

## Central Extensions of Current Groups in Two Dimensions

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**Abstract:** The paper is devoted to generalization of the theory of loop groups to the two-dimensional case. To every complex Riemann surface we assign a central extension of the group of smooth maps from this surface to a simple complex Lie group  $G$  by the Jacobian of this surface. This extension is topologically nontrivial, as in the loop group case. Orbits of coadjoint representation of this extension correspond to equivalence classes of holomorphic principal  $G$ -bundles over the surface. When the surface is the torus (elliptic curve), classification of coadjoint orbits is related to linear difference equations in one variable, and to classification of conjugacy classes in the loop group. We study integral orbits, for which the Kirillov–Kostant form is a curvature form for some principal torus bundle. The number of such orbits for a given level is finite, as in the loop group case; conjecturally, they correspond to analogues of integrable modules occurring in conformal field theory.

### Introduction

The theory of loop groups and their representations [13] has recently developed in an extensive field with deep connections to many areas of mathematics and theoretical physics. On the other hand, the theory of current groups in higher dimensions contains rather isolated results which have not revealed so far any deep structure comparable to the one-dimensional case. In the present paper we investigate the geometry of current groups in two dimensions and point out several remarkable similarities with loop groups. We believe that these observations give a few more hints about the existence of a new vast structure in dimension two.

One of important problems in the theory of loop groups is integration of central extensions of loop algebras. It is known [13] that the non-trivial one dimensional central extension of the loop algebra  $\mathfrak{g}^{S^1}$  of a compact Lie algebra  $\mathfrak{g}$  integrates to a Lie group, which is a one dimensional central extension of the loop group  $G^{S^1}$  for the corresponding compact group  $G$ . Topologically this group turns out to be a nontrivial circle bundle over  $G^{S^1}$ , which plays a crucial role in the geometric realization of representations of affine Lie algebras. The study of the coadjoint

action of this group yields a very simple description of orbits by means of the theory of ordinary differential equations [5,16], and it turns out that there is a perfect correspondence between orbits and representations [5].

The Lie algebra of vector fields on the circle  $\text{Vect}(S^1)$  arises as the Lie algebra of outer derivations of a loop algebra and has many similarities with loop algebras. It has a unique nontrivial one-dimensional central extension - the Virasoro algebra. The structure of coadjoint orbits of the Virasoro algebra can be obtained from the study of Hill's operator [7,16]. The invariants of the coadjoint action of the complex Virasoro algebra are essentially the same as those of the extended algebra of loops in  $\mathfrak{sl}_2$ .

In this paper we generalize some of these results for loop algebras and groups as well as for the Virasoro algebra to the two-dimensional case.

We define and study a class of infinite dimensional complex Lie groups which are central extensions of the group of smooth maps from a two dimensional orientable surface without boundary to a simple complex Lie group  $G$ . These extensions naturally correspond to complex curves. The kernel of such an extension is the Jacobian of the curve. The study of the coadjoint action shows that its orbits are labelled by moduli of holomorphic principal  $G$ -bundles over the curve and can be described in the language of partial differential equations.

In genus one it is also possible to describe the orbits as conjugacy classes of the twisted loop group, which leads to consideration of difference equations for holomorphic functions. This gives rise to a hope that the described groups should possess a counterpart of the rich representation theory that has been developed for loop groups.

We also define a two-dimensional analogue of the Virasoro algebra associated with a complex curve. In genus one, a study of a complex analogue of Hill's operator yields a description of invariants of the coadjoint action of this Lie algebra. The answer turns out to be the same as in dimension one: the invariants coincide with those for the extended algebra of currents in  $\mathfrak{sl}_2$ .

Note that our main constructions are purely two-dimensional. In particular, in three and more dimensions the space of orbits tends to be infinite-dimensional and topologically unsatisfactory.

### 1. Lie Algebra Extensions

Let  $G$  be a simply connected simple complex Lie group, and let  $\mathfrak{g}$  be its Lie algebra. Denote by  $\langle , \rangle$  a nonzero invariant bilinear form on  $\mathfrak{g}$ . Let  $\Sigma$  be a nonsingular two-dimensional surface of genus  $g$ . Define  $G^\Sigma$  as the group of all smooth maps from  $\Sigma$  to  $G$ . Its Lie algebra consists of all smooth maps from  $\Sigma$  to  $\mathfrak{g}$  and will be denoted by  $\mathfrak{g}^\Sigma$ . These are called *the current group* and *the current algebra*.

Now fix a complex structure on the surface  $\Sigma$ . Let  $H_\Sigma$  be the space of holomorphic differentials on  $\Sigma$ . Its dimension is equal to  $g$ . Let  $\omega$  be the identity element in  $H_\Sigma \otimes H_\Sigma^*$ . The element  $\omega$  can be regarded as a holomorphic differential on  $\Sigma$  with values in  $H_\Sigma^*$ . Define a 2-cocycle on  $\mathfrak{g}^\Sigma$  with values in  $H_\Sigma^*$  (regarded as a trivial  $\mathfrak{g}^\Sigma$ -module) by

$$\Omega(X, Y) = \int_\Sigma \omega \wedge \langle X, dY \rangle, \quad X, Y \in \mathfrak{g}^\Sigma. \tag{1.1}$$

This cocycle defines a  $g$ -dimensional central extension of  $\mathfrak{g}^\Sigma$ . This extension is non-trivial for surfaces of positive genus, and it has no nonzero trivial subextensions or quotient extensions. Denote this new Lie algebra by  $\hat{\mathfrak{g}}^\Sigma$ .

If  $\Sigma$  is a complex torus, we can single out a subalgebra  $\hat{\mathfrak{g}}_{\text{pol}}^\Sigma$  in  $\hat{\mathfrak{g}}^\Sigma$  that consists of all currents realized by trigonometric polynomials. The restriction of the above central extension to this subalgebra in its natural basis looks very simple. Namely, let  $L$  be a lattice in the complex plane  $\mathbb{C}$  such that  $\Sigma = \mathbb{C}/L$ . The Lie algebra  $\hat{\mathfrak{g}}_{\text{pol}}^\Sigma$  is spanned by elements  $x(n), x \in \mathfrak{g}, n \in L$ , where  $x(n)$  depends linearly on  $x$  for any fixed  $n$ . These elements satisfy the following commutation relations:

$$[x(n), y(m)] = [x, y](n + m) + n\delta_{n,-m}\langle x, y \rangle k, \tag{1.2}$$

where  $k$  is the central element. This extension is a natural two-dimensional counterpart of affine algebras. Similar extensions were considered in [10].

Each of the extensions defined above can be obtained as a suitable quotient of the universal central extension  $U\mathfrak{g}^\Sigma$  of the Lie algebra  $\mathfrak{g}^\Sigma$ .

**Proposition 1.1.** ([13, Section 4.2]) *The universal central extension  $U\mathfrak{g}^\Sigma$  is an extension of  $\mathfrak{g}^\Sigma$  by means of the infinite-dimensional space  $\mathfrak{a} = \Omega^1(\Sigma)/d\Omega^0(\Sigma)$  of complex-valued 1-forms on  $\Sigma$  modulo exact forms. This extension is defined by the  $\mathfrak{a}$ -valued cocycle*

$$u(\xi, \eta) = \langle \xi, d\eta \rangle \text{ mod } d\Omega^0(\Sigma), \tag{1.3}$$

where  $\xi, \eta \in \mathfrak{g}^\Sigma$ .

Note that this construction does not involve the complex structure on the surface.

