpha) e^{-\alpha x} + O(e^{-px}) \tag{2.24}$$

with some $p = p(\mathbf{F}) > \alpha$. Recall that α was introduced in (1.11) so that the eigenvalue $\rho(\alpha)$ equals one. The assertion of Theorem 3 is that class $\mathscr{F}(\gamma)$ contains exactly one solution, $\mathbf{F}^{(\gamma)} = (F_1^{(\gamma)}, \ldots, F_M^{(\gamma)})$, to (1.10).

We shall prove more: class $\mathscr{F}(\gamma)$ is actually attracted to $\mathbf{F}^{(\gamma)}$, in the course of iterating operator Λ. This fact is proved in two steps. Step one is to check that if two vectors, $\mathbf{F} = (F_1, \ldots, F_M)$ and $\mathbf{F}' = (F'_1, \ldots, F'_M)$, satisfy

$$|F_j(x) - F'_j(x)| \exp(p^0 x) \leqq h C_j(p^0) , \tag{2.25}$$

where $p^0 > \alpha$, and $\rho(p^0) < 1$, then

$$|(\Lambda_j \mathbf{F})(x) - (\Lambda_j \mathbf{F}')(x)| \exp(p^0 x) \leqq \rho(p^0) h C_j(p^0) . \tag{2.26}$$

Step two is to prove that for $\mathbf{F} = (F_1, \ldots, F_M) \in \mathscr{F}(\gamma)$,

$$(\Lambda_j \mathbf{F})(x) = F_j(x) + O(e^{-p^1 x}) , \tag{2.27}$$

where p^1 coincides with value $p(\mathbf{F})$, if $p(\mathbf{F}) < \min[2\alpha, a^0]$, and p^1 may be chosen arbitrarily close to $\min[2\alpha, a^0]$ from the left, if $p(\mathbf{F}) \geqq \min[2\alpha, a^0]$.

To check the first step, we use (2.8) and write

$$\begin{aligned}
&|(\Lambda_j \mathbf{F})(x) - (\Lambda_j \mathbf{F}')(x)| \\
&= \Theta(x) |\varphi_j([F_1 * P_{j,1}](x), \ldots, F_M * [P_{j,M}](x) \\
&\quad - \varphi_j([F'_1 * P_{j,1}](x), \ldots, [\mathrm{F}'_M * P_{j,M}](x))| \\
&\leqq \Theta(x) \sum_{k=1}^{M} \left(\frac{\partial \varphi_j}{\partial z_k} \left([F_1 * P_{j,1}](x), \ldots, [F_{k-1} * P_{j,k-1}](x), z_k , \right.\right. \\
&\quad \left.\left. [F'_{k+1} * P_{j,k+1}](x), \ldots, [F'_M * P_{j,M}](x) \right) \Big|_{z_k = \vartheta_k} \right) \\
&\quad \times |[F_k * P_{j,k}](x) - [F'_k * P_{j,k}](x)| ,
\end{aligned} \tag{2.28}$$

where ϑ_k is an intermediate point between $[F_k * P_{j,k}](x)$ and $[F'_k * P_{j,k}](x)$. By using an immediate bound

$$\frac{\partial \varphi_j}{\partial z_k}\Big([F_1 * P_{j,1}](x), \ldots, [F_{k-1} * P_{j,k-1}](x), z_k, [F'_{k+1} * P_{j,k+1}](x), \ldots, [F'_M * P_{j,M}](x)\Big)\Big|_{z_k = \vartheta_k} \leqq K(j,k),$$

we can estimate the r.h.s. of (2.28) by

$$\begin{aligned} &h\Theta(x) \sum_{k=1}^{M} K(j,k) \int [P_{j,k}](dy) e^{-p^0(x-y)} C_k(p^0) \\ &= h\Theta(x) \exp(-p^0 x) \sum_{k=1}^{M} K(j,k) \mathbf{E} e^{p^0 \xi_{j,k}} C_k(p^0) \\ &= h\Theta(x) \exp(-p^0 x) (\mathbf{A}(p^0)\mathbf{C}(p^0))_j \\ &= h\Theta(x) \exp(-p^0 x) C_j(p^0) \rho(p^0). \end{aligned}$$

To check step two, take $\mathbf{F} = (F_1, \ldots, F_M) \in \mathscr{F}(\gamma)$ and assume for definiteness that $p(\mathbf{F}) < \min[2\alpha, a^0]$ (the opposite case is treated similarly). Having in mind that

$$\varphi_j(1, \ldots, 1) = 1 \text{ and } (\partial/\partial z_k)\varphi_j(1, \ldots, 1, z_k, 1, \ldots, 1)|_{z_k = 1} = K(j,k), \quad 1 \leqq j, k \leqq M,$$

