A Cluster Expansion Approach
to a One-Dimensional Boltzmann Equation:
A Validity Result

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Abstract: We consider a stochastic particle system on the line and prove that, when the number of particles diverges and the probability of a collision is suitably rescaled, the system is well described by a one-dimensional Boltzmann equation. The result holds globally in time, without any smallness assumption.

Contents

1. Introduction

It is well known how hard the problem is of proving rigorously the validity of the Boltzmann equation for a rarefield gas from the Newton equations of a system of N interacting particles. Very few results are available up to now (see [1] and [2] in the References). The only successful way to approach this problem is to introduce the BBGKY hierarchy for the j-particle distribution functions and try to show that they converge, in the Boltzmann-Grad limit, to a factorizing solution of the Boltzmann hierarchy (that is a j-fold tensor product of the solution of the Boltzmann equation), provided that the initial state of the BBGKY hierarchy converges to a factorizing state. This program can be achieved completely once one has suitable L_∞ estimates on the j-particle distribution functions for the N-particle system. However this is a very difficult task in general. In [1] and [2] such estimates have been obtained for a system of hard balls in special situations: either for a short time or for a moderate perturbation of the vacuum. For more general situations, the problem of proving L_∞ estimates for

the j-particle distribution functions is much harder than that of proving L_∞ bounds for the solutions of the Boltzmann equation itself. This is not yet known (maybe even false) although recent progress on the existence problem for the Boltzmann equation have been achieved by compactness methods [3].

In this paper we prove the validity of a one-dimensional kinetic Boltzmann equation starting from a N-particle system. The existence theory for such a kinetic equation is well understood (see e.g. [4] and references quoted therein) and, in particular, L_∞ bounds for the solutions are available. The underlying one-dimensional model we are considering is stochastic: two particles on the line collide with probability ε and go ahead with probability $1 - \varepsilon$. In this way the number of collisions per unit time performed by a tagged particle should be finite in the limit $N \to \infty$ (if $\varepsilon \sim N^{-1}$). Notice that the stochasticity of the model involves only the collision mechanism and this is strictly pointwise. In other words the stochasticity we allow is the minimal one we need to give sense to a one dimensional problem. On the other hand the model we consider is the simplest one with these features. A model of this type has been considered, at the level of a kinetic picture, by Ianiro and Lebowitz for some stationary problem [5].

We now anticipate the leading idea in the validity proof. The j-particle distribution functions satisfy a hierarchy of equations (with the same structure as the BBGKY hierarchy for Hamiltonian systems) from which it is extremely hard to obtain L_∞ estimates globally in time. However for any (tagged) group of j-particles we can prove, for the combination $\lambda = \varepsilon N$ small, the following remarkable fact: with a large probability, each particle interacts with a finite cluster of other particles and such clusters are disjoint. As a consequence the j-particle distribution function is expressed as a j-fold tensor product of L_∞ positive functions plus a correction, which is small in L_1. This fact, combined with a trace control, allows us to prove the convergence of the one-particle distribution function to the solution of our one-dimensional Boltzmann equation and also the propagation of chaos. The tool we use to outline the above picture is a cluster expansion technique similar to those used in Equilibrium Statistical Mechanics. The analysis is perturbative (with respect to the free motion) and therefore the cluster expansion is converging for small values of λ, independently of the size of the initial datum. Moreover, we can control the local particle density. This implies, thanks to the finite speed of propagation, that any particle is influenced, in a small interval of time, by a small fraction of the total number of molecules. Hence the analysis for small λ can be applied locally in space and time to give the desired result by an inductive procedure.

Beyond the purpose of the present paper, the cluster dynamics we propose is interesting in itself and clarify the mechanism of propagation of chaos which is expected to hold in a rarefield gas. It is conceivable that the cluster dynamics is the real physical mechanism responsible for the factorization of the j-particle distribution functions in the limit $N \to \infty$, also for Hamiltonian systems, although this crucial features would be very difficult to prove.

We also mention a recent result in which the validity of the two-dimensional Broadwell model was established, assuming the existence of a smooth solution and starting from a stochastic particle system on a lattice [6]. The techniques of that paper do not apply to the present context because they were based on a lattice approximation and on the presence of a noise on the free stream. Our model is less stochastic, as we said, but is one-dimensional and we take advantage from this fact. Actually the approach presented here is not directly applicable to solve the problem posed in [6].

The paper is organized as follows. In Sect. 2 we present the model, develop some preliminary heuristic considerations and pose the problem. In Sect. 3 we discuss some

preliminary steps concerning our particle model. In Sect. 4 we present and discuss the convergence result for small λ. In Sect. 5 we face the crucial step of the paper. Here we develop our cluster expansion and obtain estimates for the j-particle distributions already used in Sect. 4. In Sect. 6 we show how to extend the previous results to λ large.

We finally remark that our approach is based on probability estimates, namely on the fact that the j-particle distribution functions are expectations with respect to a probability measure and that large and small probability events can be estimated differently. Actually it seems very difficult to obtain the same result by a purely PDE approach.

2. The Model

Let us consider a system of N particles on the line. Denote by $Z_N = (X_N, V_N) = \{x_1 v_1, \ldots, x_N v_N\}$ a state of the system, where x_i and v_i are the position and the velocity of the i-th particle and $X_N \in \mathbf{R}^N$ and $V_N \in V^N$ denote the set of all positions and velocities. $V = \{\pm 1, \pm 2\}$ is the velocity space of a single particle. A collision between two particles is defined through the following law: if v_1 and v_2 are the ingoing velocities and v_1' and v_2' are the outgoing velocities, then two cases are possible

$$\text{either} \quad (v_1, v_2) = (2, -1) \quad \text{and} \quad (v_1', v_2') = (1, -2)$$
$$\text{or} \quad (v_1, v_2) = (1, -2) \quad \text{and} \quad (v_1', v_2') = (2, -1). \tag{2.1}$$

If $v_1 = -v_2$ then the particles go ahead freely.

Notice that the above collision rule is equivalent to a specular reflection of each colliding particle. However we prefer a labelling for which the direction of the velocity does not change in the collision because in such a way any pair of particles can interact at most only once. This fact will play an important role later on. Moreover collisions of particles travelling in the same direction are considered as transparent.

The dynamics of the system is stochastic, in the sense that each pair of particles collides (independently) with probability ε and go ahead with probability $1 - \varepsilon$. Such a stochastic process can be represented in the sample space:

$$\Omega = \{0, 1\}^{\frac{N(N-1)}{2}}.$$

Hence $\omega \in \Omega$, $\omega = \omega(i, j)$ is a function defined on the set of all pairs of particles, taking values 0 and 1. The time evolution of the system is described by means of a one-parameter group of transformations

$$T_\omega^t : (\mathbf{R} \times V)^N \to (\mathbf{R} \times V)^N \qquad t \in \mathbf{R}$$

defined as follows. $T_\omega^t Z_N$ is the free motion up to the first time in which two particles (say i and j) arrive at the same position, then if $|v_i - v_j| = 3$ and $\omega(i, j) = 1$ (that is we are in a precollisional situation) the two particles collide according to the law given in (2.1), otherwise they keep their free motion. We repeat the procedure up to the next collision time and so on. Notice that the modulus of the relative velocity of any pair of particles $|v_i - v_j|$ is an invariant for such dynamics:

$$|v_i - v_j| = |v_i' - v_j'| = 3 \quad \forall i, j \in N. \tag{2.2}$$

T_ω^t is, in principle, well defined since, by (2.1), two particles having interacted once cannot interact anymore. Moreover T_ω^t is almost everywhere (a.e.) defined with respect

to the Lebesgue measure on $(\mathbf{R} \times V)^N$. Indeed it is not defined on the sets of all Z_N's which deliver triple collisions and on the set of configurations for which, for some $i, j \in N$, it is $x_i = x_j$, $|v_i - v_j| = 3$ and $\omega(i, j) = 1$, being, in this last case, not possible to distinguish between pre- and post-collisional configurations: nevertheless these two sets have measure zero.

Defining the probability of a single event $\omega \in \Omega$ by

$$p(\omega) = \prod_{i,j} \varepsilon^{\omega(i,j)} (1 - \varepsilon)^{1 - \omega(i,j)}, \tag{2.3}$$

where $\prod\limits_{i,j}$ denotes the product on all pairs, $\{\Omega, p\}$ is a probability space and T_ω^t a stochastic process on it.

Suppose now that the N-particle system is described at time zero by a symmetric probability density on $(\mathbf{R} \times V)^N$ denoted by μ_0^N. We define the time evolved density $\mu^N(\cdot, t)$ by:

$$\mu^N(Z_N, t) = \sum_\omega p(\omega) \mu_\omega^N(Z_N, t) := \mathbb{E}[\mu_\omega^N(Z_N, t)], \tag{2.4}$$

where

$$\mu_\omega^N(Z_N, t) = \mu_0^N(T_\omega^{-t} Z_N). \tag{2.5}$$

We now want to establish an evolution equation for μ_ω^N and μ^N and to this end we will proceed by heuristic arguments, which will be made rigorous in the next section.

We have, formally, for any sample $\omega \in \Omega$:

$$\left(\partial_t + \sum_{i=1}^N v_i \partial_i \right) \mu_\omega^N(Z_N, t) = \frac{3}{2} \sum_{i \neq j} \omega(i, j) \chi(i, j)$$

$$\delta(x_i - x_j) \{ \mu_\omega^N(Z_N'(i, j), t) - \mu_\omega^N(Z_N, t) \}, \tag{2.6}$$

where $3 = |v_i - v_j|$ is the relative velocity, ∂_i is the derivative with respect to x_i, $\delta(\cdot)$ is the δ-function centered in zero, $\chi(i, j) = 1$ if $|v_i - v_j| = 3$ and 0 otherwise and

$$Z_N'(i, j) = (x_1 v_1, \dots x_i v_i', \dots x_j v_j', \dots x_N v_N).$$

Taking now the expectation with respect to $p(\omega)$ in (2.6), we obtain

$$\left(\partial_t + \sum_{i=1}^N v_i \partial_i \right) \mu^N(Z_n, t)$$

$$= \frac{3}{2} \varepsilon \sum_{i \neq j} \delta(x_i - x_j) \chi(i, j) \{ \mu^N(Z_N'(i, j), t) - \mu^N(Z_N, t) \}. \tag{2.7}$$

This equation is expected to hold because the event $\omega(i, j)$ appearing in (2.6) is independent of the random variable in the curl brackets.

