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Localization for some continuous Random Schrödinger Operators

Frédéric Klopp[1]

Département de Mathématique, Bât. 425, Université de Paris-Sud, Centre d'Orsay, F-91405 Orsay Cédex, France

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Abstract: We study the spectrum of random Schrödinger operators acting on $L^2(\mathbb{R}^d)$ of the following type $H = -\Delta + W + \sum_{x\in\mathbb{Z}^d} t_x V_x$. The $(t_x)_{x\in\mathbb{Z}^d}$ are i.i.d. random variables. Under weak assumptions on V, we prove exponential localization for H at the lower edge of its spectrum. In order to do this, we give a new proof of the Wegner estimate that works without sign assumptions on V.

Résumé: Dans ce travail, nous étudions le spectre d'opérateurs de Schrödinger aléatoires agissant sur $L^2(\mathbb{R}^d)$ du type suivant $H = -\Delta + W + \sum_{x\in\mathbb{Z}^d} t_x V_x$. Les $(t_x)_{x\in\mathbb{Z}^d}$ sont des variables aléatoires i.i.d. Sous de faibles hypothèses sur V, nous démontrons que le bord inférieur du spectre de H n'est composé que de spectre purement ponctuel et, que les fonctions propres associées sont exponentiellement décroissantes. Pour ce faire nous donnons une nouvelle preuve de l'estimée de Wegner valable sans hypothèses de signe sur V.

0. Introduction

The present paper is devoted to the study of the nature of the spectrum of some random Schrödinger operators.

In the last years, random operators have been studied quite a lot. Many of these studies have focussed on one of the properties of these objects, namely localization. Though localization, i.e. the existence of dense pure point spectrum, has been mostly studied in the discrete case, that is for Schrödinger operators defined on $l^2(\mathbb{Z}^d)$ (see, e.g. [Ai-Mo, Fr-Sp, vD-K 1], or the monographs [Ca-La] and [Pa-Fi] for further references), recently people have also been interested in the localization properties for continuous random Schrödinger operators (see [Ho-Ma, Co-Hi 1, Co-Hi 2, Kl 1 or Kl 2]). In [Ho-Ma], H. Holden and F. Martinelli studied what

[1] U.R.A. 760 C.N.R.S

can be considered as the continuous model closest to the discrete Anderson model,

$$H = -\Delta + \lambda \sum_{x \in \mathbb{Z}^d} t_x \chi_x , \tag{0.1}$$

where the $(t_x)_{x \in \mathbb{Z}^d}$ are independently identically distributed random variables, and $\chi_x(\cdot) = \chi_0(\cdot - x)$ is the characteristic function of a cube of center x and of sidelength 1. They proved absence of diffusion. Then, using this work, S. Kotani and B. Simon [Ko-Si] proved localization for this model.

More recently, J.-M. Combes and P. Hislop in [Co-Hi 1] studied more general operators of the form (0.1) for a larger class of χ_0, as well as other types of random perturbations of the free Laplacian (see also [Co-Hi 2]). They proved localization in the large coupling constant limit, $\lambda \to +\infty$.

Nevertheless, the strong restrictions they had to impose on the perturbation χ_0 seem unnecessary. Indeed, it is believed that localization in the large coupling constant limit or at the edges of the spectrum should occur for almost any function χ_0 such that H makes sense. This is in a way our main result.

The model we study is of the following form:

$$H = H_0 + \sum_{x \in \mathbb{Z}^d} t_x V_x .$$

Here H_0 is some self-adjoint perturbation of $-\Delta$. The main assumption we need on H_0 is that it is lower semi-bounded. The $(t_x)_{x \in \mathbb{Z}^d}$ are supposed to be independently identically distributed random variables. And V is a function supposed to decrease exponentially at infinity.

Let us now sketch the general strategy of our proof. Let $\Lambda \subset \mathbb{Z}^d$, be a cube and consider the following approximation of H,

$$H_\Lambda = H_0 + \sum_{x \in \Lambda} t_x V_x .$$

Then, for $\Lambda \subset \Lambda' \subset \mathbb{Z}^d$, two cubes and $E \notin \sigma(H_\Lambda) \cup \sigma(H_{\Lambda'})$ (here $\sigma(H)$ is the spectrum of H), one can write a resolvent formula,

$$(H_{\Lambda'} - E)^{-1} = (H_\Lambda - E)^{-1} - (H_\Lambda - E)^{-1} \left(\sum_{x \in \Lambda' \setminus \Lambda} t_x V_x \right) (H_{\Lambda'} - E)^{-1} .$$

Such a resolvent estimate permits then to construct an induction argument “à la Fröhlich-Spencer” (see [Fr-Sp, vD-K 1 or Kl 1]) if we know a Wegner estimate for H_Λ. That is, we must be able to show that the probability of the following event

$$\mathscr{E}_0(\varepsilon) = \{t; \|(H_\Lambda - E)^{-1}\| \geqq 1/\varepsilon\}$$

decreases sufficiently quickly with $\varepsilon > 0$ when $\varepsilon \to 0$.

If E is below $\sigma(H_0)$, we may write

$$(H_\Lambda - E)^{-1} = (H_0 - E)^{-1/2}(1 + \Gamma_\Lambda(t,E))^{-1}(H_0 - E)^{-1/2} ,$$

where

$$\Gamma_\Lambda(t,E) = \sum_{x \in \Lambda} t_x (H_0 - E)^{-1/2} V_x (H_0 - E)^{-1/2} .$$

Note that $\Gamma_\Lambda(t,E)$ is a compact operator.

Now, to estimate the probability of $\mathscr{E}_0(\varepsilon)$, we only need to estimate the probability of the following event:

$$\mathscr{E}(\varepsilon) = \{t; \|(\Gamma_\Lambda(t,E) + 1)^{-1}\| \geqq 1/\varepsilon\} .$$

But to do this we can use the following property of $\Gamma_\Lambda(t,E)$:

$$\sum_{x\in\Lambda} t_x \frac{\partial}{\partial t_x} \Gamma_\Lambda(t,E) = \Gamma_\Lambda(t,E) , \tag{0.2}$$

that is, $\Gamma_\Lambda(t,E)$ is invariant under the flow of the vector field $\sum_{x\in\Lambda} t_x \frac{\partial}{\partial t_x}$. So, if we call this flow $\varphi(u,t)$ and the eigenvalues of $\Gamma_\Lambda(t,E)$, $\mu_k(t)$, then

$$\frac{d}{du}\mu_k(\varphi(u,t)) = \mu_k(\varphi(u,t)) .$$

Hence, if $\mu_k(\varphi(u,t))$ is close to -1, then $\frac{d}{du}\mu_k(\varphi(u,t))$ is close to -1; this shows that $\mu_k(\varphi(u,t))$ "moves" as u is "moved." Then, using a regularity assumption on the distribution of the random variables $(t_x)_{x\in\mathbb{Z}^d}$, we can show that the probability that an eigenvalue $\mu_k(t)$ stays in an ε-neighborhood of -1, is small with ε. This, in turn, is the same as to say that the probability of $\mathscr{E}(\varepsilon)$ has a nice decrease in ε.

Though the proofs of the Wegner estimate we present in this paper were written for i.i.d. random variables, we hope that these ideas may be adapted to correlated potentials under suitable assumptions on the conditional probabilities of the random variables (see [vD-K 2, Co-Hi 2]).