write, using Taylor's expansion,

$$\begin{aligned} (\Lambda_j \mathbf{F})(x) &= \Theta(x)\, \varphi_j\Big([F_1 * P_{j,k}](x), \ldots, [F_M * P_{j,k}](x)\Big) \\ &= \Theta(x)\Bigg[1 + \sum_{k=1}^{M} K(j,k)\Bigg(-1 + \int_{-\infty}^{x} [P_{j,k}](dy) \\ &\quad - \gamma C_k(\alpha) \int_{-\infty}^{x} [P_{j,k}](dy)[e^{-\alpha(x-y)} + O(e^{-p(x-y)})]\Bigg)\Bigg] \\ &\quad + \sum_{k_1, k_2 = 1}^{M} \Bigg(\frac{\partial^2}{\partial z_{k_1} \partial z_{k_2}} \varphi_j(1, \ldots, 1, z_{k_1}, 1, \ldots, z_{k_2}, 1, \ldots, 1)\Big|_{\substack{z_{k_1} = \vartheta_{k_1} \\ z_{k_2} = \vartheta_{k_2}}}\Bigg) \\ &\quad \times\Bigg(-1 + \int_{-\infty}^{x} [P_{j,k_1}](dy) \\ &\quad - \gamma C_{k_1}(\alpha) \int_{-\infty}^{x} [P_{j,k_1}](dy)[e^{-\alpha(x-y)} + O(e^{-p(x-y)})]\Bigg) \\ &\quad \times\Bigg(-1 + \int_{-\infty}^{x} [P_{j,k_2}](dy) \\ &\quad - \gamma C_{k_2}(\alpha) \int_{-\infty}^{x} [P_{j,k_2}](dy)[e^{-\alpha(x-y)} + O(e^{-p(x-y)})]\Bigg). \end{aligned} \tag{2.29}$$

Here $p = p(\mathbf{F})$ and ϑ_{k_i}, $i = 1, 2$, are intermediate points between 1 and

$$\int_{-\infty}^{x} [P_{(j,k_i)}](dy) - \gamma C_{k_i}(\alpha) \int_{-\infty}^{x} [P_{(j,k_i)}](dy)[e^{-\alpha(x-y)} + O(e^{-p(x-y)})].$$

To analyse the structure of the r.h.s of (2.29), write

$$1 - \int_{-\infty}^{x} [P_{j,k}](dy) = \int_{x}^{\infty} [P_{j,k}](dy)$$

$$\leqq \int_{x}^{\infty} [P_{j,k}](dy) e^{\tilde{p}(y-x)} \leqq \exp(-\tilde{p}x) \mathbf{E} e^{\tilde{p}\xi_{j,k}} = O(e^{-\tilde{p}x}) \,; \tag{2.30}$$

these relations hold for any $\tilde{p} < a^0$. Furthermore,

$$\int_{-\infty}^{x} [P_{j,k}](dy) e^{-\alpha(x-y)} = \exp(-\alpha x) \mathbf{E} e^{\alpha \xi_{j,k}} - \int_{x}^{\infty} [P_{j,k}](dy) e^{\alpha(y-x)} \,. \tag{2.31}$$

Observe that, owing to the bound $p > \alpha$, the second term in the r.h.s of (2.31) does not exceed, in the absolute value,

$$\int_{x}^{\infty} [P_{j,k}](dy) e^{p(y-x)} \leqq e^{-px} \mathbf{E} e^{p\xi_{j,k}} \,,$$

i.e. is of the order of magnitude $O(e^{-px})$.

It is plain that, because of the bound $\alpha < p < 2\alpha$, the second-derivative terms in the r.h.s. of (2.29) also give the contribution of the order of magnitude $O(e^{-px})$. Hence, the r.h.s. in (2.29) may be written as

$$\begin{aligned} &\Theta(x)[1 - \gamma \sum_{k=1}^{M} K(j,k) \mathbf{E} e^{\alpha \xi_{j,k}} C_k(\alpha) + O(e^{-px})] \\ &= \Theta(x)[1 - \gamma \rho(\alpha) C_j(\alpha) + O(e^{-px})] \\ &= \Theta(x)[-\gamma C_j(\alpha) + O(e^{-px})] \,; \end{aligned}$$

the last equality holds due to the choice of α. This means that $\boldsymbol{\Lambda}\mathbf{F} \in \mathscr{F}(\gamma)$, which completes the proof of Theorem 3.

We conclude this section with two auxiliary results that are used below, in the analysis of the travelling wave solutions. We assume here that the conditions of Theorem 3 are valid.

Theorem 2.2. *Suppose that $\boldsymbol{\Pi} = (\Pi_1, \ldots, \Pi_M)$ is a solution to* (1.10) *with the distribution functions $F_j(x) = \Pi_j((-\infty, x])$ of the form*

$$F_j(x) = 1 - D_j e^{-\alpha_j x} + O(e^{-p_j x}) \,, \quad 1 \leqq j \leqq M \,, \tag{2.32}$$

where constants $D_j > 0$, $\alpha_j \leqq \alpha$, and $p_j > \alpha_j$, $1 \leqq j \leqq M$. Then the following properties are valid: (i) $\alpha_j = a$, $1 \leqq j \leqq M$, (ii) *matrix $\mathbf{A}(a)$ possesses a non-negative eigenvector $\bar{\mathbf{C}}(a) = (\bar{C}_1(a), \ldots, \bar{C}_M(a))$ with eigenvalue one, and* (ii) $D_j = \bar{\gamma}\bar{C}_j(a)$, $1 \leqq j \leqq M$, *for some constant $\bar{\gamma} > 0$.*

Theorem 2.3. *Solutions $\boldsymbol{\Pi}^{(\gamma)}$, from Theorem* 3, *depend continuously on γ, in the sense that for any $x > 0$,*

$$\gamma \in \mathbb{R}^1_+ \mapsto F_k^{(\gamma)}(x) \,, \quad 1 \leqq k \leqq M \,,$$

are continuous functions.

Proof of Theorem 2.2. We use (2.4). Representation (2.29) yields

$$F_j(x) = \Theta(x)\Bigg[1 + \sum_{k=1}^{M} K(j,k)\Bigg(1 - \int_{-\infty}^{x} [P_{j,k}](dy)$$

$$- D_k(\alpha_k) \int_{-\infty}^{x} [P_{j,k}](dy)[e^{-\alpha(x-y)} + O(e^{-p_k(x-y)})]\Bigg)\Bigg]$$

$$+ \text{terms } O\left(\exp\left(-\min_k p_k x\right)\right), \quad 1 \leqq j \leqq M .$$

Picking max α_k gives (i), and subsequently (ii) and (iii).

