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Convergence of General Decompositions of Exponential Operators

Masuo Suzuki
Department of Physics, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan

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Abstract: A general theorem is proved concerning the convergence of decompositions of exponential operators in a Banach space (or normed space). As a corollary, the convergence of fractal decompositions is proved. The convergence of generalized Trotter-like formulas is also shown to result from the general theorem.

1. Introduction

In the present paper, we investigate the convergence of some systematic series of decompositions of exponential operators [1 ∼ 5] such as $\exp[x(A_1+A_2+\cdots+A_q)]$ for non-commutable operators $\{A_j\}$ in a Banach space. In this note we mean an operator by a bounded linear operator on a Banach space.

As is well known [1 ∼ 11], the first-order decomposition $Q(x)$ is given by

$$Q(x)=\mathrm{e}^{xA_1}\mathrm{e}^{xA_2}\ldots\mathrm{e}^{xA_{q-1}}\mathrm{e}^{xA_q}\,, \tag{1.1}$$

i.e.,

$$\mathrm{e}^{x(A_1+A_2+\cdots+A_q)}=Q(x)+\mathrm{O}(x^2)\,, \tag{1.2}$$

and the second-order symmetric decomposition is given by

$$S(x)=\mathrm{e}^{\frac{x}{2}A_1}\ldots\mathrm{e}^{\frac{x}{2}A_{q-1}}\mathrm{e}^{xA_q}\mathrm{e}^{\frac{x}{2}A_{q-1}}\ldots\mathrm{e}^{\frac{x}{2}A_1}\,, \tag{1.3}$$

i.e.,

$$\mathrm{e}^{x(A_1+\cdots+A_q)}=S(x)+\mathrm{O}(x^3)\,. \tag{1.4}$$

The above symmetry is characterized by the relation

$$S(x)S(-x)=\mathbb{1} \text{ or } S(-x)=S^{-1}(x)\,. \tag{1.5}$$

In general, the m^{th} order exponential decomposition $Q_m(x)$ is given in the form [1 ∼ 5]

$$Q_m(x)=\mathrm{e}^{x\tau_{11}A_1}\mathrm{e}^{x\tau_{12}A_2}\ldots\mathrm{e}^{x\tau_{1q}A_q}\mathrm{e}^{x\tau_{21}A_1}\mathrm{e}^{x\tau_{22}A_2}\ldots\mathrm{e}^{x\tau_{2q}A_q}\ldots\,, \tag{1.6}$$

with some appropriate parameters $\{\tau_{ij}\}$ determined by the requirement that

$$e^{x(A_1+\cdots+A_q)}=Q_m(x)+O(x^{m+1}) . \tag{1.7}$$

It is more convenient to get the m^{th} order approximant $Q_m(x)$ of $e^{x(A_1+\cdots+A_q)}$ as a decomposition in terms of $Q(x)$, $S(x)$ or an s^{th} order approximant $Q_s(x)$ as

$$Q_m(x)=Q_s(p_1x)Q_s(p_2x)\ldots Q_s(p_nx) \tag{1.8}$$

for some appropriate parameters $\{p_j\}$ which satisfy the condition

$$p_1+p_2+\cdots+p_n=1 , \tag{1.9}$$

and some other relations [$1\sim5$]. Here n depends on m and n tends to the infinity as $m\mapsto\infty$. In (1.8), $Q_1(x)=Q(x)$ and $Q_2(x)=S(x)$.

A systematic scheme to derive higher-order decompositions even up to infinite order is given by the following recursive method [$1\sim5$]:

$$Q_{2m}(x)=Q_{2m-2}(t_{m,1}x)Q_{2m-2}(t_{m,2}x)\ldots Q_{2m-2}(t_{m,r}x) \tag{1.10}$$

with $Q_2(x)=S(x)$ and

$$t_{m,1}+t_{m,2}+\cdots+t_{m,r}=1 \text{ and } t_{m,1}^{2m-1}+t_{m,2}^{2m-1}+\cdots+t_{m,n}^{2m-1}=0 . \tag{1.11}$$