Let us now derive, always formally, the kinetic equation associated to the above process. We start by the usual j-particle distribution densities, defined as:

$$f_j^N(Z_j, t) = \int dZ_{N-j} \mu^N(Z_N, t), \tag{2.8}$$

where $dZ_{N-j} = \sum\limits_{v_{j+1} \dots v_N} dx_{j+1} \dots dx_N.$

After a rather straightforward algebra, we obtain from (2.7):

$$\left(\partial_t + \sum_{i=1}^{j} v_i \partial_i\right) f_j^N = \varepsilon G_j f_j^N + \varepsilon (N - j) C_{j,j+1} f_{j+1}^N \,, \tag{2.9}$$

where

$$G_j f_j^N(Z_j, t) = \frac{3}{2} \sum_{\substack{i \neq k \\ i,k=1\ldots j}} \delta(x_i - x_k)\chi(i, k)\{f_j^N(Z_j'(i, k), t) - f_j^N(Z_j, t)\} \tag{2.10}$$

and

$$C_{j,j+1} f_{j+1}^N(Z_j, t) = 3 \sum_{i=1}^{j} \sum_{\substack{v_{j+1} \in V \\ |v_i - v_{j+1}|=3}} \{f_{j+1}^N(x_1 v_1, \ldots x_i v_i', \ldots x_i v_{j+1}', t)$$

$$- f_{j+1}^N(x_1 v_1, \ldots x_i v_i, \ldots x_i v_{j+1} t)\} \,. \tag{2.11}$$

We are interested in performing the Boltzmann-Grad limit on Eq. (2.11), that is letting $N \to \infty$ and $\varepsilon \to 0$ while $N\varepsilon \to \lambda$, λ a fixed number. Assuming that such limit of the f_j^N does exist and denoting it by f_j, these functions are expected to satisfy the following infinite hierarchy of equations:

$$\left(\partial_t + \sum_{i=1}^{j} v_i \partial_j\right) f_j = \lambda C_{j,j+1} f_{j+1} \,. \tag{2.12}$$

Moreover, if the f_j are factorizing functions, i.e.:

$$f_j(x_1 v_1, \ldots, x_j v_j, t) = \prod_{k=1}^{j} f(x_k v_k, t) \,, \tag{2.13}$$

then f solves, as it is easily seen, the Boltzmann equation associated to our model, which is

$$(\partial_t + v\partial_x)f(x, v) = 3\lambda\{f(x, v')f(x, v_1') - f(x, v)f(x, v_1)\} \,, \tag{2.14}$$

where v_1 is the only precollisional velocity associated to v, that is $|v_1 - v| = 3$.

The aim of this paper is indeed to prove that all the above arguments can be made rigorous, so that the Boltzmann-Grad limit holds in this context.

3. Considerations on the Model and Preliminary Estimates

In this section we shall make rigorous some of the considerations on the process T_ω^t developed at a heuristic level so far. Moreover, we shall give two basic estimates which will be employed in the proof of the main result.

We start by introducing the "collision manifold" $\mathscr{F}^\omega$ defined, for any ω, as:

$$\mathscr{F}^\omega = \bigcup_{i,j:\omega(i,j)=1} \mathscr{F}_{ij}^\omega \,, \tag{3.1}$$

$$\mathscr{F}_{ij}^\omega = \{Z_N : x_i = x_j, |v_i - v_j| = 3, \omega(i, j) = 1\} \,. \tag{3.2}$$

On $\mathscr{F}^\omega$ we define the measure dY whose restriction on $\mathscr{F}_{ij}^\omega$ is given by:

$$dY = \sum_{v_1 \ldots v_N} \chi(i,j) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_j \ldots dx_N. \tag{3.3}$$

Such codimension-one manifold $\mathscr{F}^\omega$ is indeed of great importance as far as we are concerned with the dynamics of our system, so that for a correct definition of the equation describing the evolution in time of μ_ω^N, we have to extend the definition of T_ω^t on $\mathscr{F}^\omega$. For this reason we have to specify whether the velocities of the colliding pairs in the configurations Z_N belonging to $\mathscr{F}^\omega$ are to be interpreted as incoming or outcoming velocities.

Denoting the free flow semigroup by

$$S^t Z_N = (x_1 + v_1 t, v_1 \ldots x_N + v_N t, v_N), \tag{3.4}$$

we define, for any $Y \in \mathscr{F}^\omega$:

$$T_\omega^t Y = S^t \pi(Y) \quad \text{for} \quad t \in (0, \alpha(Y)), \tag{3.5}$$

where

$$\alpha(Y) = \inf\{\tau > 0 : S^\tau \pi(Y) \in \mathscr{F}^\omega\} \tag{3.6}$$

and

$$\pi(Y) = Y'(i,j) \quad \text{if} \quad Y \in \mathscr{F}_{ij}^\omega. \tag{3.7}$$

Definition (3.5) implies that we consider the elements of $\mathscr{F}^\omega$ as having precollisional velocities. The set $\bigcup_{0 < t < \alpha(Y)} S^t \pi(Y)$ is of positive measure so that we can conclude that T_ω^t is defined for dY-almost all $Y \in \mathscr{F}^\omega$. Furthermore, by (3.5) it follows that

$$T_\omega^{-t} Y = S^{-t} Y \quad \text{for } t \text{ small}. \tag{3.8}$$

Now we are ready to derive an evolution equation for $\mu_\omega^N(Z_N, t)$. Notice that, by the previous arguments, we can extend $\mu_\omega^N(\cdot, t)$ on the collision manifold according to Definition (2.5).

Given Z_N, let t_1 be the smallest collision time in the interval $[0, t]$. Then by (3.5)–(3.8):

$$\mu_\omega^N(Z_N, t) = \mu_0^N(T_\omega^{-t} Z_N) = \mu_0^N(T_\omega^{-(t-t_1)} T_\omega^{-t_1} Z_N) = \mu_0^N(T_\omega^{-(t-t_1)} \pi(S^{-t_1} Z_N))$$

$$= \{\mu_\omega^N(\pi(S^{-t_1} Z_N), t - t_1) - \mu_\omega^N(S^{-t_1} Z_N, t - t_1)\}$$

$$+ \mu_\omega^N(S^{-t_1} Z_N, t - t_1).$$

If $t_1 \ldots t_k$ are the solutions of the equations $x_i - v_i t = x_j - v_j t$ with $|v_i - v_j| = 3$, $\omega(i,j) = 1$, $i, j = 1 \ldots N$ in the interval $(0, t)$, iterating the above procedure we obtain

$$\mu_\omega^N(Z_N, t) = \mu_0^N(S^{-t} Z_N) + \sum_{r=1}^k \{\mu_\omega^N(\pi(S^{-t_r} Z_N), t - t_r) - \mu_\omega^N(S^{-t_r} Z_N, t - t_r)\}$$

$$= \mu_0^N(S^{-t} Z_N) + \frac{3}{2} \sum_{i \neq j} \omega(i,j) \chi(i,j) \int_0^t d\tau \{\delta(x_i - v_i \tau - x_j + v_j \tau)$$

$$\times [\mu_\omega^N(\pi(S^{-\tau} Z_N, t - \tau) - \mu_\omega^N(S^{-\tau} Z_N, t - \tau)]\}. \tag{3.9}$$

Integrating now (3.9) against a smooth test function with compact support Φ, we have:

$$\int dZ_N \Phi(Z_N) \mu_\omega^N(Z_N, t) = \int dZ_N \Phi(S^t Z_N) \mu_0^N(Z_N)$$

$$+ 3 \int_0^t d\tau \int_{\mathscr{F}^\omega} dY \Phi(S^{t-\tau} Y) \left\{ \mu_\omega^N(\pi(Y), \tau) - \mu_\omega^N(Y, \tau) \right\}. \tag{3.10}$$

From (3.10) we deduce that the r.h.s. is an a.e. differentiable function w.r.t. the time and

$$\frac{d}{dt} \int dZ_N \Phi(Z_N) \mu_\omega^N(Z_N, t) = \int dZ_N \sum_{i=1}^N v_i \partial_i \Phi(Z_N) \mu_\omega^N(Z_N, t)$$

$$+ 3 \int_{\mathscr{F}^\omega} dY \Phi(Y) \left\{ \mu_\omega^N(\pi(Y), t) - \mu_\omega^N(Y, t) \right\}$$

$$= \int dZ_N \sum_{i=1}^N v_i \partial_i \Phi(Z_N) \mu_\omega^N(Z_N, t)$$

$$+ \frac{3}{2} \sum_{i \neq j} \omega(i,j) \int_{\mathscr{F}_{ij}} dY \Phi(Y) \left\{ \mu_\omega^N(Y'(i,j), t) - \mu_\omega^N(Y, t) \right\}, \tag{3.11}$$

where

$$\mathscr{F}_{ij} = \left\{ Z_N : x_i = x_j, |v_i - v_j| = 3, \right\}.$$

Taking now the expectation of Eq. (3.11) we obtain the searched equation for μ^N:

$$\frac{d}{dt} \int dZ_N \Phi(Z_N) \mu^N(Z_N, t) = \int dZ_N \sum_{i=1}^N v_i \partial_i \Phi(Z_N) \mu^N(Z_N, t)$$

$$+ \frac{3}{2} \sum_{i \neq j} \varepsilon \int_{\mathscr{F}_{ij}} dY \Phi(Y) \left\{ \mu^N(Y'(i,j), t) - \mu^N(Y, t) \right\}. \tag{3.12}$$

The last passage is justified by our Definition (3.5) of the flow on $\mathscr{F}^\omega$: indeed the collision between the pair (i, j) is not included in the random variable in curl brackets in (3.11) (the so-called collision operator) which turns out to be independent of $\omega(i,j)$. Notice that the other possible choice of representing the elements of $\mathscr{F}^\omega$ in terms of outgoing velocities would create problems in taking the expectation in (3.11) and moreover would give the opposite sign to the collision operator.

Let us now derive the first equation (for $j = 1$) of the hierarchy (2.9). We consider in (3.12) a test function $\Phi = \Phi(z_1)$ depending only on the first variable and using the symmetry property of μ^N we obtain:

$$\frac{d}{dt} \int dz_1 \Phi(z_1) f_1^N(z_1, t) = \int dz_1 (v_1 \partial_1 \Phi)(z_1) f_1^N(z_1, t)$$

$$+ 3(N-1)\varepsilon \int dz_1 \Phi(z_1) \left\{ f_2^N(x_1, v_1', x_1, v_2'; t) - f_2^N(x_1, v_1, x_1, v_2; t) \right\} \tag{3.13}$$

with $|v_2 - v_1| = 3$.

Furthermore it is not hard to see that f_1^N is differentiable along the stream flow and

$$\frac{d}{dt} f_1^N(x_1 + v_1 t, v_1; t) = 3(N-1)\varepsilon\{f_2^N(x_1 + v_1 t, v_1', x_1 + v_1 t, v_2'; t)$$
$$- f_2^N(x_1 + v_1 t, v_1, x_1 + v_1 t, v_2; t)\}. \tag{3.14}$$

We have now given a rigorous meaning (actually we have derived the first equation of the hierarchy (2.9) only, but this is what we need) to Eqs. (2.7) and (2.9), so that we are ready to prove two basic estimates which will be useful in Sect. 5.