Proof of Theorem 2.3. We use the fact that

$$\mathbf{F}^{(\gamma)} = \lim_{n \to \infty} \Lambda^n \mathbf{v}^{(\gamma)} ,$$

where (cf. (2.5)) $\mathbf{v}^{(\gamma)} = (v_1^{(\gamma)}, \ldots, v_M^{(\gamma)})$, and

$$v_j^{(\gamma)}(x) = (1 - \gamma C_j(\alpha)e^{-\alpha x})^+ .$$

More precisely, write

$$\mathbf{F}^{(\gamma)} = \mathbf{v}^{(\gamma)} + \sum_{n \geqq 1} (\Lambda^n \mathbf{v}^{(\gamma)} - \Lambda^{n-1}\mathbf{v}^{(\gamma)}) . \tag{2.33}$$

As previously shown, for any $\varepsilon > 0$, the series may be truncated, uniformly in γ within an a priori fixed compact set, so that the remainder does not exceed ε. Finite sums from (2.33) depend on γ continuously. Hence the result.

3. Proof of Theorems 4 and 5. Geometric Properties of the Travelling-Wave System

Proof of Theorem 4. The proof is simple and essentially repeats that of Theorem 4 from the paper [KKS 1]. First, we derive (1.14)–(1.18). Let X_j be a random variable with distribution Π_j. Then the random variable $Y_j = \xi_j + X_j$, with probability distribution $P_j * \Pi_j$, satisfy the following stochastic equation:

$$Y_j \simeq \xi_j + \max\left[0, \max_{1 \leqq k_1 \leqq l_1(j)} Y_1^{(k_1)}, \ldots, \max_{1 \leqq k_M \leqq l_M(j)} Y_M^{(k_M)}\right], \quad 1 \leqq j \leqq M \tag{3.1}$$

(ξ_j stands for the displacement of a particle of type j). Here, as in (1.10), the random numbers $l_1(j), \ldots, l_M(j)$ are distributed according to $q(j;\cdot)$, and, given a sample vector $\mathbf{l}(j) = (l_1(j), \ldots, l_M(j))$, the random variables $Y_m^{(k_m)}$, $1 \leq k_m \leqq l_m(\cdot)$, in the j^{th} equation (3.1) are independent; for a fixed m, $Y_m^{(k_m)}$ has the same distribution as the random variable Y_m in the l.h.s. of the m^{th} equation. This specifies the joint distribution of all random variables in the r.h.s of (3.1), and hence the distribution of the whole sum. Symbol $\simeq$, as in (1.10), means equality in distribution. In terms of distribution functions G_j and moment generating functions φ_j, (3.1) takes the form

$$G_j(x) = [P_j * \varphi_j((G_1)_+, \ldots, (G_M)_+)](x) , \quad x \in \mathbb{R}^1, \quad 1 \leqq j \leqq M . \tag{3.2}$$

Here and below, $(G_k)_+(x) = \Theta(x)G_k(x)$.

According to the definition of the exponential model, the probability density function of the random variable ξ is of the form

$$p_k(x) = \frac{\lambda_k \mu_k}{\lambda_k + \mu_k}((1 - \Theta(x))\exp(\lambda_k x) + \Theta(x)\exp(-\mu_k x)), \quad x \in \mathbb{R}^1 . \tag{3.3}$$

For $x \neq 0$ we have

$$p_k''(x) + (\mu_k - \lambda_k)p_k'(x) = \lambda_k \mu_k p_k(x) , \tag{3.4}$$

and for $x = 0$

$$p_k'(0+) - p_k'(0-) = -\lambda_k \mu_k . \tag{3.5}$$

Taking the second derivative of (3.2) for $x > 0$ and using (3.3) and (3.4), we get (1.14). Taking the first derivative at $x = 0$ yields (1.15). Relations (1.16) (1.17) reflect the fact that G_k are distribution functions. Finally, (1.18) is valid by direct inspection.

Conversely, suppose that (1.14)–(1.18) hold. Equations (1.14) and (1.18) may be written in the form

$$\begin{aligned} &\frac{d^2}{dx^2} G_k(x) + (\mu_k - \lambda_k)\frac{d}{dx} G_k(x) \\ &\quad - \lambda_k \mu_k \Big(G_k(x) - \varphi_k((G_1)_+(x), \ldots, (G_M)_+(x) \Big), \quad x \neq 0 . \end{aligned} \tag{3.6}$$

Taking the convolution of both sides of (3.6) with the probability density function p_k from (3.3), integrating twice by parts and using (1.16) and (1.17), leads to (3.2) and (3.1). The latter is obviously equivalent to (1.10).

We provide some auxiliary assertions about the exponential model which are used in our analysis of travelling waves. A dynamical system (1.14) is said to have an even saddle point at $\tilde{S} \in \mathbb{R}^M \times \mathbb{R}^M$ if $\tilde{S}$ is an equilibrium point, and the linearized system, around $\tilde{S}$, has M real eigenvectors with positive eigenvalues and M with negative ones.

Lemma 3.1. *Under the condition* $\mu_k > \lambda_k$, $1 \leqq k \leqq M$, *system* (1.14) *always has an even saddle equilibrium point at the origin* S.