The main feature of our proof of the Wegner estimate is that no sign assumption on V is needed, as (0.2) does not depend on this sign.

In fact to prove localization, the Wegner estimate is not entirely sufficient. Once the induction process is constructed, one also needs to show that the first step of the induction holds. Therefore, instead of using the previous argument in all its generality, we found it more convenient to apply it to two "concrete" cases. In both cases, we prove localization at the lower edges of the spectrum.

In our first model, we suppose H_0 to be a lower semi-bounded periodic Schrödinger operator and the random variables $(t_x)_{x\in\mathbb{Z}^d}$ to be unbounded, such that the almost sure spectrum of H is $\mathbb{R}$ (i.e. we work with unbounded random perturbations). For this model, we prove that, below some energy, the spectrum of H is almost surely pure point, and that the associated eigenfunctions are exponentially decreasing.

Our second model is closer to the one studied in [Co-Hi 1]. We assume that V and the $(t_x)_{x\in\mathbb{Z}^d}$ are such that the resulting random model is lower semi-bounded. To ensure the first step of the induction, we assume rapid decay of the density of states at the edge of the spectrum. Such behaviour is a weaker form of the celebrated Lifshits behaviour of the density of states, which has been proved for many models under assumptions compatible with ours (see [Ki, Ca-La or Pa-Fi] for a review and further references). Under these assumptions, we prove that there exists some energy strictly larger than the infimum of the almost sure spectrum below which the spectrum of H is pure point with probability 1, and that the associated eigenfunctions are exponentially decreasing.

1. The Main Results

A. For Unbounded Perturbations. Let $L = \bigoplus_{j=1}^{d} \mathbb{Z} u_j$ be a lattice of $\mathbb{R}^d$ (here $(u_j)_{1 \leqq j \leqq d}$ is a basis of $\mathbb{R}^d$). Denote the unit cell of L by $C_0 = \{x = \sum_{1 \leqq j \leqq d} x_j u_j;\ (x_j)_{1 \leqq j \leqq d} \in [-\frac{1}{2}, \frac{1}{2})^d\}$.

Let W be a real, bounded, L-periodic function. We define the following Schrödinger operator:

$$H_0 = -\Delta + W .$$

H_0 is self-adjoint on $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. Moreover, as W is bounded, H_0 is lower semi-bounded. $\sigma(H_0)$ denotes the spectrum of H_0. Let us assume for convenience that

$$\inf \sigma(H_0) = 0 .$$

Let V be a real measurable function satisfying

(H.1)

$\exists C_0 > 0$ and $m_0 > 0$ such that $\forall x \in \mathbb{R}^d$,

$$|V(x)| \leqq C_0 e^{-m_0|x|} .$$

Let $(t_\gamma)_{\gamma \in L}$ be a sequence of independent identically distributed random variables with common distribution density g satisfying

(H.2)

a) $\exists \varepsilon_0 > 0$ and $\rho_0 > 0$ such that, $\forall \varepsilon \in [0, \varepsilon_0]$,

$$\int_{\mathbb{R}} |g((1+\varepsilon)t) - g(t)|\, dt \leqq \left(\frac{\varepsilon}{\varepsilon_0}\right)^{\rho_0} .$$

b) Let $q_0 = \frac{d}{2}$ if $d \geqq 4$ and $q_0 = 2$ if $d \leqq 3$. $\exists k > q_0$ such that

$$\int_{\mathbb{R}} |t|^k g(t)\, dt < +\infty .$$

Remark. Assumption (H.2) a) is a regularity assumption on g that is, for example satisfied if g is derivable and $\int |x \cdot g'(x)|\, dx < +\infty$.

Let $H(t)$ be the following operator:

$$H(t) = H_0 + \sum_{\gamma \in L} t_\gamma V_\gamma , \tag{1.1}$$

where for $\gamma \in L$ and $x \in \mathbb{R}^d$, $V_\gamma(x) = V(x - \gamma)$.

By Theorem 1 of [Ki-Ma 2] (see also [Ki-Ma 3]), by assumption (H-1) and (H-2)b), we know that, with probability 1, $H(t)$ is essentially self-adjoint on $\mathscr{C}_0^\infty(\mathbb{R}^d)$. As H_0 is L-periodic and the $(t_\gamma)_{\gamma \in L}$ are i.i.d. random variables, we also know that $H(t)$ is ergodic (or metrically transitive); so, Theorem 2 of [Ki-Ma 2] (see also [Ki-Ma]) tells us that:

(i) $\sigma(H(t))$, the spectrum of $H(t)$, is a non-random set with probability 1. Let us call it Σ.
(ii) the pure point, the absolutely continuous and the singular continuous part of $\sigma(H(t))$ are non-random sets with probability 1.

Let us now assume

(*H.3*)

$$\Sigma = \mathbb{R} .$$

Remark. Using the work of W. Kirsch and F. Martinelli [Ki-Ma 2], one can show that assumption (H-3) holds under weak conditions on V and the random variables $(t_\gamma)_{\gamma \in L}$. Essentially, one needs the range of the random variables to be unbounded and the potentials V to take the right sign on some set of non-zero Lebesgue measure, the sign depending on which side the range of the $(t_\gamma)_{\gamma \in L}$ is unbounded.

Now our main result is

Theorem 1.1. *Let $H(t)$ be defined as above and assume $(H\text{-}1)$-$(H\text{-}3)$ hold. Then for any $\varepsilon > 0$, there exists $E_\varepsilon > 0$ such that, with probability* 1,

i) the spectrum of $H(t)$ in $(-\infty, -E_\varepsilon]$ is pure point,
ii) if φ is an eigenvector associated to E an eigenvalue of $H(t)$ in $(-\infty, -E_\varepsilon]$, then there exists $C_\varphi > 0$, such that, for any $x \in \mathbb{R}^d$,

$$|\varphi(x)| \leqq C_\varphi e^{-m_0(1-\varepsilon)|x|} .$$

Remark. 1) The main novelty here is that we did not assume any sign condition on V; so the proofs of the Wegner estimate used up to now (see for example [Ho-Ma, Ko-Si and Co-Hi 1]) break down. Our main point then is to prove a new Wegner estimate in this case.

2) As will turn out from our proof, we could have considered the following random Schrödinger operator:

$$H(t, \xi) = H_0 + \sum_{\gamma \in L} t_\gamma V(x - \gamma - \xi_\gamma) ,$$

where H_0, V, and $(t_\gamma)_{\gamma \in L}$ are as above, and $(\xi_\gamma)_{\gamma \in L}$ are i.i.d. $\mathbb{R}^d$-valued random variables with common support in some compact set. For such operators, Theorem 1.1 still holds.

3) The regularity assumptions on V and W used here to get Theorem 1.1, are not optimal; we choose them this way for the sake of simplicity. Moreover the exponential decay assumption on V may certainly be relaxed to some sufficiently fast polynomial decay though we did not check the computations. In this case, one may expect that the spectrum stays pure point and that the associated eigenfunctions decrease polynomially at infinity. To prove Theorem 1.1, one constructs an induction process "à la Fröhlich-Spencer" for our case. Let us just describe the main ingredients of this induction, the bulk of it being treated in Sect. 2.