In order to obtain  $\hat{\mathfrak{g}}^\Sigma$  for a specific complex structure on the surface  $\Sigma$  as a quotient of  $U\mathfrak{g}^\Sigma$ , one needs to factorize it by the subgroup of all  $\xi \in \mathfrak{a}$  such that

$$\int_{\Sigma} \tilde{\xi} \wedge \omega = 0 \tag{1.4}$$

for any lifting  $\tilde{\xi} \in \Omega^1(\Sigma)$  of the element  $\xi$ .

**Proposition 1.2.** (i) *Let  $\Sigma_1$  and  $\Sigma_2$  be two Riemann surfaces. The Lie algebras  $\hat{\mathfrak{g}}^{\Sigma_1}$  and  $\hat{\mathfrak{g}}^{\Sigma_2}$  are isomorphic if and only if  $\Sigma_1$  and  $\Sigma_2$  are conformally equivalent.*

(ii) *If  $\Sigma$  has positive genus then any automorphism  $f$  of  $\hat{\mathfrak{g}}^\Sigma$  can be uniquely represented as a composition:  $f = h \circ \phi_*$ , where  $h$  is a conjugation by an element of  $\text{Aut}(\mathfrak{g})^\Sigma$  and  $\phi_*$  is the direct image map induced by a conformal diffeomorphism  $\phi : \Sigma \rightarrow \Sigma$ .*

*Proof.* Let  $f : \hat{\mathfrak{g}}^{\Sigma_1} \rightarrow \hat{\mathfrak{g}}^{\Sigma_2}$  be any isomorphism. Then the dimensions of the centers of  $\hat{\mathfrak{g}}^{\Sigma_1}$  and  $\hat{\mathfrak{g}}^{\Sigma_2}$  must coincide, so the genera of  $\Sigma_1$  and  $\Sigma_2$  are the same. Consider the induced map  $f_0 : \mathfrak{g}^{\Sigma_1} \rightarrow \mathfrak{g}^{\Sigma_2}$  of current algebras. Since  $\Sigma_1$  and  $\Sigma_2$  are diffeomorphic, we may actually regard this map as an automorphism of the current algebra. According to [13, Section 3.4], any such map uniquely decomposes as  $f_0 = h \circ \phi_*$ , where  $h$  is a conjugation by an element of  $\text{Aut}(\mathfrak{g})^{\Sigma_1}$  and  $\phi_*$  is the direct image map induced by a conformal diffeomorphism  $\phi : \Sigma_1 \rightarrow \Sigma_2$ . In order for  $f_0$  to extend to an isomorphism of extensions,  $\phi$  must take holomorphic differentials to holomorphic differentials which forces it to be a conformal equivalence. This proves both (i) and (ii). ■

One can also classify embeddings  $\hat{g}^{\Sigma_1} \hookrightarrow \hat{g}^{\Sigma_2}$ . Indeed, any embedding  $f : g^{\Sigma_1} \hookrightarrow g^{\Sigma_2}$  is induced by a surjective map  $\varphi : \Sigma_2 \rightarrow \Sigma_1$ , and conversely, any such map defines an embedding. In order for this embedding to continue to central extensions, the map between surfaces must be holomorphic. However, in this case the continuation is not unique. Continuations correspond to monomorphisms  $\chi : H_{\Sigma_1}^* \hookrightarrow H_{\Sigma_2}^*$  with the property  $\varphi_*(\chi(v)) = v$  for all  $v \in H_{\Sigma_1}^*$ . Here  $\varphi_*$  is the map  $H_{\Sigma_2}^* \rightarrow H_{\Sigma_1}^*$  induced by  $\varphi$ .

We have shown that the group of outer automorphisms of the Lie algebra  $\hat{g}^\Sigma$  is isomorphic to the group of holomorphic automorphisms of the Riemann surface  $\Sigma$ . If  $g > 1$ , this group is finite, and it is trivial for almost every surface. In spite of it,  $\hat{g}^\Sigma$  has plenty of outer derivations.

**Proposition 1.3.** *If  $g > 1$ , the Lie algebra of outer derivations of  $\hat{g}^\Sigma$  coincides with the Lie algebra  $\text{Vect}_{0,1}(\Sigma)$  of all complex-valued vector fields on  $\Sigma$  of type  $(0, 1)$ , i.e. of the form  $u(z, \bar{z}) \frac{\partial}{\partial \bar{z}}$  for any local complex coordinate  $z, u$  being a smooth function.*

*If  $g = 1$ , the Lie algebra of outer derivations is  $\left\langle \frac{\partial}{\partial z} \right\rangle \ltimes \text{Vect}_{0,1}(\Sigma)$ .*

*Proof.* It is known that outer derivations of  $g^\Sigma$  are in one-to-one correspondence with complex vector fields on the surface  $\Sigma$ . In order for such a derivation to continue to the central extension  $\hat{g}^\Sigma$ , the vector field must annihilate holomorphic differentials. If  $g > 1$ , such a field must have type  $(0,1)$ . If  $g = 1$ , any such field is a linear combination of a  $(0,1)$ -field and the constant field  $\frac{\partial}{\partial z}$ . ■

The Lie algebra  $\text{Vect}_{0,1}(\Sigma)$  should be thought of as an analogue of the Witt algebra. It has a  $g$ -dimensional central extension which is a natural analogue of the Virasoro algebra. This is the extension by the space  $H_\Sigma$  defined by the cocycle

$$F(X, Y) = \int_\Sigma \omega \wedge \bar{\partial}X \bar{\partial}^2 Y, \quad X, Y \in \text{Vect}_{0,1}(\Sigma). \tag{1.5}$$

In this expression,  $\bar{\partial}X$  is a function and  $\bar{\partial}^2 Y$  is a differential 1-form, since the operator  $\bar{\partial}$  maps vector fields to functions and functions to 1-forms. We will denote this extension by  $\text{Vir}(\Sigma)$ . For  $\Sigma$  being the torus, the polynomial part  $\text{Vect}_{0,1}(\Sigma)_{\text{pol}}$  of  $\text{Vect}_{0,1}(\Sigma)$  is a graded Lie algebra of a very simple structure. Namely, let  $L$  be the lattice in  $\mathbb{C}$  such that  $\Sigma = \mathbb{C}/L$ . The basis of  $\text{Vect}_{0,1}(\Sigma)_{\text{pol}}$  consists of elements  $e_n, n \in L$ , which satisfy Witt's relations

$$[e_n, e_m] = (m - n)e_{n+m}. \tag{1.6}$$

The one dimensional extension we have described has an additional basis vector  $c$ -the central element, and the relations are analogous to Virasoro relations:

$$[e_n, e_m] = (m - n)e_{n+m} + m^3 \delta_{m,-n} c. \tag{1.7}$$

One can easily show that this is the universal central extension of  $\text{Vect}_{0,1}(\Sigma)_{\text{pol}}$  (i.e. it is the only possible nontrivial one-dimensional extension up to an isomorphism). The proof is similar to that for the Witt algebra and is given in [15].

## 2. Group Extensions

The following theorem describes the topology of the current group and can be deduced from the results of [13, Chapter 4].

### Theorem 2.1.

- (i)  $G^\Sigma$  is connected.
- (ii)  $\pi_1(G^\Sigma) = \mathbb{Z}$ .
- (iii)  $\pi_2(G^\Sigma) \otimes \mathbb{R} = H_2(G^\Sigma, \mathbb{R}) = \mathbb{R}^{2g}$ .