Proof of Lemma 3.1. Linearising (1.14) at S yields

$$\left.\begin{aligned} U_k'(x) &= V_k(x) , \\ V_k'(x) &= -(\mu_k - \lambda_k) V_k(x) + \lambda_k \mu_k q_k^1 U_k(x) , \end{aligned}\right. \quad x > 0, \quad 1 \leqq k \leqq M . \tag{3.7}$$

Here,

$$q_k^1 = 1 - q(k; \{l_k = 1,\ l_r = 0 \text{ for } r \neq k\}) , \quad 1 \leqq k \leqq M ;$$

in view of (1.29), $q_k^1 > 0$. Observe that system (3.7) is decoupled into M two-dimensional systems, each involving a single pair of variables U_k, V_k. Therefore, the eigenvalues of the matrix of system (3.7) are precisely the roots of the following equation:

$$\prod_{k=1}^{M} \det\begin{pmatrix} -\sigma & 1 \\ \lambda_k \mu_k q_k^1 & -(\mu_k - \lambda_k) - \sigma \end{pmatrix} = 0 ; \tag{3.8}$$

σ is the unknown here. The roots are given by

$$\sigma_k^{\pm} = -\frac{1}{2}(\mu_k - \lambda_k) \pm \frac{1}{2}\sqrt{(\mu_k - \lambda_k)^2 + 4\lambda_k\mu_k q_k^1}\,, \quad 1 \leqq k \leqq M\,. \tag{3.9}$$

This completes the proof of Lemma 3.1.

Observe that $\sigma_k^+ \leqq \lambda_k$.

The eigenvectors e_k^+ corresponding to the positive eigenvalues σ_k^+ are given, in the co-ordinates u_j, v_j, corresponding to U_j and V_j, respectively, $1 \leqq j \leqq M$, by

$$u_k = 1\,, \quad v_k = \sigma_k^+\,, \quad u_j = v_j = 0\,, \quad j \neq k\,. \tag{3.10}$$

We define the unstable ($-$)-manifold $\mathscr{U}_-(S)$, of system (1.14) at point S, as the set of points in $\mathbb{R}^M \times \mathbb{R}^M$ obtained by integrating (1.14) from S, along the cone $\mathscr{C}_-(S)$ generated by linear combinations of vectors e_k^+, $1 \leqq k \leqq M$, with non-negative coefficients. Cone $\mathscr{C}_-(S)$, in a natural sense, is tangent to $\mathscr{U}_-(S)$ at S.

The mutual "position" of the trajectories of system (1.14) and cone $\mathscr{C}_-(S)$ is characterized in the following Lemma 3.2:

Lemma 3.2. *Let* $(U_j(x),\ V_j(x),\ 1 \leqq j \leqq M)$ *be a trajectory of* (1.14), *with* $\lim_{x\to\infty} U_j(x) = 1$, $1 \leqq j \leqq M$. *If, for some* $x_0 \in \mathbb{R}^1$ *and* $k = 1, \ldots, M$, *point* $(U_k(x_0), V_k(x_0))$ *in the* u_k, v_k*-plane, has* $U_k(x_0)$, $V_k(x_0) \geqq 0$ *and lies below line* $v_k = \lambda_k u_k$ (*i.e.*, $V_k(x_0) < \lambda_k U_k(x_0)$), *then point* $(U_k(x), V_k(x))$ *lies below this line for any* $x \geqq x_0$.

Proof of Lemma 3.2. The proof of Lemma 3.2 follows the construction used in the proof of Proposition 5.1 from [KKS 1]. For the sake of brevity, we consider in detail a particular case (which may be described as the "worst" one), where $q_k^1 = 1$, and thus $\sigma_k^+ = \lambda_k$, $\sigma_k^- = -\mu_k$.

We start with some definitions. In the argument that follows, speaking of a codimension one manifold given by an equation $v_k = \Phi(u_1, \ldots, u_M; v_1, \ldots, v_M)$, we refer to the domains $v_k > \Phi(u_1, \ldots, u_M; v_1, \ldots, v_M)$, and $v_k < \Phi(u_1, \ldots, u_M; v_1, \ldots, v_M)$, as "above" and "below" the manifold, respectively.

For any trajectory $(U_j(x),\ V_j(x),\ 1 \leqq j \leqq M)$, of system (1.14), the zeros of the derivative $V_k'(x)$ lie down on a $(2M-1)$-dimensional manifold $\mathscr{M}_k^1$ given by

$$v_k = \frac{\lambda_k\mu_k}{\mu_k - \lambda_k}(u_k - \varphi_k(u_1, \ldots, u_M))\,. \tag{3.11}$$

Below this manifold the derivative is positive and above negative. If we form the function

$$Z_k : x \in \mathbb{R}^1 \mapsto V_k(x) - \lambda_k U_k(x)\,, \tag{3.12}$$

then the zeros of the derivative $Z_k'(x)$ lie down on another $(2M-1)$-dimensional manifold, $\mathscr{M}_k^2$, given by

$$v_k = \lambda_k(u_k - \varphi_k(u_1, \ldots, u_M))\,. \tag{3.13}$$

[This follows from the formula

$$Z_k'(x) = -\mu_k V_k(x) + \lambda_k\mu_k(U_k(x) - \varphi_k(U_1(x), \ldots, U_M(x)))\,,$$

which in turn is obtained from (1.14). Observe that manifolds $\mathscr{M}_k^1$ and $\mathscr{M}_k^2$ are homothetic.]