For example, for $r=5$, we have $Q_{2m}(x)=\hat{S}_{2m}(x)$ for

$$\hat{S}_{2m}(x)=\hat{S}_{2m-2}^2(t_mx)\hat{S}_{2m-2}((1-4t_m)x)\hat{S}_{2m-2}^2(t_mx) \tag{1.12}$$

with $\hat{S}_2(x)=S(x)$, where the parameter t_m is given [$1\sim5$] by

$$t_m=\frac{1}{4-4^{1/(2m-1)}} . \tag{1.13}$$

It should be noted that $t_m>0$ but $(1-4t_m)<0$. Consequently, the above decomposition (1.10) shows a fractal property [$1\sim5$] in the parameter space $\{p_{mj}\}$ defined in

$$\hat{S}_{2m}(x)=S(p_{m1}x)S(p_{m2}x)S(p_{m3}x)\ldots S(p_{mn}x) \tag{1.14}$$

with $n=n(m)=5^{m-1}$. Namely, from the above recursive scheme (1.10), for $r=5$, we obtain

$$\begin{aligned}\{p_{2j}\}&=(t_2,t_2,1-4t_2,t_2,t_2) ,\\ \{p_{3j}\}&=\{p_{2j}\}\otimes(t_3,t_3,1-4t_3,t_3,t_3)\\ &=(t_2t_3,t_2t_3,(1-4t_2)t_3,t_2t_3,t_2t_3,\ldots,t_2t_3) ,\\ \{p_{mj}\}&=\{p_{m-1j}\}\otimes(t_{m-1},t_{m-1},1-4t_{m-1},t_{m-1},t_{m-1}).\end{aligned} \tag{1.15}$$

Since $t_m\to\frac{1}{3}$ for $m\to\infty$, we find that $|p_{mj}|\sim3^{-m}\to0$ as $m\to\infty$ for any j ($j=1,2,\ldots,5$). Thus, the decomposition parameter $\{p_{mj}\}$ in (1.14) decreases exponentially in our fractal decompositions of exponential operators. This property is essential in the proof of the convergence of the fractal decompositions, as will be seen later.

Our problem is to study the convergence of the general m^{th} order decomposition $Q_m(x)$ in the limit $m\to\infty$. Is it possible to prove, for example, the convergence of (1.14) to the operator $e^{x(A_1+\cdots+A_q)}$? If the parameters $\{p_{mj}\}$ satisfied the condition

$$\sum_{j=1}^{n(m)} |p_{mj}| \text{ is bounded }, \tag{1.16}$$

as in complex decompositions [1~5], then it would be rather easy to prove the convergence. However, it is not the situation. For example, for the parameter $\{p_{mj}\}$ in (1.14), we find

$$\sum_{j=1}^{n(m)} |p_{mj}| \sim \left(\frac{1}{3}\right)^m \times 5^m = \left(\frac{5}{3}\right)^m \to \infty \text{ as } m \to \infty , \tag{1.17}$$

because $n \equiv n(m) = 5^{m-1}$. This corresponds to the fact that the dimensionality D of the above decomposition (1.12) with (1.13) is given by

$$D = \frac{\log 5}{\log 3} = 1.46\ldots > 1 . \tag{1.18}$$

Thus, the above decomposition is called the "fractal decomposition."

In general, the decomposition with parameters $\{p_{mj}\}$ is called "fractal," when the parameters $\{p_{mj}\}$ satisfy the following conditions:

$$\text{(i)} \quad p_{m1} + p_{m2} + \cdots p_{mn} = 1 , \tag{1.19}$$

$$\text{(ii)} \quad |p_{mj} + p_{mj+1} + \cdots + p_{mn}| \tag{1.20}$$

is bounded uniformly for both m and j, and

$$\text{(iii)} \quad \sum_{j=1}^{n} |p_{mj}| \to \infty \text{ as } m \to \infty . \tag{1.21}$$

Here, the parameter n depends on the index m, namely $n = n(m)$. We also say that the m^{th} order decomposition is "of index m."