Let $\Lambda_l(0)$, be a cube in L, of center 0 and side l (i.e. $\Lambda_l(0) = \{\sum_{1 \leqq j \leqq d} x_j u_j \in L; -\frac{1}{2} \leqq x_j < \frac{l}{2}\}$) and define

$$H_{\Lambda_l(0)}(t) = H_0 + \sum_{\gamma \in \Lambda_l(0)} t_\gamma V_\gamma . \tag{1.2}$$

For any realisation of $(t_\gamma)_{\gamma \in L}$, $H_{\Lambda_l(0)}(t)$ is a relatively compact perturbation of H_0; so the negative spectrum of $H_{\Lambda_l(0)}(t)$ is discrete.

For $E \notin \sigma(H_{\Lambda_l(0)}(t))$, we define

$$G_{\Lambda_l(0)}(E) = (H_{\Lambda_l(0)}(t) - E)^{-1} .$$

Let $\Lambda \subset \Lambda'$, be two cubes of L. Then, for $E \notin \sigma(H_\Lambda(t)) \cup \sigma(H_{\Lambda'}(t))$, we get the following resolvent formula:

$$G_{\Lambda'}(E) = G_\Lambda(E) + G_\Lambda(E) V_{\Lambda,\Lambda'} G_{\Lambda'}(E) ,$$

where $V_{\Lambda,\Lambda'} = \sum_{\gamma \in \Lambda' \setminus \Lambda} t_\gamma V_\gamma$.

Then, the tool to control the induction process is

Theorem 1.2 (*The Wegner Estimate*). *Under the assumptions of Theorem* 1.1, *for any* $l \geqq 2$, $E_0 > 0$, $p_0 > 0$, *there exists* $C_0 > 0$ *and* $p_0' > 0$, *such that for any* $E \in (-\infty, -E_0]$, $\varepsilon \in [0, \varepsilon_0]$ *and any* $l \geqq 2$,

$$\mathscr{P}\left(\left\{t;\ \|G_{\Lambda_l(0)}(E)\| \geqq \frac{1}{\varepsilon}\right\}\right) \leqq l^{-p_0} + C_0 \cdot l^{p_0'} \varepsilon^{\inf(1,\rho_0)} .$$

B. For Lower Semi-bounded Perturbations. Let H_0 be defined as in Subsect. A. Let V be a measurable function $L^\infty(\mathbb{R}^d)$ such that

(H.1)

V is not identically 0 in L^1_{loc}-sense and has compact support.

Let $(t_\gamma)_{\gamma \in L}$ be a sequence of independent identically distributed random variables with common distribution density g satisfying

(H.2)

a) $\forall \varepsilon \in [0, \varepsilon_0]$, $\int_{\mathbb{R}} \sup_{u \in [-1,1]} |g(t + \varepsilon u) - g(t)|\, dt \leqq (\frac{\varepsilon}{\varepsilon_0})^{\rho_0}$.

b) G, the essential support of g, is bounded.

Consider the Schrödinger operator given by formula (1.1). Then, for any realization of the random variables $(t_\gamma)_{\gamma \in L}$, $H(t)$ is self-adjoint on $L^2(\mathbb{R}^d)$ with domain $H^2(\mathbb{R}^d)$. $H(t)$ is also lower semi-bounded.

Remark. If we assume that V is non-negative, we may replace the boundedness assumption on the random variables $(t_\gamma)_{\gamma \in L}$ by a positivity assumption plus assumption (H.2) b) of Subsect. A. Then, by [Ki-Ma 2], for almost every realization of the random variables $(t_\gamma)_{\gamma \in L}$, $H(t)$ is essentially self-adjoint on $\mathscr{C}_0^\infty(\mathbb{R}^d)$. Moreover, it will be lower semi-bounded.

As in Subsect. A, $H(t)$ is ergodic. Let $E_{\inf}$, be the infimum of Σ, the almost sure spectrum of $H(t)$. Using the ergodicity of $H(t)$, we can define $N(E)$, the integrated density of states of $H(t)$ (see for example, [Ki or Pa-Fi]).

Let us assume that the following holds:

(H.3)

For any $n \in \mathbb{N}$, $(E - E_{\inf})^{-n}$. $N(E) \to 0$ when $E \to E_{\inf}$ and $E > E_{\inf}$.

Remark. 1) If we assume V non-negative and the random variables $(t_\gamma)_{\gamma \in L}$ also to be non-negative then, if $0 \in G$, $E_{\inf} = 0$ as can be seen by the results of [Ki-Ma 2] (see also [Ki]).

2) Assumption (H.3) b) is naturally implied by the celebrated Lifshifts tail behavior for the density of states at the edges of the spectrum. The Lifshits behavior has been studied quite extensively and has been proved under various conditions on the random variables $(t_\gamma)_{\gamma \in L}$ and on the operator H_0 (for example, see [Ki,

Pa-Fi or Ca-La] for a review and further references). Under these assumptions, we prove

Theorem 1.3. *Let $H(t)$ be defined as above. Assume that (H.1)-(H.3) are satisfied. Then, there exists $E_0 > E_{\inf}$ and $\gamma_0 > 0$ such that, with probability 1,*

i) the spectrum of $H(t)$ in $[E_{\inf}, E_0]$ is pure point,
ii) if φ is eigenvector associated to E an eigenvalue of $H(t)$ in $[E_{\inf}, E_0]$, then there exists $C_\varphi > 0$, such that, for any $x \in \mathbb{R}^d$,

$$|\varphi(x)| \leqq C_\varphi e^{-\gamma_0 |x|}.$$

Remark. 1) This theorem may be considered as an extension of some of the results obtained by J-M. Combes and P. Hislop in [Co-Hi 1]. We prove that one may remove the quite restrictive lower bound in their assumption (u_A).

2) Assumption (H.3) b) is weaker than the actual Lifshits tail behaviour; nevertheless, as will be seen from the proof of Theorem 1.3, it is much stronger than what is actually needed for localization. Let us now sketch the ideas of the proof of Theorem 1.3. The details are given in Sect. 3. As for Theorem 1.1, the result will be obtained via an induction process. In this case we will use the induction process designed in [Co-Hi 1]. Our main goal will then be to get a Wegner estimate in our case and to prove the initial step of the induction.

For $\Lambda \subset \mathbb{R}^d$, we denote by $H_\Lambda(t)$, the operator $H(t)$ restricted to Λ with Dirichlet boundary conditions. Let us point out the fact that, as V is supported in some compact ball B, $H_\Lambda(t)$ only depends on finitely many random variables.

We then prove

Theorem 1.4 (*The Wegner Estimate*). *There exists $E_0 > 0$, $C_0 > 0$ and $q_0 > 0$ such that, for any $l \geqq 2$, any $\varepsilon \geqq 0$ and any $E \in [0, E_0]$,*

$$\mathscr{P}(\{t;\, d(\sigma(H_{\Lambda_l}(t), E) \leqq \varepsilon)\} \leqq C_0 \varepsilon^{\rho_0} l^{q_0} .$$

Remark. [Co-Hi 1] already obtained a Wegner estimate for more general models, but only under a much stronger assumption V. On the other hand, their estimate is more accurate than ours as it permits to get information on the regularity on the density of states. Let $1 < \delta < +\infty$. As in [Co-Hi 1], for Λ_l, a cube of $\mathbb{R}^d$, let $\tilde{\Lambda}_l = \{x;\, x + B(0, \delta) \subset \Lambda_l\}$. Then define χ_l to be a non-negative $\mathscr{C}^2$ function that is 1 on $\tilde{\Lambda}_l$ and 0 outside Λ_l, and $W(\chi_l) = [-\Delta, \chi_l]$.