From now on we will assume ω fixed. We recall that $f_j^N(Z_j, t \,|\, \omega)$, the marginal distribution densities of μ_ω^N, are defined in analogy to (2.8); moreover we will sometimes be concerned with the "trace" of $f_2^N(\cdot\,|\,\omega)$ defined as

$$f_2^N(x_1, v_1, x_1, v_2; t|\omega) = \int dZ_{N-2}\mu_0^N(T_\omega^{-t}(x_1, v_1, x_1, v_2, Z_{N-2})). \tag{3.15}$$

Finally, all through the paper the measure μ_0^N will be assumed as factorizing, that is

$$\mu_0^N(Z_N) = \prod_{i=1}^N f_0(z_i) \tag{3.16}$$

with

$$\|f_0\|_{L_\infty} \le A. \tag{3.17}$$

Denote by k the number of (possible) collisions among the N particles, that is

$$k(\omega) = \sum_{i \neq j=1}^N \omega(i, j), \tag{3.18}$$

then we have the following result:

Proposition 3.1. *Let* $\omega(1, 2) = 1$ *and* $|v_2 - v_1| = 3$. *Then:*

$$\int_0^\infty dt \int dx_1 f_2^N(x_1, v_1, x_1, v_2; t \,|\, \omega) \le \frac{k(\omega)}{3}.$$

Proof.

$$\int_0^\infty dt \int dx_1 f_2^N(x_1, v_1, x_1, v_2, t \,|\, \omega) = \sum_{v_3 \ldots v_N} \int_0^\infty dt$$

$$\times \int dx_1 dx_3 \ldots dx_N \mu_0^N(T_\omega^{-t}(x_1, v_1, x_1, v_2, \ldots, x_N v_N))\chi(1, 2)$$

$$\le \int_0^\infty dt \int_{\mathscr{T}_{1,2}^\omega} dY \mu_0^N(T_\omega^{-t}(Y)) \le \int_0^\infty dt \int_{\mathscr{T}^\omega} dY \mu_0^N(T_\omega^{-t}(Y)). \tag{3.19}$$

Define

$$\beta(Y) = \inf\{\tau > 0 : S^{-\tau}Y \in \mathscr{F}^\omega\}$$
$$= \infty \quad \text{if} \quad S^{-t}Y \notin \mathscr{F}^\omega \quad \text{for all} \quad t \in (0, \infty),$$

then the mapping $R:\mathscr{F}^\omega \to \mathscr{F}^\omega$

$$R(Y) = \pi(S^{-\beta(Y)}Y) \tag{3.20}$$

is invertible and preserves the measure dY. Now we have

$$\int_0^\infty dt \int_{\mathscr{F}^\omega} dY \mu_0^N(T_\omega^{-t}Y) = \sum_{r=0}^{k(\omega)-1} \sum_{l=0}^r \int_{\mathscr{F}^\omega} dY \chi(A_r)$$

$$\times \int_0^{\beta(R^l(Y))} dt \mu_0^N(S^{-t}R^l(Y)), \tag{3.21}$$

where

$$A_r = \{Y \in \mathscr{F}^\omega : T_\omega^{-t}Y \text{ displays } r \text{ collisions if } t \in (0,\infty)\},$$

and from now on $\chi(A)$ will denote the characteristic function of the set A. Notice that if $Y \in A_r$ then $\beta(R^r(Y)) = +\infty$.

Since R is measure preserving, we have:

$$(3.21) \le \sum_{l=0}^{k(\omega)-1} \int_{\mathscr{F}^\omega} dY \int_0^{\beta(R^l(Y))} dt \mu_0^N(S^{-t}R^l(Y))$$

$$\le k(\omega) \int_{\mathscr{F}^\omega} dY \int_0^{\beta(Y)} dt \mu_0^N(S^{-t}Y). \tag{3.22}$$

Notice that Y is of the form $(x_1, v_1, \ldots x_i v_i \ldots x_i v_j \ldots x_N v_N)$ for a couple (i,j), so that by (3.22) we get

$$(3.21) \le k(\omega) \int_{\mathscr{F}^\omega} dY \int_0^\infty dt \mu_0^N(S^{-t}Y)$$

$$= k(\omega) \int_{\mathscr{F}^\omega} dY \int_0^\infty dt \mu_0^N(x_1 - v_1 t, v_1 \ldots x_i - v_i t, v_i \ldots x_i$$

$$- v_j t, v_j \ldots x_N - v_N t, v_N)$$

$$= k(\omega) \int_{\mathscr{F}^\omega} dY \int_0^\infty dt \mu_0^N(x_1, v_1, \ldots x_i, v_i \ldots x_i - (v_j - v_i)t, v_j \ldots x_N, v_N)$$

$$\le \frac{k(\omega)}{3} \int dZ_N \mu_0^N(Z_N) = \frac{k(\omega)}{3}, \tag{3.23}$$

from which we complete the proof. □

Our next goal in this section is to give an estimate on the probability density of one particle $f_1^N(\cdot \,|\, \omega)$. To this purpose we need some more definitions which will play an important role in Sect. 5.

Given an integer k, we denote by $I_k = \{1, 2, \ldots k\}$ the set of the first k integers, while I will stand for any subset of indices of I_N.

Given $\omega \in \Omega$, we denote by ch(i) ("chain" starting from i) any set $I \subset I_N$ of indices such that

$$\omega(i, i_1)\omega(i_1, i_2) \ldots \omega(i_{k-1}, i_k) = 1$$

for some ordering $i_1, \ldots, i_k$ of the set I. The union of all chains starting from the particle i will be called "cluster of i" and denoted by

$$\mathrm{cl}(i) = \bigcup \mathrm{ch}(i) . \tag{3.24}$$

Notice that the index i is not included in cl(i). Finally we denote by

$$n(I) := \sum_{\substack{h \neq k \\ h,k \in I}} \omega(h, k) , \tag{3.25}$$

the number of collisions occurring in I. We want to stress that

$$|\mathrm{cl}(i)| \leq n(\mathrm{cl}(i) \cup \{i\}) \leq \frac{N(N-1)}{2}$$

(here and after $|A|$ stands for the cardinality of the set A) so that $|\mathrm{cl}(i)|$ is the minimum number of collisions in order to set the cluster.

Proposition 3.2. *Given $\omega \in \Omega$, assume that*

$$n(\mathrm{cl}(1) \cup \{1\}) = |\mathrm{cl}(1)| = h .$$

Then

$$f_1^N(x_1, v_1; t \,|\, \omega) = \int \mathrm{d}Z_{N-1} \mu_\omega^N(Z_N, t) \leq 4A \left(\frac{3}{2}\right)^h .$$

Proof. Let $\varphi = \varphi(z_1)$ be a positive real function such that

$$\int \mathrm{d}z_1 \varphi(z_1) = 1 .$$

Then we have:

$$\int \mathrm{d}z_1 f_1^N(z_1; t \,|\, \omega)\varphi(z_1) = \int \mathrm{d}Z_N \mu_0^N(Z_N)\varphi((T_\omega^t Z_N)_1) , \tag{3.26}$$

where $(T_\omega^t Z_N)_1$ denotes position and velocity of the first particle of the configuration $T_\omega^t Z_N$.

Let us put $(T_\omega^t Z_N)_1 = (y, u)$ and consider, for fixed $v_1, z_2, \ldots, z_N$, the mapping

$$x_1 \mapsto y . \tag{3.27}$$

We will show that

$$\frac{\partial y}{\partial x_1} \geq \left(\frac{2}{3}\right)^h \tag{3.28}$$

which implies that the mapping (3.27) is invertible, so that we can conclude, by (3.26) and (3.16) and (3.17):

$$\int dz_1 f_1^N(z_1; t \mid \omega) \varphi(z_1) \leq A \int dZ_{N-1} \mu_0^N(Z_{N-1}) \int dz_1 \varphi(y, u)$$

$$\leq A \left(\frac{3}{2}\right)^h \int dZ_{N-1} \mu_0^N(Z_{N-1}) \sum_{v_1} \int dy \varphi(y, u(y, v_1, Z_{N-1}))$$

$$\leq 4A \left(\frac{3}{2}\right)^h \int dZ_{N-1} \mu_0^N(Z_{N-1}) \sum_u \int dy \varphi(y, u) = 4A \left(\frac{3}{2}\right)^h \quad (3.29)$$

which is our thesis.

Thus, the whole proof relies upon estimate (3.28), which we now prove. Let us start by analyzing $T_\omega^t Z_N$. Let us denote by $0 < t_1 < t_2 < \ldots < t_m < t$ the collision times and by (i_j, s_j) the pair of particles colliding at time t_j, $j = 1, \ldots, m$. Putting

$$b_j = \tfrac{1}{3}(v_{i_j} + v_{s_j}) \quad (3.30)$$

we have, for $\tau \in (t_1, t_2)$:

$$x_{i_1}(\tau) = (1 - b_1)x_{i_1} + b_1 x_{s_1} - v_{s_1}\tau,$$
$$x_{s_1}(\tau) = -b_1 x_{i_1} + (1 + b_1)x_{s_1} - v_{i_1}\tau, \quad (3.31)$$

since $v'_{i_1} = -v_{s_1}$ and $v'_{s_1} = -v_{i_1}$. Hence,

$$T_\omega^\tau Z_N = (X_N(\tau), V_N(\tau)) = (W_1 X_N + V'_N(1)\tau, V'_N(1)), \quad (3.32)$$

where $V'_N(1) = (v_1, \ldots, v'_{i_1}, \ldots, v'_{s_1}, \ldots v_N)$ and W_1 is the $N \times N$ matrix whose elements $(W_1)_{r,l}$ are the following:

$$(W_1)_{r,r} = \begin{cases} 1 & \text{if } r \neq i_1 \text{ or } s_1 \\ 1 - b_1 & \text{if } r = i_1 \\ 1 + b_1 & \text{if } r = s_1 \end{cases},$$

$$(W_1)_{r,l} = \begin{cases} b_1 & \text{if } (r, l) = (i_1, s_1) \\ -b_1 & \text{if } (r, l) = (s_1, i_1). \\ 0 & \text{otherwise} \end{cases} \quad (3.33)$$

By iterating m times formula (3.32) we have at time t:

$$X_N(t) = (x_1(t), \ldots, x_N(t)) = W_m W_{m-1} \ldots W_1 X_N$$

$$+ \sum_{r=1}^{m-1} W_m W_{m-1} \ldots W_{r+1} V'_N(r)(t_r - t_{r-1}) + V'_N(t_m)(t - t_m), \quad (3.34)$$

where W_j, $j = 1, \ldots, m$, are the analogues of W_1, associated to the j-th collision and are defined as in (3.33) replacing the index 1 by j everywhere. Similarly,

$$V'_N(r) = (v_1, \ldots, v'_{i_r}, \ldots, v'_{s_r}, \ldots, v_N). \quad (3.35)$$

Putting

$$W^{(m)} = W_m \ldots W_1, \quad (3.36)$$

we easily realize that

$$\frac{\partial y}{\partial x_1} = (W^{(m)})_{1,1}, \tag{3.37}$$

so that to prove (3.28) we have to give more details on the matrix $W^{(m)}$.
Since we have, by definition:

$$(W^{(m)})_{1,1} = (W_m W^{(m-1)})_{1,1} = \sum_{r=1}^{N} (W_m)_{1,r} (W^{(m-1)})_{r,1},$$

there are two possibilities:
(i) the particle 1 is not involved in the m-th collision. In this case

$$(W^{(m)})_{1,1} = (W^{(m-1)})_{1,1}, \tag{3.38}$$

(ii) the particle 1 collides with particle s_m. Then

$$(W^{(m)})_{1,1} = (1 - b_m)(W^{(m-1)})_{1,1} + b_m (W^{(m-1)})_{s_m,1}. \tag{3.39}$$

