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Operadic Formulation of Topological Vertex Algebras and Gerstenhaber or Batalin-Vilkovisky Algebras

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Abstract: We give the operadic formulation of (weak, strong) topological vertex algebras, which are variants of topological vertex operator algebras studied recently by Lian and Zuckerman. As an application, we obtain a conceptual and geometric construction of the Batalin-Vilkovisky algebraic structure (or the Gerstenhaber algebra structure) on the cohomology of a topological vertex algebra (or of a weak topological vertex algebra) by combining this operadic formulation with a theorem of Getzler (or of Cohen) which formulates Batalin-Vilkovisky algebras (or Gerstenhaber algebras) in terms of the homology of the framed little disk operad (or of the little disk operad).

1. Introduction

Recently operads have received a lot of attention from mathematicians working in different areas. Many complicated algebraic structures can now be formulated and studied conceptually using the language of operads. In the present paper, we give the operadic formulation of another type of algebraic structures – (weak, strong) topological vertex algebras – which are variants of topological vertex operator algebras defined in [LZ]. As an application, we obtain a geometric construction of the Batalin-Vilkovisky algebraic structure (or the Gerstenhaber algebra structure) on the cohomology of a topological vertex algebra (or of a weak topological vertex algebra) using this operadic formulation and a recent theorem of Getzler [Get] (or a theorem of Cohen [C, Get]).

Operads are devices to describe operations. For classical algebraic structures, the corresponding operads are so simple (geometrically they are usually constructed from one-dimensional objects) that these operads themselves do not have any interesting structure to be studied. Even though the notion of operad is very natural, it would be only a fancy language rather than a necessary and deep way to understand operations conceptually for these classical algebraic structures. But the situation changes when we consider more complicated operations. The first important example of an operad-

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like structure occurred in Stasheff's notion of A_∞-space [S1]. Unlike the operads for classical algebraic structures, it has rich geometric structure. In [May], May formalized the notion of operad and began to use it systematically in the study of iterated loop spaces. The recent studies of Gerstenhaber algebras, Batalin-Vilkovisky algebras, homotopy Lie algebras and vertex operator algebras in terms of operads (see, for example, [Get, HS, HL1, HL2]) show the power and importance of the notion of operad. See also [S2] for a recent survey.

The notion of vertex operator algebra and its various variants arise naturally in the vertex operator construction of the Monster sporadic group by Frenkel, Lepowsky and Meurman [FLM1]. It is essentially the same as the notion of chiral algebra in conformal field theory (see, e.g., [MS]). In [B] Borcherds introduces the notion of vertex algebra based on his insightful understanding of the theory of vertex operators and of [FLM1]. In [FLM2], the notion of vertex operator algebra – a modification of that of vertex algebra – is introduced and it is proved that the moonshine module constructed in [FLM1] has the structure of a vertex operator algebra such that the Monster is its automorphism group. In [FHL], the basic axiomatic theory of vertex operator algebras is developed. The difference between vertex algebras and vertex operator algebras is that vertex algebras do not have to satisfy those axioms on the grading and do not have to have a Virasoro element. We can also define some other variants of vertex operator algebras by relaxing or generalizing some other axioms. For various examples, variant and generalizations of vertex operator algebras, see, for example, [FLM2, FHL, T, FFR, H3, FZ, DL, L].

In [H1, H2, HL1, HL2, H4] it has been established that vertex operator algebras are in fact algebras over the partial operads of powers of the determinant line bundle over the partial operad K of a certain moduli space of spheres with punctures and local coordinates, satisfying a certain meromorphicity axiom. A small part of this operadic interpretation of vertex operator algebras in fact shows that vertex algebras are algebras over a partial suboperad $\widehat{K}$ of K satisfying the part of the meromorphicity axiom which still makes sense for $\widehat{K}$. In this paper, the algebraic structures we will study are variants of the topological vertex operator algebra introduced by Lian and Zuckerman: weak topological vertex algebras, topological vertex algebras and strong topological vertex algebras (see Definition 2.1). The main purpose of the present paper is to give a completely geometric formulation of these topological vertex algebras using the structure of the partial operad $\widehat{K}$.

We show that the category of topological vertex algebras (or weak topological vertex algebras) is isomorphic to a category whose objects are triples of the form $(V, Q, \{\omega_j \mid j \in \mathbb{N}\})$, where V is a $\mathbb{Z} \times \mathbb{Z}$-graded vector space, Q an operator on V of degree 1 with respect to the second $\mathbb{Z}$-grading, and ω_j a holomorphic form on $\widehat{K}(j)$ for every j in the set $\mathbb{N}$ of nonnegative integers, satisfying certain natural axioms. An immediate consequence is that the operad of the homology of the framed little disk operad (or the little disk operad) acts on the cohomology of a topological vertex algebra (or of a weak topological vertex algebra). Combined with Getzler's theorem (or Cohen's theorem) which describes Batalin-Vilkovisky algebras (or Gerstenhaber algebras) in terms of the operad of the homology of the framed little disk operad (or of the little disk operad), it gives the Batalin-Vilkovisky algebra structure (or the Gerstenhaber structure) on the cohomology of a topological vertex algebra (or of a weak topological vertex algebra). This last result has been obtained first by Lian and Zuckerman [LZ]. Also, beginning with the geometric axioms of topological conformal field theories, Getzler has shown that the cohomology of a topological conformal field theory has the structure of a Batalin-Vilkovisky algebra [Get]. The main new result of the present paper is the

equivalence between the algebraic formulation and the geometric formulation of (weak, strong) topological vertex algebras and, as a consequence, Lian- Zuckerman's result can now be understood conceptually using the results of this paper and Getzler's theorem or Cohen's theorem. This equivalence also provides examples of what Getzler begins with. In fact it can be shown further that the category of strong topological vertex algebras is isomorphic to the category of algebras over the partial operad of the differentiable chain complexes of the partial operad $\widehat{K}$ satisfying a certain meromorphicity axiom. But since the meromorphicity axiom of these algebras over the differentiable chain complexes is much more complicated than that for $\omega_j, j \in \mathbb{N}$ and the formulation in terms of these forms is already conceptual enough, we only give half of this further formulation for future topological applications; we show that these forms give a morphism of differential graded partial operads from the partial operad of the differentiable chain complexes of the partial operad $\widehat{K}$ to the endomorphism partial pseudo-operad for V.

One remark on the algebraic and geometric formulations of quantum field theories might be helpful here. In general, the geometric formulations of quantum field theories are axiomizations of the path integral approach in physics and have the advantage that they give conceptually satisfactory definitions and they also allow one to derive many important results using the geometric intuition. One famous example is the Verlinde formula. In [MS], it is shown at the physical level of rigor that the geometric axioms plus the rationality imply the Verlinde conjecture that the modular transformation corresponding to $\tau \to -1/\tau$ diagonalizes the fusion rules. If one can show that all irreducible highest weight representations at level k of an affine Lie algebra indeed give a rational conformal field theory, the Verlinde formula which has attracted many mathematicians recently will be an easy consequence. But in fact the most difficult part is the rigorous construction of a rational conformal field theory satisfying the geometric axioms. Even in the affine Lie algebra case, a complete construction of the corresponding rational conformal field theories (not just the construction of the conformal blocks) is still to be given. On the other hand, the algebraic formulations of quantum field theories correspond to the operator product expansion approach in physics and many concrete quantum field theories in the algebraic formulations can be constructed rigorously from some simple algebraic data. But the axioms in the algebraic formulations are usually complicated and are conceptually difficult to understand. It is therefore necessary and important to establish rigorously the relationship between the algebraic and geometric formulations. In some sense, establishing the equivalence between the algebraic and geometric formulations can be thought of as establishing the existence of path integrals rigorously without writing down classical fields and Lagrangians of quantum field theories explicitly, since the axioms in the geometric formulations are nothing but the properties of path integrals. The paper [H1, H2, HL1, HL2, H4] are in this spirit. The present paper is another one in this spirit.

The present paper is based on the operadic formulation of (graded) vertex algebras in terms of the partial operad $\widehat{K}$. Roughly speaking, a weak topological vertex algebra is a graded vertex algebra together with three operators $Q, g(0)$ and $g(-1)$ satisfying certain natural axioms including $Q^2 = 0$. A topological vertex algebra is a weak topological vertex algebra satisfying the additional axiom that $(g(0))^2$ is Q-exact. A strong topological vertex algebra is a topological vertex algebra satisfying the stronger axiom that $(g(0))^2 = 0$. Since a (weak, strong) topological vertex algebra is a graded vertex algebra, we have an action of the partial operad $\widehat{K}$. Here we give a heuristic interpretation of two most important axioms on $Q, g(0)$ and $g(-1)$ in the case of strong topological vertex algebras: $[Q, g(l)] = L(l)$, $l = 0, -1$, where $Q, g(0)$ and $g(-1)$

are operators mentioned above and $L(0)$ and $L(-1)$ are operators defined using the underlying graded vertex algebra structure (see (2.9) and (2.10)). The operators $L(0)$ and $L(-1)$ can be thought of as linearly representing the Lie derivatives $\mathscr{L}_0$ and $\mathscr{L}_{-1}$ on tensor fields on $\widehat{K}$ along certain directions. On $\widehat{K}$, we have $[d, i_l] = \mathscr{L}_l$, $l = 0, -1$, where d is the exterior derivation, i_l is the interior derivation along the direction corresponding to $L(l)$. In general, for a graded vertex algebra, the algebraic structure on the space of differential forms does not have to be linearly represented. If we require that the algebraic structure on the space of differential forms is also linearly represented and assume that d, i_0 and i_{-1} are represented by Q, $g(0)$ and $g(-1)$, respectively, then we must have $[Q, g(l)] = L(l)$, $l = 0, -1$, which are exactly the two axioms we would like to interpret.

The main technical work in the present paper is the proofs of the properties of the forms $\omega_j, j \in \mathbb{N}$, in Sect. 3. Our construction of ω_j [see (3.21) and (3.20)] is motivated by [Z]. In fact in the case that the (strong) topological vertex algebra is a weak vertex operator algebra of central charge 26 tensored with the graded vertex operator algebra of the ghosts, the forms ω_j in the present paper are completely analogous to the forms in [Z]. The equivalence proved in Sect. 4 between the notion of (weak, strong) topological vertex algebras and the operadic formulation in terms of these forms in fact shows that any family of forms satisfies those properties in Sect. 3 must be of the form (3.21) and (3.20). In the case of strong topological vertex algebras the axiom $(g(0))^2 = 0$ and its consequences $(g(-1))^2 = g(0)g(-1) + g(-1)g(0) = 0$ simplify the definition of the forms and consequently simplify the proofs greatly. In this paper, for completeness and for possible use in the future, we are interested in arbitrary topological vertex algebras and also in weak topological vertex algebras. Thus we cannot assume these relations. Consequently the proofs for (weak) topological vertex algebras are much more complicated than those for strong topological vertex algebras.

This paper is organized as follows: In Sect. 2 we briefly discuss the notions of graded vertex operators algebra, weak (graded) vertex operator algebra, topological vertex operator algebra, (graded) vertex algebra, and introduce the notions of weak topological vertex algebra, topological vertex algebra and strong topological vertex algebra. We also discuss the action of the partial operad $\widehat{K}$ on (graded) vertex algebras. In the subsequent section we construct the forms ω_j from the data of a (weak, strong) topological vertex algebra and prove their basic properties. We use these forms and their properties to define the notions of (weak, strong) topological $\widehat{K}$-associative algebra and show that these notions are equivalent to those of (weak, strong) topological vertex algebra in Sect. 4. In the same section, we also show that there is a family of natural symmetry-group-equivariant chain maps from the differential graded partial operad of the differentiable chain complex of $\widehat{K}$ to the differential graded endomorphism partial pseudo-operad of a weak topological vertex algebra and that a strong topological vertex algebra is an algebra over the differential graded partial operad of the differentiable chain complex of $\widehat{K}$. In Sect. 5, we first review the definitions of Batalin-Vilkovisky algebra and Gerstenhaber algebra, some basic results including Cohen's and Gretzler's theorems and some other related notions and results. Then combining the results obtained in Sect. 3 and in Sect. 4 with Getzler's theorem (or Cohen's theorem), we obtain the Batalin-Vilkovisky algebra structure (or the Gerstenhaber algebra structure) on the cohomology of a topological vertex algebra (or a weak topological vertex algebra).

2. Topological Vertex Algebras and the Action of a (Partial) Operad

In this section, we discuss various variants of vertex operator algebras, introduce the notions of (weak, strong) topological vertex algebra and describe the actions of a (partial) operad on these variants of vertex operator algebras. We will only give definitions and state the theorems. All proofs are omitted since they are either easy exercises or to be published elsewhere.

We begin with the definition of graded vertex operator algebra (or "super chiral algebra" as is called in [LZ]). The notion of graded vertex operator algebra is a natural generalization of that of vertex operator algebra and can be viewed as a specialization of the notion of an abelian intertwining algebra introduced in [DL]. We assume that the readers are familiar with the definition of vertex operator algebra, as presented in [FLM2] or [FHL]. We only give the differences between vertex operator algebras and graded vertex operator algebras.

For a *graded vertex operator algebra*, the vector space V is $\mathbb{Z} \times \mathbb{Z}$-graded (graded by weights and by *fermion numbers*), that is,

$$V = \coprod_{m,n\in\mathbb{Z}} V_{(n)}^{(m)} = \coprod_{n\in\mathbb{Z}} V_{(n)} = \coprod_{m\in\mathbb{Z}} V^{(m)}, \tag{2.1}$$

where

$$V_{(n)} = \coprod_{m\in\mathbb{Z}} V_{(n)}^{(m)}, \quad V^{(m)} = \coprod_{n\in\mathbb{Z}} V_{(n)}^{(m)}. \tag{2.2}$$

The map $Y : V \otimes V \to V[[x, x^{-1}]]$ maps $V^{(m_1)} \otimes V^{(m_2)}$ to $V^{(m_1+m_2)}[[x, x^{-1}]]$. For $v \in V^{(m)}$, we say that v has fermion number m and use $|v|$ to denote m. For the vacuum 1 and the Virasoro element ω, $|1| = |\omega| = 0$. The (Cauchy-)Jacobi identity is of the following form for u, v with homogeneous fermion numbers:

$$\begin{aligned} &x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - (-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1) \\ &\quad = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2). \end{aligned} \tag{2.3}$$

All the other data and axioms are the same as those for vertex operator algebras. We also call a quadruple $(V, Y, \mathbf{1}, \omega)$ satisfying all axioms for vertex operator algebras (or for graded vertex operator algebras) except the two grading axioms – $\dim V_{(n)} < \infty$ and $V_{(n)} = 0$ for n sufficiently small – a *weak vertex operator algebra* (or a *weak graded vertex operator algebra*).

Next we recall the definition of topological vertex operator algebra (or "topological chiral algebra") given in [LZ]. A *topological vertex operator algebra* is a weak graded vertex operator algebra V together with three distinguished elements $f \in V_{(1)}$, $q \in V_{(1)}^{(1)}$ and $g \in V_{(2)}^{(1)}$ satisfying the following axioms:

(i) Let $Y(f, x) = \sum_{n\in\mathbb{Z}} f_n x^{-n-1}$. Then for any $v \in V^{(m)}$

$$f_0 v = mv. \tag{2.4}$$

(ii) Let $Y(q, x) = \sum_{n\in\mathbb{Z}} q_n x^{-n-1}$ and $Q = q_0$. Then

$$L(n)q = 0, \quad n > 0, \tag{2.5}$$

$$Q^2 = 0. \tag{2.6}$$

(iii) Let $Y(g,x) = \sum_{n\in\mathbb{Z}} g(n)x^{-n-2}$ and ω the Virasoro element of V. Then

$$L(n)g = 0, \quad n > 0, \tag{2.7}$$

$$Qg = \omega . \tag{2.8}$$

Note that (2.4) implies $f \in V_{(1)}^{(0)}$. In fact, the construction of the Batalin-Vilkovisky algebra (or the "coboundary Gerstenhaber algebra") structure on the cohomology of a topological vertex operator algebra in [LZ] also uses the additional axiom that $(g(0))^2$ is Q-exact, though the construction of the Gerstenhaber algebra structure does not need this axiom.

Examples of topological vertex operator algebras are weak vertex operator algebras with central charge 26 tensored with the graded vertex operator algebra of ghosts and the holomorphic genus-zero parts of $N = 2$ twisted superconformal theories. These examples all satisfy the additional axioms $(g(0))^2 = (g(-1))^2 = g(0)g(-1) + g(-1)g(0) = 0$.

In the construction of the Gerstenhaber algebra or the Batalin-Vilkovisky algebra structure on the cohomology of a topological vertex operator algebra by Lian and Zuckerman [LZ], only a small part of the axioms for topological vertex operator algebras is needed. In this paper we also need only a small part of the axioms. We summarize those axioms which we do need in the definition of the notion of (weak, strong) topological vertex algebra below.