Using assumption (H.3), we show

Proposition 1.5. *There exists $l_0 \geqq 2$ and $C_0 > 0$ such that for $l \geqq l_0$ and $E \in [0, \sqrt{l}]$,*

$$\mathscr{P}\left(\left\{t;\, \sup_{\varepsilon > 0} \|[-\Delta, \chi_l] G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3}\| \leqq e^{-\gamma \cdot l}\right\}\right) \geqq 1 - l^{-(2d+1)},$$

where: i) $\gamma = C_0 \cdot l^{-\frac{1}{4}}$,

ii) $G_{\Lambda_l}(E + i\varepsilon)$ is the resolvent of $H_{\Lambda_l(t)}$ at energy $E + i\varepsilon$.

Then, adapting the induction process of [Co-Hi 1] as is explained in Sect. 3, one proves Theorem 1.3.

II. For Unbounded Perturbations: The Proof of Theorem 1.1

A. The Wegner Estimate (Theorem 1.2). Fix $E_0 > 0$ and $p_0 > 0$. For $E \leqq -E_0$, one has

$$\mathscr{P}\left(\left\{t;\ \|G_{\Lambda_l(0)}(E)\| > \frac{1}{\varepsilon}\right\}\right) \leqq \mathscr{P}\left(\left\{t;\ \|G_{\Lambda_l(0)}(E)\| > \frac{1}{\varepsilon} \quad \text{and} \quad \forall \gamma \in \Lambda_l(0), |t_\gamma| \leqq l^{r_0}\right\}\right)$$

$$+ \mathscr{P}(\{t; \exists \gamma \in \Lambda_l(0), |t_\gamma| > l^{r_0}\}) , \tag{2.1}$$

where $r_0 > 0$ is to be chosen later on.

Then by assumption (H.2) b), we know that

$$\mathscr{P}(\{t; \exists \gamma \in \Lambda_l(0), |t_\gamma| > l^{r_0}\}) \leqq C \cdot l^d \cdot l^{-r_0 k} ,$$

so, for some $r_0 > 0$ large enough,

$$\mathscr{P}(\{t; \exists \gamma \in \Lambda_l(0), |t_\gamma| > l^{r_0}\}) \leqq l^{-p_0} . \tag{2.2}$$

If we prove

Lemma 2.1. *There exists $C_0 > 0$ and $p'_0 > 0$, such that for any $l \geqq 2$, $E \in (-\infty, -E_0]$, $\varepsilon \in [0, \varepsilon_0[$ and any $l \geqq 2$,*

$$\mathscr{P}\left(\left\{t;\ \|G_{\Lambda_l(0)}(E)\| \geqq \frac{1}{\varepsilon} \text{ and } \forall \gamma \in \Lambda_l(0), |t_\gamma| \leqq l^{r_0}\right\}\right) \leqq C_0 \cdot l^{p'_0} \varepsilon^{\inf(1,\rho_0)} ,$$

then, using (2.2), we get Theorem 1.3.

Proof of Lemma 2.1. Let $E \notin \sigma(H_{\Lambda_l(0)}(t))$ and $E \leqq -E_0$, *then*

$$G_{\Lambda_l(0)}(E) = (H_0 - E)^{-1/2}\left(1 + \sum_{\gamma \in \Lambda_l(0)} t_\gamma (H_0 - E)^{-1/2} \right.$$

$$\left. \times V_\gamma (H_0 - E)^{-1/2}\right)^{-1} (H_0 - E)^{-1/2} . \tag{2.3}$$

Define

$$\Gamma(t, E) = -\sum_{\gamma \in \Lambda_l(0)} t_\gamma (H_0 - E)^{-1/2} V_\gamma (H_0 - E)^{-1/2} .$$

By our assumptions on V and W, $\Gamma(t, E)$ is compact and uniformly bounded for $E \leqq -E_0$. By (2.3), we see that, as $(H_0 - E)^{-1/2}$ is uniformly bounded for $E \leqq -E_0$,

$$\|G_{\Lambda_l(0)}(E)\| \leqq C \left\| \left(1 + \sum_{\gamma \in \Lambda_l(0)} t_\gamma (H_0 - E)^{-1/2} V_\gamma (H_0 - E)^{-1/2}\right)^{-1} \right\| .$$

Now, if we prove

Lemma 2.2. *There exists $C > 0$ and $p'_0 > 0$, such that for any $l \geqq 2$, $E \in (-\infty, -E_0]$, $\varepsilon \in [0, \varepsilon_0[$ and any $l \geqq 2$,*

$$\mathscr{P}(\{t; \operatorname{dist}(1, \Gamma(t, E)) < \varepsilon \text{ and } \forall \gamma \in \Lambda_l(0), |t_\gamma| \leqq l^{r_0}\}) \leqq C_0 \cdot l^{p'_0} \varepsilon^{\inf(1,\rho_0)} ,$$

we are done with the proof of the Wegner estimate.

Proof of Lemma 2.2. Notice that, for $\lambda \in \mathbb{R}$, $\Gamma(\lambda t, E) = \lambda \Gamma(t, E)$. Let us define the mapping φ: $\mathbb{R} \times \mathbb{R}^{\Lambda_l(0)} \to \mathbb{R}^{\Lambda_l(0)}$ by $\varphi(u,t) = e^u \cdot t$ (here t denotes a vector of the form $(t_\gamma)_{\gamma \in \Lambda_l(0)}$). φ is the flow of the vector field $t \cdot \nabla$ defined on $\mathbb{R}^{\Lambda_l(0)}$. So

$$\frac{d}{du}\Gamma(\varphi(u,t),E) = \Gamma(\varphi(u,t),E)\,. \tag{2.4}$$

Let $(\mu_k(t,u))$ (resp. $(\mu_k(t))$ denote the positive eigenvalues of $\Gamma(\varphi(u,t),E)$ (resp. $\Gamma(t,E)$) ordered in a decreasing way, then $\mu_k(t,u) = e^u \mu_k(t)$. For $\varepsilon > 0$, define $N(E,\varepsilon,t) = \#\{k;\, \mu_k(t) \geqq \varepsilon\}$, the cardinal of $\{k;\, \mu_k(t) \geqq \varepsilon\}$. Then $N(E,\varepsilon,t) < +\infty$ as $\Gamma(\varphi(u,t),E)$ is compact. So $N(E,\varepsilon,\varphi(u,t)) = N(E, e^{-u}\varepsilon, t)$ and

$$N(E,1-\varepsilon,t) - N(E,1+\varepsilon,t) = N\left(E,1,\frac{t}{1-\varepsilon}\right) - N\left(E,1,\frac{t}{1+\varepsilon}\right).$$

Let us define $I = [-l^{r_0}, l^{r_0}]$. Then following Wegner ([We]), we get

$$\begin{aligned}
&\mathscr{P}(\{t;\, \mathrm{dist}(1,\Gamma(t,E)) < \varepsilon \textit{ and } \forall \gamma \in \Lambda_l(0), |t_\gamma| \leqq l^{r_0}\}) \\
&\leqq \int_{I^{\Lambda_l(0)}} (N(E,1-\varepsilon,t) - N(E,1+\varepsilon,t))\, dP \\
&\leqq \int_{I^{\Lambda_l(0)}} \left[N\left(E,1,\frac{t}{1-\varepsilon}\right) - N\left(E,1,\frac{t}{1+\varepsilon}\right)\right] \prod_{\gamma \in \Lambda_l(0)} g(t_\gamma)\, dt_\gamma\,.
\end{aligned}$$