In Sect. 2, the main theorem on the convergence of decompositions is given under some general conditions. As a corollary of the theorem, the convergence of the decomposition (1.8) for $s = 1$ and $s = 2$ is shown under some general conditions on the parameters $\{p_{mj}\}$. In Sect. 3, the proof of the main theorem is given. Some applications of the theorem are presented in Sect. 4. Summary and discussion are given in Sect. 5.

It should also be mentioned that the above higher-order decompositions are very useful in quantum physics, statistical physics, nonlinear dynamics, astrophysics, quantum chemistry and many other fields [12~27].

2. Main Theorem

In the present section, we give a general theorem concerning the convergence of the decomposition (1.8) or more general non-uniform decompositions in the limit $m \to \infty$. First we define an approximant of index s as follows.

Definition. *A family of operators $Q_s^{(j)}(x)$ depending on $x \in \mathbb{C}$ and*

$$\| Q_s^{(j)}(x) - e^{x\mathcal{H}} \| \leqq K_s |x|^{s+1} \tag{2.1}$$

uniformly for any j for $|x|<\varepsilon$, with some $\varepsilon>0$, with a positive number s and with some positive constant K_s, all independent of j, is called an approximant of index s.

We shall be using such a uniform family of approximants for an s^{th} order decomposition of the original exponential operator $e^{x\mathscr{H}}$ with $\mathscr{H}$ given as the sum of operators $\{A_j\}_{1\leq j\leq q}$,

$$\mathscr{H}=A_1+A_2+\cdots+A_q\,. \tag{2.2}$$

Then, we have the following theorem.

Theorem. *Let $\{Q_s^{(j)}(x)\}$ be approximants of index s for the exponential operator* $\exp(x\mathscr{H})\equiv\exp[x(A_1+A_2+\cdots+A_q)]$ *with the operators $\{A_j\}$ in a Banach space. A systematic series of approximants $\{F_m(x)\}$ for* $\exp(x\mathscr{H})$ *constructed by the ordered product*

$$F_m(x)=Q_s^{(1)}(p_{m1}x)Q_s^{(2)}(p_{m2}x)\ldots Q_s^{(m)}(p_{mn}x) \tag{2.3}$$

converges to $\exp(x\mathscr{H})$, *namely*

$$\lim_{m\to\infty}\|F_m(x)-e^{x\mathscr{H}}\|=0,\text{ i.e., }\lim_{m\to\infty}F_m(x)=e^{x\mathscr{H}} \tag{2.4}$$

for all $x\in\mathbb{C}$ under the condition that

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}|p_{mj}|^{s+1}=0 \tag{2.5}$$

together with the conditions (1.19) *and* (1.20). *The limit* (2.4) *is uniform provided $|x|<\delta$ for any positive number δ, namely in any compact region of x.*

Conversely, if $Q_s^{(1)}(x)=\cdots=Q_s^{(n)}(x)=Q_s(x)$ for a strictly s^{th} order $Q_s(x)$ and $\{F_m(x)\}$ converges uniformly to $e^{x\mathscr{H}}$ for any operators $\{A_j\}$ in a Banach space, then

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}p_{mj}^{s+1}=0\,. \tag{2.6}$$

Remark 1. There are many different choices of s^{th} order approximants $Q_s^{(1)}$, $Q_s^{(2)}$, ... As the first-order approximants

$Q_1^{(1)}\equiv Q^{(1)}(x)$, $Q_2^{(2)}\equiv Q^{(2)}(x)$, ..., we may choose, for example,

$$Q^{(1)}(x)=e^{xA_1}e^{xA_2}\ldots Q^{(2)}(x)=e^{xA_2}e^{xA_1}\ldots,\ldots. \tag{2.7}$$

Furthermore $Q_s^{(j)}$ may depend also on m.

Remark 2. Here, $F_m(x)$ is not necessarily an operator of index m.

Remark 3. This theorem can be applied to fractal decompositions in which $\sum_j|p_{mj}|$ diverges in the limit $m\to\infty$. It is easy to confirm the condition (2.5) for fractal decompositions, as will be discussed later explicitly.