Before giving the definition, we need to discuss certain variants of vertex operator algebras and graded vertex operator algebras: the notions of vertex algebra and of graded vertex algebra. The notion of vertex algebra is introduced by Borcherds in [B]. In [L] they are called preVOAs. (In [DL], vertex algebra is a technical term referred to weak vertex operator algebras mentioned above.) In this paper, by a *vertex algebra* we mean a $\mathbb{Z}$-graded vector apace V equipped with a map $Y : V \otimes V \to V[[x, x^{-1}]]$, $u \otimes v \mapsto Y(u,v)v = \sum_{n\in\mathbb{Z}} u_n v x^{-n-1}$ and a *vacuum* $\mathbf{1}$ such that for any homogeneous element v, v_k maps $V_{(n)}$ to $V_{(n+\mathrm{wt}\, v-k-1)}$, satisfying all the axioms for a vertex operator algebra except the axioms $\dim V_{(n)} < \infty$, $V_{(n)} = 0$ for n sufficiently small and those axioms for the Virasoro element. The definition of *graded vertex algebra* is obvious. Given a vertex algebra or a graded vertex algebra, we can define two operators $L(0)$ and $L(-1)$ as follows: For any $v \in V_{(n)}$,

$$L(0)v = nv, \tag{2.9}$$

and for any $v \in V$,

$$L(-1)v = \lim_{x\to 0} \frac{d}{dx} Y(v,x)\mathbf{1} . \tag{2.10}$$

Using these definitions and the (Cauchy-)Jacobi identity, it is easy to verify the following identities:

$$[L(0), L(-1)] = L(-1) , \tag{2.11}$$

$$[L(-1), Y(v,x)] = Y(L(-1)v, x) = \frac{d}{dx} Y(v,x) , \tag{2.12}$$

$$\begin{aligned} [L(0), Y(v,x)] &= Y((L(0) + xL(-1))v, x) \\ &= Y(L(0)v, x) + x\frac{d}{dx} Y(v,x) . \end{aligned} \tag{2.13}$$

Definition 2.1. *A weak topological vertex algebra is a graded vertex algebra together with operators $Q, g(0)$ and $g(-1)$ mapping $V^{(m)}$ to $V^{(m+1)}$, $V^{(m-1)}$ and $V^{(m-1)}$, respectively, satisfying all the axioms for topological vertex operator algebras which still make sense, that is, satisfying*

$$Q^2 = 0, \tag{2.14}$$

$$[Q, g(0)] = L(0), \tag{2.15}$$

$$[Q, g(-1)] = L(-1) \tag{2.16}$$

$$[L(0), g(0)] = 0, \tag{2.17}$$

$$[L(0), g(-1)] = g(-1), \tag{2.18}$$

$$[L(-1), g(0)] = -g(-1), \tag{2.19}$$

$$[L(-1), g(-1)] = 0, \tag{2.20}$$

$$QY(u,x) - (-1)^{|u|}Y(u,x)Q = Y(Qu,x), \tag{2.21}$$

$$g(0)(u,x) - (-1)^{|u|}Y(u,x)g(0) = Y((g(0) + xg(-1))u,x), \tag{2.22}$$

$$g(-1)Y(u,x) - (-1)^{|u|}Y(u,x)g(-1) = Y(g(-1)u,x), \tag{2.23}$$

where the bracket $[A.B]$ of two operators A and B with homogeneous fermion numbers is defined by

$$[A, B] = AB - (-1)^{|A||B|}BA. \tag{2.24}$$

A *topological vertex algebra* is a weak topological vertex algebra such that $(g(0))^2$ is a Q-exact operator, that is, there exists an operator $U_{0,0} : V \to V$ such that

$$(g(0))^2 = [Q, U_{0,0}] \tag{2.25}$$

A *strong topological vertex algebra* is a weak topological vertex algebra satisfying

$$(g(0))^2 = 0. \tag{2.26}$$

From (2.15)–(2.20) and the Jacobi identity for the bracket $[\cdot, \cdot]$ we see that for a weak topological vertex algebra,

$$\begin{aligned}(g(-1))^2 &= \frac{[Q, [g(0), (g(-1))^2]]}{2} \\ &= -\frac{[Q, [g(-1), [g(0), g(-1)]]]}{2},\end{aligned} \tag{2.27}$$

$$\begin{aligned}[g(0), g(-1)] &= [Q, [g(0), [g(0), g(-1)]] \\ &= -[Q, [g(-1), (g(0))^2]].\end{aligned} \tag{2.28}$$

In particular, these formulas also hold for a topological vertex algebra. From these formulas we see that for a strong topological vertex algebra we also have

$$[g(0), g(-1)] = 0, \tag{2.29}$$

and

$$(g(-1))^2 = 0. \tag{2.30}$$

It is also easy to show that a topological vertex operator algebra is a weak topological vertex algebra using the (Cauchy-)Jacobi identity. When there is no confusion, we will call V a (weak, strong) topological vertex algebra.

Now we discuss the projective action of a (partial) operad on graded vertex operator algebras and in particular on topological vertex operator algebras. For the basic notions in the language of (partial) operads, see [May, HL1 or HL2]. Let $K(j), j \in \mathbb{N}$ (where $\mathbb{N}$ is the set of all nonnegative integers), be the moduli space of one-dimensional genus-zero compact connected complex manifolds with $j + 1$ ordered punctures, the zeroth negatively oriented and the others positively oriented, and with local analytic coordinates vanishing at these punctures. The family $K = \{K(j)|j \in \mathbb{N}\}$ is an associative analytic $\mathbb{C}^\times$-rescalable partial operad [HL1, HL2, H4]. The determinant line bundle over K raised to any complex power c (denoted by $\widetilde{K}^c$) is also an associative analytic $\mathbb{C}^\times$-rescalable partial operad [HL1, HL2, H4]. On the other hand, for any $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ and any graded subspace W of V, we have a (multilinear) endomorphism partial pseudo-operad $\mathscr{H}_{V,W}$ defined as follows (see [HL1, HL2]): For any $j \in \mathbb{N}$ the set $\mathscr{H}_{V,W}(j)$ is the set of all multilinear maps from V^j to $\overline{V} = \prod_{n\in\mathbb{Z}} V_{(n)}$ such that W^j is mapped to $\overline{W} = \prod_{n\in\mathbb{Z}} W_{(n)}$. (It is understood that $V^0 = W^0$ is the one-element set, so that the set $\mathscr{H}_{V,W}(0)$ is equal to $\overline{W}$.) The identity $I_{V,W}$ is the embedding map from V to $\overline{V}$. The symmetric group S_j acts on $\mathscr{H}_{V,W}(j)$ in the obvious way. To define the substitution (composition) maps, we first define a contraction operation on $\mathscr{H}_{V,W}$: Given $f \in \mathscr{H}_{V,W}(k)$ and $g \in \mathscr{H}_{V,W}(j)$ $(k, j \in \mathbb{N})$ and a positive integer $s \le k$, we say that the *contraction of f at the s-th argument and g at the zeroth argument exists* if for any $v_1, \dots, v_{k+j-1} \in V$ and $v' \in V'$, the series

$$\sum_{n\in\mathbb{Z}} \langle v', f(v_1, \dots, v_{s-1}, P_n(g(v_s, \dots, v_{s+j-1})), v_{s+j}, \dots, v_{k+j-1})\rangle \tag{2.31}$$

converges absolutely, where $P_n : \overline{V} \to V_{(n)}$ is the projection operator. In this case the (well-defined) limits for all $v_1, \dots, v_{k+j-1} \in V$, $v' \in V'$ define an element $f_s *_0 g$ of $\mathscr{H}^G_{V,W}(k + j - 1)$, the *contraction*. More generally, given any subset of $\{1, \dots, k\}$ and any element of $\bigcup_{j\in\mathbb{N}} \mathscr{H}^G_{V,W}(j)$ for each element of the subset, we have the analogous *contraction*, defined using the appropriate multisums, when they are absolutely convergent. The substitution map

$$\begin{aligned} \gamma_{V,W} : \mathscr{H}_{V,W}(k) \times \mathscr{H}_{V,W}(j_1) \times \\ \times \cdots \times \mathscr{H}_{V,W}(j_k) \to \mathscr{H}_{V,W}(j_1 + \cdots + j_k) \\ (f; g_1, \dots, g_k) \mapsto \gamma_{V,W}(f; g_1, \dots, g_k) \end{aligned} \tag{2.32}$$

is defined by this procedure, using the whole set $\{1, \dots, k\}$. The family of sets $\mathscr{H}_{V,W}(j), j \in \mathbb{N}$, equipped with the substitution maps $\gamma_{V,W}$, the identity $I_{V,W}$ and the actions of S_j on $\mathscr{H}_{V,W}(j)$, $j \in \mathbb{N}$ is a partial pseudo-operad and is denoted $\mathscr{H}_{V,W}$. A vertex associative algebra with central charge c is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ such that $\dim V_{(n)} < \infty, n \in \mathbb{Z}$ and a morphism of partial pseudo-operads from $\widetilde{K}^c$ to $\mathscr{H}_{V,W}$ satisfying certain meromorphicity axioms [HL1, HL2, H4]. It has been established [H1, H2, HL1, HL2, H4] that the category of vertex operator algebras with central charge c is isomorphic to the category of vertex associative algebras with central charge c. In particular, for any vertex operator algebra V, there is a projective action of the partial operad K. See [HL1] and [HL2] for more detailed descriptions.

These discussions can be generalized to graded vertex operator algebras without any difficulty. Given a $\mathbb{Z} \times \mathbb{Z}$-graded vector space V (graded by weights and fermion numbers) and a $\mathbb{Z} \times \mathbb{Z}$-graded subspace W, we can define the graded endomorphism partial pseudo-operad $\mathscr{H}_{V,W}$ in the same way as in the preceding paragraph except that the left actions of the symmetry groups are defined such that for any $f \in \mathscr{H}_{V,W}(j)$ and $\sigma_{i,i+1} \in S_j$ which is the transposition permuting i and $i+1$,

$$\begin{aligned}&(\sigma_{i,i+1}(f))(v_1, \dots, v_i, v_{i+1}, \dots, v_j)\\ &\quad = (-1)^{|v_i||v_{i+1}|} f(v_1, \dots, v_{i+1}, v_i, \dots, v_j)\end{aligned} \tag{2.33}$$

for any $v_1, \dots, v_j \in V$ with homogeneous fermion numbers. Using this graded endomorphism partial pseudo-operad, we can define graded vertex associative algebra in the same way as that defining vertex associative algebras. Then the proof in the case of vertex operator algebras can be adopted to show that the category of graded vertex operator algebras with central charge c is isomorphic to the category of graded vertex associative algebras with central charge c. In particular, given any graded vertex operator algebra, there is a projective action of the partial operad K.

The partial operad K has several important partial suboperads. The two which are useful in the present paper are $\widehat{K} = \{\widehat{K}(j) | j \in \mathbb{N}\}$, where $\widehat{K}(j)$ consists of those conformal equivalence classes whose members are conformally equivalent to $\mathbb{C} \cup \{\infty\}$ with the negatively oriented puncture ∞, the positively oriented punctures $z_1, \dots, z_j$, and with the standard local coordinate at ∞, the standard local coordinates multiplied by nonzero complex numbers as local coordinates at $z_1, \dots, z_j$, and $\overline{K}$ which is the same as $\widehat{K}$ except that the local coordinates at $z_1, \dots, z_j$ are also standard. The family $\widehat{K}$ is also an associative analytic $\mathbb{C}^\times$-rescalable partial operad and $\overline{K}$ is a partial suboperad of $\widehat{K}$. For any graded vertex algebra V, there is an action of the partial operad $\widehat{K}$. In fact this action can be written down easily. Let $P \in \widehat{K}(j)$. Then P can be identified with $(z_1, \dots, z_j; a_1, \dots, a_j) \in \mathbb{F}_j(\mathbb{C}) \times (\mathbb{C}^\times)^j$, where $\mathbb{F}_j(\mathbb{C}) = \{(z_1, \dots, z_j) \in \mathbb{C}^j | z_k \neq z_l, 1 \leq k < l \leq j\}$. For convenience, we call $z_1, \dots, z_j$ the *punctures* of P and $a_1, \dots, a_j$ the *local coordinates* of P. Let V be a graded vertex algebra and $W = 0$. In this case $\mathscr{H}_{V,0}(j) = \mathrm{Hom}(V^{\otimes j}, \overline{V})$. We define $\nu(P) \in \mathscr{H}_{V,0}(j)$ by

$$\begin{aligned}&\nu(P)(v_1 \otimes \cdots \otimes v_j)\\ &\quad = Y((a_1)^{-L(0)} v_1, x_1) \cdots Y((a_j)^{-L(0)} v_j, x_j)\mathbf{1}|_{x_1 = z_1, \dots, x_j = z_j}\end{aligned} \tag{2.34}$$

when $|z_1| > \cdots > |z_j|$. When this inequality does not hold, we can change the order of the vertex operators or use the iterations of the vertex operators or use analytic extension to define $\nu(P)$. We have

Proposition 2.1. *The maps $\nu : \widehat{K}(j) \to \mathscr{H}_{V,0}(j)$, $j \in \mathbb{N}$ give a morphism of partial pseudo-operad such that the image is a partial operad. In particular, the maps ν give a morphism of partial operads from $\widehat{K}$ to its image.*

The proof of this proposition uses only (2.13) and the duality properties of graded vertex algebras.

Definition 2.2. *A graded meromorphic $\widehat{K}$-associative algebra is a $\mathbb{Z}\times\mathbb{Z}$-graded vector space V of the form (2.1) together with a morphism ν of partial pseudo-operads from $\widehat{K}$ to $\mathscr{H}_{V,0}$ such that for any $v \in V_{(n)}$,*

$$\nu((0;a))v = a^{-n}v \tag{2.35}$$

where $(0;a) \in \mathbb{F}_1(\mathbb{C}) \times \mathbb{C}^\times = \widehat{K}(1)$ and such that for any $v_1, \dots, v_j \in V, v' \in V' = \coprod_{n\in\mathbb{Z}} V_{(n)}^*$ and for $P \in \widehat{K}(j) = \mathbb{F}_j(\mathbb{C}) \times (\mathbb{C}^\times)^j$ of the form $(z_1, \dots, z_j; 1, \dots, 1)$, $\langle v', \nu(P)(v_1, \dots, v_j)\rangle$ as a function of $z_1, \dots, z_j$ is meromorphic on $\mathbb{F}_j(\mathbb{C})$ with $z_i = z_k$, $i < k$, and $z_i = \infty$ as the only possible poles, and for fixed $v_i, v_k \in V$ there is an upper bound for the orders of the pole $z_i = z_k$ of the functions $\langle v', \nu(P)(v_1, \dots, v_j)\rangle$ for all $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{k-1}, v_{k+1}, \dots, v_j \in V$, $v' \in V'$.

Part of the proof of the equivalence theorem for graded vertex operator algebras in fact proves the following equivalence theorem for graded vertex algebras:

Theorem 2.2. *The functor given by $(V, Y, \mathbf{1}) \to (V, \nu)$ is an isomorphism from the category of graded vertex algebras to the category of graded meromorphic $\widehat{K}$-associative algebras.*

3. The Meromorphic Forms Constructed From a (Weak, Strong) Topological Vertex Algebra

Now we would like to see what kind of topological and geometric information the extra data and axioms in the definitions of (strong, weak) topological vertex algebra give. Let V be a weak topological vertex algebra. We define a holomorphic form $\omega_j \in \Omega^*(\widehat{K}(j), \mathscr{H}_{V,0}(j))$ valued in the space $\mathscr{H}_{V,0}(j)$ for each $j \in \mathbb{N}$. Since holomorphic forms valued at a point are multilinear skew-symmetric maps from products of the holomorphic tangent space at this point of $\mathbb{C}$, we first have to discuss the holomorphic tangent space of $\widehat{K}(j)$. Recall that $\widehat{K}(j)$ can be identified with $\mathbb{F}_j(\mathbb{C}) \times (\mathbb{C}^\times)^j$ whose elements are denoted $(z_1, \dots, z_j; a_1, \dots, a_j)$. Therefore a holomorphic tangent vector X_P at the point $P = (z_1, \dots, z_j; a_1, \dots, a_j) \in \widehat{K}(j)$ can be written in the form

$$\begin{aligned} X_P = {} & c_{-1}^{(1)}\left(-a_1^{-1}\frac{\partial}{\partial z_1}\Big|_P\right) + \cdots + c_{-1}^{(j)}\left(-a_j^{-1}\frac{\partial}{\partial z_j}\Big|_P\right) \\ & + c_0^{(1)}\left(-a_1\frac{\partial}{\partial a_1}\Big|_P\right) + \cdots + c_0^{(j)}\left(-a_j\frac{\partial}{\partial a_j}\Big|_P\right), \end{aligned} \tag{3.1}$$

where $c_{-1}^{(1)}, \dots, c_{-1}^{(j)}, c_0^{(1)}, \dots, c_0^{(j)} \in \mathbb{C}$. Let

$$X_P^{(i)} = c_{-1}^{(i)}\left(-a_i^{-1}\frac{\partial}{\partial z_i}\Big|_P\right) + c_0^{(i)}\left(-a_i\frac{\partial}{\partial a_i}\Big|_P\right), \quad 1 \le i \le j. \tag{3.2}$$

Then

$$X_P = \sum_{i=1}^{j} X_P^{(i)}. \tag{3.3}$$

We denote the space of holomorphic tangent vectors at $P \in \widehat{K}(j)$ by $T_P(\widehat{K}(j))$, the subspace of $T_P(\widehat{K}(j))$ spanned by $-a_i^{-1}\frac{\partial}{\partial z_i}|_P$ and $-a_i\frac{\partial}{\partial a_i}|_P$ by $T_P^{(i)}(\widehat{K}(j))$,

$i = 1, \dots, j$. Then $\dim_{\mathbb{C}} T_P(\widehat{K}(j)) = 2j$, $\dim_{\mathbb{C}} T_P^{(i)}(\widehat{K}(j)) = 2$, $i = 1, \dots, j$, where $\dim_{\mathbb{C}}$ means the complex dimension. And we have

$$T_P(\widehat{K}(j)) = \bigoplus_{i=1}^{j} T_P^{(i)}(\widehat{K}(j)). \tag{3.4}$$

Let $T(\widehat{K}(j))$ be the holomorphic bundle over $\widehat{K}(j)$ whose fiber at $P \in \widehat{K}(j)$ is $T_P(\widehat{K}(j))$ and $T_P^{(i)}(\widehat{K}(j))$, $i = 1, \dots, j$, the holomorphic bundles over $\widehat{K}(j)$ whose fibers at $P \in \widehat{K}(j)$ are $T_P^{(i)}(\widehat{K}(j))$. Then we also have a decomposition

$$T(\widehat{K}(j)) = \bigoplus_{i=1}^{j} T^{(i)}(\widehat{K}(j)). \tag{3.5}$$

Any holomorphic section $X^{(i)}$ of $T^{(i)}(\widehat{K}(j))$ for $i = 1, \dots, j$ is of the form

$$X^{(i)} = c_{-1}^{(i)}\left(-a_i^{-1}\frac{\partial}{\partial z_i}\right) + c_0^{(i)}\left(-a_i\frac{\partial}{\partial a_i}\right), \tag{3.6}$$

where $c_{-1}^{(i)}$ and $c_0^{(i)}$ are analytic functions on $\widehat{K}(j)$. For any holomorphic section X of $T(\widehat{K}(j))$, from (3.5) we have a decomposition

$$X = \sum_{i=1}^{j} X^{(i)}, \tag{3.7}$$

where $X^{(i)}, i = 1, \dots, j$, are holomorphic sections of $T^{(i)}(\widehat{K}(j))$, $i = 1, \dots, j$, respectively. We denote the spaces of holomorphic sections of $T^{(i)}(\widehat{K}(j))$, $i = 1, \dots, j$, and $T(\widehat{K}(j))$ by $\Gamma(T^{(i)}(\widehat{K}(j)))$, $i = 1, \dots, j$, and by $\Gamma(T(\widehat{K}(j)))$, respectively. Then by (3.7),

$$\Gamma(T(\widehat{K}(j))) = \bigoplus_{i=1}^{j} \Gamma(T^{(i)}(\widehat{K}(j))). \tag{3.8}$$

We can discuss the holomorphic tangent vector space $T_P(\overline{K}(j))$ at $P \in \overline{K}(j)$, the holomorphic tangent bundle $T(\overline{K}(j))$ and the space $\Gamma(T(\overline{K}(j)))$ of holomorphic sections of $T(\overline{K}(j))$ for each $j \in \mathbb{N}$ in the same way. In particular, we also have $T_P^{(i)}(\overline{K}(j))$ and $T^{(i)}(\overline{K}(j))$, $1 \le i \le j, j \in \mathbb{N}$ defined in the obvious way. All these vector spaces and vector bundles can be embedded naturally in the corresponding vector spaces and vector bundles for $\widehat{K}(j)$.