Set $\lambda = \#\Lambda_l(0)$, $dt_\Lambda = \prod_{\gamma \in \Lambda_l(0)} dt_\gamma$ and $\tilde{g}_\Lambda(t) = \prod_{\gamma \in \Lambda_l(0)} \tilde{g}(t_\gamma)$, where $\tilde{g} = g \cdot \chi_I$ and χ_I is the characteristic function of I, then

$$\begin{aligned}
&\mathscr{P}(\{t;\, \mathrm{dist}(1,\Gamma(t,E)) < \varepsilon \text{ and } \forall \gamma \in \Lambda_l(0), |t_\gamma| \leqq l^{r_0}\}) \\
&\leqq \int_{\mathbb{R}'} N(E,1,t)\,[(1-\varepsilon)^\lambda \tilde{g}_\Lambda((1-\varepsilon)t) - (1+\varepsilon)^\lambda \tilde{g}_\Lambda((1+\varepsilon)t)]\, dt_\Lambda \\
&= \int_{\mathbb{R}'} N(E,1,t) \left[\sum_{k=0}^{2\lambda-1} \left(1 - \frac{\lambda-k}{\lambda}\varepsilon\right)^\lambda \tilde{g}_\Lambda\left(\left(1-\frac{\lambda-k}{\lambda}\varepsilon\right)t\right) \right. \\
&\quad \left. - \left(1 - \frac{\lambda-k-1}{\lambda}\varepsilon\right)^\lambda \tilde{g}_\Lambda\left(\left(1 - \frac{\lambda-k-1}{\lambda}\varepsilon\right)t\right)\right] dt_\Lambda\,.
\end{aligned} \tag{2.5}$$

Taking into account the following lemma:

Lemma 2.3. *There exists $C > 1$ such that, for $E \leqq -E_0$, if $\forall \gamma \in \Lambda_l(0), |t_\gamma| \leqq 2 \cdot l^{r_0}$, then*

$$\begin{aligned}
N(E,1,t) &\leqq \#\left\{E \leqq E_0 \textit{ such that } E \textit{ is an eigenvalue of } H_0 - \sum_{\gamma \in \Lambda_l(0)} |t_\gamma V_\gamma|\right\} \\
&\leqq C \int_{\mathbb{R}^d} \left(\sum_{\gamma \in \Lambda_l(0)} |t_\gamma V_\gamma(x)| \right)^{\frac{d}{2}} dx \leqq C^2 l^{(2+r_0)\frac{d}{2}}\,.
\end{aligned}$$

We get, using (2.5),

$$\begin{aligned}
&\mathscr{P}(\{t;\ \mathrm{dist}(1,\Gamma(t,E))<\varepsilon \text{ and } \forall\gamma\in\Lambda_l(0), |t_\gamma|\leqq l^{r_0}\})\\
&\leqq C^2 l^{(2+r_0)\frac{d}{2}} \sum_{k=0}^{2\lambda-1}\int_{\mathbb{R}^\lambda}\left|\left(1-\frac{\lambda-k}{\lambda}\varepsilon\right)^\Lambda \tilde{g}_\Lambda\left(\left(1-\frac{\lambda-k}{\lambda}\varepsilon\right)t\right)\right.\\
&\quad\left.-\left(1-\frac{\lambda-k-1}{\lambda}\varepsilon\right)^\Lambda \tilde{g}_\Lambda\left(\left(1-\frac{\lambda-k-1}{\lambda}\varepsilon\right)t\right)\right| dt_\Lambda\\
&\leqq C^2 l^{(2+r_0)\frac{d}{2}} \sum_{k=0}^{2\lambda-1}\int_{\mathbb{R}^\prime}\left|\tilde{g}_\Lambda(t)-\left(\frac{1-\frac{\lambda-k-1}{\lambda}\varepsilon}{1-\frac{\lambda-k}{\lambda}\varepsilon}\right)^\lambda \tilde{g}_\Lambda\left(\frac{1-\frac{\lambda-k-1}{\lambda}\varepsilon}{1-\frac{\lambda-k}{\lambda}\varepsilon}t\right)\right| dt_\Lambda\,.
\end{aligned} \tag{2.6}$$

Let us call $\varepsilon_k = 1-\left(1-\frac{\lambda-k-1}{\lambda}\varepsilon\right)\left(1-\frac{\lambda-k}{\lambda}\varepsilon\right)$. Reordering the points of $\Lambda_l(0)$ by $\Lambda_l(0)=\{t_j;\ 1\leqq j\leqq\lambda\}$, we get,

$$\begin{aligned}
&\int_{\mathbb{R}^\prime} |\tilde{g}_\Lambda(t)-(1-\varepsilon_k)^\lambda \tilde{g}_\Lambda((1-\varepsilon_k)t)|\,dt_\Lambda\\
&\leqq \int_{\mathbb{R}^\lambda}\left(\sum_{j=0}^{\lambda-1}\left|(1-\varepsilon_k)^j\prod_{n=0}^{\lambda-j}\tilde{g}(t_n)\prod_{n=\lambda-j+1}^{\lambda}\tilde{g}((1-\varepsilon_k)t_n)\right.\right.\\
&\quad\left.\left.-(1-\varepsilon_k)^{j+1}\prod_{n=0}^{\lambda-j-1}\tilde{g}(t_n)\prod_{n=\lambda-j}^{\lambda}\tilde{g}((1-\varepsilon_k)t_n)\right|\right)dt_\Lambda\\
&\leqq \int_{\mathbb{R}^\lambda}\left(\sum_{j=0}^{\lambda-1}(1-\varepsilon_k)^j\prod_{n=0}^{\lambda-j-1}\tilde{g}(t_n)\prod_{n=\lambda-j+1}^{\lambda}\tilde{g}((1-\varepsilon_k)t_n)\right.\\
&\quad\left.|\tilde{g}(t_j)-(1-\varepsilon_k)\tilde{g}((1-\varepsilon_k)t_j)|\right)dt_\Lambda \leqq \lambda\int_{\mathbb{R}}|\tilde{g}(t)-(1-\varepsilon_k)\tilde{g}((1-\varepsilon_k)t)|\,dt
\end{aligned}$$

by a change in the t variables; here we used the semi-group properties of φ, though we did not use φ explicitly as in [Kl 2]. Using the regularity assumption on g, (H.2) a), we get

$$\begin{aligned}
\int_{\mathbb{R}}|\tilde{g}((1+\varepsilon)t)-\tilde{g}(t)|\,dt &\leqq \int_{\mathbb{R}}|\chi_l((1+\varepsilon)t)-\chi_l(t)|g(t)\,dt\\
&\quad+\int_{\mathbb{R}}|g((1+\varepsilon)t)-g(t)|\,\chi_l((1+\varepsilon)t)\,dt\\
&\leqq \int_{l^{r_0}}^{+\infty}\left|\frac{1}{1+\varepsilon}g((1+\varepsilon)t)-g(t)\right|dt\\
&\quad+\int_{-\infty}^{-l^{r_0}}\left|\frac{1}{1+\varepsilon}g((1+\varepsilon)t)-g(t)\right|dt+\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\rho_0}\\
&\leqq C\cdot(\varepsilon+\varepsilon^{\rho_0})\,.
\end{aligned}$$

So,

$$\int_{\mathbb{R}^\lambda} |\tilde{g}_\Lambda(t) - (1-\varepsilon_k)^\lambda \tilde{g}_\Lambda((1-\varepsilon_k)t)|\, dt_\Lambda \leqq C\lambda \varepsilon_k^{\inf(1,\rho_0)}$$

$$\leqq C l^{p_0'} \varepsilon^{\inf(1,\rho_0)}$$

for some C independent of l, of $0 \leqq k \leqq 2\lambda - 1$ and of ε small enough. Plugging this into (2.6), we get the announced result.