Remark 4. The parameter n in Eqs. (2.3) and (2.5) increases as m increases, namely $n=n(m)\to\infty$ as $m\to\infty$. The parameters $\{p_{mj}\}$ go to zero as m goes to infinity, as is required from (2.5). However, it does not necessarily imply the boundedness of the summation $\sum_j|p_{mj}|$, as is easily seen from the example (1.12) in which $\sum_j|p_{mj}|^3\sim5^m\times(3^{-m})^3=(5/27)^m\to0$ but $\sum_j|p_{mj}|\sim(5/3)^m\to\infty$.

Remark 5. Actually the above theorem is more general. This holds for any kind of $Q_s(x)$ for $s>0$ and for any exponential operator $\exp(x\mathscr{H})$ (not necessarily of the type $\exp[x(A_1+A_2+\cdots+A_q)]$). For example, $(1+xA)(1+xB)$, $(1+xA)(1-xB)^{-1}$ and $(1-xA)^{-1}(1-xB)^{-1}$ can be used as $Q_1(x)$.

Corollary 1. *If we construct* $F_m(x)$ *as*

$$F_m(x)=Q^{(1)}(p_{m1}x)Q^{(2)}(p_{m2}x)\ldots Q^{(n)}(p_{mn}x) \tag{2.8}$$

with $\{Q^{(j)}(x)\}$ *of the form* (1.1), *namely of first order, then*

$$\lim_{m\to\infty} F_m(x)=e^{x\mathscr{H}} \tag{2.9}$$

under the condition that

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}|p_{mj}|^2=0\,, \tag{2.10}$$

together with (1.20).

Conversely, if $Q^{(1)}(x)=\cdots=Q^{(n)}(x)=Q(x)$ and Eq. (2.9) holds, then

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}p_{mj}^2=0\,. \tag{2.11}$$

Furthermore, for the real decomposition (i.e., for real $\{p_{mj}\}$) satisfying Eq. (1.20), Eq. (2.11) is a necessary and sufficient condition of the convergence (2.8).

Corollary 2. *For the decomposition*

$$S_{2m}(x)=S(p_{m1}x)S(p_{m2}x)\ldots S(p_{mn}x)\,, \tag{2.12}$$

we have

$$\lim_{m\to\infty} S_{2m}(x)=e^{x(A_1+\cdots+A_q)} \tag{2.13}$$

under the condition that

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}|p_{mj}|^3=0\,, \tag{2.14}$$

together with (1.20).

Conversely, if (2.13) holds, then

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}p_{mj}^3=0\,. \tag{2.15}$$

Corollary 3. *We consider the following non-uniform decomposition*

$$E_n(A,B;\{t_j\},\{s_j\})=e^{t_1A}e^{s_1B}e^{t_2A}e^{s_2B}\ldots e^{t_nA}e^{s_nB} \tag{2.16}$$

for the operators A and B in a Banach space. This converges to the exponential operator e^{A+B}, *namely*

$$\lim_{n\to\infty} E_n(A,B;\{t_j\},\{s_j\})=e^{A+B}\,, \tag{2.17}$$

if

(i) $$\sum_{j=1}^{n} t_j = \sum_{j=1}^{n} s_j = 1\,, \tag{2.18}$$

(ii) $$|p_j + p_{j+1} + \cdots + p_{2n-1}| \text{ is bounded} \tag{2.19}$$

for any j and n,

and

$$\lim_{n\to\infty} \sum_{k=1}^{2n-1} |p_k|^2 = 0\,. \tag{2.20}$$

Here,

$$p_{2k-1} = t_k + \sum_{j=1}^{k-1} (t_j - s_j) \quad \text{and} \quad p_{2k} = \sum_{j=1}^{k} (s_j - t_j)\,. \tag{2.21}$$