We define $4j$ operators $g^{(i)}(0)$, $g^{(i)}(-1)$, $L^{(i)}(0)$ and $L^{(i)}(-1)$, $i = 1, \dots, j$, on $V^{\otimes j}$ by

$$\begin{aligned} &g^{(i)}(0)(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j) \\ &\quad = (-1)^{|v_1|+\cdots|v_{i-1}|} v_1 \otimes \cdots \otimes g(0)v_i \otimes \cdots \otimes v_j, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &g^{(i)}(-1)(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j) \\ &\quad = (-1)^{|v_1|+\cdots|v_{i-1}|} v_1 \otimes \cdots \otimes g(-1)v_i \otimes \cdots \otimes v_j, \end{aligned} \tag{3.10}$$

$$L^{(i)}(0)(v_1 \otimes \cdots \otimes v_j) = v_1 \otimes \cdots \otimes L(0)v_i \otimes \cdots \otimes v_j, \tag{3.11}$$

$$L^{(i)}(-1)(v_1 \otimes \cdots \otimes v_j) = v_1 \otimes \cdots \otimes L(-1)v_i \otimes \cdots \otimes v_j. \tag{3.12}$$

From these definitions and the formulas (2.11), (2.17)–(2.20), we see that they satisfy

$$\begin{aligned}[g^{(k)}(p), g^{(l)}(q)] &= g^{(k)}(p)g^{(l)}(q) + g^{(l)}(q)g^{(k)}(p) = 0\,,\\ p, q &= 0, -1, \quad k \neq l\,, \end{aligned} \tag{3.13}$$

$$\begin{aligned}[L^{(k)}(p), L^{(l)}(q)] &= L^{(k)}(p)L^{(l)}(q) - L^{(l)}(q)L^{(k)}(p)\\ &= \delta_{kl}(p-q)L^{(k)}(p+q)\,, \qquad p, q = 0, -1\,, \end{aligned} \tag{3.14}$$

$$\begin{aligned}[L^{(k)}(p), g^{(l)}(q)] &= L^{(k)}(p)g^{(l)}(q) - g^{(l)}(q)L^{(k)}(p)\\ &= \delta_{kl}(p-q)L^{(k)}(p+q)\,, \qquad p, q = 0, -1\,. \end{aligned} \tag{3.15}$$

For a fixed $i, 1 \leq i \leq j$, let $X_P^{(i)} \in T_P^{(i)}$. We define the operators $g(X_P^{(i)})$ on $V^{\otimes j}$ by replacing $-a_i \dfrac{\partial}{\partial a_i}\Big|_P$ and $-a_i^{-1}\dfrac{\partial}{\partial z_i}\Big|_P$ in $X_P^{(i)}$ by $g^{(i)}(0)$ and $g^{(i)}(-1)$, respectively, and define the operators $L(X_P^{(i)})$ on $V^{\otimes j}$ by replacing $-a_i \dfrac{\partial}{\partial a_i}\Big|_P$ and $-a_i^{-1}\dfrac{\partial}{\partial z_i}\Big|_P$ in $X_P^{(i)}$ by $L^{(i)}(0)$ and $L^{(i)}(-1)$, respectively. For $X_P \in T_P(\widehat{K}(j))$ of the form (3.3), we define

$$g(X_P) = \sum_{i=1}^{j} g(X_P^{(i)})\,, \tag{3.16}$$

$$L(X_P) = \sum_{i=1}^{j} L(X_P^{(i)})\,. \tag{3.17}$$

For holomorphic sections $X^{(i)}$ of $T^{(i)}(\widehat{K}(j))$ and X of $T(\widehat{K}(j))$, we can define $g(X^{(i)})$, $g(X)$, $L(X^{(i)})$ and $L(X)$ in the obvious way. Using (2.12) and the definition of $\nu(P)$, we have

$$X\nu(P) = \nu(P)L(X)\,. \tag{3.18}$$

Given $X_{1,P}, \ldots, X_{n,P} \in T_P(\widehat{K}(j))$, we use the following notation:

$$g(X_{1,P}) \wedge \cdots \wedge g(X_{n,P}) = \frac{1}{n!}\sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}\sigma} g(X_{\sigma(1),P}) \cdots g(X_{\sigma(n),P})\,. \tag{3.19}$$

We are ready to define the form ω_j. For any nonnegative integer n, we define a holomorphic form $\omega_{j,n}$ of degree n by

$$\omega_{j,n}|_P(X_{1,P}, \ldots, X_{n,P}) = \nu(P)g(X_{1,P}) \wedge \cdots \wedge g(X_{n,P})\,, \tag{3.20}$$

where $P \in \widehat{K}$ and $X_{1,P}, \ldots, X_{n,P}$ are tangent vectors at P. From the definition, $\omega_{j,n}$ is skew-symmetric and is holomorphic in P. Thus it indeed gives a holomorphic form. It is obvious that $\omega_{j,n} = 0$ for $n > 2j = \dim_{\mathbb{C}} \widehat{K}(j)$. The holomorphic form ω_j is defined to be the sum of $\omega_{j,n}$ for all n, that is,

$$\omega_j = \sum_{n=0}^{2j} \omega_{j,n}\,. \tag{3.21}$$

From the definition and (2.35) we have

$$\omega_{1,0}|_{(0,a)}v = a^{-n}v \tag{3.22}$$

for any $v \in V_{(n)}$.

The differential forms $\omega_j, j \in \mathbb{N}$, have three important properties. The first is its meromorphicity.

Proposition 3.1. *For any integers $m, i_1, \dots, i_m, j_1, \dots, j_{n-m}$ satisfying $1 \le m \le n$, $1 \le i_1 < \cdots < i_m \le j$, $1 \le j_1 < \cdots < j_{n-m} \le j$, and for any $v_1, \dots, v_j \in V, v' \in V'$,*

$$\left\langle v', \omega_{j,n}\left(-a_{i_1}^{-1}\frac{\partial}{\partial z_{i_1}}, \dots, -a_{i_1}^{-1}\frac{\partial}{\partial z_{i_m}}, -a_{j_1}\frac{\partial}{\partial a_{j_1}}, \dots, -a_{j_{n-m}}\frac{\partial}{\partial a_{j_{n-m}}}\right)(v_1 \otimes \cdots \otimes v_j)\right\rangle \tag{3.23}$$

as a function of P is meromorphic on $\mathbb{F}_j(\mathbb{C})$ with $z_i = z_k$, $i < k$, and $z_i = \infty$ as the only possible poles, and for fixed $v_i, v_k \in V$ there is an upper bound for the orders of the pole $z_i = z_k$ of the functions (3.23) for all $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{k-1}, v_{k+1}, \dots, v_j \in V$, $v' \in V'$.

Proof. This proposition follows directly from the definition of $\omega_{j,n}$, Definition 2.2 and Theorem 2.2. □

The second property concerns the exterior derivatives of these meromorphic forms. Before stating the second property, we have to define the action of the operator Q on the space $\mathscr{H}_{V,0}(j)$. The fermion number grading on V induces $\mathbb{Z}$-gradings on $V^{\otimes j}$ and $\overline{V}$. These $\mathbb{Z}$-gradings induce a $\mathbb{Z}$-grading on $\mathrm{Hom}(V^{\otimes j}, \overline{V}) = \mathscr{H}_{V,0}(j)$. We still call these gradings the *fermion number gradings*. The operator Q on V extends to an operator (still denoted Q) on $\overline{V}$. For any $v_1, \dots, v_j \in V$ with homogeneous fermion numbers, we define

$$Q_{V^{\otimes j}}(v_1 \otimes \cdots \otimes v_j) = \sum_{i=1}^{j}(-1)^{|v_1|+\cdots+|v_{i-1}|}\, v_1 \otimes \cdots \otimes Qv_i \otimes \cdots \otimes v_j\,. \tag{3.24}$$

Using the linearity we extend $Q_{V^{\otimes j}}$ to an operator on $V^{\otimes j}$. Let $h \in \mathscr{H}_{V,0}(j) = \mathrm{Hom}(V^{\otimes j}, \overline{V})$ with fermion number n. We define $Q_{\mathscr{H}_{V,0}}h \in \mathscr{H}_{V,0}(j) = \mathrm{Hom}(V^{\otimes j}, \overline{V})$ by

$$\begin{aligned}(Q_{\mathscr{H}_{V,0}}(h))(v_1 \otimes \cdots \otimes v_j) &= Q(h(v_1 \otimes \cdots \otimes v_j)) \\ &\quad - (-1)^n h(Q_{V^{\otimes}}(v_1 \otimes \cdots \otimes v_j))\,.\end{aligned} \tag{3.25}$$

Using (2.21), the formula

$$[Q, L(0)] = 0 \tag{3.26}$$

(a consequence of (2.15)) and the definition of ν, we have

$$Q\nu(P) = \nu(P)Q_{V^{\otimes j}}\,. \tag{3.27}$$

From the condition that Y maps $V^{(m_1)} \otimes V^{(m_2)}$ to $V^{(m_1+m_2)}[[x, x^{-1}]]$ and the definition of ν, we see that the fermion number of $\nu(P)$ for any $P \in \widehat{K}$ is 0. Thus (3.27) is equivalent to

$$Q_{\mathscr{H},V,0}(\nu(P)) = 0\,. \tag{3.28}$$

Proposition 3.2. *The differential form ω_j satisfies the following equation:*

$$d\omega_j = Q_{\mathscr{K}_{V,0}}(\omega_j). \tag{3.29}$$

Proof. The formula (3.29) is equivalent to

$$d\omega_{j,n} = Q_{\mathscr{K}_{V,0}}(\omega_{j,n+1}) \tag{3.30}$$

for any $n \geq 0$. Since when evaluated at $P \in \widehat{K}(j)$, both sides of (3.30) are multilinear skew-symmetric maps from $(T_P(\widehat{K}(j)))^n$ to $\mathbb{C}$, we need only to show

$$d\omega_{j,n}\left(X_1^{(i_1)}, \dots, X_{n+1}^{(i_{n+1})}\right) = Q_{\mathscr{K}_{V,0}}\left(\omega_{j,n+1}\left(X_1^{(i_1)}, \dots, X_{n+1}^{(i_{n+1})}\right)\right) \tag{3.31}$$

for $X_k^{(i_k)} \in \Gamma(T^{(i_k)}(\widehat{K}(j)))$, $k = 1, \dots, n+1$, $1 \leq i_1 \leq \cdots \leq i_{n+1} \leq j$, equal to either $-a_{i_k}^{-1}\dfrac{\partial}{\partial z_{i_k}}$ or $-a_{i_k}\dfrac{\partial}{\partial a_{i_k}}$. Since $\dim_{\mathbb{C}} T_P^{(i_k)}(\widehat{K}(j)) = 2$, $k = 1, \dots, n+1$, both sides of (3.31) are zero if three of $i_1, \dots, i_{n+1}$ are the same. Thus we can assume that there are at least two different numbers in any three of $i_1, \dots, i_{n+1}$. Using the definition of the exterior derivative and the definition of $\omega_{j,n}$, we have

$$\begin{aligned}
&(d\omega_{j,n})\left(X_1^{(i_1)}, \dots, X_{n+1}^{(i_{n+1})}\right)\\
&\quad = \sum_{k=1}^{n+1}(-1)^{k+1} X_k^{(i_k)}\omega_{j,n}\left(X_1^{(i_1)}, \dots, X_{k-1}^{(i_{k-1})}, X_{k+1}^{(i_{k+1})}, \dots, X_{n+1}^{(i_{n+1})}\right)\\
&\qquad + \sum_{1\leq k<l\leq n+1}(-1)^{k+l}\omega_{j,n}\left([X_k^{(i_k)}, X_l^{(i_l)}], X_1^{(i_1)}, \dots, X_{k-1}^{(i_{k-1})}, X_{k+1}^{(i_{k+1})},\right.\\
&\qquad \left.\dots, X_{l-1}^{(i_{l-1})}, X_{l+1}^{(i_{l+1})}, \dots, X_{n+1}^{(i_{n+1})}\right)\\
&\quad = \sum_{k=1}^{n+1}(-1)^{k+1} X_k^{(i_k)}\nu(P)\frac{1}{n!}\\
&\qquad \cdot \sum_{\sigma\in S_{n+1},\sigma(k)=k}(-1)^{\operatorname{sgn}\sigma} g\left(X_{\sigma(1)}^{(i_{\sigma(1)})}\right)\cdots g\left(X_{\sigma(k-1)}^{(i_{\sigma(k-1)})}\right)\\
&\qquad \cdot g\left(X_{\sigma(k+1)}^{(i_{\sigma(k+1)})}\right)\cdots g\left(X_{\sigma(n+1)}^{(i_{\sigma(n+1)})}\right) + \sum_{1\leq l<k\leq n+1}(-1)^{l+k}\nu(P)\frac{1}{n!}\\
&\qquad \cdot \sum_{p\neq l,k}\ \sum_{\substack{\sigma\in S_{n+2},\sigma(l+1)=l+1\\ \sigma(k+1)=k+1,\sigma^{-1}(1)=p}}(-1)^{\operatorname{sgn}\sigma} g\left(X_{\sigma(1)-1}^{(i_{\sigma(1)-1})}\right)\\
&\qquad \cdots g\left(X_{\sigma(p-1)-1}^{(i_{\sigma(p-1)-1})}\right)g\left([X_l^{(i_l)}, X_k^{(i_k)}]\right)g\left(X_{\sigma(p+1)-1}^{(i_{\sigma(p-1)-1})}\right)\\
&\qquad \cdots g\left(X_{\sigma(l)-1}^{(i_{\sigma(l)-1})}\right)\cdot g\left(X_{\sigma(l+2)-1}^{(i_{\sigma(l+2)-1})}\right)\\
&\qquad \cdots g\left(X_{\sigma(k)-1}^{(i_{\sigma(k)-1})}\right)g\left(X_{\sigma(k+2)-1}^{(i_{\sigma(k+2)-1})}\right)\\
&\qquad \cdots g\left(X_{\sigma(n+2)-1}^{(i_{\sigma(n+2)-1})}\right).
\end{aligned} \tag{3.32}$$

For convenience we introduce the following notation: Since we assume that $1 \leq i_1 \leq \cdots \leq i_{n+1} \leq j$ and that there are at least two different numbers in any three of $i_1, \ldots, i_{n+1}$, for any given $k, 1 \leq k \leq n+1$, there are at most two of $i_1, \ldots, i_{n+1}$ including i_k equal to i_k and if there are two of $i_1, \ldots, i_{n+1}$ equal to i_k, these two must be successive. Let $j_1, \ldots, j_q$ satisfying $1 \leq j_1 < \cdots < j_q \leq j$ be the distinct numbers in $\{i_1, \ldots, i_{n+1}\}$. For any m, $1 \leq m \leq q$, if j_m is equal to only one number i_k in $\{i_1, \ldots, i_{n+1}\}$, we let $g^{(j_m),1} = g\left(X_k^{(i_k)}\right)$ and $g^{(i),2}$ the identity operator on $V^{\otimes j}$; if $j_m = i_k = i_{k+1}$, we let $g^{(j_m),1} = g\left(X_k^{(i_k)}\right)$ and $g^{(j_m),2} = g\left(X_{k+1}^{(i_{k+1})}\right)$. We also define another bracket of two operators A and B of homogeneous fermion numbers by

$$[A, B]_1 = AB - (-1)^{(|A||B|-1)}BA. \tag{3.33}$$

Then using (3.13), the k^{th} term in the first sum of the right-hand side of (3.32) is equal to: (i)

$$(-1)^{k+1} X_k^{(i_k)} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots \widehat{\left(\frac{[g^{(j_m),1}, g^{(j_m),2}]_1}{2}\right)} \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \tag{3.34}$$

(where $\widehat{\ }$ means that the factor is omitted) if i_k is the only number in $\{i_1, \ldots, i_{n+1}\}$ equal to j_m; (ii)

$$(-1)^{k+1} X_k^{(i_k)} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots g^{(j_m),2} \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \tag{3.35}$$

if $j_m = i_k = i_{k+1}$; (iii)

$$(-1)^{k+1} X_k^{(i_k)} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots g^{(j_m),1} \cdots \frac{[g^{(j_q),1} g^{(j_q),2}]_1}{2} \tag{3.36}$$

if $j_m = i_{k-1} = i_k$. Since $X_l^{(i_l)}, l = 1, \ldots, n+1$, are equal to either $-a_{i_l}^{-1} \dfrac{\partial}{\partial z_{i_l}}$ or $-a_{i_l} \dfrac{\partial}{\partial a_{i_l}}$, $g^{(j_r),1}$ and $g^{(j_r),2}$, $r = 1, \ldots, q$, are constant as functions on $\widehat{K}(j)$ by the definitions. Therefore, using (3.18), we have: (i)

$$\begin{aligned} & X_k^{(i_k)} \nu(P) \frac{[g^{(j_1),1} g^{(j_1),2}]_1}{2} \cdots \widehat{\left(\frac{[g^{(j_m),1}, g^{(j_m),2}]_1}{2}\right)} \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \\ & = \nu(P) L\left(X_k^{(i_k)}\right) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots \widehat{\left(\frac{[g^{(j_m),1}, g^{(j_m),2}]_1}{2}\right)} \\ & \quad \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \end{aligned} \tag{3.37}$$

if i_k is the only number in $\{i_1, \dots, i_{n+1}\}$ equal to j_m; (ii)

$$\begin{aligned} & X_k^{(i_k)} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots g^{(j_m),2} \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \\ & = \nu(P) L\left(X_k^{(i_k)}\right) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots g^{(j_m),2} \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \end{aligned} \tag{3.38}$$

if $j_m = i_k = i_{k+1}$; (iii)

$$\begin{aligned} & X_k^{(i_k)} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots g^{(j_m),1} \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \\ & = \nu(P) L\left(X_k^{(i_k)}\right) \frac{[g^{(j_i),1}, g^{(j_1),2}]_1}{2} \cdots g^{(j_m),1} \\ & \quad \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \end{aligned} \tag{3.39}$$

if $j_m = i_{k-1} = i_k$. Using (3.15) we see that the k^{th} term of the right-hand side of (3.32) is equal to

$$(-1)^{k+1} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots L\left(X_k^{(i_k)}\right) \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \tag{3.40}$$

in the case (i), is equal to

$$\begin{aligned} & (-1)^{k+1} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots L\left(X_k^{(i_k)}\right) g^{(j_m),2} \\ & \quad \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2}, \end{aligned} \tag{3.41}$$

in the case (ii) and is equal to

$$\begin{aligned} & (-1)^{k+1} \nu(P) \frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots L\left(X_k^{(i_k)}\right) g^{(j_m),1} \\ & \quad \cdots \frac{[g^{(j_q),1}, g^{(j_q),2}]_1}{2} \end{aligned} \tag{3.42}$$

in the case (iii). Using the axioms (2.15) and (2.16), we have

$$\begin{aligned} L\left(X_k^{(i_k)}\right) &= \left[Q_{V\otimes J}, g\left(X_k^{(i_k)}\right)\right] \\ &= Q_{V\otimes J} g\left(X_k^{(i_k)}\right) + g\left(X_k^{i_k)}\right) Q_{V\otimes J}, \end{aligned} \tag{3.43}$$

where $Q_{V^{\otimes j}} : V^{\otimes j} \to V^{\otimes j}$ is defined by (3.24). Thus the k^{th} term of the right-hand side of (3.32) is equal to