Define the (plain) cone $\mathscr{C}_{-,k}(S)$ as the intersection of the hyperplane $v_k = \lambda_k\mu_k$ with the non-negative orthant in $\mathbb{R}^M \times \mathbb{R}^M$. Cone $\mathscr{C}_{-,k}(S)$ intersects $\mathscr{M}_k^1$ at the

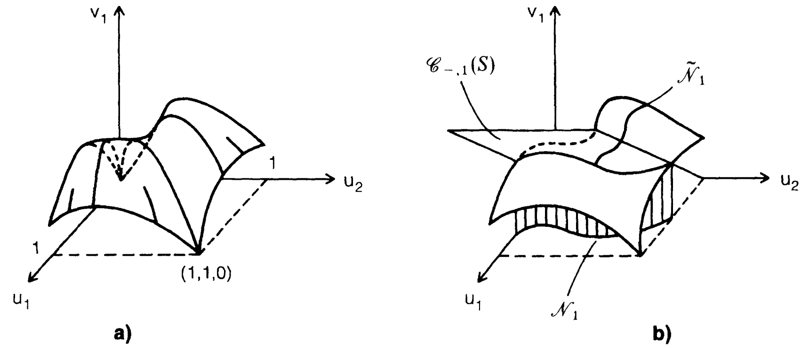

Fig. 3a. Manifold $\mathscr{M}_1^i$ in $\mathbb{R}^2 \times \mathbb{R}^1$, $i = 1, 2$. **b.** Manifolds $\mathscr{M}_1^1$, $\mathscr{C}_{-,1}(S)$ and $\tilde{\mathscr{N}}_1$ in $\mathbb{R}^2 \times \mathbb{R}^1$; manifold $\mathscr{N}_1$ in $\mathbb{R}_+^2$.

origin S. Observe that $\mathscr{C}_{-,k}(S)$ is tangent to $\mathscr{M}_k^2$ at the origin and has no other intersection with $\mathscr{M}_k^2$.

Apart from the origin, cone $\mathscr{C}_{-,k}(S)$ intersects $\mathscr{M}_k^1$ along the $2(M-1)$-dimensional manifold $\tilde{\mathscr{N}}_k$ given by

$$v_k = \lambda_k u_k\ , \quad u_k = \frac{\lambda_k}{\mu_k}\varphi_k(u_1, \ldots, u_M)\ ;$$

the orthogonal projection of $\tilde{\mathscr{N}}_k$ to the $(u_1, \ldots, u_M)$-space $\mathbb{R}^M$ forms a compact $(M-1)$-dimensional manifold $\mathscr{N}_k$ in $\mathbb{R}^M$. [It is worth noting that this projection is one-to-one.] More precisely, we consider the connected component of $\mathscr{N}_k$ situated in the non-negative orthant $\mathbb{R}_+^M$ of $\mathbb{R}^M$ and use for it the same notation $\mathscr{N}_k$. Observe that $\mathscr{N}_k$ partitions $\mathbb{R}_+^M$ into two connected regions, one of which is compact and contains the origin. We denote this region by $\mathrm{Int}\,\mathscr{N}_k$. See Figs. 3a, 3b where the case $M = 2$ and $k = 1$ is considered, and the (non-essential) v_2-direction is omitted.

Now suppose that, in a trajectory $(U_j(x), V_j(x), x \in \mathbb{R}^1, 1 \leqq j \leqq M)$, of (1.14), we have $U_k(x_0), V_k(x_0) \geqq 0$ and $V_k(x_0) < \lambda_k U_k(x_0)$, for some $x_0 \in \mathbb{R}^1$. We can assume that $(U_1(x_0), \ldots, U_M(x_0))$ belongs to $\mathrm{Int}\,\mathscr{N}_k$. Assume that this trajectory reaches cone $\mathscr{C}_{-,k}$ at some point $x_1 > x_0$ and assume that x_1 is the first such point. It means that

$$V_k(x) < \lambda_k U_k(x) \quad \text{for } x_0 \leqq x < x_1\ ,$$

and

$$V_k(x_1) = \lambda_k U(x_1)\ .$$

We conclude that the tangent vector to the curve $(U_k(x), V_k(x), x \in \mathbb{R}^1)$, at $x = x_1$, obeys

$$\left.\frac{dv_k}{du_k}\right|_{\substack{u_k = U_k(x_1)\\ v_k = V_k(x_1)}} = \frac{V_k'(x_1)}{U_k'(x_1)} > \lambda_k\ .$$

[In fact, equality $\dfrac{V_k'(x_1)}{U_k'(x_1)} = \lambda_k$ is impossible, because then the point $(U_j(x_1), V_j(x_1), 1 \leqq j \leqq M)$ must lie down on $\mathscr{M}_k^2$, i.e., must coincide with the origin S.] So

$$V_k'(x_1) > \lambda_k U_k'(x_1) > 0\ .$$

Thus, point $(U_j(x_1),\ V_j(x_1),\ 1 \leqq j \leqq M)$ is below $\mathscr{M}_k^1$, that is, point $(U_1(x_1), \ldots, U_M(x_1)) \in \operatorname{Int} \mathscr{N}_k$. But in our solution, the curve $(U_k(x), V_k(x), x \in \mathbb{R}^1)$ eventually reaches point $u_k = 1$, $v_k = 0$; it means that this curve intersects the line $v_k = \lambda_k u_k$ at least once after x_1. If $x_2 > x_1$ is the next point of intersection, then

$$v_k(x) > \lambda_k U_k(x) \quad \text{for } x_1 < x < x_2 \,. \tag{3.14}$$

Our function Z_k (see above) vanishes at both x_0 and x_1. Therefore, $Z_k'(\tilde{x}) = 0$ for some $\tilde{x} \in (x_1, x_2)$. That is point $(U_j(\tilde{x}),\ V_j(\tilde{x}),\ 1 \leqq j \leqq M)$ lies down on $\mathscr{M}_k^2$. But this contradicts (3.14), which completes the proof of Lemma 3.2.