The proof of Corollary 3 is given as follows. It is impossible to express Eq. (2.16) as a uniform decomposition $Q(p_1)Q(p_2)\ldots Q(p_\tau)$ with $Q(p) = e^{pA}e^{pB}$. However, (2.16) can be rearranged in the following non-uniform decomposition [4]

$$E_n(A, B; \{t_j\}, \{s_j\}) = Q(p_1)\tilde{Q}(p_2)Q(p_3)\tilde{Q}(p_4)\ldots Q(p_{2n-1}) \tag{2.22}$$

with the tilde operator

$$\tilde{Q}(p) = Q^{-1}(-p) = e^{pB}e^{pA}\,, \tag{2.23}$$

and with the relations (2.21). Then, our main theorem can be applied to (2.22) and we arrive at Corollary 3. From (2.20), we find that, at least, $\lim t_j = \lim s_j = 0$ for $n \to \infty$.

Conversely, we assume the convergence of (2.22). Then we obtain the following necessary condition:

$$\lim_{n\to\infty} \sum_{k=1}^{n} (p_{2k-1}^2 - p_{2k}^2) = 0\,. \tag{2.24}$$

In order to derive this result, it should be noted that if we can choose $R(p)$ such that

$$Q(p) = e^{p(A+B)}e^{p^2 R(p)}\,, \tag{2.25}$$

then

$$\tilde{Q}(p) = Q^{-1}(-p) = e^{-p^2 R(-p)}e^{p(A+B)}\,. \tag{2.26}$$

Thus, the tilde operators $\{\tilde{Q}(p_{2k})\}$ contribute to the second-order correction with negative sign as $\{-p_{2k}^2\}$ and consequently we obtain the necessary condition (2.24).

3. Proof of the Theorem

Let $Q_s^{(j)}(x)$ be an exponential decomposition which is an s^{th} order approximant of $e^{x\mathcal{H}}$. We can put $Q_s^{(j)}(x)$ in the form

$$Q_s^{(j)}(x) = e^{x\mathcal{H}}e^{x^{s+1}R_{s+1}^{(j)}(x)} \tag{3.1}$$

for an appropriate operator $R_{s+1}^{(j)}(x)$ with (2.2). Indeed we obtain the following lemma.

Lemma 1. *There exists an operator $R_{s+1}^{(j)}(x)$ satisfying* (3.1) *at least in the region $|x|<x_s$ (x_s a suitable positive number) which is defined by the inequality*

$$\| e^{-x\mathscr{H}} Q_s^{(j)}(x) - \mathbb{1} \| < 1 \tag{3.2}$$

for any j.

The proof of this lemma is easily given as follows [7]. The operator $x^{s+1}R_{s+1}^{(j)}(x)$ exists under the condition (3.2) as follows,

$$x^{s+1}R_{s+1}^{(j)}(x) = \log(\mathrm{e}^{-x\mathscr{H}} Q_s^{(j)}(x)) = \log[\mathbb{1} + (\mathrm{e}^{-x\mathscr{H}} Q_s^{(j)}(x) - \mathbb{1})] . \tag{3.3}$$

Next we show that there exists a positive x_s. From (2.1) and from the condition of the theorem, $Q_s^{(j)}(x)$ is of index s, and we have

$$\begin{aligned} \| \mathrm{e}^{-x\mathscr{H}} Q_s^{(j)}(x) - \mathbb{1} \| &\leqq \mathrm{e}^{|x|\,\|\mathscr{H}\|} \| Q_s^{(j)}(x) - \mathrm{e}^{x\mathscr{H}} \| \\ &\leqq K_s |x|^{s+1} \mathrm{e}^{|x|\,\|\mathscr{H}\|} \end{aligned} \tag{3.4}$$

for small $|x|$ and for any j. Thus, at least, in the region $|x|<x_s$ where x_s is given by

$$K_s x_s^{s+1} \mathrm{e}^{x_s \|\mathscr{H}\|} = 1 , \tag{3.5}$$

there exists the operator $x^{s+1}R_{s+1}^{(j)}(x)$. For more precise determination of the region of x for Eq. (3.2) to hold, see Appendix. Furthermore, it is easy to show that $\| R_{s+1}^{(j)}(x) \|$ is bounded as