This ends the proof of Lemma 2.1 and thus also the proof of the Wegner estimate.

Proof of Lemma 2.3. Define

$$|\Gamma|(t,E) = \sum_{\gamma \in \Lambda_l(0)} |t_\gamma| (H_0 - E)^{-1/2} |V_\gamma| (H_0 - E)^{-1/2} .$$

For $E \leqq -E_0$, we get

$$|\Gamma|(t, -E_0) \geqq \Gamma(t,E) \geqq \Gamma(t,E) ,$$

so, if we define $|N|(E,t)$ to be the number of eigenvalues of $|\Gamma|(t,E)$ that are larger than 1, then, by the min-max principle,

$$|N|(-E_0,t) \geqq |N|(E,t) \geqq N(E,1,t) .$$

By a Birman-Schwinger principle, one sees that

$$|N|(-E_0,t) = \#\left\{E \leqq -E_0 \text{ such that } E \text{ is an eigenvalue of } H_0 - \sum_{\gamma \in \Lambda_l(0)} |t_\gamma V_\gamma|\right\} .$$

Now, to get Lemma 2.3, one just uses the Cwikel-Lieb-Rosenblum bound for the number of negative bound states for Schrödinger operators (see, e.g. [Re-Si]), and the assumption that, for $\gamma \in \Lambda_l(0)$, $|t_\gamma| \leqq l^{r_0}$.

B. The Induction Process and the Proof of Theorem 1.1. We will not give the proof of the induction in all its details as once the main features are explained, the details of the computations are the same as in [Kl 1].

The main tool of the induction process is the resolvent estimate (1.2). Let us rewrite it in a different, more manageable form; define, for $a \in L$, $C_a \subset \mathbb{R}^d$ to be the cube of center a and sidelength 1 (i.e. $C_a = \{x;\ x - a = \sum_{1 \leqq j \leqq d} t_j u_j$, where $-1/2 \leqq t_j < 1/2\}$). Then for $(a,b) \in L^2$ and $E \notin \sigma(H_\Lambda(t))$ (here Λ is a cube in L),

$$|G|_\Lambda(E;a,b) = \| \chi_a G_\Lambda(E) \chi_b \|_{L^2(\mathbb{R}^d)} .$$

Then (1.2) implies, for $\Lambda \subset \Lambda'$,

$$|G|_{\Lambda'}(E;a,b) \leqq |G|_\Lambda(E;a,b) + \sum_{c \in L} |G|_\Lambda(E;a,c) |V|_{\Lambda,\Lambda'}(c) |G|_{\Lambda'}(E;c,b) , \quad (2.6)$$

where $|V|_{\Lambda,\Lambda'}(c) = \sup_{x \in C_c} |V_{\Lambda,\Lambda'}(x)|$.

Now, as in [Kl 1], we define the regular and the non-resonant cubes

Definition. *Let* $\beta \in]0,1[$, $E \in \mathbb{R}$, $l > 0$ *and* $x \in L$. $\Lambda_l(x)$ *is* (E,β)-N.R (i.e. *non-resonant) if*

$$\| G_{\Lambda_l(x)}(E) \|_{L^2(\mathbb{R}^d)} \leqq e^{l^\beta} .$$

Definition. *Let $\beta \in]0,1[$, $E \in \mathbb{R}$, $\varepsilon \in]0,1[$, $l > 0$ and $x \in L$. $\Lambda_l(x)$ is $(E, m, \beta, \varepsilon)$-regular if*

(a) *$\Lambda_l(x)$ is (E, β)-N.R,*
(b) *moreover*

$$\sum_{y \in L;\ \frac{\varepsilon l}{2} \leqq |y-x|} |G|_{\Lambda_l(x)}(E; x, y) e^{m|y-x|} < 1 .$$

(Here, for $x = \sum x_j u_j \in L$, $|x| = \sup |x_j|$).

Then, as in [Kl 1], we prove

Lemma 2.4. *Let $p > \sup(4, d)$, $\beta > 0$ such that $\beta p < \inf(4, d)$, $\varepsilon \in]0, \frac{1}{2}[$, $\alpha \in]1, \inf(\frac{p}{4}, \frac{p}{d})[$ and $0 < \delta < \inf(1 - \alpha\beta, \alpha - 1)$.*

Let $H(t)$ be defined by (1.1), V satisfy (H.1) and $(t_\gamma)_{\gamma \in L}$ satisfy (H.2).

Then there exists $l_0 > 1$ such that, if for $L_0 \geqq l_0$, one has, for some $m_{L_0} \in]0, m_0(1-\varepsilon)[$, $\forall (x,y) \in L \times L$ such that $|x - y| > L_0(1+\varepsilon)$,

$$P(\{\forall E \leqq -E_0, \Lambda_{L_0}(x) \text{ or } \Lambda_{L_0}(y) \text{ is } (E, m_{L_0}, \beta, \varepsilon)\text{-regular}\}) > 1 - L_0^{-p} ,$$

then, defining the sequence L_k by $L_{k+1} = L_k^\alpha$, we get for $k \geqq 0$ and for any $(x,y) \in L \times L$ such that $|x - y| > L_{k+1}(1+\varepsilon)$,

$$P(\{\forall E \leqq -E_0, \Lambda_{L_{k+1}}(x) \text{ or } \Lambda_{L_{k+1}}(y) \text{ is } (E, m_{L_{k+1}}, \beta, \varepsilon) - \text{régular}\}) > 1 - L_{k+1}^{-p} ,$$

where

$$m_{L_{k+1}} = m_{L_k} - 2(m_0 + 1) L_k^{-\delta} \leqq m_0(1 - \varepsilon) .$$

(P is the probability measure defined by the random variables $(t_x)_{x \in L}$.)

The proof of this lemma is exactly the same as in [Kl 1]. One can first prove an exact replica of Lemma 2.2 of [Kl 1] (in the present case, it is even simpler as, if $\Lambda \cap \Lambda' = \emptyset$, $H_\Lambda(t)$ and $H_{\Lambda'}(t)$ are independent (in the probabilistic sense) of each other). Then the resolvent formula (2.5) being the same as the resolvent formula (4.4) of [Kl 1], we also get a replica of Lemma 2.3 of [Kl 1]. Putting both of these arguments together, we get Lemma 2.4. We define

Definition. *Let $E \in \mathbb{R}$. E is a generalized eigenvalue of $H(t)$ if there exists a generalized eigenfunction i.e. a polynomially bounded solution to the equation.*

$$(H(t) - E)\varphi = 0 .$$

Using again our resolvent formula, we prove

Lemma 2.5. *Let $p > \sup(4, d)$, $1 < \alpha < \frac{p}{d}$, $\varepsilon \in]0, \frac{1}{2}[$ et $m \in]0, m_0(1-\varepsilon)[$. Let $L_0 > 0$. Define the sequence L_k by $L_{k+1} = L_k^\alpha$ for $k \geqq 0$.*

Let $H(t)$ be defined by (1.1), V satisfy (H.1) and $(t_\gamma)_{\gamma \in L}$ satisfy (H.2).