It turns out that the Lie algebra  $\hat{\mathfrak{g}}^\Sigma$  can be integrated to a complex Lie group.

**Theorem 2.2.** *There exists a central extension  $\hat{G}^\Sigma$  of the current group  $G^\Sigma$  by means of the Jacobian variety of  $\Sigma$  whose Lie algebra is  $\hat{\mathfrak{g}}^\Sigma$ .*

*Proof.* This group can be constructed by the procedure described in [13, Chapter 4]. Namely, consider the left invariant holomorphic 2-form on  $G^\Sigma$  equal to  $\Omega$  on the tangent space at the identity. It shall be denoted by the same letter. Integrals of  $\Omega$  over integer 2-cycles in  $G^\Sigma$  fill the lattice  $L = H_1(\Sigma, \mathbb{Z})$  in  $H_2^*$ . Therefore, there exists a holomorphic principal bundle over  $G^\Sigma$  with fiber  $J = H_2^*/L$  and with a holomorphic connection  $\theta$  whose curvature form is  $2\pi\Omega$ . Note that  $J$  is the Jacobian of the surface  $\Sigma$ . Define the group  $\hat{G}^\Sigma$  as the group of all transformations of the constructed bundle that preserve the connection  $\theta$  and project to left translations on the group  $G^\Sigma$ . It is a central extension of  $G^\Sigma$  by  $J$ . One has an exact sequence.

$$1 \rightarrow J \rightarrow \hat{G}^\Sigma \rightarrow G^\Sigma \rightarrow 1, \tag{2.1}$$

and it follows that the Lie algebra of  $\hat{G}^\Sigma$  is isomorphic to  $\mathfrak{g}^\Sigma$ . ■

Obviously, this theorem will remain valid if we replace the current group  $G^\Sigma$  with its universal covering  $\tilde{G}^\Sigma$ . Denote the extension obtained in this way by  $\tilde{\hat{G}}^\Sigma$ .

The following theorem characterizes the universal central extension of the current group.

**Theorem 2.3.** [13, Section 4.10] (i) *The universal central extension  $UG^\Sigma$  of the group  $G^\Sigma$  is an extension of  $\tilde{G}^\Sigma$  by means of the infinite-dimensional abelian group  $\mathcal{A}$  of complex-valued 1-forms on  $\Sigma$  modulo closed 1-forms with integer periods.*

(ii)  $\pi_n(UG^\Sigma) \otimes \mathbb{R} = 0$  for  $n < 3$ .

(iii) *The group  $\mathcal{A}$  is homotopy equivalent to a  $2g$ -dimensional torus. The natural map  $\pi_2(G^\Sigma) \rightarrow \pi_1(\mathcal{A})$  associated to the fiber bundle  $UG^\Sigma \rightarrow G^\Sigma$  is an isomorphism up to torsion.*

**Proposition 2.4.** *The universal central extension  $UG^\Sigma$  is homotopy equivalent to  $\tilde{\hat{G}}^\Sigma$ .*

*Proof.* Consider a homomorphism  $\varepsilon : \mathcal{A} \rightarrow J$  defined by

$$\varepsilon(a) = \int_\Sigma \tilde{a} \wedge \omega \text{ mod } L, \tag{2.2}$$

where  $\tilde{a}$  is any lifting of  $a \in \mathcal{A}$  into the space of 1-forms on  $\Sigma$ . It is easy to check that this map is well defined, i.e. independent of the choice of  $\tilde{a}$ . Obviously,  $\varepsilon$

is a homotopy equivalence. This implies that the corresponding map of extensions  $\tilde{\epsilon} : UG^\Sigma \rightarrow \tilde{G}^\Sigma$  of the group  $\tilde{G}^\Sigma$  is also a homotopy equivalence.

This theorem implies that  $\tilde{G}^\Sigma$  is a nontrivial  $J$ -bundle on  $\tilde{G}^\Sigma$  and represents the homotopically nontrivial part of the universal central extension.

In fact, the universal extension can be constructed rather explicitly as a central extension of  $\tilde{G}^\Sigma$  by the additive group of a vector space, as follows.

Let  $K$  be the maximal compact subgroup of  $G$ , and let  $\tilde{K}^\Sigma$  be the subgroup of all elements in  $\tilde{G}^\Sigma$  whose projections to  $G^\Sigma$  are  $K$ -valued currents. Obviously,  $\tilde{K}^\Sigma$  is a central extension of  $K^\Sigma$  by  $T^{2g} \times \mathbb{Z}$ , where  $T^{2g}$  is the real  $2g$ -dimensional torus.

Choose a metric on  $\Sigma$  compatible to the complex structure. A metric induces an inner product on the space of differential forms. Define  $d^*$  to be the conjugate operator to the de Rham differential  $d$ . This operator maps 2-forms to 1-forms. Let  $\mathfrak{a}$  be the space of 1-forms on  $\Sigma$  modulo exact 1-forms, and let  $W$  be the image of the projection  $p : \text{Im}d^* \rightarrow \mathfrak{a}$ .

**Lemma 2.5.** *The subspace  $W$  has codimension  $2g$  in  $\mathfrak{a}$  and is complementary to the subspace  $H^1(\Sigma, \mathbb{R}) \subset \mathfrak{a}$ .*

*Proof.* Let  $\Delta : \Omega^1(\Sigma) \rightarrow \Omega^1(\Sigma)$  be the Laplacian associated with the metric:  $\Delta = dd^* + d^*d$ . Since  $\Delta$  is self-adjoint, any form  $\alpha \in \Omega^1(\Sigma)$  can be represented in the form  $\alpha = \alpha_1 + \alpha_2$ , where  $\alpha_1 \in \text{Ker}\Delta, \alpha_2 \in \text{Im}\Delta$ . Then  $d\alpha_1 = 0$  by Hodge’s theorem, and  $\alpha_2 = \Delta\beta = d^*d\beta + dd^*\beta = d^*\beta_1 + d\beta_2$  by construction. Thus,  $p(\alpha) = p(\alpha_1) + p(d^*\beta_1)$ . Since  $p(\alpha_1) \in H^1(\Sigma, \mathbb{R})$  and  $p(d^*\beta_1) \in W$ , we have shown that any vector in  $\mathfrak{a}$  can be written as a sum of a vector in  $W$  and a vector in  $H^1$ .

Now let us show that  $W \cap H^1(\Sigma, \mathbb{R}) = 0$ . Let  $w$  be an element of this intersection. Let  $\tilde{w}$  be an inverse image of  $w$  in  $\text{Im}d^*$ . Then  $d\tilde{w} = 0$  and  $d^*\tilde{w} = 0$ , so  $w$  is harmonic. By Hodge’s theorem, harmonic forms are in one-to-one correspondence with cohomology classes, so any harmonic form in  $\text{Im}d^*$  has to be zero. Thus,  $w = 0$ . ■

Define a 2-cocycle on  $\tilde{K}^\Sigma$  with values in  $W$  as follows:

$$C(\tilde{a}, \tilde{b}) = p(d^*\langle b^{-1}db, da \cdot a^{-1} \rangle), \quad \tilde{a}, \tilde{b} \in \tilde{K}^\Sigma, \tag{2.3}$$

where  $a, b \in K^\Sigma$  are images of  $\tilde{a}, \tilde{b}$ .

This cocycle defines a central extension of  $\tilde{K}^\Sigma$  by  $W$ . Let us show that this new group is isomorphic to the universal central extension of  $K^\Sigma$ .