$$\| R_{s+1}^{(j)}(x) \| = \frac{1}{|x|^{s+1}} \| \log\{\mathbb{1} + (\mathrm{e}^{-x\mathscr{H}} Q_s^{(j)}(x) - \mathbb{1})\} \| \leqq \hat{K}_s . \tag{3.6}$$

with some appropriate upper bound $\hat{K}_s$ larger than K_s in some fixed region $|x| \leqq \hat{x}_s$ (which is smaller than x_s). Here, $\hat{x}_s$ is the solution of the following nonlinear equation

$$(1 - K_s |\hat{x}_s|^{s+1} \mathrm{e}^{|\hat{x}_s|\,\|\mathscr{H}\|}) \mathrm{e}^{\hat{K}_s |\hat{x}_s|^{s+1}} = 1 . \tag{3.7}$$

Thus, $\hat{x}_s$ depends on $\hat{K}_s$ ($>K_s$), and it is smaller than x_s. Conversely $\hat{K}_s$ is given by $\hat{K}_s = K_s(\hat{x}_s)$. Here we have used the inequality (3.4) and the following norm inequality.

Lemma 2. *For an arbitrary operator A in a Banach space,*

$$\| \log(\mathbb{1} + xA) \| \leqq -\log(1 - |x|\, \|A\|) \tag{3.8}$$

in the region $|x|\,\|A\| < 1$.

Using the representation (3.1) of $Q_s^{(j)}(x)$, the operator $F_m(x)$ can be written as

$$\begin{aligned} F_m(x) &= Q_s^{(1)}(p_1 x) Q_s^{(2)}(p_2 x) \ldots Q_s^{(n)}(p_n x) \\ &= \mathrm{e}^{p_1 x\mathscr{H}} \mathrm{e}^{(p_1 x)^{s+1} R_{s+1}^{(1)}(p_1 x)} \mathrm{e}^{p_2 x\mathscr{H}} \mathrm{e}^{(p_2 x)^{s+1} R_{s+1}^{(2)}(p_2 x)} \ldots \\ &\quad \mathrm{e}^{p_{n-1} x\mathscr{H}} \mathrm{e}^{(p_{n-1} x)^{s+1} R_{s+1}^{(n-1)}(p_{n-1} x)} \mathrm{e}^{p_n x\mathscr{H}} \mathrm{e}^{(p_n x)^{s+1} R_{s+1}^{(n)}(p_n x)} \end{aligned} \tag{3.9}$$

with the abbreviation $p_j = p_{mj}$.

First we exchange the factor $\exp(p_n x\mathscr{H})$ with $\exp(p_{n-1} x)^{s+1} R_{s+1}^{(n-1)}(p_{n-1} x))$ in (3.9). For this purpose, we introduce an operator $W_{s+1}^{(j)}(x, y)$ satisfying the relation

$$\mathrm{e}^{x^{s+1} R_{s+1}^{(j)}(x)} \mathrm{e}^{y\mathscr{H}} = \mathrm{e}^{y\mathscr{H}} \mathrm{e}^{x^{s+1} W_{s+1}^{(j)}(x, y)} . \tag{3.10}$$

Then we have the following lemma.

Lemma 3. *There exists an operator* $W_{s+1}^{(j)}(x, y)$ *satisfying the relation* (3.10) *in the region* $|x|<x_s$. *It is also bounded as*

$$\| W_{s+1}^{(j)}(x, y) \| \leqq e^{2|y| \|\mathscr{H}\|} \| R_{s+1}^{(j)}(x) \| . \tag{3.11}$$