One of the consequences of Lemma 3.2 is that the integral curves of problem (1.14)–(1.17) are "correctly" parametrized by their initial points on the M-dimensional plain manifold $\mathscr{I}$ in $\mathbb{R}^M \times \mathbb{R}^M$, given by the linear equations $v_k = \lambda_k u_k$ and the inequalities $0 \leqq u_k \leqq 1$, $1 \leqq k \leqq M$. [By an integral curve we mean, here and below, a trajectory as a locus in $\mathbb{R}^M \times \mathbb{R}^M$, without taking into account the "time" variable $x \in \mathbb{R}^1$.] We essentially use this fact in our analysis of travelling wave solutions (see Sect. 4). In particular, the intersection $\mathscr{U}_-(S) \cap \mathscr{S}(T)$ lies below any cone $\mathscr{C}_{-,k}(S)$, $1 \leqq k \leqq M$, and hence does not reach $\mathscr{I}$.

We now pass to the branching diffusion model.

Proof of Theorem 5. The proof of Theorem 5 essentially repeats the argument from Sect. 2 of paper [McK 1]. We give it here for the sake of completeness. See also [CHTWW]. Following [McK 1], denote by τ the first time, after time zero, when a particle splits in the course of the branching diffusion. Then, according to whether $\tau < t$ or $\tau \geqq t$, we decompose

$$\begin{aligned} f_k(t,x) &= \mathbf{P}_k(\tau > t) \int \Pr_k(\mathrm{w}_k(t) + x \in dy) f_k^0(y) \\ &\quad + \int_0^t \mathbf{P}_k(\tau \in dt') \int \Pr_k(\mathrm{w}_k(t') + x \in dy) \\ &\quad\quad \times \sum_{\mathbf{l} = (l_1, \ldots, l_M) \in \mathbf{Z}_+^M} q(k; \mathbf{l}) \prod_{j=1}^M (f_j(t-t', y))^{l_j} \\ &= e^{-\varpi_k t} (e^{-(t/2) D_k} f_k^0)(x) \\ &\quad + \varpi_k \int_0^t e^{-\varpi_k t'} \sum_{\mathbf{l}} \prod_j (e^{-(t'/2) D_k} (f_j(t-t', \cdot))^{l_j})(x) dt' \,. \end{aligned} \tag{3.15}$$

Here $\mathbf{P}_k$ denotes the probability distribution of the branching diffusion process started with a type k particle (our use of this distribution is reduced to the exponential law with mean ϖ_k^{-1}), $\Pr_k$ is the distribution of the Wiener process $w_k(\cdot)$ with drift β_k and diffusion coefficient Σ_k, and D_k is the generators of process $w_k(\cdot)$. As in [McK 1], system (1.21) emerges after the change $t - t' \to t'$ in the integral in the r.h.s. of (3.15) and differentiation.

4. Proof of Theorem 6

It is convenient to preface the proof of Theorem 6 with a lemma summarizing the information gained from our analysis of the exponential model.

Lemma 4.1. *Under condition* (1.5), (1.6), (1.28), (1.29) and (1.32), *there exists a map*

$$\Upsilon: \theta \in [0,\infty) \mapsto (\Upsilon_1(\theta), \ldots, \Upsilon_M(\theta)) \in [0,1] \times \ldots \times [0,1] \tag{4.1}$$

such that

(I) *For any* $\theta \geqq 0$, *there exists a (unique) solution,* $(W_k(x;\theta),\ V_k(x;\theta);\ k = 1, \ldots, M)$, *to the problem*

$$\begin{aligned} W_k'(x) &= V_k(x)\,, \\ V_k'(x) &= (-2(\beta_k + c_k)/\Sigma_k^2)V_k(x) \\ &\quad + (2\varpi_k/\Sigma_k^2)[W_k(x) - \varphi_k(W_1(x), \ldots, W_M(x))]\,, \\ &\quad x > 0\,,\ k = 1, \ldots, M\,, \end{aligned} \tag{4.2}$$

with the initial-boundary value

$$W_k(0) = \Upsilon_k(\theta)\,, \quad V_k(0) = \lambda_k \Upsilon_k(\theta)\,, \tag{4.3}$$

$$0 \leqq W_k \leqq 1 \text{ and } V_k(x) \geqq 0\,, \quad x > 0\,, \tag{4.4}$$

$$W_k(x) = 1 - L_k\theta\exp(-\alpha x) + \eta_k(x)\,, \tag{4.5}$$

where $\eta_k(x), \eta_k'(x) = O(e^{-px})$, *as* $x \to \infty$, *for some* $p > \alpha$. *Here* (i) λ_k *is the positive root of the quadratic equation*

$$\frac{2(\beta_k + c_k)}{\Sigma_k^2} = \frac{2\varpi_k}{\lambda_k\Sigma_k^2} - \lambda_k\,.$$

(ii) α *is the smallest positive root of the equation*

$$\kappa(a) = 1\,, \tag{4.6}$$

and (iii) $\mathbf{L}(a) = (L_1(\alpha), \ldots, L_M(\alpha)) > 0$ *is the eigenvector of matrix* $\mathbf{B}_c(\alpha)$ *with eigenvalue* $\kappa(\alpha) = 1$.