Assume, for $k \geqq 0$ and for any $(x, y) \in L^2$ such that $|x - y| > L_k(1 + \varepsilon)$, one has

$$P(\{\forall E \leqq E_0, \Lambda_{L_k}(x) \text{ or } \Lambda_{L_k}(y) \text{ is } (E, m, \beta, \varepsilon) - \text{regular}\}) > 1 - L_k^{-p} .$$

Then, with probability 1, if φ is a generalized eigenfunction of $H(t)$ associated to $E \leqq E_0$, one has

$$\limsup_{x \to +\infty} \frac{\log|\varphi(x)|}{|x|} \leqq -m(1-\varepsilon) .$$

The proof of this lemma is the same as the one of Lemma 2.6 of [Kl 1]. The only difference is that one first proves that the generalized eigenfunction φ satisfies: $\|\varphi\|_{L^2(C_\gamma)} \leqq Ce^{-m(1-\varepsilon)|\gamma|}$ for any $\gamma \in L$ (here C_γ is the unit cell of L centered in γ). Then, one proves the above announced exponential decrease using a subsolution estimate (cf. Theorem C.1.2 of [Si]) and noticing that by our assumptions on V and on $(t_\gamma)_{\gamma \in L}$, with probability 1, $\sum_{\gamma \in l} t_\gamma V_\gamma$ is polynomially bounded.

Now, using [Ki-Ma 3] and [Si], we know that, under our assumption on V and the random variables $(t_\gamma)_{\gamma \in L}$, with probability 1, almost every energy in the spectrum of $H(t)$ is a generalized eigenvalue. Combining this with Lemmas 2.3 and 2.4, to end the proof of Theorem 1.1, it is sufficient to show that, for any $L_0 > 0$, there exists $E_0 > 0$, such that,

$$P(\{\forall E \leqq -E_0,\ \Lambda_{L_0}(x) \text{ or } \Lambda_{L_0}(y) \text{ is } (E, m_0(1-\varepsilon), \beta, \varepsilon) - \text{regular}\}) > 1 - L_0^{-p}\,.$$

By a Combes–Thomas argument,

$$\begin{aligned} P(\{\exists E \leqq -E_0,\ &\Lambda_{L_0}(x) \text{ and } \Lambda_{L_0}(y) \text{ are not } (E, m_0(1-\varepsilon), \beta, \varepsilon) - \text{regular}\}) \\ &\leqq P(\{\exists E \leqq -E_0 + 2m_0;\ E \in \sigma(H_{\Lambda_{l_0}}(x))\}) \\ &\leqq C L_0^d P(\{t_0 \leqq -E_0 + 2m_0\}) \\ &< L_0^{-p} \end{aligned}$$

for $E_0 > 0$ large enough using our assumptions on the probability distribution of the random variable t_0.

III. For Lower Semi-Bounded Perturbations

A. Proof of Theorem 1.3. Let us assume that Theorem 1.4 and Proposition 1.5 are proved. Then we may use the induction process designed in [Co-Hi 1]. More precisely, we see that Lemma A.2 [Co-Hi 1] still holds under our assumptions. Let us now define the sequences $(\gamma_k)_{k \geq 0}$ as in Lemma A.3 [Co-Hi 1], where $\gamma_0 = C_0 l_0^{-1/4}$ (see Proposition 1.5). By (A.17) of [Co-Hi 1] and the subsequent commentary, for some $K_0 > 0$, $K_1 > 0$, $K_2 > 0$ (independent of l_0),

$$\gamma_{k+1} \geqq K_0 \gamma_0 - K_1 \sum_{j=0}^{k} C_j\,, \tag{3.1}$$

where

$$C_j \leqq K_2 \left(\frac{1}{l_j} + \frac{\log l_{j+1}}{l_{j+1}} \right).$$

But $l_{j+1} = l_j^{3/2}$ so, for l_0 large enough,

$$\frac{\log l_{j+1}}{l_{j+1}} \leqq \frac{1}{l_j}\,.$$

By (3.1), there exists $K_3 > 0$, $K_4 > 0$ (independent of l_0),

$$\gamma_{k+1} \geqq K_0\gamma_0 - K_3\frac{1}{l_0}\left(1 + \sum_{j=0}^{k}\frac{l_0}{l_j}\right)$$

$$\geqq K_0\gamma_0 - K_4\frac{1}{l_0}$$

$$\geqq \frac{K_0 C_0}{2} l_0^{-\frac{1}{4}},$$

for l_0 large enough.

Then, applying Theorem 2.3 of [Co-Hi 1] and following their argument, we get Theorem 1.3.

B. Proof of Theorem 1.4 (*the Wegner Estimate*). Consider $V = \sum_{\gamma\in L} V_\gamma$ and the L-periodic Schrödinger operator $H_T = H_0 + T\cdot\tilde{V}$. Let $E_0(T)$ be the infimum of the spectrum of H_T.

Lemma 3.1. *For some* $T_0 \in G$, $E_0(T_0) > E_{\text{inf}}$.

Proof. Using the Floquet reduction for periodic Schrödinger operators (see [Bi or Ki-Si]), we know that $E_0(T)$ is simple; moreover as it can be expressed as a Floquet eigenvalue, it is analytic in T for $T \in \mathbb{R}$. Let us recall that G is the essential support of g, the common density of the random variables $(t_\gamma)_{\gamma\in L}$ and, that it is not empty and cannot have isolated points (by the regularity assumption on g).

We claim, that, for some $T \in G$, $E_0(T) > E_{\text{inf}}$. Indeed, assume the contrary; by [Ki-Ma 2], we know that, for all $T \in G$, $E_0(T) \geqq E_{\text{inf}}$. So, for all $T \in G$, $E_0(T) = E_{\text{inf}}$. Hence, for all $T \in \mathbb{R}$, $E_0(T) = E_{\text{inf}}$ by analyticity of $E_0(T)$. But this is impossible; indeed, as V is not equal to 0 almost-everywhere, there exists some $\varphi \in \mathscr{C}_0^\infty$ such that $\langle V\varphi, \varphi\rangle > 0$ or $\langle V\varphi, \varphi\rangle < 0$, then $\langle H_T(\varphi), \varphi\rangle \to -\infty$ as $T \to -\infty$ or $+\infty$. Hence, the infimum of the spectrum of H_T, $E_0(T)$ tend also to $-\infty$ which contradicts our assumption. So for some $T \in G$, $E_0(T_0) > E_{\text{inf}}$.

Let us rewrite $H(t)$,

$$H(t) = H(T_0) + \sum_{\gamma\in L} \tilde{t}_\gamma V_\gamma ,$$

where $\tilde{t}_\gamma = t_\gamma - T_0$.