The proof of this lemma is given as follows. From (3.10), we obtain

$$\begin{aligned} W_{s+1}^{(j)}(x, y) &= x^{-(s+1)} \log (e^{-y\mathscr{H}} e^{x^{s+1} R_{s+1}^{(j)}} e^{y\mathscr{H}}) \\ &= x^{-(s+1)} \log \exp (x^{s+1} e^{-y\delta_{\mathscr{H}}} R_{s+1}^{(j)}(x)) \\ &= e^{-y\delta_{\mathscr{H}}} R_{s+1}^{(j)}(x) , \end{aligned} \tag{3.12}$$

where $\delta_{\mathscr{H}}$ is the inner derivation defined by

$$\delta_{\mathscr{H}} R = [\mathscr{H}, R] = \mathscr{H}^{\times} R \tag{3.13}$$

with Kubo's notation $\mathscr{H}^{\times}$ [28]. Thus, $W_{s+1}^{(j)}(x, y)$ exists in the region $|x|<x_s$ and the norm of $W_{s+1}^{(j)}(x, y)$ is bounded as (3.11), because

$$\| W_{s+1}^{(j)}(x, y) \| = \| e^{-y\mathscr{H}} R_{s+1}^{(j)}(x) e^{y\mathscr{H}} \| \leqq e^{2|y| \|\mathscr{H}\|} \| R_{s+1}^{(j)}(x) \| . \tag{3.14}$$

Now we exchange the factor $\exp(p_n x\mathscr{H})$ with the exponential operator $\exp((p_{n-1}x)^{s+1} R_{s+1}^{(n-1)}(p_{n-1}x))$ as follows:

$$e^{(p_{n-1}x)^{s+1} R_{s+1}^{(n-1)}(p_{n-1}x)} e^{p_n x\mathscr{H}} = e^{p_n x\mathscr{H}} e^{(p_{n-1}x)^{s+1} W_{s+1}^{(n-1)}(p_{n-1}x, p_n x)} , \tag{3.15}$$

using Lemma 3 in the region $|p_{n-1}x|<x_s$, namely

$$|x|<x_s/|p_{n-1}| . \tag{3.16}$$

Then the factor $\exp(p_n x\mathscr{H})$ can be combined with $\exp(p_{n-1}x\mathscr{H})$ in (3.9) as $\exp[(p_{n-1}+p_n)x\mathscr{H}]$. Next we exchange this new factor $\exp[(p_{n-1}+p_n)x\mathscr{H}]$ with the operator $\exp[(p_{n-2}x)^{s+1} R_{s+1}^{(n-2)}(p_{n-2}x)]$ as

$$e^{(p_{n-2}x)^{s+1} R_{s+1}^{(n-2)}(p_{n-2}x)} e^{(p_{n-1}+p_n)x\mathscr{H}} = e^{(p_{n-1}+p_n)x\mathscr{H}} e^{x^{s+1} W_{n-1}^{(n-2)}(p_{n-2}x, (p_{n-1}+p_n)x)} , \tag{3.17}$$

using again Lemma 3.

Thus, we can repeat this procedure $(n-1)$ times and consequently we arrive at the following result

$$F_m(x) = e^{(p_1+\cdots+p_n)x\mathscr{H}} e^{Y_1(x)} e^{Y_2(x)} \ldots e^{Y_n(x)} , \tag{3.18}$$

where

$$Y_j(x) = (p_j x)^{s+1} W_{s+1}^{(j)}(p_j x, (p_{j+1}+\cdots+p_n)x) , \tag{3.19}$$

for $1 \leqq j \leqq n-1$, and

$$Y_n(x) = (p_n x)^{s+1} R_{s+1}^{(n)}(p_n x) . \tag{3.20}$$

Now, we can derive the following norm inequality

$$\begin{aligned} \| e^{-x\mathscr{H}} F_m(x) - \mathbb{1} \| &= \left\| \prod_{j=1}^{n} e^{Y_j(x)} - \mathbb{1} \right\| \\ &\leqq \exp\left(\sum_{j=1}^{n} \| Y_j(x) \| \right) - 1 \\ &\leqq \left(\sum_{j=1}^{n} \| Y_j(x) \| \right) \exp\left(\sum_{j=1}^{n} \| Y_j(x) \| \right) . \end{aligned} \tag{3.21}$$