We are going to show that in case (ii), under our assumptions, we have

$$(W^{(m-1)})_{s_m,1} = 0. \tag{3.40}$$

Indeed:

$$(W^{(m-1)})_{s_m,1} = \sum_{r=1}^{N} (W_{m-1})_{s_m,r} (W^{(m-2)})_{r,1},$$

and iterating this formula we obtain

$$(W^{(m-1)})_{s_m,1} = \sum_l \alpha_l (W_1)_{l,1}, \tag{3.41}$$

where α_l are suitable coefficients and the sum is taken over the indices l of particles $\sigma_1, \ldots, \sigma_p$, each of them being linked to particle 1 by a chain. An element $(W_1)_{l,1}$ is different from zero only if there is a collision between the particle 1 and one of the σ_i's, $i = 1, \ldots, p$, but this is not allowed by our hypothesis since a collision of this kind would necessarily imply $n(\mathrm{cl}(1)) > |\mathrm{cl}(1)|$. Thus (3.40) is proven and we can conclude by (3.39):

$$(W^{(m)})_{1,1} \geq \frac{2}{3}(W^{(m-1)})_{1,1} \tag{3.42}$$

which, by iteration, implies the thesis by (3.37). □

4. The Main Result and the Strategy of the Proof

From now on c will denote any positive constant independent of N and t. The main result in this paper is the following theorem.

Theorem 4.1. *Suppose $f_0 \in L_\infty$ is an initial datum for the Boltzmann equation (2.14) and $f = f(x, v, t)$ is the solution of the initial value problem. Let $\{f_j^N\}$ be the j-particle distributions associated to the measure $\mu^N(Z_N, t)$, solution to Eq. (2.7) with*

initial condition $\mu_0^N(Z_N) = \prod_{i=1}^{N} f_0(z_i)$. *Then, there exists* $\lambda_0 > 0$ *(independent of t and* f_0) *such that for* $\lambda = N\varepsilon < \lambda_0$, *for all j and* $t \geq 0$, *the following holds:*

$$\lim_{N \to \infty} \int \left| f_j^N(Z_j, t) - \prod_{i=1}^{j} f(z_i, t) \right| dZ_j = 0. \tag{4.1}$$

The proof of the above theorem follows from the following fundamental proposition, which we shall illustrate in this section and prove in the next one.

Proposition 4.1. *Under the same assumptions as in Theorem 4.1, there exist a function* $\tilde{f}^N : \mathbf{R} \times V \to \mathbf{R}^+$ *and a sequence of functions* $g_j^N : \mathbf{R}^j \times V^j \to \mathbf{R}^+$, $j = 1, \ldots, N$ *such that the following holds:*

$$f_j^N(z_1, \ldots, z_j; t) = \prod_{k=1}^{j} \tilde{f}^N(z_k, t) + g_j^N(z_1, \ldots, z_j; t) \tag{4.2}$$

with the properties

(i) $$\tilde{f}^N \leq c \quad (c \text{ independent of } t \text{ and } N), \tag{4.3}$$

(ii) $$\|g_j^N\|_{L_1} \leq \frac{c^j}{N}, \tag{4.4}$$

(iii) $$\sum_{\substack{v_1, v_2 \\ |v_1 - v_2| = 3}} \int_0^\infty dt \int dx |g_2^N(x, v_1, x, v_2)| \leq \frac{c}{N}. \tag{4.5}$$

Proof of Theorem 4.1. By (2.9), (2.14) and (4.2),

$$\|f(t) - f_1^N(t)\|_{L_1} \leq 6(\lambda - \varepsilon) \int_0^t d\tau$$

$$\times \int dx \Bigg\{ \sum_{\substack{v, v_1 \\ |v - v_1| = 3}} f(x, v; \tau) |f(x, v_1; \tau) - \tilde{f}^N(x, v_1; \tau)|$$

$$+ \tilde{f}^N(x, v_1; \tau) |f(x, v; \tau) - \tilde{f}^N(x, v; \tau)| \Bigg\}$$

$$+ 6\varepsilon \int_0^t d\tau \int dx \sum_{\substack{v, v_1 \\ |v - v_1| = 3}} f(x, v; \tau) f(x, v_1; \tau)$$

$$+ 6(\lambda - \varepsilon) \int_0^t d\tau \int dx \sum_{\substack{v, v_1 \\ |v - v_1| = 3}} |g_2^N(x, v, x, v_1; \tau)|, \tag{4.6}$$

the factor 6 in (4.6) coming from the trivial observation that $\sum_{v, v_1} = \sum_{v', v_1'}$.

Since f is bounded in L_∞ (see [4]) by (4.3), (4.4) and (4.5) we obtain for $t \le T$ (T arbitrarily fixed):

$$\|f(t) - f_1^N(t)\|_{L_1} \le \frac{c(1+T)}{N} + c \int_0^t d\tau \|f(\tau) - \tilde{f}^N(\tau)\|_{L_1}$$

$$\le \frac{c(1+T)}{N} + c \int_0^t d\tau \|f(\tau) - f_1^N(\tau)\|_{L_1} + c \int_0^T d\tau \|g_1^N(\tau)\|_{L_1}$$

$$\le \frac{c(1+T)}{N} + c \int_0^t d\tau \|f(\tau) - f_1^N(\tau)\|_{L_1}, \qquad (4.7)$$

so that for any $t \in [0, T]$ we have by the Gronwall's Lemma,

$$\|f(t) - f_1^N(t)\|_{L_1} \to 0. \qquad (4.8)$$

Finally, since by (4.4) it is also

$$\|f(t) - \tilde{f}^N(t)\|_{L_1} \to 0, \qquad (4.9)$$

we can conclude the proof. $\square$

Remark. In the proof of Theorem 4.1 we have made use of the fact that there exists a L_∞-solution to the Boltzmann equation 2.14. However this is not a necessary assumption, since proceeding as in this proof we could show that the sequence $\{f_1^N\}$ is a Cauchy sequence in L_1 and its limit satisfies Eq. (2.14).

Let us now argue about the meaning and the implications of Proposition 4.1. We know, following Lanford's arguments [1], that an estimate of the j-particle distributions of the kind $\|f_j^N\|_{L_\infty} < c^j$ would imply the convergence, but it seems very hard to deduce it for a large time interval, even for λ small, starting directly from the hierarchy of Eqs. (2.9). Thus we change point of view. The leading idea in this paper is to show that the dynamics of our N-particle system can be interpreted as a "cluster dynamics" in the following sense: for λ small the cluster of any particle i is, with large probability, minimal (i.e. $n(\text{cl}(i)) = |\text{cl}(i)|$) and this implies that any tagged group of particles have non-overlapping clusters. Moreover, the probability of having $|\text{cl}(i)|$ large is exponentially small and this fact, together with the previous considerations allow us to say that the f_j^N are to be interpreted as "almost" factorizing, but for a small (order $1/N$) error.

5. Proof of Proposition 4.1

For any subset I of the integers we put

$$Z_I = \{z_i\}_{i \in I}. \qquad (5.1)$$

We have, by definition:

$$f_j^N(Z_j;t) = \mathbf{E}\left[\int dZ_{N-j} \mu_0^N(T_\omega^{-t} Z_N)\right]$$

$$= \sum_{h\geq 0} \sum_{\substack{I \subset I_j^c \\ |I|=h}} \sum_{k\geq h} \mathbf{E}[\chi(cl(I_j) = I)\chi(n(I \cup I_j) = k)$$

$$\times \int dZ_{N-j} \mu_0^N(T_\omega^{-t} Z_N)],\qquad(5.2)$$

where, as before, $n(I \cup I_j) = \sum_{i,k\in I\cup I_j} \omega(i,k)$ and $cl(I) = \bigcup_{i\in J} cl(i)/I$.

Let us decompose f_j^N as:

$$f_j^N = \bar{f}_j^N + R_j^N,\qquad(5.3)$$

where $\bar{f}_j^N$ represents the contribution to f_j^N by those ω's for which $n(I \cup I_j)$ is minimal that is:

$$\bar{f}_j^N(Z_j;t) = \sum_{h\geq 0} \sum_{\substack{I \subset I_j^c \\ |I|=h}} \mathbf{E}\left[\chi(cl(I_j) = I)\chi(n(I \cup I_j) = h)\right.$$

$$\left. \times \int dZ_{N-j} \mu_0^N(T_\omega^{-t} Z_N)\right],\qquad(5.4)$$

while R_j^N is defined accordingly.

The fact that $n(I \cup I_j) = h$ necessarily implies that I has to be the disjoint union of sets, i.e. $I = \bigcup_{s=1}^{j} J_s$, with $|J_s| = h_s$ and $\sum_{s=1}^{j} h_s = h$. Therefore:

$$\bar{f}_j^N(Z_j;t) = \sum_{h_1,\ldots,h_j} \sum_{\substack{J_1,\ldots,J_j \\ |J_s|=h_s \\ J_r\cap J_s=0, r\neq s}} \mathbf{E}\left[\prod_{s=1}^{j} \chi(cl(s) = J_s), \chi(n(\bar{J}_s) = h_s)\right.$$

$$\left. \times \int dZ_{J_s} \mu_0^{h_s+1}(T_\omega^{-t} Z_{\bar{J}_s})\right],\qquad(5.5)$$

where $\bar{J}_s = J_s \cup \{s\}$.

Indeed:

$$\int dZ_{N-j} \mu_0^N(T_\omega^{-t} Z_N) = \int dZ_I \mu_0^{h+j}(T_\omega^{-t} Z_{I\cup I_j})$$

$$= \prod_{s=1}^{j} \int dZ_{J_s} \mu_0^{h_s+1}(T_\omega^{-t} Z_{\bar{J}_s}),\qquad(5.6)$$

due to the definition of cluster and to the fact that clusters of different particles have no intersection.