By Lemma 3.1, for some $\delta > 0$,

$$H(T_0) - E_{\text{inf}} > \delta .$$

Taking the restriction to Λ_l, a cube of side l (with Dirichlet boundary conditions), we get,

$$H^D_{\Lambda_l}(t) = H^D_{\Lambda_l}(T_0) + \sum_{\gamma\in\Lambda_{l+R}} \tilde{t}_\gamma V_\gamma \kappa_{\Lambda_l} ,$$

as $\operatorname{supp} V \subset B(0,R)$ (here κ_{Λ_l} is the characteristic function of Λ_l).

Then, for $E < E_{\text{inf}} + \delta$,

$$(H^D_{\Lambda_l}(t) - E)^{-1} = (H^D_{\Lambda_l}(T_0) - E)^{-1/2}(1 + \Gamma_{\Lambda_l}(t,E))^{-1}(H^D_{\Lambda_l}(T_0) - E)^{-1/2} ,$$

where

$$\Gamma_{\Lambda_l}(t,E) = \sum_{\gamma\in\Lambda_{l+R}} \tilde{t}_\gamma (H^D_{\Lambda_l}(T_0) - E)^{-1/2} V_\gamma \kappa_{\Lambda_l} (H^D_{\Lambda_l}(T_0) - E)^{-1/2} .$$

Hence, for $E < E_{\inf} + \delta/2$,

$$\begin{aligned} P(\{t;\ \mathrm{dist}(E,\sigma(H^D_{\Lambda_l}(t))) \leqq \varepsilon\}) &= P\left(\left\{t;\ \|(H^D_{\Lambda_l}(t) - E)^{-1}\| \geqq \frac{1}{\varepsilon}\right\}\right) \\ &\leqq P\left(\left\{t;\ \|(1 + \Gamma_{\Lambda_l}(t,E))^{-1}\| \geqq \frac{\delta^2}{4}\frac{1}{\varepsilon}\right\}\right) \\ &= P\left(\left\{t;\ \mathrm{dist}(-1, \sigma(\Gamma_{\Lambda_l}(t,E))) \leqq \varepsilon\frac{4}{\delta^2}\right\}\right). \end{aligned}$$

The last term may now be estimated by the method used in Sect. 2 to prove Theorem 1.2; this gives us the Wegner estimate in this case.

C. Proof of Proposition 1.5. The idea of the proof is to use the rapid decrease of the density of states $N(E)$ at the lower edge of the spectrum to show that the probability that $H^D_{\Lambda_l}(t)$ has a small eigenvalue, is small. This idea is the heuristical argument used by E. Lifshits [Li] to prove physically localization in disordered media. This technique has also proven to be quite efficient mathematically (see, e.g. [Ho-Ma] or [Sp]).

Denote by $N^D_{\Lambda_l}(E,t)$, the number of eigenvalues of $H^D_{\Lambda_l}(t)$ smaller than E. Then, it is well known (see [Ki-Ma 3]) that, for $l \geqq 1$,

$$\frac{\mathbb{E}(N^D_{\Lambda_l}(E,t))}{\sharp\Lambda_l} \leqq N(E),$$

here $\mathbb{E}$ denotes the expectation taken with respect to the random variables $(t_\gamma)_{\gamma\in L}$.

If $\lambda^D_0(t)$ denotes the lowest eigenvalue for $H^D_{\Lambda_l}(t)$, then

$$\mathscr{P}(\{t;\ \lambda^D_0(t) \leqq E\}) \leqq \mathbb{E}(N^D_{\Lambda_l}(E,t)) \leqq \sharp\Lambda_l \cdot N(E).$$

If we now choose $E = E_{\inf} + 4/\sqrt{l}$, then, by assumption (H.3) b), for some l large enough, we get,

$$\mathscr{P}(\{t;\ \lambda^D_0(t) \leqq E\}) \leqq l^{-(2d+1)}.$$

Hence, with probability at least $1 - l^{-(2d+1)}$, $\sigma(H^D_{\Lambda_l}(t)) \cap (-\infty, E_{\inf} + 4/\sqrt{l}) = \emptyset$ that is, for $E \in [E_{\inf}, E_{\inf} + 2/\sqrt{l}]$, $\mathrm{dist}(E, \sigma(H^D_{\Lambda_l}(t))) \geqq 2/\sqrt{l}$.

Using a Combes-Thomas argument (see, for example, [Si] Sect. B.7, or [Ho-Ma]), we get, for $\varepsilon > 0$ and $|x-y| \geqq \sqrt{l}$,

$$|(H^D_{\Lambda_l}(t) - (E + i\varepsilon))^{-1}(x,y)| \leqq e^{-l^{-1/4}|x-y|}. \tag{3.2}$$

This gives Proposition 1.5. Indeed, define

$$(H^D_{\Lambda_l}(t) - (E + i\varepsilon))^{-1} = G_{\Lambda_l}(E + i\varepsilon).$$

Then

$$\begin{aligned} [-\nabla, G_{\Lambda_l}(E + i\varepsilon)] &= G_{\Lambda_l}(E + i\varepsilon)[H^D_{\Lambda_l}(t), \nabla]\, G_{\Lambda_l}(E + i\varepsilon) \\ &= G_{\Lambda_l}(E + i\varepsilon)\, \nabla V_{\Lambda_l}(t)\, G_{\Lambda_l}(E + i\varepsilon), \end{aligned}$$

where $V_{\Lambda_l}(t) = \sum_{\gamma \in \Lambda_{l+R}} \tilde{t}_\gamma V_\gamma \kappa_{\Lambda_l}$. One computes

$$\begin{aligned}
&[-\Delta, \chi_l] G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3} \\
&\quad = \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\nabla\chi_{l/3} + \nabla\chi_l[-\nabla, G_{\Lambda_l}(E + i\varepsilon)]\chi_{l/3} \\
&\quad = \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\nabla\chi_{l/3} + \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\nabla V_{\Lambda_l}(t) G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3} \\
&\quad = \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\nabla\chi_{1/3} + \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\chi_{l_r}\nabla V_{\Lambda_l}(t) G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3} \\
&\qquad + \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\nabla V_{\Lambda_l}(t)(1 - \chi_{l_r}) G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3} ,
\end{aligned} \tag{3.3}$$

for some $0 < r$ which will be chosen later on.

As $\chi_l \equiv 1$ in $|x| \leqq l(1 - \delta)$ and $\chi_l \equiv 0$ if $|x| \geqq l$ and by (3.2), we get

$$\| \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\chi_{l_r} \| \leqq e^{-(1-\delta-r)l^{-3/4}} ,$$

$$\|(1 - \chi_{l_r}) G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3} \| \leqq e^{-(r-1/3)l^{-3/4}} ,$$

and

$$\| \nabla\chi_l G_{\Lambda_l}(E + i\varepsilon)\nabla\chi_{l/3} \| \leqq e^{-(-\delta+2/3)l^{-3/4}} .$$

Now, choosing $\delta < 2/3$ and $r < 1/3$, we get, for some $C_0 > 0$ and l large enough,

$$\sup_{\varepsilon > 0} \| [-\Delta, \chi_l] G_{\Lambda_l}(E + i\varepsilon)\chi_{l/3} \| \leqq e^{-\gamma \cdot l} ,$$

where $\gamma = C_0 l^{-1/4}$.

This ends the proof of Proposition 1.5.

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