Since both groups are simply connected, it is enough to establish an isomorphism between their Lie algebras. Both Lie algebras are central extensions of  $\mathfrak{k}^\Sigma$ , where  $\mathfrak{k}$  is the Lie algebra of  $K$ . The kernels of these extensions are  $\tilde{\mathfrak{a}} = W \oplus H_{\Sigma\mathbb{R}}$  and  $\mathfrak{a}$ , respectively, and the corresponding cocycles are  $\tilde{u} = C \oplus \Omega_{\mathbb{R}}$  and  $u$  (if  $V$  is a complex vector space then  $V_{\mathbb{R}}$  denotes the space  $V$  regarded as a real vector space). All we need is to construct an isomorphism  $\rho : \mathfrak{a} \rightarrow \tilde{\mathfrak{a}}$  such that  $\rho(u) = \tilde{u}$ . It is easy to check that such an isomorphism is provided by Lemma 2.5.

The universal extension of  $\tilde{G}^\Sigma$  can now be obtained from the constructed extension of  $K^\Sigma$  by complexification.

*Remark 2.1.* (0,1)-vector fields on the surface are realized as  $J$ -invariant holomorphic vector fields on  $\hat{G}^\Sigma$  (= first order differential operators on holomorphic functions). Any such field  $v$  satisfies the condition  $v(gh) = v(g)h + gv(h)$  for all  $g, h \in \hat{G}^\Sigma$ .

*Remark 2.2.* A very challenging problem is to obtain a more explicit construction of  $\tilde{G}^\Sigma$  than we have given, by presenting a 2-dimensional counterpart of the famous Wess–Zumino–Witten construction for loop groups [19,9].

It would also be interesting to obtain a direct construction of the universal central extension of  $G^\Sigma$  from  $\tilde{G}^\Sigma$  by an analogue of formula (2.3) without referring to the maximal compact subgroup.

### 3. Orbits of the Coadjoint Action

Let  $\eta$  be a holomorphic differential on  $\Sigma$ . Denote by  $E_\eta$  the one-dimensional central extension of  $\mathfrak{g}^\Sigma$  by means of the cocycle

$$\Omega_\eta(X, Y) = \int_\Sigma \eta \wedge \langle X, dY \rangle . \tag{3.1}$$

Obviously,  $E_\eta$  is the quotient of  $\hat{\mathfrak{g}}^\Sigma$  by the kernel of  $\eta$  as a linear function on  $H_\Sigma^*$ .

Denote by  $E^*$  the space of all operators  $D = \lambda\bar{\partial} + \xi$ , where  $\lambda \in \mathbb{C}$  and  $\xi$  is a  $\mathfrak{g}$ -valued (0,1) form on  $\Sigma$ , i.e. a 1-form which can be written as  $u(z, \bar{z})d\bar{z}$  for any local complex coordinate  $z, u$  being a  $\mathfrak{g}$ -valued smooth function. For any representation  $V$  of  $\mathfrak{g}$ , these operators take  $V$ -valued functions on  $\Sigma$  to  $V$ -valued (0,1)-forms according to

$$D\psi = \lambda\bar{\partial}\psi + \xi\psi ,$$

which allows to define an action of  $G^\Sigma$  on  $E^*$ :

$$h \circ (\lambda\bar{\partial} + \xi) = \lambda\bar{\partial} + h^{-1}\bar{\partial}h + \text{Ad } h \circ \xi, h \in G^\Sigma . \tag{3.2}$$

Consider a pairing between  $E^*$  and  $E_\eta$  given by

$$(\lambda\bar{\partial} + \xi, \mu k + X) = \lambda\mu + \int_\Sigma \eta \wedge \langle \xi, X \rangle , \tag{3.3}$$

where  $k$  is a fixed central element in  $E_\eta$  and  $\mu \in \mathbb{C}$ .

**Proposition 3.1.** *Pairing (3.3) is  $G^\Sigma$ -invariant and nondegenerate.*

The proof is straightforward.

The constructed pairing allows us to interpret  $E^*$  as the smooth part of the dual  $E'_\eta$  to the Lie algebra  $E_\eta$ , i.e. as the proper coadjoint representation.

Our goal now is to study the orbits of the action of  $G^\Sigma$  in  $E^*$ . Hyperplanes  $\lambda = \text{const}$  are invariant under this action. Let us fix a nonzero value of  $\lambda$  and examine the orbits contained in the corresponding hyperplane  $\mathcal{H}_\lambda$ .

We will use the following classical construction [4].

To an operator  $D = \lambda\bar{\partial} + \xi$  one can naturally associate a holomorphic principal  $G$ -bundle on  $\Sigma$  as follows. Let  $U_i, i \in I$ , be a cover of  $\Sigma$  by proper open subsets. For every  $i$  fix a solution  $\psi_i : U_i \rightarrow G$  of the partial differential equation

$$\lambda\bar{\partial}\psi_i \cdot \psi_i^{-1} + \xi = 0 \tag{3.4}$$

(the left-hand side of (3.4) being a  $\mathfrak{g}$ -valued  $(0,1)$  form on  $U_i$ ). Such solutions can always be found. Now for  $i, j \in I$  define transition functions  $\phi_{ij} : U_i \cap U_j \rightarrow G$  by  $\phi_{ij} = \psi_i^{-1}\psi_j$ . It is easy to see that these functions are holomorphic. Thus they define a holomorphic principal  $G$ -bundle on  $\Sigma$ , which we denote by  $B(D)$ . Moreover, it follows that the holomorphic bundles  $B(D_1)$  and  $B(D_2)$  associated to two operators  $D_1 = \lambda\bar{\partial} + \xi_1$  and  $D_2 = \lambda\bar{\partial} + \xi_2$  are equivalent if and only if there exists an element  $h \in G^\Sigma$  such that  $h \circ D_1 = D_2$ . This  $h$  will be exactly the gauge transformation establishing the equivalence between  $B(D_1)$  and  $B(D_2)$ .

Conversely, for any holomorphic principal  $G$ -bundle  $B$  on  $\Sigma$  there exists an operator  $D = \lambda\bar{\partial} + \xi$  such that  $B = B(D)$ . Indeed, any principal  $G$ -bundle on  $\Sigma$  is topologically trivial, since  $G$  is simply connected. If we choose a global trivialization then the local holomorphic trivializations over open sets  $U_i$  will be expressed by smooth functions  $\psi_i : U_i \rightarrow G$ . Since the transition functions  $\phi_{ij} = \psi_i^{-1}\psi_j$  are holomorphic on  $U_i \cap U_j$ , we have  $\bar{\partial}\psi_i \cdot \psi_i^{-1} = \bar{\partial}\psi_j \cdot \psi_j^{-1}$  on  $U_i \cap U_j$ . Therefore there exists a  $\mathfrak{g}$ -valued 1-form  $\xi$  on  $\Sigma$  such that  $\xi = -\lambda\bar{\partial}\psi_i \cdot \psi_i^{-1}$  on  $U_i$  for all  $i \in I$ . Let  $D = \lambda\bar{\partial} + \xi$ . Then  $B = B(D)$ .

This reasoning proves the following

**Proposition 3.2.** *Orbits of the action of  $G^\Sigma$  in  $\mathcal{H}_\lambda$  are in one-to-one correspondence with equivalence classes of holomorphic principal  $G$ -bundles on  $\Sigma$ . The correspondence is  $D \leftrightarrow B(D)$ .*

*Remark 3.1.* The relation between differential operators and holomorphic principal bundles on  $\Sigma$  was used in computations of partition functions of the gauged Wess–Zumino–Witten model in [6] and [20].