We will show that $\tilde{f}_j^N$ is "almost" a product of functions. More precisely let us define:

$$\tilde{f}^N(z_s;t) = \sum_{h \geq 0} \sum_{J_s:|J_s|=h} \mathbf{E}\left[\chi(\mathrm{cl}(s) = J_s)\chi(n(\bar{J}_s) = h)\right.$$

$$\left. \times \int \mathrm{d}Z_{J_s}\mu_0^{h+1}(T_\omega^{-t}Z_{\bar{J}_s})\right] \tag{5.7}$$

and

$$h_j^N(Z_J;t) = \bar{f}_j^N(Z_J;t) - \prod_{s=1}^{j}\tilde{f}^N(z_s;t). \tag{5.8}$$

Before proving that h_j^N is small in L_1 we need to get a L_∞-estimate of $\tilde{f}^N$. To this purpose we note that by Proposition 3.2:

$$\tilde{f}(z_1,t) \leq \sum_{h \geq 0} \sum_{I:|I|=h} \left(\frac{3}{2}\right)^{h+1} \mathbf{E}[\chi(\mathrm{cl}(1) = I)\chi(n(\bar{I}) = h)]$$

$$\leq \sum_{h \geq 0} \sum_{I:|I|=h} \left(\frac{3}{2}\right)^{h+1} \mathbf{E}[\chi(\mathrm{cl}(1) = I)]. \tag{5.9}$$

Now, to estimate the above expectation we need to introduce one more definition. Given the set $I = \mathrm{cl}(1)$ with $|I| = h$, we call a "graph" of I [denoted by $G(I)$] a way to settle h links among the $h + 1$ particles of $\bar{I}$, in such a way that each particle is connected to the first through a chain.
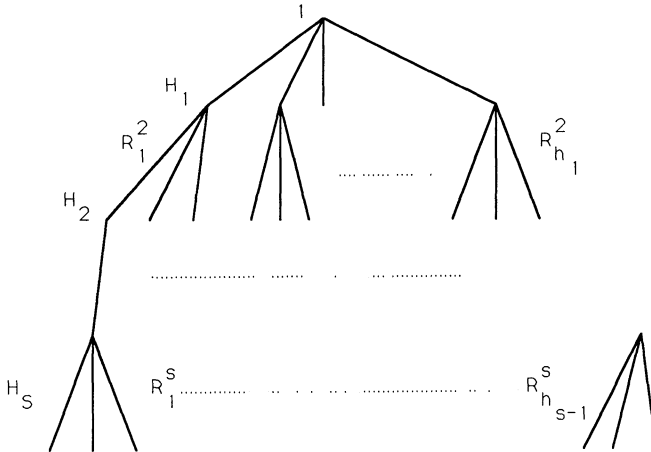

Fig. 1.

A graph is specified when a number $s \leq h$ is given, together with s groups of particles $H_1,\ldots,H_s$ such that $|H_i| = h_i$ and arranged as follows: H_1 is connected with particle 1, H_2 with particles of H_1 and so on up to s, so that $\sum_{i=1}^{s} h_i = h$. Moreover $H_i = \bigcup_{l=1}^{h_{i-1}} R_l^i$, where R_l^i is the group of particles of H_i which is connected with the

l-th particle of the group H_{i-1}, $|R_l^i| = r_l^i$ and $\sum_{l=1}^{h_{i-1}} r_l^i = h_i$. Notice that h_i is always different from zero, while r_l^i may possibly be zero.

Given a graph $G(I)$ we denote by $\chi(G(I))$ the characteristic function of the event "$G(I)$ is realized," that is $\omega(i,j) = 1$ if i and j are connected by a link in the graph. We want to stress the fact that if $\mathrm{cl}(1) = I$, then at least a graph $G(I)$ is realized. Therefore:

$$\chi(\mathrm{cl}(1) = I) \le \sum_{G(I)} \chi(G(I)) \tag{5.10}$$

which implies

$$\mathbf{E}(\chi(\mathrm{cl}(1)) = I) \le \sum_{G(I)} \varepsilon^h . \tag{5.11}$$

Moreover

$$\sum_{G(I)} 1 = \sum_{s=1}^{h} \sum_{\substack{h_1,\ldots,h_s \\ h_i > 0 \\ \sum h_i = h}} \binom{h}{h_1} \binom{h - h_1}{h_2} \cdots \binom{h - \sum_{i=1}^{s-1} h_i}{h_s}$$

$$\times \prod_{l=2}^{s} \sum_{\substack{r_1^l \ldots r_{h_l-1}^l \\ \sum r_i^l = h_l}} \binom{h_l}{r_1^l} \binom{h_l - r_1^l}{r_2^l} \cdots \binom{h_l - \sum_{i=1}^{h_{l-1}-1} r_i^l}{r_{h_l-1}^l}$$

$$= \sum_{s=1}^{h} \sum_{\substack{h_1,\ldots,h_s \\ h_i > 0 \\ \sum h_i = h}} \frac{h!}{h_1! \ldots h_s!} \prod_{l=2}^{s} \sum_{\substack{r_1^l \ldots r_{h_l-1}^l \\ \sum r_i^l = h_l}} \frac{h_l!}{r_1^l! \ldots r_{h_l-1}^l!}$$

$$\le \sum_{s=1}^{h} \sum_{\substack{h_1,\ldots,h_s \\ h_i > 0 \\ \sum h_i = h}} h! e^h \le h! c^h . \tag{5.12}$$

The last passage in (5.12) follows from the fact that:

$$\sum_{\substack{h_1 \ldots h_s \\ \sum h_i = h}} 1 = \sum_{\substack{h_1 \ldots h_s \\ \sum h_i = h}} e^{h_i} e^{-h_i} \le e^h \left(\sum_{h_i > 0} e^{-h_i} \right)^s = e^h \left(\frac{e}{e-1} \right)^s .$$

Hence (5.11) and (5.12) imply that

$$\mathbf{E}(\chi(\mathrm{cl}(1) = I) \le (\varepsilon c)^h h! , \tag{5.13}$$

which, inserted in (5.9) gives us:

$$\tilde{f}^N(z_1, t) \le \sum_{h \ge 0} \binom{N}{h} (\varepsilon c)^h h! \le \sum_{h \ge 0} \frac{N!}{(N-h)!} (c\lambda)^h \frac{1}{N^h} \le \sum_{h \ge 0} (c\lambda)^h \le c \tag{5.14}$$

for $\lambda \le \lambda_0$, a sufficiently small constant. Thus 4.3 is proven.

Let us now go back to (5.3) and (5.8): to prove (4.4) in Proposition 4.1 we have to show that $g_j^N := R_j^N + h_j^N$ is small in the L_1-norm. We start by estimating R_j^N. It is by definition:

$$R_j^N(Z_j, t) = \sum_{\substack{h \geq 0}} \sum_{\substack{I \subset I_j^c \\ |I| = h}} \sum_{k > h} \mathbf{E}\left[\chi(\mathrm{cl}(I_j)) = I)\chi(n(I \cup I_j) = k) \right.$$

$$\left. \times \int \mathrm{d}Z_{N-j} \mu_0^N(T_\omega^{-t} Z_N) \right]. \tag{5.15}$$

Let $J_1, \ldots, J_1$ be defined as follows:

$$\begin{aligned} J_1 &= \mathrm{cl}(1)\,, \\ J_k &= \mathrm{cl}(k)/J_{k-1}\,, \qquad k = 2, \ldots, j\,. \end{aligned} \tag{5.16}$$

Then, by definition, the J_k's are disjoint "subclusters," $J_k \subseteq \mathrm{cl}(k)$, union of some chains starting from k.

Thus, by (5.15) (using the fact that $\int \mathrm{d}Z_N \mu_0^N(T_\omega^{-t} Z_N) = 1$) we have:

$$\|R_j^N(Z_j, t)\|_{L_1} \leq \sum_{h \geq 0} \sum_{\substack{I \subset I_j^c \\ |I| = h}} \sum_{k > h} \sum_{\substack{h_1, \ldots, h_j \\ \sum h_i = h}}$$

$$\times \sum_{\substack{J_1, \ldots, J_j \\ |J_r| = h_r \\ \sum_{r=1} J_r = 1 \\ J_r \cap J_s = \phi \, \forall r \neq s}} \mathbf{E}\left[\prod_{r=1}^r \chi(\mathrm{cl}(r) \supseteq J_r)\chi(n(I \cup I_j) = k) \right]. \tag{5.17}$$

Now, since

$$\chi(\mathrm{cl}(r) \supseteq J_r) \leq \sum_{G(J_r)} \chi(G(J_r))\,, \tag{5.18}$$

we have

$$\mathbf{E}\left[\prod_{r=1}^j \chi(\mathrm{cl}(r) \supseteq J_r)\chi(n(I \cup I_j) = k) \right]$$

$$\times \sum_{G(J_1)\ldots G(J_j)} \mathbf{E}[\chi(G(J_1)\ldots\chi(G(J_j))\chi(n(I \cup I_j) = k)]\,. \tag{5.19}$$

Since in the last sum j graphs are to be fixed, also $h_1, \ldots, h_j$ links are fixed, such that $\sum_{i=1}^j h_i = h$. It only remains to fix $k - h$ links, which can be settled in $\binom{\mathcal{N}}{k-h}$ ways, being $\mathcal{N} = \dfrac{(h+j)(h+j-1)}{2}$. Thus

$$(5.19) \leq \sum_{G(J_1)\ldots G(J_j)} \varepsilon^k \binom{\mathcal{N}}{k-h} \leq \prod_{r=1}^j h_r! c^{h_r} \binom{\mathcal{N}}{k-h} \varepsilon^k \tag{5.20}$$

by (5.12) and this implies:

$$\|R_j^N(Z_j,t)\|_{L_1} \leq \sum_{h\geq 0}\sum_{k>h}\binom{N}{h}\sum_{\substack{h_1,\ldots,h_j\\ \sum h_i=h}}\frac{h!}{h_1!\ldots h_j!}$$

$$\times \prod_{r=1}^{j}h_r!c^{hr}\binom{\mathcal{N}^{\cdot}}{k-h}\varepsilon^k$$

$$\leq \sum_{h\geq 0}\frac{N!}{(N-h)!}\sum_{\substack{h_1,\ldots,h_j\\ \sum h_i=h}}\left(\frac{c\lambda}{N}\right)^h\sum_{r=1}^{\mathcal{N}^{\cdot}}\left(\frac{\lambda}{N}\right)^r\binom{\mathcal{N}^{\cdot}}{r}$$

$$\leq \sum_{h\geq 0}(c\lambda)^h\sum_{r=1}^{\mathcal{N}^{\cdot}}\left(\frac{\lambda}{N}\right)^r\binom{\mathcal{N}^{\cdot}}{r} \tag{5.21}$$

since $k>h$. Recalling that

$$\frac{\mathcal{N}^{\cdot}}{N} \leq \frac{(h+j)^2}{N} \leq h+j, \tag{5.22}$$

we have:

$$\sum_{r=1}^{\mathcal{N}^{\cdot}}\left(\frac{\lambda}{N}\right)^r\binom{\mathcal{N}^{\cdot}}{r} = \left(1+\frac{\lambda}{N}\right)^{\mathcal{N}^{\cdot}} - 1 \leq \exp\left(\frac{\lambda\mathcal{N}^{\cdot}}{N}\right) - 1$$

$$\leq \frac{\lambda\mathcal{N}^{\cdot}}{N}\exp\left(\frac{\lambda\mathcal{N}^{\cdot}}{N}\right) \leq \lambda\frac{(h+j)^2}{N}\exp(\lambda(h+j)), \tag{5.23}$$

which, inserted in (5.21) gives us finally:

$$\|R_j^N(Z_j,t)\|_{L_1} \leq \frac{c^j}{N}\sum_{h\geq 0}(c\lambda)^h \leq \frac{c^j}{N}. \tag{5.24}$$