(II) *Solutions* $(W_1(x;\theta), \ldots W_M(x;\theta))$ *are monotone decreasing in* θ: *for any* $x \geqq 0$, *the inequality* $0 \leqq \theta' < \theta''$ *implies*

$$W_k(x;\theta') \geqq W_k(x;\theta'')\,, \quad k = 1, \ldots, M\,.$$

(III) *For any* $x \geqq 0$, *the functions* $\theta \mapsto W_k(x;\theta)$, $1 \leqq k \leqq M$, *are continuous in* $\theta \in (0,\infty)$.

Proof of Lemma 4.1. For definiteness, assume that $q_k^1 = 1$ (cf. proof of Lemma 3.2). Under condition (1.32), we have the family of trajectories of problem (1.14)–(1.17) (in the notation that matches the one in (1.24)), $\{(W_k(x;\theta),\ V_k(x;\theta),\ x \geqq 0,\ k = 1, \ldots, M), \theta \geqq 0\}$, figuring in Lemma 3.1. These solutions have all but one of the properties we need: they do not reach the origin S. Our aim is to check that as $\theta \to \infty$, the corresponding integral curves approach manifold $\mathscr{U}(S)$. A limiting curve will then give a travelling wave profile. [By a limiting curve we mean a limit point, not necessarily the limit.] The main step here is to check that the initial points $(U_k(0),\ V_k(0),\ k = 1, \ldots, M)$ converge, as $\theta \to \infty$, to S along manifold $\mathscr{I} = \{(u_k, v_k, k = 1, \ldots, M): 0 \leqq u_k \leqq 1, v_k = \lambda_k u_k\}$. See Fig. 4, where the projection of the integral curves under consideration is drawn, to a two-dimensional u, v-plane. [*Warning*: Figure 4 does not give a ground for a conjecture that point T is an attracting node for system (1.24): it only suggests that T "acts" as an attracting node

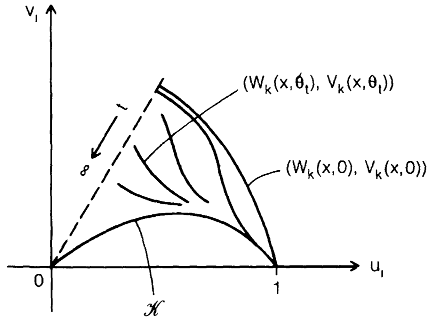

Fig. 4. Projection of the integral curves ($W_k(x;\theta)$, $V_k(x;\theta)$, $x \geqq 0$, $k = 1,2,$) onto plane H_i

along any direction that "points" to the orthant $u_k \leqq 1$, $v_k \geqq 0$, $k = 1, \ldots, M$.]

The formal statement is provided in Lemma 4.2 below.

Lemma 4.2. *Trajectories* ($W_k(x;\theta)$, $V_k(x;\theta)$, $x \geqq 0$, $k = 1, \ldots, M$) *converge, as* θ *runs along an indefinitely increasing sequence, to a curve* $\mathscr{K}$ *that* (a) *joins S and T,* (b) *lies down in the intersection of* $\mathscr{W}(S) \cap \mathscr{S}(T)$ *with the strip* $0 \leqq W_k \leqq 1$, $V_k \geqq 0$, $1 \leqq k \leqq M$, *and* (c) *is an integral curve for system* (4.1). *The convergence is in the following sense: or any fixed* $y \in (0,1)$ *and any* $x \in \mathbb{R}^1$, *there exists an indefinitely increasing sequence* $\{\theta_t, t = 1, 2, \ldots,\}$ *such that there exists the limits*

$$\lim_{t \to \infty} W_k(\hat{x}(y, \theta_t) + x; \theta_t) = \bar{W}_k(y; x) , \tag{4.7}$$

and

$$\lim_{t \to \infty} V_k(\hat{x}(y, \theta_t) + x; \theta_t) = \bar{V}_k(y; x) ; \tag{4.8}$$

the locus ($\bar{W}_k(y;x)$, $\bar{V}_k(y;x)$, $x \in \mathbb{R}^1$, $k = 1, \ldots, M$) *forms curve* $\mathscr{K}$. *Here* $x(y,\theta)$ *is defined as a unique solution to the equation*

$$W_1(x(y,\theta);\theta) = y . \tag{4.9}$$

Proof of Lemma 4.2. We first note that, according to assertions (I) and (II) of Lemma 2.1, we have, for each $x \geqq 0$,

$$\hat{W}_k(x) = \inf_{\theta > 0} W_k(x;\theta) = \lim_{\theta \to \infty} W_k(x;\theta) \geqq 0 . \tag{4.10}$$

We want to show that the limits $\hat{W}_k(0)$, $1 \leqq k \leqq M$, are equal to zero.

Clearly, all functions $\hat{W}_1(x), \ldots, \hat{W}_M(x)$, $x > 0$, are smooth in x, and setting

$$\hat{V}_k(x) = \lim_{\theta \to \infty} V_k(x;\theta) , \quad 1 \leqq k \leqq M , \tag{4.11}$$

yields

$$\hat{V}_k(x) = \hat{W}'_k(x) , \quad 1 \leqq k \leqq M, \quad x > 0 .$$

Moreover, ($\hat{W}_k(x)$, $\hat{V}_k(x)$, $x \geqq 0$, $k = 1, \ldots, M$) gives a solution to (1.24), with the initial-value condition

$$\hat{V}_k(0) = \lambda_k \hat{W}_k(0) , \tag{4.12}$$

and with the bounds $\hat{V}_k \geqq 0$, $k = 1, \ldots, M$.