*Remark 3.2.* Holomorphic sections of the bundle  $B(D)$ , i.e. solutions of (3.4), are known as a special case of generalized analytic functions which were introduced in the fifties by L. Bers and I. Vekua [2,18]. If  $\xi = 0$ , they become usual analytic functions.

Holomorphic principal  $G$ -bundles for  $G = SL_n(\mathbb{C})$  were classified by Atiyah [1] for  $g = 1$  and by Narasimhan and Sheshadri [11,12], for  $g > 1$ . Their results were generalized to the case of any simple group  $G$  by Ramanathan [14]. We summarize here some of them.

Let  $\Pi$  be the fundamental group of  $\Sigma$  and  $K$  be a maximal compact subgroup of  $G$ . Let  $\rho : \Pi \rightarrow K$  be any homomorphism. This homomorphism defines a flat  $G$ -bundle  $B_\rho$  over  $\Sigma$  with a canonical holomorphic structure. Bundles coming from this construction are called unitary. It is known [14] that the conjugacy class of the representation  $\rho$  is completely determined by the equivalence class of  $B_\rho$  as a holomorphic principal bundle.

The following theorem shows that almost all holomorphic principal bundles are flat and unitary.

**Theorem 3.3.** [14] *Let  $\{B_t\}_{t \in \mathcal{T}}$  be a holomorphic family of holomorphic principal  $G$ -bundles parametrized by a complex space  $\mathcal{T}$ . Then the subset  $\mathcal{T}_0$  of such  $t \in \mathcal{T}$  that the bundle  $B_t$  is flat and unitary is Zariski open in  $\mathcal{T}$ .*

Let us now define an appropriate notion of “almost everywhere.”

Let  $V$  be any complex topological vector space. We say that a set  $Z \subset V$  is Zariski open if for any finite dimensional complex manifold  $\mathcal{T}$  and any holomorphic map  $f : \mathcal{T} \rightarrow V$  the set of points  $t \in \mathcal{T}$  such that  $f(t) \in Z$  is Zariski open in  $\mathcal{T}$ . Obviously, this defines a topology in  $V$ . We will say that some property holds almost everywhere in  $V$  if it holds on a nonempty Zariski open subset of  $V$ . Note that every nonempty Zariski open subset in  $V$  is dense, open, and connected in the usual topology.

It is clear that a flat bundle  $B$  can be obtained from an operator  $D = \lambda\bar{\partial} + \xi$  with  $\xi$  being a  $\mathfrak{g}$ -valued antiholomorphic differential on  $\Sigma : B = B(D)$ . Therefore, the above theorem in fact states that almost every (in Zariski sense) differential operator of the form  $\lambda\bar{\partial} + \xi$  can be reduced to an operator with an antiholomorphic  $\xi$  by means of a gauge transformation from  $G^\Sigma$ . This implies that the union  $S$  of  $G^\Sigma$ -orbits of all operators  $\lambda\bar{\partial} + \xi$  with  $\xi$  being antiholomorphic is a Zariski open subset of  $\mathcal{H}_\lambda$ .

*Remark 3.3.* It is easy to construct an example of a holomorphic bundle which is not flat and unitary. For instance, take any holomorphic line bundle  $\beta$  on  $\Sigma$  of degree 1 and form a rank 2 bundle  $\beta \oplus \beta^*$ , where  $\beta^*$  is the dual bundle to  $\beta$ . This bundle has degree 0. Let  $B = \text{Aut}_0(\beta \oplus \beta^*)$  be the  $SL_2(\mathbb{C})$ -bundle of automorphisms of  $\beta \oplus \beta^*$  having determinant one. It is easy to see that this bundle is not flat: the associated bundle  $\beta \oplus \beta^*$  has a nonzero holomorphic section  $s$  which vanishes at a point  $z \in \Sigma$ ; this section cannot satisfy any equation  $\lambda\bar{\partial}s + \xi s = 0$  with  $\xi$  being antiholomorphic. Another example would be  $\text{Aut}_0(\mathcal{A})$ , where  $\mathcal{A}$  is the Atiyah’s bundle of rank 2 over a complex torus which is a semidirect sum of two trivial line bundles. This bundle is flat, but not unitary. However, according to the above theorem, these bundles are exceptional and can be made flat and unitary by an arbitrarily small perturbation of transition functions.

The space of equivalence classes of flat and unitary  $G$ -bundles is a complex manifold with singularities. Denote this manifold by  $\mathcal{M}$ . This manifold can be thought of as the space of coadjoint orbits of generic position.

We can now classify all holomorphic invariants of the action of  $\mathfrak{g}^\Sigma$  in  $\mathcal{H}_\lambda$ .

Let  $\mathfrak{b}$  be a complex Lie algebra and  $V$  be a topological representation of  $\mathfrak{b}$ . Define a holomorphic invariant of the action of  $\mathfrak{b}$  in  $V$  as a holomorphic map from a Zariski open subset in  $V$  to a fixed finite-dimensional complex manifold which is invariant under the action of  $\mathfrak{b}$ . The coefficient  $\lambda$  is an example of a holomorphic invariant of the coadjoint action of  $\mathfrak{g}^\Sigma$ . Another example would be the  $\mathcal{M}$ -valued function  $b$  defined on a Zariski open subset in  $E^*$  which takes an operator  $D$  to the equivalence class of  $B(D)$ . The orbit classification result implies

**Proposition 3.4.** *Any holomorphic invariant of the action of  $\mathfrak{g}^\Sigma$  in  $E^*$  is a function of  $b$  and  $\lambda$ .*

### 4. Orbital Structure in Genus 1

In this section we assume the surface  $\Sigma$  to be a complex torus  $\mathbb{C}/L, L = \{ p + q\tau \mid p, q \in \mathbb{Z} \}, \text{Im}\tau > 0$ . In this case it is possible to give a more explicit description of the space of orbits.

The coadjoint representation  $E^*$  of the group  $\hat{G}^\Sigma$  can be identified with the space of differential operators  $\lambda \frac{\partial}{\partial \bar{z}} + \xi$ , where  $\xi$  is a  $\mathfrak{g}$ -valued function on the torus. These operators act on  $V$ -valued functions on the torus for any representation  $V$  of  $G$ . According to the previous section, orbits of the coadjoint action of  $G^\Sigma$  restricted to a hyperplane  $\mathcal{H}_\lambda$  correspond to equivalence classes of holomorphic principal  $G$ -bundles over the complex torus  $\Sigma$ . Almost every such bundle is flat and unitary, i.e. comes from a homomorphism from the fundamental group to the maximal compact subgroup  $K$  in  $G$ . The fundamental group of the torus is  $\Pi = \mathbb{Z}^2$ , so we may assume that this homomorphism lands in a maximal torus  $T \subset K$ . Since all maximal tori are conjugate, we can assume  $T$  to be a fixed maximal torus in  $K$ . The images of the two generators of  $\Pi$  can be any two elements in  $T$ . Thus, we get a covering of the set of non-equivalent unitary representations of  $\Pi$  by the product  $T \times T$ . Two elements of this product correspond to isomorphic representations of  $\Pi$  if and only if one of them can be obtained from the other by the action of an element of the Weyl group  $W$  of  $K$ . Therefore, the set of equivalence classes of unitary representations of  $\Pi$  is  $(T \times T)/W$ .

Thus, we have found that topologically the space  $\mathcal{M}$  of orbits of generic position is  $(T \times T)/W$ . However, this realization does not tell us anything about the complex structure on  $\mathcal{M}$ . Therefore let us describe a different realization of  $\mathcal{M}$ .