$$
\begin{aligned}
&(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \cdots Q_{V^{\otimes j}} g\left(X_k^{(i_k)}\right) \\
&\quad \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} + (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \\
&\quad \cdots g\left(X_k^{(i_k)}\right) Q_{V^{\otimes j}} \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} \\
&= (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \cdots Q_{V^{\otimes j}} g^{(j_m),1} \\
&\quad \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} + (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \\
&\quad \cdots g^{(j_m),1} Q_{V^{\otimes j}} \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} \\
&= (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \cdots Q_{V^{\otimes j}} \frac{\left[g^{(j_m),1}, g^{(j_m),2}\right]_1}{2} \\
&\quad \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} + (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \\
&\quad \cdots \frac{\left[g^{(j_m),1}, g^{(j_m),2}\right]_1}{2} Q_{V^{\otimes j}} \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2}
\end{aligned} \tag{3.44}
$$

in the case (i), is equal to

$$
\begin{aligned}
&(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \cdots Q_{V^{\otimes j}} g\left(X_k^{(i_k)}\right) g^{(j_m),2} \\
&\quad \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} + (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \\
&\quad \cdots g\left(X_k^{(i_k)}\right) Q_{V^{\otimes j}} g^{(j_m),2} \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} \\
&= (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \cdots Q_{V^{\otimes j}} g^{(j_m),1} g^{(j_m),2} \\
&\quad \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2} + (-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1}, g^{(j_1),2}\right]_1}{2} \\
&\quad \cdots g^{(j_m),1} Q_{V^{\otimes j}} g^{(j_m),2} \cdots \frac{\left[g^{(j_q),1}, g^{(j_q),2}\right]_1}{2}
\end{aligned} \tag{3.45}
$$

in the case (ii) and is equal to

$$\begin{aligned}
&(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1},g^{(j_1),2}\right]_1}{2}\cdots\left[L\left(X_k^{(i_k)}\right),g^{(j_m),1}\right]\\
&\quad\cdots\frac{\left[g^{(j_q),1},g^{(j_q),2}\right]_1}{2}+(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1},g^{(j_1),2}\right]_1}{2}\\
&\quad\cdots g^{(j_m),1}L\left(X_k^{(i_k)}\right)\cdots\frac{\left[g^{(j_q),1},g^{(j_q),2}\right]_1}{2}\\
&=(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1},g^{(j_1),2}\right]_1}{2}\cdots\left[L\left(X_k^{(i_k)}\right),g^{(j_m),1}\right]\\
&\quad\cdots\frac{\left[g^{(j_q),1},g^{(j_q),2}\right]_1}{2}+(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1},g^{(j_1),2}\right]_1}{2}\\
&\quad\cdots g^{(j_m),1}Q_{V\otimes j}g^{(j_m),2}\cdots\frac{\left[g^{(j_q),1},g^{(j_q),2}\right]_1}{2}\\
&\quad+(-1)^{k+1}\nu(P)\frac{\left[g^{(j_1),1},g^{(j_1),2}\right]_1}{2}\cdots g^{(j_m),1},g^{(j_m),2}Q_{V\otimes j}\\
&\quad\cdots\frac{\left[g^{(j_q),1},g^{(j_q),2}\right]_1}{2}
\end{aligned}\tag{3.46}$$

in the case (iii).

We prove that

$$\begin{aligned}
&Q_{V\otimes j}g^{(j_m),1}g^{(j_m),2}-g^{(j_m),1}g^{(j_m),2}Q_{V\otimes j}\\
&\quad=Q_{V\otimes j}\frac{\left[g^{(j_m),1},g^{(j_m),2}\right]_1}{2}-\frac{\left[g^{(j_m),1},g^{(j_m),2}\right]_1}{2}Q_{V\otimes j}
\end{aligned}\tag{3.47}$$

for any m such that $i_{k-1}=i_k=j_m$. Using (3.43), we have

$$\begin{aligned}
&Q_{V\otimes j}g^{(j_m),1},g^{(j_m),2}-g^{(j_m),1},g^{(j_m),2}Q_{V\otimes j}\\
&\quad=Q_{V\otimes j}\left[g^{(j_m),1},g^{(j_m),2}\right]_1+Q_{V\otimes j}g^{(j_m),2},g^{(j_m),1}\\
&\qquad-g^{(j_m),2}g^{(j_m),1}Q_{V\otimes j}-\left[g^{(j_m),1},g^{(j_m),2}\right]_1Q_{V\otimes j}\\
&\quad=Q_{V\otimes j}\left[g^{(j_m),1},g^{(j_m),2}\right]_1-\left[g^{(j_m),1},g^{(j_m),2}\right]_1Q_{V\otimes j}\\
&\qquad+Q_{V\otimes}g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)-g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes j}\\
&\quad=Q_{V\otimes j}\left[g^{(j_m),1},g^{(j_m),2}\right]_1-\left[g^{(j_m),1},g^{(j_m),2}\right]_1Q_{V\otimes j}\\
&\qquad-g\left(X_k^{(i_k)}\right)Q_{V\otimes j}g\left(X_{k-1}^{(i_{k-1})}\right)+L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)\\
&\qquad-g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes j}\\
&\quad=Q_{V\otimes j}\left[g^{(j_m),1},g^{(j_m),2}\right]_1-\left[g^{(j_m),1},g^{(j_m),2}\right]_1Q_{V\otimes j}\\
&\qquad+g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes j}-g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right)\\
&\qquad+L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)-g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes j}
\end{aligned}$$

$$= Q_{V\otimes J}\left[g^{(j_m),1}, g^{(j_m),2}\right]_1 - \left[g^{(j_m),1}, g^{(j_m),2}\right]_1 Q_{V\otimes j} - g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right) + L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right). \tag{3.48}$$

Since

$$g\left(\left[X_{k-1}^{(i_{k-1})}, X_k^{(i_k)}\right]\right) = -\left[L\left(X_k^{(i_k)}\right), g\left(X_{k-1}^{(i_{k-1})}\right)\right] = -L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right) + g\left(X_{k-1}^{(i_{k-1})}\right)L\left(X_k^{(i_k)}\right), \tag{3.49}$$

$$g\left(\left[X_{k-1}^{(i_{k-1})}, X_k^{(i_k)}\right]\right) = -\left[L\left(X_k^{(i_k)}\right), g\left(X_k^{(i_k)}\right)\right] = L\left(X_{k-1}^{(i_{k-1})}\right)g\left(X_k^{(i_k)}\right) - g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right), \tag{3.50}$$

we have

$$-L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right) + g\left(X_{k-1}^{(i_{k-1})}\right)L\left(X_k^{(i_k)}\right) = L\left(X_{k-1}^{(i_{k-1})}\right)g\left(X_k^{(i_k)}\right) - g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right) \tag{3.51}$$

or

$$-g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right) + L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right) = g\left(X_{k-1}^{(i_{k-1})}\right)L\left(X_k^{(i_k)}\right) - L\left(X_{k-1}^{(i_{k-1})}\right)g\left(X_k^{(i_k)}\right). \tag{3.52}$$

Thus

$$\begin{aligned} &-g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right) + L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right) \\ &= \frac{1}{2}\Big(-g\left(X_k^{(i_k)}\right)L\left(X_{k-1}^{(i_{k-1})}\right) + L\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right) \\ &\quad + g\left(X_{k-1}^{(i_{k-1})}\right)L\left(X_k^{(i_k)}\right) - L\left(X_{k-1}^{(i_{k-1})}\right)g\left(X_k^{(i_k)}\right)\Big). \end{aligned} \tag{3.53}$$

Using (3.43), the right-hand side of (3.53) becomes

$$\begin{aligned} &\frac{1}{2}\Big(-g\left(X_k^{(i_k)}\right)\left(Q_{V\otimes J}g\left(X_{k-1}^{(i_{k-1})}\right) + g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes J}\right) \\ &\quad + \left(Q_{V\otimes j}g\left(X_k^{(i_k)}\right) + g\left(X_k^{(i_k)}\right)Q_{V\otimes j}\right)g\left(X_{k-1}^{(i_{k-1})}\right) \\ &\quad + g\left(X_{k-1}^{(i_{k-1})}\right)\left(Q_{V\otimes j}g\left(X_k^{(i_k)}\right) + g\left(X_k^{(i_k)}\right)Q_{V\otimes J}\right) \\ &\quad - \left(Q_{V\otimes J}g\left(X_{k-1}^{(i_{k-1})}\right) + g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes J}\right)g\left(X_k^{(i_k)}\right)\Big) \\ &= \frac{1}{2}\Big(-g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right)Q_{V\otimes j} + Q_{V\otimes j}g\left(X_k^{(i_k)}\right)g\left(X_{k-1}^{(i_{k-1})}\right) \\ &\quad + g\left(X_{k-1}^{(i_{k-1})}\right)g\left(X_k^{(i_k)}\right)Q_{V\otimes J} - Q_{V\otimes j}g\left(X_{k-1}^{(i_{k-1})}\right)g\left(X_k^{(i_k)}\right)\Big) \\ &= -Q_{V\otimes J}\frac{\left[g\left(X_{k-1}^{(i_{k-1})}\right), g\left(X_k^{(i_k)}\right)\right]_1}{2} + \frac{\left[g\left(X_{k-1}^{(i_{k-1})}\right), g\left(X_k^{(i_k)}\right)\right]_1}{2}Q_{V\otimes j} \\ &= -Q_{V\otimes j}\frac{\left[g^{(j_m),1}, g^{(j_m),2}\right]_1}{2} + \frac{\left[g^{(j_m),1}, g^{(j_m),2}\right]_1}{2}Q_{V\otimes j}. \end{aligned} \tag{3.54}$$

Substituting (3.54) into the right-hand side of (3.48), we obtain (3.47).

Now we calculate the second sum in the right-hand side of (3.32). Since $X_l^{(i_l)}$, $l = 1, \dots, n+1$ are equal to either $-a_{i_l}^{-1}\dfrac{\partial}{\partial z_{i_l}}$ or $-a_{i_l}\dfrac{\partial}{\partial a_{i_l}}$,

$$g\left(\left[X_l^{(i_l)}, X_k^{(i_k)}\right]\right) = 0 \tag{3.55}$$

if $i_l \neq i_k$. Therefore we need to consider only those terms in the sum such that $i_l = i_k$, that is, $l = k-1$ and $i_{k-1} = i_k = j_m$ for some m, $1 \leq m \leq q$. From (3.15), we have

$$\begin{aligned} g\left(\left[X_{k-1}^{(i_{k-1})}, X_k^{(i_k)}\right]\right) &= -g\left(\left[X_k^{(i_k)}, X_{k-1}^{(i_{k-1})}\right]\right) \\ &= -\left[L\left(X_k^{(i_k)}\right), g\left(X_{k-1}^{(i_{k-1})}\right)\right] \\ &= -\left[L\left(X_k^{(i_k)}\right), g^{(j_m),1}\right]. \end{aligned} \tag{3.56}$$

Thus these terms are of the form

$$\begin{aligned} &-\nu(P)g\left(\left[X_{k-1}^{(i_{k-1})}, X_k^{(i_k)}\right]\right)\frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots \widehat{\left(\frac{[g^{(j_m),1}, g^{(j_m),2}]_1}{2}\right)} \\ &\quad \cdots \frac{[g^{(j_{n+1}),1}, g^{(j_{n+1}),2}]_1}{2} \\ &= (-1)^{k-1}\nu(P)\frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \\ &\quad \cdots \frac{[g^{(j_{m-1}),1}, g^{(j_{m-1}),2}]_1}{2} g\left(\left[X_k^{(i_{k-1})}, X_l^{(i_k)}\right]\right)\frac{g[^{(j_{m+1}),1}, g^{(j_{m+1}),2}]_1}{2} \\ &\quad \cdots \frac{[g^{(j_{n+1}),1}, g^{(j_{n+1}),2}]_1}{2} \\ &= (-1)^{k}\nu(P)\frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \\ &\quad \cdots \frac{[g^{(j_{m-1}),1}, g^{(j_{m-1}),2}]_1}{2}\left[L\left(X_k^{(i_k)}\right), g^{(j_m),1}\right]\frac{[g^{(j_{m+1}),1}, g^{(j_{m+1}),2}]_1}{2} \\ &\quad \cdots \frac{[g^{(j_{n+1}),1}, g^{(j_{n+1}),2}]_1}{2}. \end{aligned} \tag{3.57}$$

Substituting (3.44)–(3.46) and (3.57) into (3.32) and using (3.47), we obtain

$$\begin{aligned} &(d\omega_{j,n})\left(X_1^{(i_1)}, \cdots, X_{n+1}^{(i_{n+1})}\right) \\ &\quad = \nu(P)Q_{V\otimes j}\frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots \frac{[g^{(j_{n+1}),1}, g^{(j_{n+1}),2}]_1}{2} \\ &\qquad + (-1)^{n+2}\nu(P)\frac{[g^{(j_1),1}, g^{(j_1),2}]_1}{2} \cdots \frac{[g^{(j_{n+1}),1}, g^{(j_{n+1}),2}]_1}{2}Q_{V\otimes j} \end{aligned}$$

$$= \nu(P)Q_{V^{\otimes j}}\frac{1}{(n+j)!}\sum_{\sigma\in S_{n+1}}(-1)^{\mathrm{sgn}\sigma}g\Big(X_{\sigma(1)}^{(i_{\sigma(1)})}\Big)\cdots g\Big(X_{\sigma(n+1)}^{(i_{\sigma(n+1)})}\Big)$$
$$+(-1)^{n+2}\nu(P)\frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}(-1)^{\mathrm{sgn}\sigma}g\Big(X_{\sigma(1)}^{(i_{\sigma(1)})}\Big)\cdots g\Big(X_{\sigma(n+1)}^{(i_{\sigma(n+1)})}\Big)Q_{V^{\otimes j}}$$
$$= Q\nu(P)\frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}(-1)^{\mathrm{sgn}\sigma}g\Big(X_{\sigma(1)}^{(i_{\sigma(1)})}\Big)\cdots g\Big(X_{\sigma(n+1)}^{(i_{\sigma(n+1)})}\Big)$$
$$+(-1)^{n+2}\nu(P)\frac{1}{(n+1)!}\sum_{\sigma\in S_{n+1}}(-1)^{\mathrm{sgn}\sigma}g\Big(X_{\sigma(1)}^{(i_{\sigma(1)})}\Big)\cdots g\Big(X_{\sigma(n+1)}^{(i_{\sigma(n+1)})}\Big)Q_{V^{\otimes j}}$$
$$= Q_{\mathscr{H}_{V,0}}\Big(\omega_{j,n+1}\Big(X_1^{(i_1)},\cdots,X_{n+1}^{(i_{n+1})}\Big)\Big). \quad \square \tag{3.58}$$

The third property describes how the partial operad $\widehat{K}$ (or $\overline{K}$) acts on $\omega_j, j\in\mathbb{N}$. In fact it is this property which characterizes the three classes of topological vertex algebras. Let $P_0\in\widehat{K}(k), P_1\in\widehat{K}(j_1),\dots,P_k\in\widehat{K}(j_k)$. Assume that $\gamma_{\widehat{K}}(P_0;P_1,\dots,P_k)$ exists. Since there is a natural isomorphism from $T_{P_0}(\widehat{K}(k))\oplus T_{P_1}(K(j_1))\oplus\cdots\oplus T_{P_k}(\widehat{K}(j_k))$ to $T_{(P_0,\dots,P_k)}(\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k))$ (the holomorphic tangent space of $\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)$ at $(P_0,\dots,P_k)$), the holomorphic tangent spaces $T_{P_0}(\widehat{K}(k))$, $T_{P_1}(\widehat{K}(j_1)),\dots,T_{P_k}(\widehat{K}(j_k))$ can all be embedded in $T_{(P_0,\dots,P_k)}(\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k))$. We denote these embeddings by $e^{(0)},\dots,e^{(k)}$, respectively. For convenience we sometimes use the notation $j_0=k$. Let $X_{li}\in T_{P_l}(\widehat{K}(j_l))$, $i=1,\dots,n_l$, $l=0,\dots,k$. Then $(\gamma_{\widehat{K}})_*(e_*^{(l)}(X_{li}))$, $i=1,\dots,n_l$, $l=0,\dots,k$, are elements of $T_{\gamma_{\widehat{K}}(P_0;P_1,\dots,P_k)}(\widehat{K}(j_1+\cdots+j_k))$. For any vector bundle B, we use $\mathscr{T}(B)$ to denote the direct sum of all tensor powers of B. Then the maps $e^{(l)}$, $l=0,\dots,k$, and $(\gamma_{\widehat{K}})_*$ can be extended naturally to maps from $\mathscr{T}(T(\widehat{K}(j_l)))$ to $\mathscr{T}(T(\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)))$ and a map from $\mathscr{T}(T(\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)))$ to $\mathscr{T}(T(\widehat{K}(j_1+\cdots+j_k)))$, respectively, and we still use $e^{(l)}$, $l=0,\dots,k$, and $(\gamma_{\widehat{K}})_*$ to denote these maps. Let $e^{(0)}+\cdots+e^{(k)}$ be the map from $\mathscr{T}(T(\widehat{K}(j_0)))\times\cdots\times\mathscr{T}(T(\widehat{K}(j_k)))$ to $\mathscr{T}(T(\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)))$ defined by taking sum of the images of $e^{(l)}$, $l=0,\dots,k$. Then $(\gamma_{\widehat{K}})_*\circ(e^{(0)}+\cdots+e^{(k)})$ is a map from $\mathscr{T}(T(\widehat{K}(j_0)))\times\cdots\times\mathscr{T}(T(\widehat{K}(j_k)))$ to $\mathscr{T}(T(\widehat{K}(j_1+\cdots+j_k)))$. It is easy to see that the family $\{\mathscr{T}(T(\widehat{K}(j)))\mid j\in\mathbb{N}\}$, equipped with these maps, the zero vector in the fiber of $\mathscr{T}(T(\widehat{K}(1)))$ at the identity I of $\widehat{K}$ and the obvious actions of the symmetry groups, is a partial operad. We denote this partial operad by $\mathscr{T}(T(\widehat{K}))$.

The manifold $\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)$ has natural fibrations whose fibers are obtained by fixing certain punctures and the corresponding local coordinates of the projection images of its elements to $\widehat{K}(k),\widehat{K}(j_1),\dots,\widehat{K}(j_k)$. We call these fibrations *coordinate fibrations*. We also have fibrations of $\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)$ whose fibers are obtained by fixing certain punctures and the corresponding local coordinates of the images of its elements under the map $\gamma_{\widehat{K}}:\widehat{K}(k)\times\widehat{K}(j_1)\times\cdots\times\widehat{K}(j_k)\to\widehat{K}(j_1+\cdots+j_k)$. We call these fibrations *pull-back coordinate fibrations*.

We also denote the space of holomorphic functions on a complex manifold M by $\mathrm{Hol}(M)$.