We will also need to control the term

$$\int_0^\infty \mathrm{d}t \int \mathrm{d}x\, R_2^N(x,v_1,x,v_2;t)$$

in order to prove (4.5). This can be done by exactly performing the same procedure as before, with the only difference that, instead of bounding as in (5.17) the L_1-norm of

$$\int \mathrm{d}Z_{N-2}\mu_0^N(T_\omega^{-t}(x,v_1,x,v_2,x_3,v_3\ldots x_N,v_N)) \tag{5.25}$$

by one, we make use of Proposition 3.1. This implies that in all passages made from (5.17) up to (5.21) an extra divergent factor appears, which is $k \leq (h+j)^2$, but this does not affect the convergence of the geometric series, so that:

$$\sum_{v_1}\int_0^\infty \mathrm{d}t \int \mathrm{d}x\, R_2^N(x,v_1,x,v_2;t) \leq \frac{c}{N}. \tag{5.26}$$

Now, let us estimate the L_1-norm of h_j^N (see (5.8)). We have, by definition

$$h_j^N = \mathscr{T}_2^j - \mathscr{T}_1^j\,, \tag{5.27}$$

where

$$\mathscr{T}_1^j = \sum_{h_1,\ldots,h_j} \sum_{\substack{J_1,\ldots,J_j \\ |J_i|=h_i}}^* \prod_{r=1}^j \mathbf{E}[\gamma(z_r,h_r,J_r;t)] \tag{5.28}$$

$$\mathscr{T}_2^j = \sum_{h_1,\ldots,h_j} \sum_{\substack{J_1,\ldots,J_j \\ |J_i|=h_i \\ J_r \cap J_s=\phi,r\neq s}} \left\{ \mathbf{E}\left[\prod_{r=1}^j \gamma(z_r,h_r,J_r;t)\right] \right.$$

$$\left. - \prod_{r=1}^j \mathbf{E}[\gamma(z_r,h_r,J_r;t)] \right\}, \tag{5.29}$$

where $\sum^*$ is the sum over sets $J_1,\ldots,J_j$ such that at least two of them have non-void intersection and

$$\gamma(z_r,h_r,J_r;t) = \chi(A_r)\xi_r(\omega,z_r)\,, \tag{5.30}$$

$$A_r = \{\omega: \mathrm{cl}(r) = J_r, n(\bar{J}_r = h_r)\}\,, \tag{5.31}$$

$$\xi_r(\omega,z_r) = \int \mathrm{d}Z_{J_r} \mu_0^{h_r+1}(T_\omega^{-t} Z_{\bar{J}_r})\,. \tag{5.32}$$

We start by a bound on $\mathscr{T}_1^j$. It is:

$$\|\mathscr{T}_1^j\|_{L_1} \leq \sum_{h_1,\ldots,h_j} \sum_{\substack{J_1,\ldots,J_j \\ |J_i|=h_i}}^* \prod_{r=1}^j \mathbf{E}[\chi(A_r)]$$

$$\leq \sum_{h_1,\ldots,h_j} \sum_{r,s} \sum_{\substack{J_1,\ldots,J_j \\ |J_i|=h_i \\ J_r \cap J_s=\phi}} \prod_{r=1}^j \mathbf{E}[\chi(A_r)]\,. \tag{5.33}$$

Now we have, using (5.13):

$$\sum_{h_r,h_s} \sum_{\substack{J_r,J_s \\ |J_i|=h_i,i=r,s \\ J_r\cap J_s\neq\phi}} \mathbf{E}[\chi(A_r)]\,\mathbf{E}[\chi(A_s)]$$

$$\leq \sum_{h_r,h_s} \sum_{\substack{J_r \\ |J_r|=h_r}} \sum_{i\in J_r} \sum_{\substack{J_s:i\in J_s \\ |J_s|=h_s}} \mathbf{E}[\chi(A_r)]\,\mathbf{E}[\chi(A_s)]$$

$$\leq \sum_{h_r,h_s} \binom{N}{h_r}\binom{N}{h_s-1} h_r h_r! h_s! c^{h_r+h_s}\varepsilon^{h_r+h_s}$$

$$\leq \frac{1}{N}\sum_{h_r,h_s} h_s h_s \frac{(N!)^2}{(N-h_r)!(N-h_s+1)!}\frac{1}{N^{h_r}}\frac{1}{N^{h_s-1}}(c\lambda)^{h_r+h_s}$$

$$\leq \frac{1}{N}\left(\sum_h h(c\lambda)^h\right)^2 \leq \frac{c^2}{N}\,. \tag{5.34}$$

Hence, by (5.33) and again (5.13):

$$\|\mathscr{T}_1^j\|_{L_1} \leq \frac{c^2}{N} j^2 \left(\sum_{h_r} \sum_{\substack{I_r \\ |I_r|=h_r}} \mathbf{E}[\chi(A_r)] \right)^{j-2} \leq \frac{c^j}{N}. \tag{5.35}$$

Let us pass to $\mathscr{T}_2^j$. To simplify the algebra we consider for the moment $j = 2$. We set, for $I, J \subset I_N$,

$$\Omega(I, J) = \{\omega(i, k): i \in I, k \in J\}. \tag{5.36}$$

First of all we notice that the function ξ_r in (5.32) depends on ω through its restriction on $\Omega(\bar{J}_r, \bar{J}_r)$ and we will denote such restriction still by ξ_r. Now

$$\mathbf{E}[\gamma(z_r, h_r, J_r; t)] = \sum_{\omega_r \in \Omega(\bar{J}_r, \bar{J}_r)} p(\omega_r)\xi_r(\omega_r, z_r)\chi(\bar{A}_r)$$

$$\times \sum_{\eta_r \in \Omega(\bar{J}_r, \bar{J}_r^c)} p(\eta_r)\chi(\eta_r = 0) \tag{5.37}$$

for $r = 1, 2$, where

$$\bar{A}_r = \{\omega_r \in \Omega(\bar{J}_r, \bar{J}_r): \mathrm{cl}(r) = J_r, n(\bar{J}_r) = h_r\}. \tag{5.38}$$

The sum over η_r in (5.37) can be computed, giving $(1 - \varepsilon)^{h_r(N-h_r)}$. On the other side

$$\mathbf{E}[\gamma(z_1, h_1, J_1; t)\gamma(z_2, h_2, J_2; t)]$$

$$= \sum_{\omega_1 \in \Omega(\bar{J}_1, \bar{J}_1)} p(\omega_1)\xi_1(\omega_1)\chi(\bar{A}_1) \sum_{\omega_2 \in \Omega(\bar{J}_2, \bar{J}_2)} p(\omega_2)\xi_2(\omega_2)\chi(\bar{A}_2)$$

$$\times \sum_{\eta \in \Omega(\bar{J}_1 \cup \bar{J}_2, (\bar{J}_1 \cup \bar{J}_2)^c)} p(\eta)\chi(\eta = 0) \sum_{\gamma \in \Omega(\bar{J}_1 \bar{J}_2)} p(\gamma)\chi(\gamma = 0). \tag{5.39}$$

The last two sums in (5.39) give $(1 - \varepsilon)^{(h_1+h_2)(N-h_1-h_2)}(1 - \varepsilon)^{h_1 h_2}$ and by (5.38):

$$\mathbf{E}[\chi(A_1)\xi_1(\omega, z_1)\chi(A_2)\xi_2(\omega, z_2)]$$

$$= \mathbf{E}[\chi(A_1)\xi_1(\omega, z_1)]\,\mathbf{E}[\chi(A_2)\xi_2(\omega, z_2)]\,(1 - \varepsilon)^{-h_1 h_2}. \tag{5.40}$$

Hence, inserting (5.40) in (5.29) we obtain

$$\|\mathscr{T}_2^2\|_{L_1} \leq \sum_{h_1,h_2} \sum_{J_1,J_2} \prod_{r=1}^{2} \mathbf{E}[\chi(A_r)]\,[(1 - \varepsilon)^{-h_1 h_2} - 1]. \tag{5.41}$$

Noticing that

$$(1 - \varepsilon)^{-h_1 h_2} - 1 \leq \frac{\varepsilon h_1 h_2}{(1 - \varepsilon)^{h_1 h_2 + 1}} \leq 2\varepsilon h_1 h_2 \left(1 - \frac{\lambda}{N} \right)^{-N \frac{h_1 h_2}{N}}$$

$$\leq \frac{2\lambda h_1 h_2}{N} e^{\lambda h_1}, \tag{5.42}$$

proceeding as usual we obtain, by (5.41), (5.42) and (5.13):

$$\|\mathscr{T}_2^2\|_{L_1} \leq \frac{1}{N} \left(\sum_{h \geq 0}(c\lambda)^h \right)^2 \leq \frac{c}{N}. \tag{5.43}$$

For the term $\mathscr{T}_2^j$ we proceed analogously and prove that

$$\mathbf{E}\left[\prod_{r=1}^{j}\chi(A_r)\xi_r\right] - \prod_{r=1}^{j}\mathbf{E}[\chi(A_r)\xi_r] = \prod_{r=1}^{j}\mathbf{E}[\chi(A_r)\xi_r]\left[(1-\varepsilon)^{-\prod_{r<s}h_r h_s} - 1\right]. \quad (5.44)$$

Following (5.42):

$$(1-\varepsilon)^{-\prod_{r<s}h_r h_s} - 1 \le \frac{2\lambda h^2}{N}e^{\lambda h}, \qquad h = \sum_{r=1}^{j}h_r, \quad (5.45)$$

we can conclude that

$$\|\mathscr{T}_2^j\|_{L_1} \le \frac{c^j}{N}. \quad (5.46)$$

To achieve the proof of the proposition we have to show that (4.5) holds. Indeed, proceeding as in the proof of estimate (5.26), by the use of Proposition 3.1 we can conclude that

$$\int_0^{\infty}\mathrm{d}t\int\mathscr{T}_r^2(x,v_1,x,v_2;t)\mathrm{d}x \le \frac{c}{N}, \qquad r = 1,2. \quad (5.47)$$

6. From Small to Large λ

In this section we remove the assumption on the smallness of λ we used so far. In doing this we pay a price: the quality of the convergence we are able to prove is weaker. This is, probably, a technical limitation only.

The idea underlying this extension is the following. Look at the particle system at time $t + \Delta t$, with Δt sufficiently small, and at a small interval Λ on the line. Due to the boundedness of the velocities, the particles inside Λ are influenced only by the particles in an interval, say Λ_0, a little bit larger than Λ, if Δt is small. Suppose that at the time t we have convergence to the solution of the Boltzmann equation. This means that, with large probability, the number of particles in Λ_0, say $N(\Lambda_0)$, is of the order αN, where $\alpha = \sum_v \int_{\Lambda_0}\mathrm{d}x f(x,v;t)$. Since we have L_{∞} bounds for f, α can be made arbitrarily small, provided that Λ and Δt are sufficiently small. They will be fixed in such a way that $\alpha\lambda < \lambda_0$, where λ_0 is the value under which the cluster expansion is convergent and $\lambda = \varepsilon N$ (in general larger than λ_0). On the other hand the particle system at time $t + \Delta t$ restricted in Λ is the same as if the particles outside Λ_0 were absent at time t, so that we are in a situation for which the number of particles is αN and consequently the effective λ is $\alpha N\varepsilon = \alpha\lambda < \lambda_0$.