Let  $D = \lambda \frac{\partial}{\partial \bar{z}} + \xi$ . If the bundle  $B(D)$  is flat (which happens, as we know, for almost every  $\xi$ ) then the equation  $D\psi = 0$  on a  $G$ -valued function  $\psi$  will have a solution  $\psi(z)$  with the properties  $\psi(z + 1) = \psi(z), \psi(z + \tau) = \psi(z)A$ , where  $A$  is a fixed element of  $G$ . If the bundle  $B(D)$  is also unitary, the element  $A$  will be semisimple. Then we may assume that it belongs to a fixed complex maximal torus  $T_{\mathbb{C}} \subset G$ .

The element  $A$  completely determines the bundle  $B(D)$ . Indeed, let  $a \in \mathfrak{g}$  satisfy  $A = \exp(a)$ , and set  $F(z) = \psi(z)\exp\left(-a\frac{z-\bar{z}}{\tau-\bar{\tau}}\right)$ . Obviously  $F$  is doubly periodic, i.e.  $F \in G^\Sigma$ , and we have

$$F \circ \left( \lambda \frac{\partial}{\partial \bar{z}} + \xi \right) = \lambda \frac{\partial}{\partial \bar{z}} - \frac{a}{\tau - \bar{\tau}}. \tag{4.1}$$

Moreover, different elements of  $T_{\mathbb{C}}$  may correspond to equivalent bundles. Let  $r$  be the rank of  $G$  and  $\mathfrak{t}$  be the Lie algebra of  $T$ . If  $h_j \in \mathfrak{t}, 1 \leq j \leq r$  denote the standard basis of the Cartan subalgebra of  $\mathfrak{g}$  then  $\exp(2\pi i h_j) = 1$ , and the solution  $\psi(z)$  of the equation  $D\psi = 0$  can be replaced by another solution  $\psi_n(z) = \psi(z)\exp(2\pi i z(\sum_{j=1}^r n_j h_j))$ , where  $n_j \in \mathbb{Z}$ . This solution satisfies  $\psi_n(z + 1) = \psi_n(z), \psi_n(z + \tau) = \psi_n(z)A_n$ , where  $A_n = A \exp(2\pi i \tau(\sum_{j=1}^r n_j h_j))$ . Obviously,  $A_n$  corresponds to the same bundle as  $A$ . Furthermore, if we conjugate  $A$  by any element of the Weyl group  $W$ , we will obtain an element  $A'$  of  $T_{\mathbb{C}}$  that corresponds to the same bundle as  $A$ .

Let  $Q^\vee$  be the lattice generated by  $h_j$  (the dual weight lattice of  $G$ ). We have shown that the bundle  $B(D)$  is completely determined by the projection of  $A$  into the

complex space  $T_{\mathbb{C}}/(W \ltimes \exp(2\pi i \tau Q^{\vee})) = t_{\mathbb{C}}/(W \ltimes (Q^{\vee} \oplus \tau Q^{\vee}))$ . It is easy to see that different points of this space correspond to non-isomorphic bundles. Therefore, we have

**Proposition 4.1.** *The space  $\mathcal{M}$  of equivalence classes of flat and unitary holomorphic  $G$ -bundles over the complex torus is isomorphic to  $t_{\mathbb{C}}/(W \ltimes (Q^{\vee} \oplus \tau Q^{\vee}))$ .*

An alternative way to obtain the classification of generic coadjoint orbits of  $\hat{G}^{\Sigma}$  is based on a study of conjugacy classes of the twisted loop group. Let us give a brief description of this method.

By a loop group  $G_h^{S^1}$  we mean the group of holomorphic maps from the cylinder  $\mathbb{C}/\mathbb{Z}$  to  $G$ . The abelian group  $\mathbb{C}/\mathbb{Z}$  acts by automorphisms on  $G_h^{S^1}$  through translations of the argument. From the semidirect product  $\check{G}_h^{S^1} = \mathbb{C}/\mathbb{Z} \ltimes G_h^{S^1}$  associated to this action. Let us analyze the conjugacy classes of  $\check{G}_h^{S^1}$ . For  $(z, f), (\tau, g) \in \check{G}_h^{S^1}$  we have

$$(z, f)^{-1}(\tau, g)(z, f) = (\tau, h), \quad h(u) = f(u - \tau)^{-1}g(u - z)f(u), \quad u \in \mathbb{C}/\mathbb{Z}. \quad (4.2)$$

This implies that  $\tau$  is preserved under conjugations, so we may assume that it is fixed, and study conjugacy classes inside the coset  $C_{\tau} = \{(\tau, g) | g \in G_h^{S^1}\}$ . We will treat the case  $\tau \notin \mathbb{R}/\mathbb{Z}$ . In this case we may assume, without loss of generality, that  $\text{Im}\tau > 0$ .

Let  $\xi$  be a smooth function on the torus  $\Sigma_{\tau}$ . Consider the differential equation

$$\lambda \frac{\partial \psi}{\partial \bar{z}} + \xi \psi = 0 \quad (4.3)$$

with respect to a  $G$ -valued function  $\psi$  on the cylinder  $\mathbb{C}/\mathbb{Z}$ .

Let  $\psi_0(z)$  be a solution of Eq. (4.3). Then  $\psi_0(z + \tau)$  is also a solution, since the equation is invariant under a translation by  $\tau$ . Therefore,  $v_0(z) = \psi_0(z)^{-1}\psi_0(z + \tau)$  is a holomorphic  $G$ -valued function on the cylinder. If we choose another solution of (4.3), say,  $\psi_1$ , it will have the form  $\psi_1(z) = \psi_0(z)\mu(z)$ , where  $\mu$  is holomorphic. Therefore, the function  $v_1(z) = \psi_1(z)^{-1}\psi_1(z + \tau)$  can be expressed as follows:  $v_1(z) = \mu(z)^{-1}v_0(z)\mu(z + \tau)$ . This implies that the conjugacy class of the element  $(\tau, v_0)$  is independent on the choice of the solution  $\psi_0$  of (4.3). Thus we have canonically associated a conjugacy class of  $\check{G}_h^{S^1}$  to any equation of form (4.3). It is clear that every conjugacy class in  $C_{\tau}$  comes from a certain equation. Thus we have established a one-to-one correspondence between orbits of the action of  $G^{\Sigma}$  in  $\mathcal{H}^{\lambda}$  and conjugacy classes of  $\check{G}_h^{S^1}$  in  $C_{\tau}$ .

Now the result that generic orbits of  $G^{\Sigma}$  in  $\mathcal{H}_{\lambda}$  are parametrized by points of the space  $\mathcal{M}$  can be deduced from the classical theory of difference equations. This theory was developed in the beginning of the twentieth century, by G.D. Birkhoff, R.D. Carmichael, C.R. Adams, W.J. Trjitzinsky and others [3,17].

Consider the equation

$$F(z + \tau) = A(z)F(z), \quad (4.4)$$

where  $A(z)$  is an entire periodic  $G$ -valued function with period 1.

**Proposition 4.2.** *For almost every  $A(z)$  (i.e. for  $A(z)$  belonging to a Zariski open subset in  $G_h^{S^1}$ ) there exists a solution of (4.4) of the form*

$$F(z) = \Phi(z)\exp(Pz), \tag{4.5}$$

where  $P \in \mathfrak{g}$  and  $\Phi$  is an entire  $G$ -valued function with period 1.