Proposition 3.3. *Let V be a strong topological vertex algebra and $\gamma_{\mathscr{H}_{V,0}}$ be the substitution map (composition map) for the endomorphism partial pseudo-operad $\mathscr{H}_{V,0}$. Then the forms ω_j, $j \in \mathbb{N}$, define a morphism from $\mathscr{T}(T(\widehat{K}))$ to $\mathscr{H}_{V,0}$. More precisely, we have*

$$\begin{aligned}
&\omega_{j_1+\cdots+j_k,\, n_0+\cdots+n_k}\ \Big((\gamma_{\widehat{K}})_*(e_*^{(0)}\Big(X_{01}\Big)),\ \ldots,\ (\gamma_{\widehat{K}})_* \\
&\quad\times\Big(e_*^{(k)}\Big(X_{kn_k}\Big)\Big)\Big)\Big|_{\gamma_{\widehat{K}}(P_0;\,P_1,\,\ldots,\,P_k)} \\
&\quad=\gamma_{\mathscr{H}_{V,0}}\Big(\omega_{k,n_0}\Big(e_*^{(0)}(X_{01}),\ \ldots,\ e_*^{(0)}(X_{0n_0})\Big)\Big|_{P_0}; \\
&\qquad\omega_{j_1,n_1}\Big(e_*^{(1)}(X_{11}),\ \ldots,\ e_*^{(1)}(X_{1j_1})\Big)\Big|_{P_1},\ \ldots,\ \omega_{j_k,n_k}\Big(e_*^{(k)}(X_{k1}), \\
&\qquad\ldots,\ e_*^{(k)}(X_{kn_k})\Big)\Big|_{P_k}\Big)
\end{aligned} \tag{3.59}$$

for $P_l \in \widehat{K}$, $l = 0, \ldots, k$, and $X_{li} \in T_{P_l}(\widehat{K}(j_l))$, $i = 1, \ldots, n_l$, $l = 0, \ldots, k$, and

$$\sigma(\omega_j|_P) = \omega|_{\sigma(P)} \tag{3.60}$$

for any element σ of the symmetry group S_j and any element P of $\widehat{K}(j)$. If V is a topological vertex algebra, then there exist maps

$$\begin{aligned}
&h, \alpha : (\Gamma(T(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k))))^{\otimes(n_0+\cdots+n_k)} \\
&\quad\to \mathscr{H}_{V,0}(j_1 + \cdots + j_k) \otimes \mathrm{Hol}(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)),
\end{aligned}$$

where α is a sum of maps which are holomorphic forms when restricted to certain tensor factors of

$$(\Gamma(T(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \in \widehat{K}(j_k))))^{\otimes(n_0+\cdots+n_k)}$$

and are exact form when these holomorphic forms are restricted to fibers of certain coordinate fibrations or pull-back coordinate fibrations of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$, such that modulo $Q_{\mathscr{H}_{V,0}} \circ h$ and α, the forms ω_j, $j \in \mathbb{N}$, is a morphism as in the case of strong topological vertex algebras. More precisely, in this case (3.60) and

$$\begin{aligned}
&\omega_{j_1+\cdots+j_k,n_0+\cdots+n_k}\ ((\gamma_{\widehat{K}})_*(e_*^{(0)}(X_{01})),\ \ldots,\ (\gamma_{\widehat{K}})_*(e_*^{(k)}(X_{kn_k})))|_{\gamma_{\widehat{K}}(P_0;P_1,\,\ldots,\,P_k)} \\
&\quad=\gamma_{\mathscr{H}_{V,0}}(\omega_{k,n_0}(e_*^{(0)}(X_{01}),\ \ldots,\ e_*^{(0)}(X_{0n_0}))|_{P_0}; \\
&\qquad\omega_{j_1,n_1}(e_*^{(1)}(X_{11}),\ \ldots,\ e_*^{(0)}(X_{1j_1}))|_{P_1}, \\
&\qquad\ldots,\ \omega_{j_k,n_k}(e_*^{(k)}(X_{k1}),\ \ldots,\ e_*^{(k)}(X_{kn_k}))|_{P_k}) \\
&\qquad+ Q_{\mathscr{H}_{V,0}}(h((\gamma_{\widehat{K}})_*(e_*^{(0)}(X_{01})),\ \ldots,\ (\gamma_{\widehat{K}})_*(e_*^{(k)}(X_{kn_k})))|_{(P_0,\,\ldots,\,P_k)}) \\
&\qquad+ \alpha((\gamma_{\widehat{K}})_*(e_*^{(0)}(X_{01})),\ \ldots,\ (\gamma_{\widehat{K}})_*(e_*^{(k)}(X_{kn_k})))|_{(P_0,\,\ldots,\,P_k)}
\end{aligned} \tag{3.61}$$

hold. Moreover,

$$h = 0 \quad \text{when } n_0 = 0 \quad \text{or} \quad n_1 = \cdots = n_k = 0 \tag{3.62}$$

and

$$\alpha = 0 \quad \text{when } n_0, n_1, \ldots, n_k = 0, 1\,. \tag{3.63}$$

If V is a weak topological vertex algebra, (3.59) *holds when $n_0 = 0$ or $n_1 = \cdots = n_k = 0$ and* (3.61) *and* (3.63) *hold for $P_l \in \overline{K}$, $l = 0, \ldots, k$ and $X_{li} \in T_{P_l}(\overline{K}(j_l))$, $i = 1, \ldots, n_l$, $l = 0, \ldots, k$.*

Proof. In all three cases, property (3.60) follows from the fact that ν is a morphism of partial operad.

We first prove (3.59) for strong topological vertex algebras. For convenience we use the notations $Y_{li} = e_*^{(l)}(X_{li})$, $i = 1, \ldots, n_l$, $l = 0, \ldots, k$. Also in the proof below, we regard $P_0, \ldots, P_k$ as variables and omit the symbols $|_{P_0}, \ldots, |_{P_k}, |_{\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k)}$ and $|_{(P_0, \ldots, P_k)}$. By the definition of $\omega_{j_1+\cdots+j_k, n_0+\cdots+n_k}$, we have

$$\begin{aligned}&\omega_{j_1+\cdots+j_k, n_0+\cdots+n_k}\ ((\gamma_{\widehat{K}})_*(Y_{01}), \ldots, (\gamma_{\widehat{K}})_*(Y_{kn_k}))\\ &\quad = \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))g((\gamma_{\widehat{K}})_*(Y_{01})) \wedge \cdots \wedge g((\gamma_{\widehat{K}})_*(Y_{kn_k})).\end{aligned} \tag{3.64}$$

Since for strong topological vertex algebras, $g((\gamma_{\widehat{K}})_*(Y_{ls}))$ and $g((\gamma_{\widehat{K}})_*((Y_{mt}))$ anticommute with each other for $s = 1, \ldots, n_l$, $t = 1, \ldots, n_m, l, m = 0, \ldots, k$, the right-hand side of (3.64) can be written as

$$\begin{aligned}&\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))(g((\gamma_{\widehat{K}})_*(Y_{01})) \wedge \cdots \wedge g((\gamma_{\widehat{K}})_*(Y_{0n_0})))\\ &\quad \cdots (g((\gamma_{\widehat{K}})_*(Y_{1n_k})) \wedge \cdots \wedge g((\gamma_{\widehat{K}})_*(Y_{kn_k}))).\end{aligned} \tag{3.65}$$

On the other hand,

$$\begin{aligned}&\gamma_{\mathscr{Y}_{V,W}}(\omega_{k,n_0}(Y_{01}, \ldots, Y_{0n_0}); \omega_{j_1,n_1}(Y_{11}, \ldots, Y_{1n_1}), \ldots, \omega_{j_k,n_k}(Y_{k1}, \ldots, Y_{kn_k}))\\ &\quad = \gamma_{\mathscr{Y}_{V,W}}(\nu(P_0)g(Y_{01}) \wedge \cdots \wedge g(Y_{0n_0});\\ &\qquad \nu(P_1)g(Y_{11}) \wedge \cdots \wedge g(Y_{1n_1}),\\ &\qquad \ldots, \nu(P_k)g(Y_{k1}) \wedge \cdots \wedge g(Y_{kn_k})).\end{aligned} \tag{3.66}$$

To prove that (3.65) is equal the right-hand side of (3.66), we need only to prove

$$\begin{aligned}&\nu(g_{\widehat{K}}(P_0; P_1, \ldots, P_k))(g((\gamma_{\widehat{K}})_*(Y_{01})) \ldots g((\gamma_{\widehat{K}})_*(Y_{0n_0})))\\ &\qquad \cdots (g((\gamma_{\widehat{K}})_*(Y_{1n_k})) \ldots g((\gamma_{\widehat{K}})_*(Y_{kn_k})))\\ &\quad = \gamma_{\mathscr{Y}_{V,W}}(\nu(P_0)g(Y_{01}) \ldots g(Y_{0n_0}); \nu(P_1)g(Y_{11}) \ldots g(Y_{1n_1}),\\ &\qquad \ldots, \nu(P_k)g(Y_{k1}) \ldots g(Y_{kn_k})).\end{aligned} \tag{3.67}$$

To prove (3.67), we first note that the bracket formulas of $g(0)$, $g(-1)$ with vertex operators and with $L(0), L(-1)$ are the same as those of $L(0), L(-1)$ with vertex operators and with $L(0), L(-1)$ themselves, except suitable signs coming from the difference between the fermion numbers of $g(0), g(-1)$ and of $L(0), L(-1)$. Therefore if we can prove

$$\begin{aligned}&\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))L((\gamma_{\widehat{K}})_*(Y_{01})) \ldots L((\gamma_{\widehat{K}})_*(Y_{kn_k}))\\ &\quad = \gamma_{\mathscr{Y}_{V,W}}(\nu(P_0)L(Y_{01}) \ldots L(Y_{0n_0});\\ &\qquad \nu(P_1)L(Y_{11}) \ldots L(Y_{1n_1}), \ldots, \nu(P_k)L(Y_{k1}) \ldots L(Y_{kn_k}))\end{aligned} \tag{3.68}$$

without using the bracket formulas for $L(0)$ and $L(-1)$, (3.67) will be a consequence. To prove (3.68), we first use (3.18) and the fact that ν is a morphism of partial pseudo-operads to obtain

$$\begin{aligned}
&\nu(\gamma_{\widehat{K}}(P_0; P_1, \dots, P_k))L((\gamma_{\widehat{K}})_*(Y_{01})) \dots L((\gamma_{\widehat{K}})_*(Y_{kn_k})) \\
&\quad = (\gamma_{\widehat{K}})_*(Y_{01}) \dots (\gamma_{\widehat{K}})_*(Y_{kn_k})\nu(\gamma_{\widehat{K}}(P_0; P_1, \dots, P_k)) \\
&\quad = (\gamma_{\widehat{K}})_*(Y_{01}) \dots (\gamma_{\widehat{K}})_*(Y_{kn_k})\gamma_{\mathscr{H}_{V,W}}(\nu(P_0); \nu(P_1), \dots, \nu(P_k)). \qquad (3.69)
\end{aligned}$$

Note that as differential operators $(\gamma_{\widehat{K}})_*(Y_{li})$'s are the same as Y_{li}'s. We denote these differential operators as ∂_{li}. Then

$$\begin{aligned}
&(\gamma_{\widehat{K}})_*(Y_{01}) \dots (\gamma_{\widehat{K}})_*(Y_{kn_k})\gamma_{\mathscr{H}_{V,W}}(\nu(P_0); \nu(P_1), \dots, \nu(P_k)) \\
&\quad = \partial_{01} \dots \partial_{kn_k}\gamma_{\mathscr{H}_{V,W}}(\nu(P_0); \nu(P_1), \dots, \nu(P_k)) \\
&\quad = \gamma_{\mathscr{H}_{V,W}}(\partial_{01} \dots \partial_{0n_0}\nu(P_0); \partial_{11} \dots \partial_{1n_1}\nu(P_1), \dots, \partial_{k1} \dots \partial_{kn_k}\nu(P_k)) \\
&\quad = \gamma_{\mathscr{H}_{V,W}}(\nu(P_0)L(Y_{01}) \dots L(Y_{0n_0}); \\
&\qquad \nu(P_1)L(Y_{11}) \dots L(Y_{1n_1}), \dots, \nu(P_k)L(Y_{k1}) \dots L(Y_{kn_k}))\,, \qquad (3.70)
\end{aligned}$$

where we have used (3.18) and

$$\partial_{li}\nu(P_m) = 0, \quad i = 1, \dots, n_l, \quad l \neq m\,. \qquad (3.71)$$

This proves (3.59). Note that the proof above also proves that (3.65) is equal to the right-hand side of (3.66) even for weak topological vertex algebras.

To prove the case for topological vertex algebras, we first notice that from the definition that

$$(\gamma_{\widehat{K}})_*(Y_{li}) \in \bigoplus_{j=j_1+\cdots+j_{l-1}+1}^{j_1+\cdots+j_l} \Gamma(T^{(i)}(\widehat{K}(j_1 + \cdots + j_k)))\,,$$

$i = 1, \dots, n_l$. Thus by (3.13) $g((\gamma_{\widehat{K}})_*(Y_{li}))$ and $g((\gamma_{\widehat{K}})_*(Y_{mi}))$ anti-commute with each other if $l \neq m$ and $l, m \neq 0$. We take X_{li} to be in $\Gamma(T^{(p)}(\widehat{K}(j_l)))$ for some p, $1 \leq p \leq n_l$. Then if there exist l and p such that more than two of $X_{li}, i = 1, \dots, n_l$, are in $\Gamma(T^{(p)}(\widehat{K}(j_l)))$, both the left-hand side and the first term in the right-hand side of (3.61) are zero. Thus we can define the values of h and α on these tangent vector fields to be zero. The formula (3.61) holds in these cases. Thus we can assume that for any l and any p there are at most two of $X_{li}, i = 1, \dots, n_l$, in $\Gamma(T^{(p)}(\widehat{K}(j_l)))$. In these case the left-hand side of (3.61) can be written as a sum with suitable sign of terms of the form

$$\nu(\gamma_{\widehat{K}}(P_0; P_1, \dots, P_k))A^{(1)} \dots A^{(j_1+\cdots+j_k)}\,, \qquad (3.72)$$

where $A^{(q)}, q = 1, \dots, j_1 + \cdots + j_k$, are operators on $V^{\otimes(j_1+\cdots+j_k)}$ equal to the skew-symmetrization of the product of at most four of $g(((\gamma_{\widehat{K}})_*(Y_{li}))^{(q)})$ such that at most two of X_{li} are in the set $\{X_{0i} \mid i = 1, \dots, n_0\}$ and at most two of X_{li} are in the set $\{X_{li} \mid i = 1, \dots, n_l, l = 1, \dots, k\}$. We need the following lemma.

Lemma 3.4. *For any q satisfying $1 \le q \le j_1+\cdots+j_k$, we can write $q = j_1+\cdots j_{l-1}+i$ for some l, i satisfying $1 \le l \le k, l \le i \le j_l$, respectively. Then $A^{(q)}$ can be written as a sum of three terms: the first term is the product of the skew-symmetrization of the product of those $g(((\gamma_{\widehat{K}})_*(Y_{ms}))^{(q)})$'s with X_{ms}'s in $\{X_{01} \mid s = 1, \dots, n_0\}$ and the skew-symmetrization of the product of those $g(((\gamma_{\widehat{K}})_*(Y_{ms}))^{(q)})$'s with X_{ms}'s in $\{X_{ls} | s = 1, \dots, n_l, l = 1, \dots, k\}$; the second term is Q-exact; and the third term satisfies the property that the composition of $\nu(\gamma_{\widehat{K}}(P_0; P_1, \dots, P_k))$ with it is an exact form evaluated at those tangent vector fields defining $A^{(q)}$ on any leave of the fibration of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ obtained by fixing all punctures and local coordinates of elements of $\widehat{K}(j_m), m = 0, \dots, k$, except the l^{th} ones of elements of $\widehat{K}(k)$ and the i^{th} ones of elements of $\widehat{K}(j_l)$.*

Proof. For simplicity, we prove only the case that $A^{(q)}$ is the skew-symmetrization of the product of three of $g(((\gamma_{\widehat{K}})_*(Y_{ms}))^{(q)}$ with only one of them obtained from a tangent vector field on $\widehat{K}(k)$. Let $P_m = (z_{m1}, \dots, z_{mj_m}; a_{m1}, \dots, a_{mj_m}), m = 0, \dots, k$. We assume that the corresponding three X_{ms}'s are $-a_{0l}\dfrac{\partial}{\partial a_{0l}}$, $-a_{li}^{-1}\dfrac{\partial}{\partial z_{li}}$ and $-a_{li}\dfrac{\partial}{\partial a_{li}}$. The other cases can be proved in the same way. For convenience, we denote $e_*^{(0)}\left(-a_{0l}\dfrac{\partial}{\partial a_{0l}}\right)$, $e_*^{(l)}\left(-a_{li}^{-1}\dfrac{\partial}{\partial z_{li}}\right)$ and $e_*^{(l)}\left(-a_{li}\dfrac{\partial}{\partial a_{li}}\right)$ by Y_{01}, Y_{l1} and Y_{l2}, respectively. It is easy to see that

$$\begin{aligned}\gamma_{\widehat{K}}(P_0; P_1, \dots, P_k) = (&z_{01} + a_{01}z_{11}, \dots, z_{01} + a_{01}z_{1j_1}, \dots,\\ &z_{0k} + a_{0k}z_{k1}, \dots, z_{0k} + a_{0k}z_{kj_k};\\ &a_{01}a_{11}, \dots, a_{01}a_{1j_1}, \dots, a_{0k}a_{k1}, \dots, a_{0k}a_{kj_k}),\end{aligned} \tag{3.73}$$

$$g(((\gamma_{\widehat{K}})_*(Y_{01}))^{(q)}) = z_{li}g^{(q)}(-1) + g^{(q)}(0), \tag{3.74}$$

$$g(((\gamma_{\widehat{K}})_*(Y_{l1}))^{(q)} = g^{(q)}(-1), \tag{3.75}$$

$$g(((\gamma_{\widehat{K}})_*(Y_{l2}))^{(q)}) = g^{(q)}(0). \tag{3.76}$$

Let

$$U_{-1,-1} = \frac{[g(0), [g(-1), g(-1)]]}{2} = -[g(-1), [g(0), g(-1)]], \tag{3.77}$$

$$U_{0,-1} = [g(0), [g(0), g(-1)] = \frac{[g(-1), [g(0), g(0)]]}{2}. \tag{3.78}$$

Also for topological vertex algebras, we fix a choice of $U_{0,0}$ in (2.25). Let $U_{-1,-1}^{(q)}$, $U_{0,-1}^{(q)}$ and $U_{0,0}^{(q)}$ be the operators on $V^{\otimes(j_1+\cdots+j_k}$) equal to $U_{-1,-1}$, $U_{0,-1}$, respectively, acting on the q^{th} tensor factor. For topological vertex algebras, the bracket $[g(Z_1), g(Z_2)]$ is Q-exact for any tangent vectors $Z_1, Z_2 \in T_{\gamma_{\widehat{K}}(P_0;P_1,\dots,P_k)}$ $(\widehat{K}(j_1 + \cdots + j_k))$. The bracket $[g(Z_1), g(Z_2)]$ can be written as a linear combination of $[g^{(r)}(-1), g^{(r)} - (1)]$, $[g^{(r)}(0), g^{(r)}(-1)]$ and $[g^{(r)}(0), g^{(r)}(0)]$, $r = 1, \dots, j_1 + \cdots + j_k$. For any such Z_1 and Z_2 we define an operator U_{Z_1,Z_2} on $V^{\otimes(j_1+\cdots+j_k)}$ by replacing $[g^{(r)}(-1), g^{(r)}(-1)]$,

$[g^{(r)}(0), g^{(r)}(-1)]$ and $[g^{(r)}(0), g^{(r)}(0)]$, $r = 1, \dots, j_1 + \cdots + j_k$, in the linear expansion of $[g(Z_1), g(Z_2)]$ by $U^{(r)}_{-1,-1}$, $U^{(r)}_{0,-1}$ and $U^{(r)}_{0,0}$, respectively. Then

$$[g(Z_1), g(Z_2)] = [Q_{V^{\otimes(j_1+\cdots+j_k)}}, U_{Z_1,Z_2}] \tag{3.79}$$

by (2.25), (2.27) and (2.28).