At this point we use the following simple assertion[4]

Proposition 4.3. *Under assumption* (1.5), *system* (1.24) *does not have equilibrium points in strip* (1.27) *other than S and T.*

Proof. Recall, any equilibrium point $(w_1, \ldots, w_M; v_1, \ldots, v_M)$ must have $v_1 = \cdots = v_M = 0$ and

$$w_k = \varphi_k(w_1, \ldots, w_M)\,, \quad 1 \leqq k \leqq M\,. \tag{4.13}$$

We have two equilibrium points, S and T. Assume that there exists an equilibrium point, $(\mathbf{w}', \mathbf{0})$, with $\mathbf{w}' = (w'_1, \ldots, w'_M)$ and $0 \leqq w'_k \leqq 1$, different from S. We want to show that there is no other equilibrium point $(\mathbf{w}, \mathbf{0})$, $\mathbf{w} = (w_1, \ldots, w_M)$, with $w_k \geqq w'_k$; it will imply that this equilibrium point coincides with T.

The idea is to use the (strict) monotonicity and convexity of functions $\varphi_k(z_1, \ldots, z_M)$, $1 \leqq k \leqq M$. More precisely, we use two properties: (i) the restriction, of any of these functions, to an interval (finite or infinite) contained in the orthant $\{z_k \geqq 0, 1 \leqq k \leqq M\}$ is a strictly convex function (of one variable), and (ii) apart from the origin, there is no equilibrium point $(\mathbf{w}, \mathbf{0})$, $\mathbf{w} = (w_1, \ldots, w_M)$, with $0 \leqq w_k \leqq 1$ and $\min w_k = 0$. [The last property follows easily if we take $w_{i_0} = \max w_k$: equality $\varphi_{i_0}(w_1, \ldots, w_M) = w_{i_0}$ is impossible under the condition $\min w_k = 0$.]

Hence, our equilibrium point $\mathbf{w}' = (w'_1, \ldots, w'_M)$ must lie in the interior of the unit cube $0 \leqq w'_k \leqq 1$, $1 \leqq k \leqq M$.

For definiteness, assume that $\mathbf{w}'$ is one of the "closest" equilibrium points to S, in the sense that, apart from S there is no other equilibrium point $(\mathbf{w}, \mathbf{0})$, $\mathbf{w} = (w_1, \ldots, w_M)$, with $0 \leqq w_k \leqq w'_k$. Now take an equilibrium point $(\mathbf{w}'', \mathbf{0})$, $\mathbf{w}'' = (w''_1, \ldots, w''_M)$, with $w''_k \geqq w'_k$ (point T is an example of such an equilibrium point). Then functions φ_k restricted to an interval that joins $\mathbf{w}''$ and $\mathbf{w}'$ are convex and, at the end points of this interval, they coincide with their "associated" linear functions obtained by joining the corresponding values w''_k and w'_k. Hence, on the prolongation of this interval, they strictly exceed their linear "associates". But this leads to a contradiction because the same functions are convex on the interval joining the origin and $\mathbf{w}'$. This completes the proof of Proposition 4.3.

We now can complete the proof of Lemma 4.2. Since system (1.24) has no equilibrium points in $0 \leqq u_k \leqq 1$, $v_k \geqq 0$, $k = 1, \ldots, M$, other than S and T, we have

$$\lim_{x \to \infty} \hat{W}_1(x) = \cdots = \lim_{x \to \infty} \hat{W}_M(x) = 1\,. \tag{4.14}$$

Furthermore, relations

$$\lim_{x \to \infty} \frac{\hat{V}_k(x)}{\hat{W}_1(x)} \leqq \alpha\,, \quad k = 1, \ldots, M\,, \tag{4.15}$$

must hold. By Theorem 2.2, the limit in (4.15) equals α. Furthermore, matrix $\mathbf{B}(\alpha)$ has no eigenvectors with positive components other than $\mathbf{L}(\alpha)$. This means that solution $(\hat{W}_k, \hat{V}_k, k = 1, \ldots, M)$ belongs to $\mathscr{F}^{(\gamma)}$ for some $\gamma > 0$. But this contradicts the fact that $\mathscr{F}^{(\theta)}$ contains, for every $\theta \geqq 0$, a unique solution, $(W_k(\,\cdot\,;\theta), V_k(\,\cdot\,;\theta)$; $k = 1, \ldots, M)$. Hence, all limits in (4.10) must be zero.

[4] Our attention to this fact was pointed by Vitaly Volpert.

Therefore, the initial points of the integral curves under consideration approach S as $\theta \to \infty$. Now fix $y \in (0, 1)$. The points $(W_k(\hat{x}(y, \theta); \theta)$, $V_k(\hat{x}(y, \theta); \theta)$, $k = 1, \ldots, M)$ vary in a compact set, separated from S and T. Hence, there exists an indefinitely increasing sequence of values of θ along which limits (4.7)–(4.10) exists for $x = 0$. The limit point $(\bar{W}_k(y; 0), \bar{V}_k(y; 0), k = 1, \ldots, M)$ must lie down on $\mathscr{U}(S)$, otherwise it will contradict the foregoing conclusion. Since the vector field determining system (1.24) is analytic and has no equilibrium points in the strip $0 \leqq u_k \leqq 1, k = 1, \ldots, M$, it is easy to see that limits (4.7)–(4.9) exist for any $x \in \mathbb{R}^1$ and the limit point $(\bar{W}_k(y; x), \bar{V}_k(y; x), k = 1, \ldots, M) \in \mathscr{U}(S)$. Furthermore, $x \in \mathbb{R}^1 \mapsto (\bar{W}_k(y; x), \bar{V}_k(y; x), k = 1, \ldots, M)$ is a solution to (1.24). It also lies down in the intersection of $\mathscr{S}(T)$ with above strip. This completes the proof of Lemma 4.2, and hence of Theorem 6.

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