This proposition, for  $G = SL_n$ , can be deduced from the Fundamental Existence Theorem in the theory of linear difference equations [17].

Clearly, Proposition 4.2 implies the classification result for generic orbits.

Despite we did not use the notion of a vector bundle in this argument, it was implicitly present in our considerations. Indeed, E. Looijenga [8] observed that the conjugacy classes of  $G_h^{S^1}$  which lie inside  $C_\tau$  are in one-to-one correspondence with equivalence classes of holomorphic principal  $G$ -bundles over the complex torus  $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ . This correspondence is constructed as follows.

The complex torus  $\Sigma_\tau$  can be obtained from the annulus  $\{z \in \mathbb{C}/\mathbb{Z} \mid 0 \leq \text{Im}z \leq \text{Im}\tau\}$  by gluing together the boundary components according to the rule  $z \leftrightarrow z + \tau$ . In order to define a holomorphic bundle on the torus, it is enough to present a  $G$ -valued holomorphic transition function  $A(z)$  in a neighborhood of the seam  $\text{Im}z = 0$ . Therefore, we can naturally associate a holomorphic  $G$ -bundle to every element  $(\tau, g) \in C_\tau$  by setting  $A(z) = g(z)$ . Now observe that Eq (4.2) expresses exactly the fact that the equivalence class of this bundle does not depend on the choice of the element inside the conjugacy class, and that different conjugacy classes give rise to non-equivalent bundles. It remains to make sure that every holomorphic  $G$ -bundle over  $\Sigma_\tau$  comes from a certain conjugacy class in  $C_\tau$ . To see this, let us pick a bundle  $B$  over  $\Sigma_\tau$ , and pull it back to the cylinder  $\mathbb{C}/\mathbb{Z}$ . Of course, the obtained bundle  $\tilde{B}$  will be trivial. Let us pick a global holomorphic section  $\chi(z)$  of  $\tilde{B}$ . Then  $\chi(z + \tau)$  is another holomorphic section. Therefore,  $g(z) = \chi(z)^{-1}\chi(z + \tau)$  is a holomorphic function. Evidently, the bundle  $B$  is associated to the conjugacy class of  $(\tau, g)$ .

To conclude this section, let us discuss the coadjoint action of  $\text{Vir}(\Sigma)$  in the case when  $\Sigma$  is a complex torus. Obviously, this Lie algebra cannot be integrated to a Lie group—this is impossible even in the one-dimensional case as long as we work over  $\mathbb{C}$ . Therefore, we cannot define orbits of the coadjoint action in the usual way. What we can do, though, is define and fully describe holomorphic invariants of the coadjoint action.

Coadjoint orbits of the real Virasoro algebra have been described in [7,16]. The smooth part of the coadjoint representation can be interpreted as the space of Hill’s operators  $\lambda \frac{d^2}{dx^2} + q(x)$  taking densities of weight  $-1/2$  to densities of weight  $3/2$  on the circle. The coefficient  $\lambda$  is invariant under the coadjoint action, so we may restrict this action to the hyperplane  $\mathcal{H}_\lambda$  of operators with a fixed value of this coefficient. Orbits of the group  $\text{Diff}_+(S^1)$  lying in this hyperplane for  $\lambda \neq 0$  are labelled by conjugacy classes of the universal covering of the group  $SL_2(\mathbb{R})$ .

For the complex Virasoro algebra, orbits are not defined since the algebra does not integrate to a Lie group. However, one can study holomorphic invariants of the coadjoint action which carry the same information as orbits. Repeating the above argument with obvious modifications, one finds that besides  $\lambda$  there is essentially a unique holomorphic invariant of the coadjoint action—the monodromy of the corresponding Hill’s operator. This invariant takes values in the space of conjugacy classes of  $SL_2(\mathbb{C})$ . Since almost every element in  $SL_2(\mathbb{C})$  is diagonalizable, we may assume that the monodromy invariant takes values in  $\mathbb{C}^*/\sim$  where  $\sim$  is the equivalence relation that identifies  $z$  with  $z^{-1}$ . Any holomorphic invariant of the

coadjoint action will then be a function of the coefficient  $\lambda$  and the monodromy invariant.

It is surprising that the same procedure works in two dimensions. Define a density of weight  $\mu \in \mathbb{C}$  on the torus  $\Sigma$  as a formal expression  $u(d\bar{z})^\mu$  where  $u$  is a smooth function on  $\Sigma$ . The action of  $(0,1)$ -vector fields on densities is given by

$$v \frac{\partial}{\partial \bar{z}} \circ u(d\bar{z})^\mu = \left( v \frac{\partial u}{\partial \bar{z}} + \mu u \frac{\partial v}{\partial \bar{z}} \right) (d\bar{z})^\mu . \tag{4.6}$$

The smooth part of the coadjoint representation of  $\text{Vir}(\Sigma)$  can now be identified with the space of two-dimensional Hill’s operators  $H = \lambda \frac{\partial^2}{\partial \bar{z}^2} + q(z, \bar{z})$  taking densities of weight  $-1/2$  to densities of weight  $3/2$  on the torus.

The monodromy of such an operator can be defined as follows. Assume  $\lambda \neq 0$ . Consider the differential equation

$$Hu = 0 . \tag{4.7}$$

This equation is equivalent to the system

$$\frac{\partial}{\partial \bar{z}} U + QU = 0, \quad Q = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}, U = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad u' = \frac{\partial u}{\partial \bar{z}} . \tag{4.8}$$

This system, as we know, defines a holomorphic  $SL_2(\mathbb{C})$ -bundle on the torus, which is almost always flat and unitary. Therefore, for almost every  $H$  there is a point on  $\mathcal{M}$  corresponding to this bundle. Let us call this point the monodromy of  $H$ .

**Proposition 4.3.** *The monodromy is a holomorphic invariant of the coadjoint action of  $\text{Vir}(\Sigma)$ .*

*Proof.* The proof is by giving another definition of the monodromy. Let  $\mathcal{S}(H)$  be the set of all  $s \in \mathbb{C}$  such that there exist two linearly independent solutions  $u_1, u_2$  of (4.7) which are functions on the cylinder  $\mathbb{C}/\mathbb{Z}$  and satisfy the conditions

$$u_1(z + \tau) = su_1(z), \quad u_2(z + \tau) = s^{-1}u_2(z) . \tag{4.9}$$

$\mathcal{S}(H)$  is non-empty whenever the bundle associated to  $H$  is flat and unitary, i.e. for almost every  $H$ . Also, if  $s \in \mathcal{S}(H)$  then for any  $n \in \mathbb{Z}$   $s^{(n)} = \text{sexp}(2\pi i n \tau) \in \mathcal{S}(H)$ . Indeed, if we consider new solutions  $u_1^{(n)} = u_1 \exp(2\pi i n z), u_2^{(n)} = u_2 \exp(-2\pi i n z)$  of (4.8), they will satisfy (4.9) with  $s$  replaced by  $s^{(n)}$ . Finally, if  $s \in \mathcal{S}(H)$  then  $s^{-1} \in \mathcal{S}(H)$ . Moreover, it is easy to check that conversely, if  $s_1, s_2 \in \mathcal{S}(H)$  then either  $s_2 = s_1^{(n)}$  for some integer  $n$  or  $s_2^{-1} = s_1^{(n)}$ .