Therefore, using the previous analysis, we expect convergence at time $t + \Delta t$, if we look at the region Λ. Then it is not difficult to show that the local convergence in each region Λ of small size implies the convergence in any region, so that all the argument can be iterated to reach an arbitrary time, since Δt is a priori fixed.

To make rigorous the above argument we need a local characterization of a physical state of the system (and we shall do it later) and the lemma below:

Lemma 6.1. *Let $\{\mu^N\}_{N>1}$ be a sequence of densities of symmetric probability measures on $(\mathbf{R}\times V)^N$. Denote by f_j^N their marginals. Then the following two statements are equivalent:*

i) *For all j and all φ bounded, $\varphi : \mathbf{R} \times V \to \mathbf{R}$,*

$$\int f_j^N(Z_j) \prod_{i=1}^{j} \varphi(Z_i) \mathrm{d}Z_j \xrightarrow[N \to \infty]{} \langle f, \varphi \rangle^j$$

with $f \in L_1(\mathbf{R} \times V)$.

ii)
$$\lim_{N \to \infty} \mathbf{E}_N(\langle \nu^N, \varphi \rangle^j) = \langle f, \varphi \rangle^j .$$

Here $\langle \cdot \rangle$ denotes the scalar product, $\langle \nu^N, \varphi \rangle = \dfrac{1}{N} \sum_{i=1}^{N} \varphi(z_i)$, and $\mathbf{E}_N$ is the expectation with respect to μ^N.

Proof. By definition, for $N > j$:

$$\mathbf{E}_N(\langle \nu^N, \varphi \rangle^j) = \int \mathrm{d}\mu^N \frac{1}{N^j} \sum_{i_1=1}^{N} \dots \sum_{i_j=1}^{N} \varphi(z_{i_1}) \dots \varphi(z_{i_j})$$

$$= \frac{N(N-1)\dots(N-j+1)}{N^j}$$

$$\times \int \mathrm{d}z_j f_j^N(z_j) \prod_{i=1}^{j} \varphi(z_i) + O\left(\frac{1}{N}\right). \tag{6.1}$$

Indeed the first term in the right-hand side fof (6.1) arises from the contribution in the sums in which all the i_k's are different and the second one from the contribution in which at least two indices coincide.

We now establish a proposition which, combined with the results of the previous section, allows us to prove the convergence for any λ.

Consider now our particle system distributed, at time zero, according to a sequence of symmetric densities $\{\mu^N\}_{N \geq 1}$ such that, for all bounded continuous φ of compact support and for all j:

$$\mathbf{E}_N(\langle \nu^N, \varphi \rangle^j) \xrightarrow[N \to \infty]{} \langle f, \varphi \rangle^j \tag{6.2}$$

with $\|f\|_{L_\infty} \leq M$.

Proposition 6.2. *Assume hypothesis (6.2) and consider an interval Λ with $|\Lambda| = length$ of $\Lambda = \delta$, $\delta = \dfrac{\lambda_0}{100\lambda M}$ (here λ_0 is a value for which the cluster expansion is converging and $\lambda = \varepsilon N$). Define*

$$\Lambda_0 = \{x \pm 2T \,|\, x \in \Lambda\}, \qquad T = \delta .$$

Then, for all $\eta > 0$ sufficiently small, $Z_j \in (\Lambda \times V)^j$, $t \leq T$ the following hold:

$$\Omega_j^N(Z_j t) + l_j^{N,-}(Z_j, t) \leq f_j^N(Z_j, t) \leq \Omega_j^N(Z_j, t) + l_j^{N,+}(Z_j, t), \tag{6.3}$$

where Ω_j^N is a positive function such that:

$$\int_{(\Lambda \times V)^j} \Omega_j^N(Z_j, t) \varphi_j(Z_j) \mathrm{d}Z \xrightarrow[N \to \infty]{} 0 \tag{6.4}$$

for all bounded continuous $\varphi_j : (\Lambda \times V)^j \to \mathbf{R}$ *and*

$$l_j^{N,\pm}(Z_j, t) = \int dZ_{[(\alpha \pm \eta)N]-j} \, \mathbf{E}(\gamma^{[(\alpha \pm \eta)N]}(T_\omega^{-t} Z_{[(\alpha \pm \eta)N]})) \qquad (6.5)$$

being [a] *the integer part of a and*

$$\gamma^k(Z_k) = \chi(Z_k \in (\Lambda_0 \times V)^k) \prod_{i=1}^{k} f(z_i) \cdot \alpha^{-k} \qquad (6.6)$$

with

$$\alpha = \int\limits_{\Lambda_0 \times V} dz f(z). \qquad (6.7)$$

Before giving the proof of Proposition 6.2 we discuss its meaning and conse-quences.

From Proposition 6.2 we see that the j-particle distributions $f_j^N(\cdot, t)$, for t sufficiently small, are controlled, in the region $(\Lambda \times V)^j$, by the j-particle distributions $l_j^{N,\pm}$ of a particle system whose initial distribution is γ^N, i.e. a product measure supported in Λ_0, associated to a system of $H = [(\alpha\eta)N]$ particles.

Notice that

$$\alpha N \varepsilon \leq \lambda 4M(\delta + 4\delta) < \lambda_0$$

so that, for a sufficiently small η, (for instance $\eta < \lambda_0/2\lambda$) the cluster expansion, associated to the initial measure γ^H, is convergent. By Theorem 4.1:

$$\lim_{N \to \infty} l_j^{N,\pm} = (h^\pm)^{\otimes j}$$

in $L_1(\mathbf{R} \times V)$. $(h^\pm)$ is the solution of the Boltzmann equation (2.14) with initial datum $h_0 = h^\pm(t = 0) = \chi_{\Lambda_0}(x) f(x, v) \alpha^{-1}$ and λ replaced by $\lambda(\alpha \pm \eta)$. Since η is arbitrary, we proved:

$$\lim_{N \to \infty} \int\limits_{(\Lambda \times V)^j} dz_j f_j^N(z_j, t) \varphi_j(z_j) = \int\limits_{(\Lambda \times V)^j} dz_j h^{\otimes j}(z_j, t) \varphi_j(z_j)$$

for all continuous bounded $\varphi_j : (\Lambda \times V)^j \to \mathbf{R}$, where $h(t)$ solves Eq. (2.14) with λ replaced by $\alpha\lambda$ and initial datum h_0 (indeed, thanks to the L_∞ bounds, it is not difficult to show that $h(t)$ is L_1-continuous with respect to λ). Finally by a simple scaling argument and by virtue of the boundedness of the velocities we conclude that $h(x, v, t) = f(x, v, t)$ for $x \in \Lambda$ and $t \leq T$. Therefore we proved the local convergence:

$$\lim_{N \to \infty} \int\limits_{(\Lambda \times \Lambda)^j} dz_j f_j^N(Z_j, t) \varphi_j(Z_j) = \int\limits_{(\Lambda \times V)^j} dZ_j f^{\otimes j}(z_j, t) \varphi_j(Z_j) \qquad (6.8)$$

for all intervals Λ such that $|\Lambda| < \delta$ and all $\varphi_j : (\Lambda \times V)^j \to \mathbf{R}$ continuous and bounded.

We now show how convergence (6.2), assumed at time zero, also holds at time T. Indeed take $\varphi : \mathbf{R} \times V \to \mathbf{R}$, continuous bounded, supported in Λ with $|\Lambda| < 2\delta$. Let

Λ_i, $|\Lambda_i| < \delta$, $i = 1, 2$ be two disjoint intervals such that $\Lambda_1 \cup \Lambda_2 = \Lambda$, and φ_i the restriction of φ to Λ_i. Then:

$$|\mathbf{E}_N(\langle \nu^N, \varphi \rangle^j - \langle f, \varphi \rangle^j)|$$

$$= \left| \sum_{k=0}^{j} \binom{j}{k} \mathbf{E}_N(\langle \nu^N, \varphi_1 \rangle^k \langle \nu^N, \varphi_2 \rangle^{j-k} - \langle f, \varphi_1 \rangle^k \langle f, \varphi_2 \rangle^{j-k})^j \right|$$

$$\leq \sum_{k=0}^{j} \binom{j}{k} \{ |\mathbf{E}_N(\langle \nu^N, \varphi_1 \rangle^k (\langle \nu^N, \varphi_2 \rangle^{j-k} - \langle f, \varphi_2 \rangle^{j-k}))|$$

$$+ \langle f, \varphi_2 \rangle^{j-k} |\mathbf{E}_N(\langle \nu^N, \varphi_1 \rangle^k - \langle f, \varphi_1 \rangle^k)| \} . \tag{6.9}$$

By Lemma 6.1 and (6.8), the 2$^{\text{nd}}$ term in the curl bracket vanishes as $N \to \infty$. The first term is bounded by:

$$\mathbf{E}_N(\langle \nu^N, \varphi_1 \rangle^{2k})^{1/2} (\mathbf{E}_N(\langle \nu^N, \varphi_2 \rangle^{2(j-k)}) + \langle f, \varphi_2 \rangle^{2(j-k)}$$

$$- 2\langle f, \varphi_2 \rangle^{j-k} \mathbf{E}_N(\langle \nu^N, \varphi_2 \rangle^{j-k})^{1/2} . \tag{6.10}$$

Also (6.10) vanishes as $N \to \infty$. In fact, for all j, $\mathbf{E}_N(\langle \nu^N, \varphi_2 \rangle^j) \to \langle f, \varphi_2 \rangle^j$, again by Lemma 6.1 and (6.8), so that (6.2) holds at time T, because, using the above argument, we can deal with any bounded continuous function φ of compact support. Finally, since the time T is a priori fixed, we can iterate the argument to prove:

Theorem 6.3. *Suppose $f_0 \in L_\infty$ be an initial datum for the Boltzmann equation (2.14) and let $f = f(x, v, t)$ be the unique solution satisfying*

$$\sup_{0 \leq t \leq t^*} \sup_{x, v} f(x, v, t) \leq M \tag{6.11}$$

for an arbitrary (but fixed) $t^* > 0$, with M a suitable positive constant depending on f_0 and t^*. Let $\mu_0^N = f_0^{\otimes N}$ be the initial condition for the particle system and $\lambda = \varepsilon N$ arbitrary. Then:

$$\lim_{N \to \infty} \langle f_j^N(t), \varphi_j \rangle = \langle f^{\otimes j}(t), \varphi_j \rangle \tag{6.12}$$

for all j and all continuous $\varphi_j : (\mathbf{R} \times V)^j \to \mathbf{R}$ of compact support.

Remark 1. The L_1 convergence of the previous section has been replaced by the weak convergence (6.12). Actually (6.12) has been proved for factorizing test functions φ_j, but it can be recovered by a standard approximation argument.