We have

$$A^{(q)} = (z_{li}g^{(q)}(-1) + g^{(q)}(0)) \wedge g^{(q)}(-1) \wedge g^{(q)}(0)\,. \tag{3.80}$$

where the right-hand side is the skew-symmetrization of

$$(z_{li}g^{(q)}(-1) + g^{(q)}(0))g^{(q)}(-1)g^{(q)}(0)\,.$$

It is a straightforward calculation to obtain

$$\begin{aligned}
&(z_{li}g^{(q)}(-1) + g^{(q)}(0)) \wedge g^{(q)}(-1) \wedge g^{(q)}(0)\\
&\quad = (z_{li}g^{(q)}(-1) + g^{(q)}(0))\frac{[g^{(q)}(-1), g^{(q)}(0)]_1}{2}\\
&\qquad - \frac{1}{2}g^{(q)}(0)[(z_{li}g^{(q)}(-1) + g^{(q)}(0)), g^{(q)}(-1)]\\
&\qquad + \frac{1}{2}g^{(q)}(-1)[(z_{li}g^{(q)}(-1) + g^{(q)}(0)), g^{(q)}(0)]\\
&\qquad - \frac{1}{3}[[g^{(q)}(-1), (z_{li}g^{(q)}(-1) + g^{(q)}(0))], g^{(q)}(0)]\\
&\qquad + \frac{1}{3}[[g^{(q)}(0), (z_{li}g^{(q)}(-1) + g^{(q)}(0))], g^{(q)}(-1)]\,.
\end{aligned} \tag{3.81}$$

Now from (3.80), (3.81), (3.77), (3.78), (3.74)–(3.76), (3.43) and (3.79) we obtain

$$\begin{aligned}
A^{(q)} &= (z_{li}g^{(q)}(-1) + g^{(q)}(0)) \wedge g^{(q)}(-1) \wedge g^{(q)}(0)\\
&= g(((\gamma_{\widehat{K}})_*(Y_{01}))^{(q)})\frac{[g(((\gamma_{\widehat{K}})_*(Y_{l1}))^{(q)}), g(((\gamma_{\widehat{K}})_*(Y_{l2}))^{(q)})]_1}{2}\\
&\quad + \frac{1}{2}[Q_{V^{\otimes(j_1+\cdots+j_k)}}, g(((\gamma_{\widehat{K}})_*(Y_{l2}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l1})}]\\
&\quad - \frac{1}{2}[Q_{V^{\otimes(j_1+\cdots+j_k)}}, g(((\gamma_{\widehat{K}})_*(Y_{l1}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l2})}]\\
&\quad - \frac{1}{2}L(((\gamma_{\widehat{K}})_*(Y_{l2}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l1})}\\
&\quad + \frac{1}{2}L(((\gamma_{\widehat{K}})_*(Y_{l1}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l2})}\\
&\quad - U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),[(\gamma_{\widehat{K}})_*(Y_{l1})),(\gamma_{\widehat{K}})_*(Y_{l1})]}\,.
\end{aligned} \tag{3.82}$$

To prove the lemma in this case, we need only to show that the sum of the last three terms in (3.82) composed with $\nu(\gamma_{\widehat{K}}(P_0; P_1, \dots, P_k))$ is an exact form on the fibers of the fibration described in the lemma evaluated at

$$(\gamma_{\widehat{K}})_*(Y_{01}) \otimes (\gamma_{\widehat{K}}) \otimes (\gamma_{\widehat{K}})_*(Y_{l1})) \otimes (\gamma_{\widehat{K}})_*(Y_{l2})\,.$$

We define a 2-form β on $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ by:

$$\beta\left(e_*^{(0)}(X_1), e_*^{(0)}(X_2)\right) = 0 \tag{3.83}$$

if $X_1, X_2 \in \Gamma(T(\widehat{K}(k)))$;

$$\beta\left(e_*^{(l)}(X_1), e_*^{(m)}(X_2)\right) = 0 \tag{3.84}$$

if $X_1 \in \Gamma(T(\widehat{K}(j_l)))X_2 \in \Gamma(T(\widehat{J}(j_m)))$, $l, m = 1, \ldots, k$;

$$\beta(e_*^{(0)}(X_1), e_*^{(l)}(X_2)) = \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))U_{(\gamma_{\widehat{K}})_*(e_*^{(0)}(X_1)),(\gamma_{\widehat{K}})_*(e_*^{(l)}(X_2))} \tag{3.85}$$

if $X_1 \in \Gamma(T(\widehat{K}(k)))$, $X_2 \in \Gamma(T(\widehat{K}(j_l)))$, $l = 1, \ldots, k$. Using the facts that $X_1, X_2 \in \Gamma(T(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)))$ can always be expanded as linear combinations of those tangent vector fields occurred in (3.83), (3.84) and (3.85) and that β should be skew-symmetric, (3.83), (3.84) and (3.85) give a well-defined form. Now we restrict β to fibers of the fibration of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ given by z_{ms} = constant, a_{ms} = constant, $m \neq 0, l$ or $m = 0$, $s \neq l$, $s \neq i$. On such a fiber, we have

$$\begin{aligned}
&(d\beta)(Y_{01}.Y_{l1}, Y^{l2}) \\
&\quad = Y_{01}\beta(Y_{l1}, Y_{l2}) - Y_{l1}(Y_{01}, Y_{l2}) + Y_{l2}\beta(Y_{01}, Y_{l1}) \\
&\qquad - \beta([Y_{01}, Y_{l1}], Y_{l2}) + \beta([Y_{01}, Y_{l2}], Y_{l1}) - \beta([Y_{l1}, Y_{l2}], Y_{01}) \\
&\quad = -\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))L(((\gamma_{\widehat{K}})_*(Y_{l1}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l2})} \\
&\qquad - \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))Y_{l1}U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l2})} \\
&\qquad + \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))L(((\gamma_{\widehat{K}})_*(Y_{l2}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l1})} \\
&\qquad + \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))Y_{l2}U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l1})} \\
&\qquad + \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))U_{(\gamma_{\widehat{K}})_*(Y_{01}),[(\gamma_{\widehat{K}})_*(Y_{l1}),(\gamma_{\widehat{K}})_*(Y_{l2})]} \\
&\quad = -\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))L(((\gamma_{\widehat{K}})_*(Y_{l1}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l2})} \\
&\qquad + \nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))L(((\gamma_{\widehat{K}})_*(Y_{l2}))^{(q)})U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),(\gamma_{\widehat{K}})_*(Y_{l1})} \\
&\qquad + 2\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))U^{(q)}_{(\gamma_{\widehat{K}})_*(Y_{01}),[(\gamma_{\widehat{K}})_*(Y_{l1}),(\gamma_{\widehat{K}})_*(Y_{l2})]}.
\end{aligned} \tag{3.86}$$

Thus we see that composed with $\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))$ the sum of the last three term in (3.82) is equal to $-\frac{1}{2}d\beta(Y_{01}, Y_{l1}, Y_{l2})$. This proves the lemma in this case. □

Using this lemma, we can write (3.72) as a sum of terms of the following types: the composition of $\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))$ with the product of the first terms of all $A^{(q)}$, $q = 1, \ldots, j_1 + \cdots + j_k$; the composition of $\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))$ with the product of the Q-exact terms of all $A^{(q)}$, $q = 1, \ldots, j_1 + \cdots + j_k$; compositions of $\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))$ with products of some of the $A^{(q)}$'s and the Q-exact terms of the other $A^{(q)}$'s; compositions of $\nu(\gamma_{\widehat{K}}(P_0; P_1, \ldots, P_k))$ with products of terms in the

$A^{(q)}$'s such that at least one of them is the third term in some $A^{(q)}$. Recall that the left-hand side of (3.61) is a sum of expressions of the form (3.72) with suitable signs. Adding terms of the first type for all expressions of the form (3.72) with the signs, we obtain (3.65). The proof for the case of strong topological vertex algebras shows that this is equal to the first term of the right-hand side of (3.61). The sum of all the terms of the second type with the signs gives a $Q_{\mathscr{K}_{V,0}}$-exact operator. Using the same method used in the proof of Proposition 3.2, it is easy to show that the sum of all the terms of the third type is a sum of $Q_{\mathscr{K}_{V,0}}$-exact operators and maps from

$$(\Gamma(T(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)))^{\otimes(n_0+\cdots+n_k)}$$

to

$$\mathscr{H}_{V,0}(j_1 + \cdots + j_k) \otimes \mathrm{Hol}(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k))$$

evaluated at X_{ms}, $s = 1, \ldots, n_m$, $m = 0, \ldots, k$, satisfying the property that when fixing some of X_{ms}'s and viewed only as functions of the remaining X_{ms}'s, these maps are holomorphic forms on $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ and when restricted to fibers of certain pull-back coordinate fibrations of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$, these forms are exact. The sum of all terms of the fourth type, by definition and Lemma 3.4, is a sum of maps from

$$(\Gamma(T(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)))^{\otimes(n_0+\cdots+n_k)}$$

to

$$\mathscr{H}_{V,0}(j_1 + \cdots + j_k) \otimes \mathrm{Hol}(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k))$$

evaluated at X_{ms}, $s = 1, \ldots, n_m$, $m = 0, \ldots, k$, satisfying the property that when fixing some of X_{ms}'s and viewed only as functions of the remaining X_{ms}'s, these maps are holomorphic forms on $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ and when restricted to fibers of the coordinate fibrations of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ described in Lemma 3.4, these forms are exact. Thus we see that the sum of all the terms of the second, third and the fourth typed gives h and α.

This concludes the proof of (3.61) in the case of topological vertex algebras. The identities (3.62) and (3.63) are obvious. The proof for weak topological vertex algebras is the same as that for topological vertex algebras. □

4. The Operadic Formulation of (Weak, Strong) Topological Vertex Algebras and the Equivalence Theorem

In Sect. 5, we will see that the meromorphic forms ω_j and their three properties give all the nice topological properties. Therefore it is natural to introduce the following notion of (weak, strong) topological $\widehat{K}$-associative algebra in terms of meromorphic forms and the three properties:

Definition 4.1. *A topological $\widehat{K}$-associative algebra is a $\mathbb{Z} \times \mathbb{Z}$-graded space V (graded by weight and fermion number) together with a differential Q of fermion number 1 on*

V, that is, an operator Q of fermion number 1 on V satisfying $Q^2 = 0$, and with a $\mathrm{Hom}(V^{\otimes j}, \overline{V})$-valued holomorphic form ω_j on $K(j)$ for every $j \in \mathbb{N}$ satisfying (3.22), Proposition 3.1, (3.29), (3.60)–(3.63). Weak topological $\widehat{K}$-associative algebras and strong topological $\widehat{K}$-associative algebras are defined in the same way except that the axiom (3.61)–(3.63) are replaced by the corresponding statements for weak and strong topological vertex algebras in Proposition 3.3, respectively.

We denote a (weak, strong) topological $\widehat{K}$-associative algebra by (V, Q, ω), where V is the underlying $\mathbb{Z} \times \mathbb{Z}$-graded vector space, Q the differential on V and $\omega = \{\omega_j \mid j \in \mathbb{N}\}$. Obviously, a topological $\widehat{K}$-associative algebra is a weak topological $\widehat{K}$-associative algebra and a strong topological $\widehat{K}$-associative algebra is a topological $\widehat{K}$-associative algebra.

Proposition 4.1. *Let $(V, Q, \{\omega_j \mid j \in \mathbb{N}\})$ be a weak topological $\widehat{K}$-associative algebra. The pair $(V, \{\omega_{j,0} \mid j \in \mathbb{N}\})$ is a graded meromorphic $\widehat{K}$-associative algebra.*

Proof. The proposition is an immediate consequence of (3.22), Proposition 3.1, (3.61) in the case $n_0 = 0$ and (3.60). □

As expected, the notion of (weak, strong) topological $\widehat{K}$-associative algebra is equivalent to the notion of (weak, strong) topological vertex algebra introduced in Sect. 2. The precise statement is the following theorem:

Theorem 4.2. *The category of (weak, strong) topological $\widehat{K}$-associative algebras and the category of (weak, strong) topological vertex algebras are isomorphic.*

Proof. We already know that the category of graded meromorphic $\widehat{K}$-associative algebras and the category of graded vertex algebras are isomorphic. The results in the previous section can be summarized as saying that given a (weak, strong) topological vertex algebra, we can construct a (weak, strong) topological $\widehat{K}$-associative algebra using the graded meromorphic $\widehat{K}$-associative algebra corresponding to the graded vertex algebra underlying the original (weak, strong) topological vertex algebra. Thus we need only the show that given a (weak, strong) topological $\widehat{K}$-associative algebra we can find operators $Q, g(0)$ and $g(-1)$ on the corresponding graded vertex algebra such that the corresponding graded vertex algebra together with these operators is a (weak, strong) topological vertex algebra, and to show that the procedure to obtain a (weak, strong) topological $\widehat{K}$-associative algebra from a (weak, strong) topological vertex algebra and the procedure to obtain a (weak, strong) topological vertex algebra from a (weak, strong) topological $\widehat{K}$-associative algebra are inverse to each other.

Given a weak topological $\widehat{K}$-associative algebra (V, Q, ω), let $(V, Y, \mathbf{1})$ be the graded vertex algebra corresponding to the graded $\widehat{K}$-associative algebra $(V, \{\omega_{j,0} \mid j \in \mathbb{N}\})$. We already have the differential Q. The operator $g(0)$ and $g(-1)$ are defined by

$$g(0) = \omega_{1,1}\left(-a_1 \frac{\partial}{\partial a_1}\right)\Big|_I, \tag{4.1}$$

$$g(-1) = \omega_{1,1}\left(-a_1^{-1} \frac{\partial}{\partial z_1}\right)\Big|_I, \tag{4.2}$$

where I is the identity of the partial operad $\widehat{K}$. From

$$d\omega_{1,0} = Q_{\mathscr{H}_{V,0}}(\omega_{1,1}), \tag{4.3}$$

we obtain

$$\begin{aligned}
-a_1\frac{\partial}{\partial a_1}\omega_{1,0}\Big|_I &= d\omega_{1,0}\Big(-a_1\frac{\partial}{\partial a_1}\Big)\Big|_I \\
&= (Q_{\mathscr{H}_{V,0}}(\omega_{1,1})\Big(-a_1\frac{\partial}{\partial a_1}\Big)\Big|_I \\
&= Q\omega_{1,1}\Big(-a_1\frac{\partial}{\partial a_1}\Big)\Big|_I + \omega_{1,1}\Big(-a_1\frac{\partial}{\partial a_1}\Big)Q\Big|_I \\
&= [Q, g(0)]. \qquad (4.4)
\end{aligned}$$

But from the definition of $L(0)$ for $(V, Y, \mathbf{1})$ we have

$$-a_1\frac{\partial}{\partial a_1}\omega_{1,0}\Big|_I = \omega_{1,0}|_I L(0) = L(0). \tag{4.5}$$

Thus (2.15) is proved. Similarly we can prove (2.16). The formula (2.21) follows immediately from $Q_{\mathscr{H}_{V,0}}(\omega_{2,0}) = 0$. To prove (2.22) we let $P = (z, 0; 1, 1) \in \widehat{K}(2)$ and from (3.61) in the case $n_0 = 0$ we obtain

$$\begin{aligned}
&\gamma_{\mathscr{H}_{V,0}}\Big(\omega_{2,0}|_P; \omega_{1,1}\Big(-a_1\frac{\partial}{\partial a_1} + z\Big(-a_1^{-1}\frac{\partial}{\partial z_1}\Big)\Big)\Big|_I, \omega_{1,0}|_I\Big) \\
&\quad + \gamma_{\mathscr{H}_{V,0}}\Big(\omega_{2,0}|_P; \omega_{1,0}|_I, \omega_{1,1}\Big(-a_2\frac{\partial}{\partial a_2}\Big)\Big|_I\Big) \\
&= \omega_{2,1}((\gamma_{\widehat{K}})_*\Big(e_*^{(1)}\Big(-a_1\frac{\partial}{\partial a_1}\Big) + ze_*^{(1)}\Big(-a_1^{-1}\frac{\partial}{\partial z_1}\Big)\Big)\Big)\Big|_P \\
&\quad + \omega_{2,1}\Big((\gamma_{\widehat{K}})_*\Big(e_*^{(2)}\Big(-a_2\frac{\partial}{\partial a_2}\Big)\Big)\Big)\Big|_P \\
&= \omega_{2,1}\Big((\gamma_{\widehat{K}})_*e_*^{(1)}\Big(-a_1\frac{\partial}{\partial a_1}\Big) + ze_*^{(1)}\Big(-a_1^{-1}\frac{\partial}{\partial z_1}\Big) \\
&\quad + e_*^{(2)}\Big(-a_2\frac{\partial}{\partial a_2}\Big)\Big)\Big)\Big|_P \\
&= \gamma_{\mathscr{H}_{V,0}}\Big(\omega_{1,1}\Big(-a_1\frac{\partial}{\partial a_1}\Big)\Big|_I; \omega_{2,0}|_P\Big). \qquad (4.6)
\end{aligned}$$

When both sides of (4.6) acting on $v_1 \otimes v_2$ we obtain

$$Y((g(0) + zg(-1))v_1, z)v_2 - (-1)^{|v_1|}Y(v_1, z)g(0)v_2 = g(0)Y(v_1, z)v_2 \tag{4.7}$$

which is exactly (2.22). The formula (2.23) can be proved similarly. The formulas (2.17)–(2.20) can also be proved similarly using I instead of P. Thus we have proved that $(V, Y, \mathbf{1})$ together with $Q, g(0)$ and $g(-1)$ is a weak topological vertex algebra. If

(V, Q, ω) is a topological $\widehat{K}$-associative algebra, then by (3.61) and (3.63) we have

$$\omega_{2,1}\left((\gamma_{\widehat{K}})_*\left(e_*^{(0)}\left(-a_1\frac{\partial}{\partial a_1}\Big|_I\right)\right),(\gamma_{\widehat{K}})_*\left(e_*^{(1)}\left(-a_1\frac{\partial}{\partial a_1}\Big|_I\right)\right)\right)$$
$$= \gamma_{\mathscr{H}_{V,0}}\left(\omega_{1,1}\left(-a_1\frac{\partial}{\partial a_1}\right)\Big|_I,\omega_{1,1}\left(-a_1\frac{\partial}{\partial a_1}\right)\Big|_I\right) + Q_{\mathscr{H}_{V,0}}(h). \tag{4.8}$$

It is easy to see that

$$(\gamma_{\widehat{K}})_*\left(e_*^{(0)}\left(-a_1\frac{\partial}{\partial a_1}\Big|_I\right)\right) = (\gamma_{\widehat{K}})_*\left(e_*^{(1)}\left(-a_1\frac{\partial}{\partial a_1}\Big|_I\right)\right). \tag{4.9}$$

Thus we left-hand side of (4.8) is zero and we conclude that the first term in the right-hand side of (4.8) is $Q_{\mathscr{H}_{V,0}}(= Q)$-exact. From the definition of $g(0)$ and $\gamma_{\mathscr{H}_{V,0}}$, we see that this ($Q$-exact) first term in the right-hand side of (4.8) is equal to $(g(0))^2$. This proves that $(g(0))^2$ is Q-exact and thus $(V, Y, \mathbf{1})$ together with Q, $g(0)$ and $g(-1)$ is a topological vertex algebra. If (V, Q, ω) is a strong topological vertex algebra, then (4.8) holds with $h = 0$. Thus the same argument shows that $(g(0))^2 = 0$ and thus $(V, Y, \mathbf{1})$ together with Q, $g(0)$ and $g(-1)$ is a strong topological vertex algebra.