Define  $\hat{s}$  as the projection of  $s$  into the space  $\mathbb{C}^*/\sim$  where  $\sim$  is the equivalence relation that identifies  $s$  with  $s^{(n)}$  for any  $n$ , and  $s$  with  $s^{-1}$ . If  $s \in \mathcal{S}(H)$  then  $\hat{s}$  depends only on  $H$ . It is obvious from the definition that  $\hat{s}$  is a holomorphic invariant. It remains to observe that  $\mathbb{C}^*/\sim$  is isomorphic to  $\mathcal{M}$ , and that  $\hat{s}$  is nothing else but the monodromy of  $H$ .

*Remark 4.1.* 1. A little more extra work is needed to prove the following:

Any holomorphic invariant of the coadjoint action of  $\text{Vir}(\Sigma)$  is a function of  $\lambda$  and the monodromy.

This statement motivates a definition of coadjoint orbits for  $\text{Vir}(\Sigma)$  as level sets of the monodromy inside hyperplanes  $\lambda = \text{const}$  in the space of two-dimensional Hill’s operators.

*Remark 4.2.* Observe that the space of “orbits” for  $\text{Vir}(\Sigma)$  is the same as for  $SL_2(\mathbb{C})^\Sigma$ . A similar coincidence takes place also in the one-dimensional theory, where it reflects the deep relationship between the representation theories of  $\text{Vir}_{\mathbb{C}}$  and  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ .

### 5. Geometry of Orbits

Define an analogue of Kirillov–Kostant structure on orbits of the coadjoint action of the group  $G^\Sigma$ . We will assume that  $\Sigma$  is a complex torus. The construction is the same as for classical Lie groups.

Let  $X_1, X_2$  be tangent vectors to an orbit  $\mathcal{O}$  in  $E^*$  at a point  $f$ . Then there exist elements  $\tilde{X}_1, \tilde{X}_2 \in \mathfrak{g}^\Sigma$  such that

$$\frac{d}{dt}(e^{t\tilde{X}_j} \circ f) = X_j. \tag{5.1}$$

Set

$$K(X_1, X_2) = f([\tilde{X}_1, \tilde{X}_2]). \tag{5.2}$$

Despite the liftings  $\tilde{X}_1, \tilde{X}_2$  are not unique, Eq. (5.2) presents a well defined anti-symmetric bilinear form on the tangent space to  $\mathcal{O}$  at  $f$  for every  $f \in \mathcal{O}$ . Thus,  $K$  is a holomorphic differential 2-form on  $\mathcal{O}$ .

The following properties of  $K$  are proved similarly to the classical orbit method.

**Proposition 5.1.**

- (i)  $K$  is closed.
- (ii)  $K$  is nondegenerate.
- (iii)  $K$  is invariant under the action of  $G^\Sigma$ .

Let us now define and classify integral orbits, by analogy with the theory of loop groups [13],[5]. For loop groups, an integral orbit is defined as an orbit on which the form  $K/2\pi$  has integral periods. A theorem in [13, Section 4.5] states that an orbit is integral if and only if there exists a circle bundle on it with curvature  $K$ .

**Definition 5.1.** Let us say that an orbit  $\mathcal{O} \subset \mathcal{H}_\lambda$  is integral if for any  $C \in H_2(\mathcal{O})$ ,

$$\frac{1}{2\pi} \int_C K \in L. \tag{5.3}$$

**Proposition 5.2.**

- (i) An orbit  $\mathcal{O}$  is integral if and only if there exists a holomorphic principal  $\Sigma$ -bundle  $\mathcal{E}$  over  $\mathcal{O}$  with a connection  $\theta$  whose curvature is  $K$ .
- (ii) If  $\mathcal{O}$  is an integral orbit and  $\mathcal{E}$  is the corresponding principal bundle then the action of  $G^\Sigma$  on  $\mathcal{O}$  can be uniquely lifted to an action of  $\hat{G}^\Sigma$  on  $\mathcal{E}$  preserving the connection  $\theta$ .

The proof is similar to that for loop groups.

Classification of integral orbits is similar to the loop group case.

Denote by  $\mathcal{O}(\lambda, m)$  the orbit in  $\mathcal{H}_\lambda$  corresponding to the point  $m \in \mathcal{M}$ . We assume that  $\lambda \neq 0$ . Let us call  $\lambda$  the level and  $m$  the weight of the orbit, by analogy with loop groups.

We also assume that  $\tau$  is generic, i.e. that  $\Sigma$  has exactly two holomorphic automorphisms modulo translations.

Denote by  $Z$  the center of  $G$ . Let  $\lambda \in \mathbb{C}^*$ ,  $m \in \mathcal{M}$ , and let  $\tilde{m} \in T \times T$  be a lifting of  $m$ .

**Proposition 5.3.** *An orbit  $\mathcal{O}(\lambda, m)$  is integral if and only if  $\lambda$  is an integer, and  $\lambda \tilde{m} \in Z \times Z \subset T \times T$ . Thus, the number of integral orbits at each level is finite.*

This statement follows immediately from the two previous propositions.

The number of integral orbits at each level can be easily calculated for any particular group. For instance, if  $G = SL_2$  then it is equal to  $2\lambda^2 + 2$ .

## Conclusion

We would like to conclude our paper with a few remarks about the relation of geometry of orbits and representation theory. This relation was very fruitful in the case of compact simple Lie groups and the corresponding loop groups (see [5]). Thus one should expect that the integral orbits of the two-dimensional current groups (see Sect. 5) give rise to certain (projective) representations of this group. In spite of the strong similarity, the orbital theory of the two-dimensional current group is different from the finite-dimensional groups in one important aspect: the target group  $G$  has to be complex rather than compact. In order to pursue the analogy, one has to recall that the orbits for the complex finite-dimensional groups  $G$  correspond in general to Harish–Chandra modules rather than to highest weight modules. Therefore, the general philosophy of orbital theory asserts that the class of (projective) representations of 2-dimensional current groups associated to the orbits studied in this paper should generalize to Harish–Chandra modules. We note however that the appropriate class of representations is not well understood even in the case of loop groups, and it is an interesting direction for future research.

Coming back to integrable highest weight modules, one can ask if there is a way to construct them by quantizing the orbits of complex rather than compact groups. In fact there is a special subset of complex orbits corresponding to spherical modules, which can be viewed as a counterpart of the class of highest weight representations. Spherical modules are characterized by the property that they contain a vector invariant with respect to the diagonal subgroup. This vector is an analogue of the highest weight vector. The duality between spherical and highest weight modules suggest that the orbits of the complex group  $G$  might be used for the latter class of representations. Since the orbits have dimension twice bigger than in the compact case, one needs to define a proper counterpart of Kähler structure on the orbits. The natural candidate is the hyperkähler structure that is known to exist on orbits of complex finite dimensional simple groups and complex loop groups [21]. We conjecture that this structure can also be defined on orbits of two-dimensional current groups, and that it is a key to their quantization.

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**Note added in proof.** The subject of this paper was developed further in a recent paper [EK], where the two-dimensional version of the Gelfand–Dickey Poisson structure was introduced, and it was demonstrated that the local structure of its symplectic leaves is the same as that for coadjoint orbits of the central extension of the two-dimensional current group.

It was also shown there that the codimension of a coadjoint orbit corresponding to a holomorphic principal  $G$ -bundle  $B$  (see Proposition 3.2) is equal to the dimension of the space of holomorphic global sections of its adjoint bundle,  $\text{ad}(B)$ , whose fiber is the Lie algebra of  $G$ .

[EK] Etingof, P.I., Khesin, B.A.: Affine Gelfand–Dickey brackets and holomorphic vector bundles. To appear in Geom. and Funct. Anal., 1994 (hep-th 9312123)