Remark 2. In the above theorem we used the existence and uniqueness of the solution $f \in L_\infty([0, t^*]; L_\infty(\mathbf{R} \times V))$ just to control the number of particles (uniformly in time up to an arbitrary fixed time) in an interval of a given size. This can also be done differently by means of the H-theorem. Indeed it is easy to show that the Entropy per particle

$$H^N(t) = \frac{1}{N} \int \mathrm{d}Z_N \mu^N(t) \log \mu^N(t) \tag{6.13}$$

is a decreasing function of time. By the subaddivitity of the entropy:

$$\int f_1^N(z, t) \log f_1^N(z, t) \mathrm{d}z \leq H^N(t) \leq H^N(0)$$

$$= \int f_0 \log f_0 \mathrm{d}z \leq C < +\infty . \tag{6.14}$$

From the above inequality we obtain

$$\int_{\Lambda} f_1^N(z,t) \to 0 \quad \text{as} \quad |\Lambda| \to 0 \tag{6.15}$$

uniformly in time.

This observation, combined with the remark in Sect. 4 allows us to avoid to assume the existence of the limit solution, but just to prove it, together with the validity proof.

Remark 3. It is not difficult, at this point, to deal with a particle system in a bounded box with suitable (for instance periodic) boundary conditions. We do not give here the details to avoid an even longer and heavy exposition.

Before proving Proposition 6.2 we need some preliminary definitions.

Let $\Lambda_0 \subset \mathbf{R}^1$ be a bounded interval and μ^N be a distribution density of a N-particle system.

Denote (for $Z_n \in (\Lambda_0 \in V)^n$) by:

$$\mu_{\Lambda_0}^{N,n}(Z_n) = \binom{N}{n} \int_{(\Lambda_0 \times V)^{N-n}} dZ_{N-n} \mu^N(Z_N), \tag{6.16}$$

the probability density of finding n particles in Λ_0 in the configuration Z_n and by

$$p_{\Lambda_0}^N(n) = \int_{(\Lambda_0 \times V)^n} dZ_n \mu_{\Lambda_0}^{N,n}(Z_n), \tag{6.17}$$

the probability of finding n particles in Λ_0.

Finally we set

$$\mu_{\Lambda_0}^N(Z_n \mid n) \mu_{\Lambda_0}^{N,n}(Z_n) \frac{1}{p_{\Lambda_0}^N(n)}, \tag{6.18}$$

which is the conditional probability (of having n particles in Λ_0) density and

$$\Phi_{\Lambda,0}^{N,k}(Z_k \mid n) = \int_{(\Lambda_0 \times V)^{n-k}} dZ_{n-k} \mu_{\Lambda_0}^N(Z_n \mid n) \tag{6.19}$$

its marginals.

Proof of Proposition 6.2. Fix $\varphi_j : (\Lambda \times V)^j \to \mathbf{R}^+$ continuous and bounded and $t \geq T$. Then

$$\langle \varphi_j, f_j^N(t) \rangle \equiv \int \varphi_j(Z_j) f_j^N(Z_j, t) dZ_j = \int dZ_N \mu^N(Z_N) \mathbf{E}(\varphi(T_\omega^t Z_N)_j)$$

(where $(Z_N)_j$ denotes the first j particles of Z_N and μ^N is the initial sequence for which (6.2) holds)

$$= \sum_{k \geq j} p_{\Lambda_0}^N(k) \int_{(\Lambda_0)^k} dZ_k \mu_{\Lambda_0}^N(Z_k \mid k) \Psi(Z_k), \tag{6.20}$$

where we set

$$\Psi(Z_k) = \mathbf{E}(\varphi(T_\omega^t Z_k)_j),$$

since the first j particles of $T^t_\omega Z_N$ depend on the particles of Z_N which are in Λ_0 only.

We now estimate the probability of having a number of particles in Λ_0 (such a random variable is denoted by $N(\Lambda_0)$) much different than αN. We have, by Tchebichev inequality:

$$\sum_{|k-\alpha N|\geq\beta N} p^N_{\Lambda_0}(k) = \mathrm{Prob}(|N(\Lambda_0) - \alpha N| \geq \beta N)$$

$$\leq \left(\frac{1}{\beta N}\right)^2 \mathbf{E}_N(|N(\Lambda_0) - \alpha N|^2)$$

$$= (\beta N)^{-2}\{\mathbf{E}(N(\Lambda_0)^2) + \alpha^2 N^2 - 2\alpha N\mathbf{E}_N(N(\Lambda_0))\}. \quad (6.21)$$

Here $\mathbf{E}_N$ denotes the expectation with respect to μ_N. By (6.2) $\dfrac{\mathbf{E}_N(N(\Lambda_0)^2)}{N^2} \to \alpha^2$,

$\dfrac{\mathbf{E}_N(N(\Lambda_0))}{N} \to \alpha$ as $N \to \infty$, so that the right-hand side of (6.21) is bounded by

$\dfrac{\xi_1(N)}{\beta^2}$, where $\xi_1(N) \to 0$ as $N \to \infty$. In conclusion, denoting from now on by $\xi_i(N)$,

$i = 1, 2, \ldots$ vanishing sequences in the limit $N \to \infty$,

$$\langle\varphi_j, f^N_j(t)\rangle = \xi_2(N) + \sum_{|k-\alpha N|<\eta N} p^N_{\Lambda_0}(k) \int_{(\Lambda_0)^k} \mathrm{d}Z_k \mu^N_{\Lambda_0}(Z_k \mid k)\Psi(Z_k). \quad (6.22)$$

Observe now that for $k \in [(\alpha-\eta)N, (\alpha+\eta)N]$, with a sufficiently small η, the cluster expansion is convergent, that is:

$$\langle f^N_j(t), \varphi_j\rangle = \xi_2(N) + \sum_{k=[(\alpha-\eta)N]}^{[(\alpha+\eta)N]} p^N_{\Lambda_0}(k) \sum_{h=1}^{k} \sum_{\substack{I\subset I_k\backslash I_j \\ |I|=h}} \mathbf{E}\left[\chi(\mathrm{cl}(I_j) = I)\right.$$

$$\left. \times \int \mathrm{d}Z_{I\cup I_j} \Phi^{N,h+j}_{\Lambda_0}(Z_{I\cup I_j} \mid k)\varphi((T^t_\omega Z_{I\cup I_j})_j)\right], \quad (6.23)$$

from which we argue that

$$\xi_4(N) + S^-(N) \leq \langle f^N_j(t), \varphi_j\rangle \leq \xi_3(N) + S^+(N), \quad (6.24)$$

where

$$S^\pm(N) = \sum_{h=1}^{[(\alpha\pm\eta)N]} \sum_{\substack{I\subset I_{[(\alpha\pm\eta)N]}\backslash I_j \\ |I|=h}} \mathbf{E}(\chi(\mathrm{cl}(I_j) = I)$$

$$\times \int \mathrm{d}Z_{I\cup I_j} \sum_{k>h+j} p^N_{\Lambda_0}(k)\Phi^{N,h+j}(Z_{I\cup I_j} \mid k)\phi((T^t_\omega Z_{I\cup I_j})_j). \quad (6.25)$$

On the other hand, for a positive function φ supported in Λ_0 we have:

$$
\mathbf{E}_N\left(\left(\frac{1}{N}\sum_{i=1}^{N}\varphi(z_i)\right)^j\right)
$$

$$
=\sum_{k=1}^{N}\int_{(\Lambda_0\times V)^k}\frac{1}{N^j}\sum_{i_1\ldots i_j}\varphi(z_{i_1})\ldots\varphi(z_{i_j})\mu_{\Lambda_0}^{N,k}(Z_k)\mathrm{d}Z_k
$$

$$
=\sum_{k=j}^{N}p_{\Lambda_0}^{N}(k)\frac{k(k-1)\ldots(k-j+1)}{N^j}
$$

$$
\times\int_{(\Lambda_0\times V)^j}\varphi(z_1)\ldots\varphi(z_j)\Phi^{N,j}(Z_j\mid k)\mathrm{d}Z_j+O\left(\frac{1}{N}\right). \tag{6.26}
$$

We know that $p_{\Lambda_0}^{N}(k)$ is concentrated around αN as follows by the arguments after (6.21). In conclusion

$$
(\alpha-\beta)^j\sum_{k=j}^{N}p_{\Lambda_0}^{N}(k)\int_{(\Lambda_0\times V)^j}\varphi(z_1)\ldots\varphi(z_j)\Phi^{N,j}(Z_j\mid k)\mathrm{d}Z_j+\xi_5(N)
$$

$$
\leq\mathbf{E}_N\left(\left(\frac{1}{N}\sum_{i=1}^{N}\varphi(z_i)\right)^j\right)
$$

$$
\leq(\alpha+\beta)^j\sum_{k=j}^{N}p_{\Lambda_0}^{N}(k)\int_{(\Lambda_0\times V)^j}\varphi(z_1)\ldots\varphi(z_j)
$$

$$
\times\Phi^{N,j}(Z_j\mid k)\mathrm{d}Z_j+\xi_6(N). \tag{6.27}
$$

Choosing a suitable sequence $\beta=\beta_N\to 0$ as $N\to\infty$, making use of Lemma 6.1 and a standard approximation argument we conclude that:

$$
\langle f_j^{N},\varphi_j\rangle=\alpha^j\sum_{k=1}^{N}p_{\Lambda_0}^{N}(k)\int_{(\Lambda_0\times V)^j}\varphi_j(Z_j)\Phi^{N,j}(Z_j\mid k)\mathrm{d}Z_j+\xi_7(N) \tag{6.28}
$$

for all φ_j continuous and supported in $(\Lambda_0\times V)^j$.

Inserting Eq. (6.28) in (6.25) we obtain:

$$
S^{\pm}(N)=\sum_{h=1}^{[(\alpha\pm\eta)N]}\sum_{\substack{I\subset I_{[(\alpha\pm\eta)N]\setminus I_j}\\|I|=h}}\mathbf{E}\left(\chi(\mathrm{cl}(I_j)=I)\alpha^{-(j+h)}\right.
$$

$$
\left.\times\int\mathrm{d}Z_{I\cup I_j}f_{h+j}^{N}(Z_{I\cup I_j})\varphi((T_{\omega}^{t}(Z_{I\cup I_j})_j)\right)+\xi_8(N). \tag{6.29}
$$

Finally, by (6.2) and the dominated convergence theorem, which we can apply thanks to the convergence of the cluster expansion, we obtain:

$$
S^{\pm}(N) = \sum_{h=1}^{[(\alpha \pm \eta)N]} \sum_{\substack{I \subset I_{[(\alpha \pm \eta)N] \setminus I_j} \\ |I|=h}} \left\{ \mathbf{E}(\chi(\mathrm{cl}(I_j) = I)\alpha^{-(j+h)} \right.
$$

$$
\left. \times \int \mathrm{d}z_{I \cup I_j} \, \varphi_j((T_\omega^{-t}(Z_{I \cup I_j}))_j) f^{\otimes(j+h)}(Z_{I \cup I_j}) \right\} + \xi_9^{\pm}(N)
$$

$$
= \langle l_j^{N,\pm}, \varphi_j \rangle + \xi_9^{\pm}(N).
$$

This concludes the proof. □

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Communicated by J. L. Lebowitz