Beginning with a (weak, strong) topological vertex algebra we obtain a (weak, strong) topological $\widehat{K}$-associative algebra using the construction in the previous section. From this (weak, strong) topological $\widehat{K}$-associative algebra we obtain a (weak, strong) topological vertex algebra using the construction above. Theorem 2.2, the construction of ω_j and (4.1), (4.2) imply that this (weak, strong) topological vertex algebra is the same as the original one. Beginning with a (weak, strong) topological $\widehat{K}$-associative algebra we obtain a (weak, strong) topological vertex algebra. From this (weak, strong) topological vertex algebra we obtain a (weak, strong) topological $\widehat{K}$-associative algebra using the construction in the previous section. Using axioms in Proposition 3.3 and the fact that $\widehat{K}$ is generated by $\widehat{K}(0)$, $\widehat{K}(1)$ and $\widehat{K}(2)$, this (weak, strong) topological $\widehat{K}$-associative algebra is the same as the original one. □

Remark 4.1. The above theorem in fact shows that for (weak, strong) topological vertex algebras, the forms $\omega_{j,n}$ must be of the form (3.20). Thus we have a conceptual interpretation of the definition of $\omega_{j,n}$.

A natural question is whether (weak, strong) topological vertex algebras are in fact algebras over a certain operad. The answer is affirmative for strong topological vertex algebras. The category of strong topological vertex algebras is isomorphic to the category of algebras over the differential graded partial operad of the differentiable chain complexes of $\widehat{K}(j)$, $j \in \mathbb{N}$, satisfying a certain meromorphicity axiom. Since this meromorphicity axiom is much more complicated to formulate than the (equivalent) one for the forms ω_j, $j \in \mathbb{N}$ and since the notion of strong topological $\widehat{K}$-associative algebra is conceptually simple, we will not try to give this meromorphicity axiom and therefore we will not give the complete further reformulation of the notion of strong topological vertex algebra in this paper. But for the topological application in the next section, in the rest of the section we prove a proposition which in the case of strong topological vertex algebras says that a strong topological vertex algebra has the structure of an algebra over the differential graded partial operad of the differentiable chain complexes of $\widehat{K}(j)$, $j \in \mathbb{N}$.

We need the notion of graded (partial) operad and differential graded (partial) operad. A *graded operad* is an operad in the category of $\mathbb{Z}$-graded vector spaces. A *differential graded operad* is a graded operad $\mathscr{C} = \{\mathscr{C}(j) \mid j \in \mathbb{N}\}$ together with an operator δ of degree -1 on each $\mathscr{C}(j)$ such that $\delta^2 = 0$. *Graded partial operads, graded partial pseudo-operads, differential graded partial operads and differential graded partial pseudo-operads* are defined in the same way. Let $(\mathscr{C}_1, \delta_1)$ and $(\mathscr{C}_2, \delta_2)$ be two differential graded operads. A *morphism* from $(\mathscr{C}_1, \delta_1)$ to $(\mathscr{C}_2, \delta_2)$ is a morphism ψ from the operad $\mathscr{C}_1$ to the operad $\mathscr{C}_2$ such that $\delta_2\psi = \psi\delta_1$. *Morphisms* for differential graded partial operads and for differential graded partial pseudo-operads are defined in the same way.

We consider the family of the differentiable chain complexes $C^S(\widehat{K}(j)), j \in \mathbb{N}$ with coefficients in $\mathbb{C}$. The natural chain maps

$$\begin{aligned}\zeta : C^S(\widehat{K}(k)) \otimes C^S(\widehat{K}(j_1)) \otimes \cdots \otimes C^S(\widehat{K}(j_k))\\ \to C^S(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k))\end{aligned} \tag{4.10}$$

(see, for example, [Mas] for details) composed with the (partial) maps

$$\begin{aligned}(\gamma_{\widehat{K}})_* : C^S(\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k))\\ \to C^S(\gamma_{\widehat{K}}(\widehat{K}(k); \widehat{K}(j_1), \ldots, \widehat{K}(j_k)))\end{aligned} \tag{4.11}$$

give (partial) chain maps

$$\begin{aligned}\gamma_{C^S(\widehat{K})} : C^S(\widehat{K}(k)) \otimes C^S(\widehat{K}(j_1)) \otimes \cdots \otimes C^S(\widehat{K}(j_k))\\ \to \ C^S(\gamma_{\widehat{K}}(\widehat{K}(k); \widehat{K}(j_1), \ldots, \widehat{K}(j_k)))\,.\end{aligned} \tag{4.12}$$

We have

Proposition 4.3. *The family $C^S(\widehat{K}) = \{C^S(\widehat{K}(j)) \mid j \in \mathbb{N}\}$ together with the maps $\gamma_{C^S(\widehat{K})}$ and the boundary operator ∂ is a differential graded partial operad.*

Proof. The proposition follows easily from the definition and properties of ζ (see, for example, [Mas]) and the fact that $\gamma_{\widehat{K}}$ are substitution (composition) maps for the partial operad $\widehat{K}$. □

Since $C^S(\widehat{K})$ is a differential graded partial operad, the homology $H_*(\widehat{K})$ is a graded partial operad.

Let (V, Q, ω) be a weak topological $\widehat{K}$-associative algebra. We consider the endomorphism partial pseudo-operad $\mathscr{H}_{V,0} = \{\mathrm{Hom}(V^{\otimes j}, \overline{V}) \mid j \in \mathbb{N}\}$. The following proposition is obvious:

Proposition 4.4. *The endomorphism operad partial pseudo-operad $\mathscr{H}_{V,0}$ together with the negation of the fermion number grading and the differential $Q_{\mathscr{H}_{V,0}}$ is a differential graded partial pseudo-operad.*

Let I^n be the unit n-cube in $\mathbb{R}^n$ and $T : I^n \to \widehat{K}(j)$ a differentiable singular n-cube. We define

$$\phi(T) = \int_T \omega_{j,n} = \int_{I^n} T^*\omega_{j,n}\,. \tag{4.13}$$

It is obvious that for degenerate differentiable singular n-cube T, $\phi(T) = 0$. From the definition of $C^S(\widehat{K}(j))$, we see that (4.13) can be extended to obtain a homomorphism $\phi : C^S(\widehat{K}(j)) \to \mathscr{H}_{V,0}(j)$ of vector spaces for every $j \in \mathbb{N}$.

Proposition 4.5. *Let (V, Q, ω) be a weak topological $\widehat{K}$-associative algebra. The homomorphism ϕ is an S_j-equivariant chain map from $C^S(\widehat{K}(j))$ to $\mathscr{H}_{V,0}(j)$ for every $j \in \mathbb{N}$. If (V, Q, ω) is a strong topological $\widehat{K}$-associative algebra, the homomorphisms ϕ is a morphism of differential graded partial pseudo-operads from $C^S(\widehat{K})$ to $\mathscr{H}_{V,0}$. In particular, in this case, V is a $C^S(\widehat{K})$-algebra.*

Proof. We first show that ϕ is a chain map. Let s be any differentiable singular chain in $C_n^S(\widehat{K}(j))$. Then using (3.29) and the Stokes' theorem, we have

$$\begin{aligned} Q_{\mathscr{H}_{V,0}}\phi(s) &= Q_{\mathscr{H}_{V,0}} \int_s \omega_{j,n} = \int_s Q_{\mathscr{H}_{V,0}} \omega_{j,n} \\ &= \int_s d\omega_{j,n} = \int_{\partial s} \omega_{j,n} = \phi(\partial s) . \end{aligned} \tag{4.14}$$

The S_j equivariance of ϕ follows easily from (3.60): Let s be any differentiable singular chain in $C_n^S(\widehat{K}(j))$ and $\sigma \in S_j$. Then by (3.60)

$$\sigma(\phi(s)) = \int_s \sigma(\omega_{j,n}) = \int_{\sigma(s)} \omega_{j,n} = \phi(\sigma(s)) . \tag{4.15}$$

This proves the first assertion of the proposition. To prove the second assertion, let $T_0 : I^{n_0} \to \widehat{K}(k)$ and $T_l : I^{n_l} \to \widehat{K}(j_l)$, $l = 1, \ldots, k$ be differentiable singular cubes. For convenience, we let $j_0 = k$. For weak topological $\widehat{K}$-associative algebras, we consider only those T_l such that the images of T_l are in $\overline{K}(j_l)$, $l = 0, \ldots, l$, We denote by $[T_l]$ the elements of $C_{nl}^S(\widehat{K}(j_l))$, $l = 0, \ldots, k$, containing $T_l, l = 0, \ldots, k$, respectively. From these differentiable singular cubes we obtain another differentiable singular cube

$$T = \gamma_{\widehat{K}} \circ (T_{n_0} \times \cdots \times T_{n_k}) : I^{n_0+\cdots+n_k} \to \widehat{K}(j_1 + \cdots + j_k) \tag{4.16}$$

when the right-hand side exists. If we denote the chain containing T by $[T]$, then by definition

$$\gamma_{C^s(\widehat{K})}([T_0]; [T_1], \ldots, [T_k]) = [T] . \tag{4.17}$$

Let $\dfrac{\partial}{\partial x_{li}}$, $i = 1, \ldots, n_l$, $l = 0, \ldots, k$, be the vector fields on $\mathbb{R}^{n_l}$ parallel to the i-th axis, where we use x_{li} to denote the coordinates of $\mathbb{R}^{n_l}$. Then

$$\begin{aligned} \phi([T_l]) &= \int_{I^{n_l}} T_l^* \omega_{j_l,n_l} \\ &= \int_{I^{n_l}} T_l^* \omega_{j_l,n_l}\left(\frac{\partial}{\partial x_{l1}}, \ldots, \frac{\partial}{\partial x_{ln_l}}\right) dx_{l1} \ldots dx_{ln_l} \\ &= \int_{I^{n_l}} \omega_{j_l,n_l}\left((T_l)_*\left(\frac{\partial}{\partial x_{l1}}\right), \ldots, (T_l)_*\left(\frac{\partial}{\partial x_{ln_l}}\right)\right) dx_{l1} \ldots dx_{ln_l} \end{aligned} \tag{4.18}$$

for $l = 0, \dots, k$. We have

$$\begin{aligned}
&\phi(\gamma_{C^S(\widehat{K})}([T_0]; [T_1], \dots, [T_k]) = \phi([T_1])\\
&= \int_{I^{n_0+\cdots+n_k}} T^*\omega_{j_1+\cdots+j_k, n_0+\cdots+n_k}\\
&= \int_{I^{n_0+\cdots+n_k}} T^*\omega_{j_1+\cdots+j_k, n_0+\cdots+n_k} \left(\frac{\partial}{\partial x_{01}}, \dots, \frac{\partial}{\partial x_{kn_k}}\right) dx_{01} \dots dx_{kn_k}\\
&= \int_{I^{n_0+\cdots+n_k}} \omega_{j_1+\cdots+j_k, n_k+\cdots+n_k} \left(T_*\left(\frac{\partial}{\partial x_{01}}\right), \dots, T_*\left(\frac{\partial}{\partial x_{kn_k}}\right)\right) dx_{01} \dots dx_{kn_k}\\
&= \int_{I^{n_0+\cdots+n_k}} \omega_{j_1+\cdots+j_k, n_0+\cdots+n_k} \left((\gamma_{\widehat{K}})_*\left(e_*^{(0)}((T_0)_*\left(\frac{\partial}{\partial x_{01}}\right))\right)\right),\\
&\qquad \cdots, (\gamma_{\widehat{K}})_*\left(e_*^{(k)}\left((T_k)_*\left(\frac{\partial}{\partial x_{kn_k}}\right)\right)\right)\Big)\Big) dx_{01} \dots dx_{kn_k}. \qquad (4.19)
\end{aligned}$$

For the topological application in he next section, we use (3.61) for an arbitrary topological vertex algebra instead of (3.59) only for strong topological vertex algebras to obtain that the right-hand side of (4.19) is equal to

$$\begin{aligned}
&\int_{I^{n_0+\cdots+n_k}} \gamma_{\mathscr{H}_{V,0}}\left(\omega_{k,n_0}\left((T_0)_*\left(\frac{\partial}{\partial x_{01}}\right), \dots, (T_0)_*\left(\frac{\partial}{\partial x_{0n_0}}\right)\right);\right.\\
&\qquad \omega_{j_1,n_1}\left((T_1)_*\left(\frac{\partial}{\partial x_{11}}\right), \dots, (T_k)_*\left(\frac{\partial}{\partial x_{1n_1}}\right)\right), \dots,\\
&\qquad \left.\omega_{j_k,n_k}\left((T_k)_*\left(\frac{\partial}{\partial x_{k1}}\right), \dots, (T_k)_*\left(\frac{\partial}{\partial x_{kn_k}}\right)\right)\right) dx_{01} \dots dx_{kn_k}\\
&+ \int_{I^{n_0+\cdots+n_k}} Q_{\mathscr{H}_{V,0}}\left(h\left((\gamma_{\widehat{K}})_*\left(e_*^{(0)}\left((T_0)_*\left(\frac{\partial}{\partial x_{01}}\right)\right)\right)\right), \dots,\right.\\
&\qquad \left.(\gamma_{\widehat{K}})_*\left(e_*^{(k)}\left((T_k)_*\left(\frac{\partial}{\partial x_{kn_k}}\right)\right)\right)\right) dx_{01} \dots dx_{kn_k}\\
&+ \int_{I^{n_0+\cdots+n_k}} \alpha\left((\gamma_{\widehat{K}})_*\left(e_*^{(0)}\left((T_0)_*\left(\frac{\partial}{\partial x_{01}}\right)\right)\right), \dots,\right.\\
&\qquad \left.(\gamma_{\widehat{K}})_*\left(e_*^{(k)}\left((T_k)_*\left(\frac{\partial}{\partial x_{kn_k}}\right)\right)\right)\right) dx_{01} \dots dx_{kn_k}. \qquad (4.20)
\end{aligned}$$

For convenience let

$$\begin{aligned}
\widetilde{h} = &\int_{I^{n_0+\cdots+n_k}} h\left((\gamma_{\widehat{K}})_*\left(e_*^{(0)}\left((T_0)_*\left(\frac{\partial}{\partial x_{01}}\right)\right)\right), \dots,\right.\\
&\left.(\gamma_{\widehat{K}})_*\left(e_*^{(k)}\left((T_k)_*\left(\frac{\partial}{\partial x_{kn_k}}\right)\right)\right)\right) dx_{01} \dots dx_{kn_k}, \qquad (4.21)
\end{aligned}$$

$$\widetilde{\alpha} = \int_{I^{n_0+\cdots+n_k}} \alpha\Big((\gamma_{\widehat{K}})_*\Big(e_*^{(0)}\Big((T_0)_*\Big(\frac{\partial}{\partial x_{01}}\Big)\Big)\Big), \dots,$$
$$(\gamma_{\widehat{K}})_*\Big(e_*^{(k)}\Big((T_k)_*\Big(\frac{\partial}{\partial x_{kn_k}}\Big)\Big)\Big)\Big)\, dx_{01} \dots dx_{kn_k}. \tag{4.22}$$

Using (4.18), (4.21) and (4.22), (4.20) is equal to

$$\gamma_{\mathscr{K}_{V,0}}\Big(\int_{I^{n_0}} \omega_{k,n_0}\Big((T_0)_*\Big(\frac{\partial}{\partial x_{01}}\Big), \dots, (T_0)_*\Big(\frac{\partial}{\partial x_{0n_0}}\Big)\Big)\, dx_{01} \dots dx_{0n_0};$$
$$\int_{I^{n_1}} \omega_{j_1,n_1}\Big((T_1)_*\Big(\frac{\partial}{\partial x_{11}}\Big), \dots, (T_k)_*\Big(\frac{\partial}{\partial x_{1n_1}}\Big)\Big)\, dx_{11} \dots dx_{1n_1}, \dots,$$
$$\int_{I^{n_k}} \omega_{j_k,n_k}\Big((T_k)_*\Big(\frac{\partial}{\partial x_{k1}}\Big), \dots, (T_k)_*\Big(\frac{\partial}{\partial x_{kn_k}}\Big)\Big)\, dx_{k1} \dots dx_{kn_k}\Big)$$
$$+ Q_{\mathscr{K}_{V,0}}(\widetilde{h}) + \widetilde{\alpha}$$
$$= \gamma_{\mathscr{K}_{,V0}}(\phi([T_0]); \phi([T_1]), \dots, \phi([T_k])) + Q_{\mathscr{K}_{V,0}}(\widetilde{h}) + \widetilde{\alpha}. \tag{4.23}$$

For strong topological $\widehat{K}$-associative algebras, the last two terms of the right-hand side of (4.23) are zero. Thus from (4.19), (4.20) and (4.23) ϕ is a morphism of differential graded partial operads. This finishes the proof. □

5. Cohen's and Getzler's Theorem and the Gerstenhaber or Batalin-Vilkovisky Algebra Structure

In this section we give an application of the results obtained in the previous two sections. We first recall the definition of Batalin-Vilkovisky algebras and Gerstenhaber algebras, results on these algebras and other related concepts and results. In particular, we state Cohen's and Getzler's theorem which describes Gerstenhaber and Batalin-Vilkovisky algebras using the operads of the holomogies of the little disk operad and the framed little disk operad, respectively. Then by using Getzler's theorem (or Cohen's theorem) and Proposition 4.5, we show that the cohomology of a topological vertex algebra (or of a weak topological vertex algebra) has the structure of a Batalin-Vilkovisky algebra (or of a Gerstenhaber algebra).

Definition 5.1. *A Batalin-Vilkovisky algebra is a graded commutative algebra A together with an operator Δ of degree -1 satisfying $\Delta^2 = 0$ and*

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac)$$
$$- (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c) \tag{5.1}$$

for any homogeneous elements a, b, c of A.

In [LZ], Batalin-Vilkovisky algebras are called "coboundary Gerstenhaber algebras." In his study of the cohomology theory of associative rings and algebras [Ger], Gerstenhaber first discovered the following algebraic structure:

Definition 5.2. *A Gerstenhaber algebra is a graded commutative algebra A together with a bracket $[\cdot,\cdot] : A \otimes A \to A$ such that $[A^{(n)}, A^{(m)}] \subset A^{(n+m-1)}$ (where $A^{(n)}, A^{(m)}, n, m \in \mathbb{Z}$, are homogeneous components of A), satisfying*

$$[a,b] = -(-1)^{(|a|-1)(|b|-1)}[b,a], \tag{5.2}$$

$$[a,[b,c]] = [[a,b],c] + (-1)^{(|a|-1)(|b|-1)}[b,[a,c]], \tag{5.3}$$

$$[a,bc] = [a,b]c + (-1)^{|a|(|b|-1)}b[a,c] \tag{5.4}$$

for any homogeneous elements a, b, c of A.

(In [Get], Gerstenhaber algebras are called "braided algebras.") In fact a Batalin-Vilkovisky algebra is a special type of Gerstenhaber algebra. We have the following proposition proved by Getzler [Get] and Penkava and Schwarz [PS]:

Proposition 5.1 (Getzler, Penkava-Schwarz). *A Batalin-Vilkovisky algebra is a Gerstenhaber algebra A equipped with an operator Δ of degree -1 such that $\Delta^2 = 0$ satisfying the following relation between the bracket and Δ:*

$$[a,b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta a)b - a(\Delta b). \tag{5.5}$$

Furthermore in a Batalin-Vilkovisky algebra, Δ satisfies the formula

$$\Delta[a,b] = [\Delta a, b] + (-1)^{|a|-1}[a, \Delta b]. \tag{5.6}$$

Gerstenhaber algebras and Batalin-Vilkovisky algebras can also be described using certain operads. For Gerstenhaber algebras we consider the little disk operad $\mathscr{D} = \{\mathscr{D}(j) \mid j \in \mathbb{N}\}$ of Boardman and Vogt [BoV] (see also [May]) where $\mathscr{D}(j)$ is the space of all maps from the disjoint union of j copies of the unit disk D to the unit disk such that on each copy of the disk they restrict to a composition of a translation and multiplication by a positive number and such that the images of different copies of the disk are disjoint. Using the Künneth theorem it is easy to see that the homology of a topological operad is a graded operad. Since the little disk operad is homotopically equivalent to the family of configuration spaces $\{\mathbb{F}_j(\mathbb{C}) \mid j \in \mathbb{N}\} = \overline{K}$, the homology of the little disk operad and the homology of this family of configuration spaces are isomorphic. In [C] Cohen proves the following theorem without stating it explicitly:

Theorem 5.2 (Cohen). *The category of algebras over the operad $\{H_*(\mathbb{F}_j(\mathbb{C})) \mid j \in \mathbb{N}\} = H_*(\overline{K})$ (or the category of algebras over the operad of the homology of the little disk operad) is equivalent to the category of Gerstenhaber algebras.*

For Batalin-Vilkovisky algebras, we have to consider the framed little disk operad $\mathscr{P} = \{\mathscr{P}(j) \mid j \in \mathbb{N}\}$ (see [Get]), where $\mathscr{P}(j)$ is the space of all maps from the disjoint union of j copies of the unit disk D to the unit disk such that on each copy of the disk they restrict to a composition of a translation and multiplication by an element of $\mathbb{C}^\times$ and such that the images of different copies of the disk are disjoint. In [Get], Getzler proves the following theorem based on Cohen's theorem above (Getzler and Jones also have another proof of Cohen's theorem):

Theorem 5.3 (Getzler). *The category of algebras over the operad $H_*(\mathscr{P})$ is equivalent to the category of Batalin-Vilkovisky algebras.*

Note that every element of $\mathscr{D}(j)$ is determined by the centers $z_1, \dots, z_j$ of the images of the j copies of the unit disk and the j positive real numbers $a_1, \dots, a_j$ multiplied to the j copies of the unit disk and similarly every element of $\mathscr{P}(j)$ is determined by the centers $z_1, \dots, z_j$ of images of the j copies of the unit disk and the j nonzero complex numbers $a_1, \dots, a_j$ multiplied to the j copies of the unit disk. Therefore $\mathscr{D}$ can be embedded in $\mathscr{P}(j)$ and $\mathscr{P}(j)$ can be embedded in $\widehat{K}(j) = \mathbb{F}_j(\mathbb{C}) \times \mathbb{C}^\times)^j$. The proof of the following proposition is a direct verification:

Proposition 5.4. *The little disk operad $\mathscr{D}$ is a suboperad of the framed little disk operad $\mathscr{D}$. The framed little disk operad $\mathscr{P}$ is a suboperad of the partial operad $\widehat{K}$.*

If we define the composition maps for the families $C^S(\mathscr{D})$ and $C^S(\mathscr{P})$ of the differentiable chain complexes $C^S(\mathscr{D}(j)), j \in \mathbb{N}$, and of $C^S(\mathscr{P}(j)), j \in \mathbb{N}$, in the same way as that for $C^S(\widehat{K})$, by Proposition 5.4 we have

Corollary 5.5. *The families $C^S(\mathscr{D})$ and $C^S(\mathscr{P})$ are differential graded suboperad of the differential graded partial operad $C^S(\widehat{K})$.*

Now we consider a topological $\widehat{K}$-associative algebra (or a weak topological $\widehat{K}$-associative algebra) $(V, Q, \{\omega_j \mid j \in \mathbb{N}\})$. By Proposition 4.5, we have an S_j-equivariant chain map ϕ from $C^S(\widehat{K}(j))$ to $\mathscr{H}_{V,0}(j)$ for every $j \in \mathbb{N}$. These maps composed with the morphism $C^S(\mathscr{P}) \to C^S(\widehat{K})$ (or $C^S(\mathscr{D}) \to C^S(\widehat{K})$) induced from the embedding of $\mathscr{P}$ (or of $\mathscr{D}$)) in $\widehat{K}$ give S_j-equivariant chain maps $\phi|_{\mathscr{P}}$ (or $\phi|_{\mathscr{D}}$) from $C^S(\mathscr{P}(j))$ (or $C^S(\mathscr{D}(j))$) to $\mathscr{H}_{V,0}(j), j \in \mathbb{N}$. Except for strong topological vertex algebras, these maps are not morphisms of partial pseudo-operads. But we have

Proposition 5.6. *Let (V, Q, ω) be a topological $\widehat{K}$-associative algebra (or a weak topological $\widehat{K}$-associative algebra). Then the graded vector space $H^*(V)$ has the structure of an algebra over the operad $H_*(\mathscr{P})$ (or over the operad $H_*(\mathscr{D})$).*

Proof. We only prove the case for topological $\widehat{K}$-associative algebras. The proof for weak topological $\widehat{K}$-associatived algebras is similar. From (4.19), (4.20) and (4.23), we see that for any differentiable singular cycles s_l in $C^S_{n_l}(\widehat{K}(j_l))$, $l = 0, \dots, k$, we have

$$\begin{aligned}\phi(\gamma_{C^S(\widehat{K})}(s_0; s_1, \dots, s_k)) &= \gamma_{\mathscr{H}_{V,0}}(\phi(s_0); \phi(s_1), \dots, \phi(s_k)) \\ &\quad + Q_{\mathscr{H}_{V,0}}(\widehat{h}) + \widehat{\alpha}, \end{aligned} \tag{5.7}$$

where $\widehat{h}, \widehat{\alpha} \in \mathscr{H}_{V,0}(j_1 + \cdots + j_k)$ and $\widehat{\alpha}$ is a sum of integrals of functions over the image of $s_0 \times \cdots \times s_k$ where, when restricted to fibers of certain coordinate fibrations or pull-back coordinate fibrations of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$, these functions to be integrated are exact forms evaluated at the projections of the tangent vector fields on $s_0 \times \cdots \times s_k$ to the tangent space of the fibers. Since $C^S(\mathscr{P})$ is a suboperad of $C^S(\widehat{K})$, for any $s_l \in C^S_{n_l}(\mathscr{P})$, $l = 0, \dots, k$, we have

$$\begin{aligned}&\phi|_{C^S(\mathscr{P})}(\gamma_{C^S(\mathscr{P})}(s_0; s_1, \dots, s_k)) \\ &\quad = \gamma_{\mathscr{H}_{V,0}}(\phi|_{C^S(\mathscr{P})}(s_0); \phi|_{C^S(\mathscr{P})}(s_1), \dots, \phi|_{C^S(\mathscr{P})}(s_k)) \\ &\qquad + Q_{\mathscr{H}_{V,0}}(\widehat{h}) + \widehat{\alpha}, \end{aligned} \tag{5.8}$$

where $\widehat{h}$ and $\widehat{\alpha}$ are the same as in (5.7). From [Get] we know that the operad $H_*(\mathscr{P})$ is generated by $H_*(\mathscr{P}(1))$ and $H_*(\mathscr{P}(2))$. Using this fact and induction, it is easy to show that we can always find cycle $s_0', \dots, s_k'$ homologous to $s_0, \dots, s_k$, respectively, such that the cycle $s_0' \times \cdots \times s_k'$ is a sum of cycles which when intersected with fibers of any coordinate fibration or pull-back coordinate fibration of $\widehat{K}(k) \times \widehat{K}(j_1) \times \cdots \times \widehat{K}(j_k)$ are cycles with dimensions equal to the degrees of the exact forms on these fibers given by Proposition 3.3. Thus if we use $s_0', \dots, s_k'$ instead of $s_0, \dots, s_k$ in (5.8), the term $\widehat{\alpha}$ is zero. Passing to the homology and using the fact that $s_0', \dots, s_k'$ are homologous to $s_0, \dots, s_k$, respectively, we obtain

$$\begin{aligned}
&(\phi|_{C^S(\mathscr{P})})_*(\gamma_{H^*(\mathscr{P})}([s_0]; [s_1], \dots, [s_k])) \\
&\quad = \gamma_{H^*(\mathscr{H}_{V,0})}((\phi|_{C^S(\mathscr{P})})_*([s_0]); (\phi|_{C^S(\mathscr{P})})_*([s_1]), \\
&\qquad \dots, (\phi|_{C^S(\mathscr{P})})_*([s_k])),
\end{aligned} \tag{5.9}$$

where $[s_0], \dots, [s_k]$ are the homology classes containing $s_0, \dots, s_k$ respectively. Therefore $(\phi|_{C^S(\mathscr{P})})_*$ is a morphism of partial pseudo-operad from $H_*(\mathscr{P})$ to $H_*(\mathscr{H}_{V,0})$. It is easy to see that the graded partial pseudo-operad $H_*(\mathscr{H}_{V,0})$ is isomorphic to the endomorphism graded partial pseudo-operad $\mathscr{H}_{H^*(V),0}$ of the cohomology $H^*(V)$ of V. Thus we can identify $\mathscr{H}_{H^*(V),0}$ with $H_*(\mathscr{H}_{V,0})$. From (2.15) we see that elements of $H^*(V)$ in fact can be uniquely represented by elements of $V_{(0)}$. This implies that the endomorphism graded partial pseudo-operad $\mathscr{H}_{H^*(V),0}$ is in fact equal to the endomorphism graded operad $\mathscr{E}_{H^*(V)}$ and we have a morphism $(\phi|_{C^S(\mathscr{P})})_*$ of graded operads from $H_*(\mathscr{P})$ to $\mathscr{E}_{H^*(V)}$. Thus $H^*(V)$ is an algebra over $H_*(\mathscr{P})$. □

Combining this proposition and Theorem 4.2 with Getzler's theorem (Theorem 5.3) or Cohen's theorem (Theorem 5.2), we obtain:

Theorem 5.7. *Let V be a topological vertex algebra (or a weak topological vertex algebra). The cohomology $H^*(V)$ of V has the structure of a Batalin-Vikovisky algebra (or of Gerstenhaber algebra).*

We now would like to derive the concrete expressions of the operator Δ, the product and the bracket on $H^*(V)$. We only discuss the case for topological vertex algebras. For weak topological vertex algebra, the discussion is similar. From [Get] we see that the operator Δ is given by the image of the generator of $H_1(\mathscr{P}(1))$ under ψ and the product is given by the image of the unique element of $H_0(\mathbb{F}_2(\mathbb{C})) \subset H_0(\mathscr{P}(2))$ under the morphism ψ. Thus from the definition of ψ we see that $\Delta = g(0)$. Since any element of $\mathbb{F}_2(\mathbb{C})$ is a cycle representing the unique element of $H_0(\mathbb{F}_2(\mathbb{C}))$, we can take the cycle to be of the form $P = (z, 0)$. Let $u, v \in V$ and $[u], [v] \in H^*(V)$ the cohomology class containing u, v, respectively. Then the product $[u]\,[v]$ of $[u]$ and $[v]$ is $[Y(u, z)v]$ which is the cohomology class containing $Y(u, z)v \in V$. Since elements of $H^*(V)$ can be represented uniquely by elements of $V_{(0)}$, we can assume u and v have weight 0. Since $Y(u, z)v = \sum_{n \in \mathbb{Z}} u_n v z^{-n-1}$ and $\operatorname{wt} u_n v = \operatorname{wt} u - n - 1 + \operatorname{wt} v = -n - 1$, $[u_n v] = 0$ for $n \neq -1$. Thus $[Y(u, z)v] = [u_{-1} v]$. Since the bracket can be obtained from Δ and the product, we have recovered the construction of Lian and Zuckerman completely.

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Note added in proof. In [KSV], Kimura, Stasheff and Voronov give a conceptual construction of the homotopy Lie algebra structure on the state space of a topological conformal field theory based on the geometric formulation of topological conformal field theory and the existence of string vertices proved by Zwiebach. Combining their result and the geometric formulation of strong topological vertex algebras, we obtain a homotopy Lie algebra structure on the underlying graded vector space of a strong topological vertex algebra or on the tensor product of the underlying graded vector space with itself of a strong topological vertex algebra.

[KSV] Kimura, T., Stasheff, J.D., Voronov, A.A.: On operad structure of moduli spaces and string theory. Preprint, hep-th/9307115, 1993

References

[BoV] Boardman, J.M., Vogt, R.M.: Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, No. 347. Berlin, Heidelberg, New York: Springer 1973

[B] Borcherds, R.E.: Vertex algebras, Kac-Moody algebras, and the monster. Proc. Natl. Acad. Sci. USA **83**, 3068–3071 (1986)

[C] Cohen, F.R.: The homology of $\mathscr{C}_{n+1}$-spaces, $n \geq 0$. In: The homology of iterated loop spaces. Lecture Notes in Mathematics, No. 533. Berlin, Heidelberg, New York: Springer 1976, pp. 207–351

[DL] Dong, C., Lepowsky, J.: Generalized vertex algebras and relative vertex operators. Research monograph, to be published, 1993

[FFR] Feingold, A.J., Frenkel, I.B., Ries, J.F.X.: Spinor construction of vertex operator algebras, triality and $E_8^{(1)}$. Contemp. Math., Vol. **121**, Providence, RI: Am. Math. Soc., 1991

[FHL] Frenkel, I.B., Huang, Y.-Z., Lepowsky, J.: On axiomatic approaches to vertex operator algebras and modules. Preprint, 1989; Memoirs Am. Math. Soc. **104**, 1993

[FLM1] Frenkel, I.B., Lepowsky, J., Meurman; A.: A natural representation of the Fischer-Griess monster with the modular function J as character. Proc. Natl. Acad. Sci USA **81**, 3256–3260 (1984)

[FLM2] Frenkel, I.B., Lepowsky, J., Meurman, A.: Vertex operator algebras and the monster. Pure and Appl. Math. **134**, Boston: Academic Press 1988

[FZ] Frenkel, I.B., Zhu, Y.: Vertex operator algebras associated to representations of affine and Virasoro algebras. Duke Math. **66**, 123–156 (1992)

[Ger] Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. (2) **78**, 267–288 (1963)

[Get] Getzler, E.: Batalin-Vilkovisky algebras and two-dimensional topological field theory. Commun. Math. Phys., to appear

[H1] Huang, Y.-Z.: On the geometric interpretation of vertex operator algebras. Ph.D. thesis, Rutgers University, 1990; Operads and the geometric interpretation of vertex operator algebras, I. Preprint, to appear

[H2] Huang, Y.-Z.: Geometric interpretation of vertex operator algebras. Proc. Natl. Acad. Sci. USA **88**, 9964–9968 (1991)

[H3] Huang, Y.-Z.: Vertex operator algebras and conformal field theory. Intl. J. Mod. Phys. **A7**, 2109–2151 (1992)

[H4] Huang, Y.-Z.: Operads and the geometric interpretation of vertex operator algebras. II. In preparation

[HL1] Huang, Y.-Z., Lepowsky, J.: Vertex operator algebras and operads. The Gelfand Mathematical Seminars, 1990–1992. Corwin, L., Gelfand, I., Lepowsky J., (eds.), Boston: Birkhäuser, 1993, pp. 145–161

[HL2] Huang, Y.-Z., Lepowsky, J.: Operadic formulation of the notion of vertex operator algebra. Preprint, to appear, 1993

[HS] Hinich, V., Schecchtman, V.: Homotopy Lie algebras. Preprint, 1992
[L] Lian, B.H.: On the classification of simple vertex operator algebras. Preprint Commun. Math. Phys., to appear
[LZ] Lian, B.H., Zuckerman, G.J.: New perspectives on the **BRST**-algebraic structure of string theory. Preprint, hep-th/9211072, 1992
[Mas] Massey, W.S.: A basic course in algebraic topology. Graduate Texts in Math., Vol. 127. Berlin, Heidelberg, New York: Springer 1991
[May] May, J.P.: The geometry of iterated loop spaces. Lecture Notes in Mathematics, No. 271. Berlin, Heidelberg, New York: Springer 1972
[MS] Moore, G., Seiberg, N.: Classical and quantum conformal field theory, Comm. Math. Phys. **123**, 177–254 (1989)
[PS] Penkava, M., Schwarz, A.: On some algebraic structures arising in string theory. Preprint, hep-the/9212072, 1992
[S1] Stasheff, J.D.: Homotopy associativity of H-spaces. I. Trans. Am. Math. Soc. **108**, 215–292 (1963); Homotopy associativity of H-spaces. II. Trans. Am. Math. Soc. **108**, 293–312 (1963)
[S2] Stasheff, J.D.: Closed string field ltheory, strong homotopy Lie algebras and the operad actions of moduli spaces. Preprint, hep-th/9304061, 1993
[T] Tsukada, H.: Vertex operator superalgebras. Comm. Alg. **18**, 2259–2274 (1990)
[Z] Zwiebach, B.: Closed string field theory: quantum action and the **B-V** master equation. Nucl. Phys. **B390**, 33–152 (1993)

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