Then, from (3.19) and (3.20), we obtain

$$\sum_{j=1}^{n} \| Y_j(x) \| \leqq \sum_{j=1}^{n-1} |p_j x|^{s+1} \| W_{s+1}^{(j)}(p_j x, (p_{j+1}+\cdots+p_n)x) \| + |p_n x|^{s+1} \| R_{s+1}^{(n)}(p_n x) \| . \tag{3.22}$$

Here it should be noted that the parameter $|p_{j+1}+\cdots+p_n|$ in (3.22) is bounded owing to the condition of the theorem and consequently

$$e^{2|p_{j+1}+\cdots+p_n|\,|x|\,\|\mathscr{H}\|} \leqq E(x) \tag{3.23}$$

with some upper bound $E(x)$ for any j and m. Since $\| R_{s+1}^{(j)}(x) \| \leqq \hat{K}_s$, as shown in (3.6), we obtain the inequality

$$\sum_{j=1}^{n} \| Y_j(x) \| \leqq \left(\sum_{j=1}^{n} |p_j|^{s+1} \right) (\hat{K}_s |x|^{s+1} E(x)) . \tag{3.24}$$

From (3.21), we arrive finally at

$$|e^{-x\mathscr{H}} F_m(x) - \mathbb{1} \| \leqq \left(\sum_{j=1}^{n} |p_j|^{s+1} \hat{K}_s |x|^{s+1} E(x) \right) \exp\left(\sum_{j=1}^{n} |p_j|^{s+1} \hat{K}_s |x|^{s+1} E(x) \right) . \tag{3.25}$$

The condition (2.5) of the theorem yields the desired result

$$\lim_{m\to\infty} \| e^{-x\mathscr{H}} F_m(x) - \mathbb{1} \| = 0 , \tag{3.26}$$

that is,

$$\lim_{m\to\infty} \| F_m(x) - e^{x\mathscr{H}} \| = 0 \tag{3.27}$$

uniformly in any compact region of x, because all the parameters $|p_j|$ go to zero and consequently $|p_j x| < \hat{x}_s < x_s$ for any x, in the limit $m \to \infty$.

If we assume the uniform convergence of $F_m(x)$ to $e^{x\mathscr{H}}$, then the product $e^{Y_1(x)} \ldots e^{Y_n(x)}$ has to approach the unit operator $\mathbb{1}$ for any value of x. The $(s+1)^{\text{th}}$-order term of the product with respect to x is given by

$$\sum_{j=1}^{n} (p_j x)^{s+1} R_{s+1}^{(j)}(0) , \tag{3.28}$$

because $W_{s+1}^{(j)}(0,0) = R_{s+1}^{(j)}(0)$. This is reduced to

$$\left(\sum_{j=1}^{n} p_j^{s+1} \right) x^{s+1} R_{s+1}(0) \tag{3.29}$$

in the case when $Q_s^{(1)}(x) = \cdots = Q_s^{(n)}(x) = Q_s(x)$ and consequently $R_{s+1}^{(1)}(0) = \cdots = R_{s+1}^{(n)} = R_{s+1}(0)$. Thus we obtain

$$\lim_{m\to\infty} \sum_{j=1}^{n(m)} p_{mj}^{s+1} = 0 , \tag{3.30}$$

using $p_j = p_{mj}$, because $R_{s+1}(0) \neq 0$. If $R_{s+1}(0) = 0$, then $R_{s+1}(x)$ is a decomposition of higher order than the s^{th} order. This contradicts our assumption.

The above necessary condition will be also easily derived by considering the following two-dimensional solvable Lie algebra $\{A, B\}$ satisfying the commutation relation [5, 11]

$$[A, B] = \alpha B; \quad \alpha \neq 0 . \tag{3.31}$$

It is well known that the following relations hold:

$$e^{x(A+B)} = e^{xA} e^{f(\alpha x)xB} = e^{f(-\alpha x)xB} e^{xA} \tag{3.32}$$

with

$$f(x) = \frac{1-e^{-x}}{x} = 1 - \frac{x}{2} + \cdots . \tag{3.33}$$

The representation (3.1) in our case is written as

$$e^{xA} e^{xB} = e^{x(A+B)} e^{x^2 R(x)} . \tag{3.34}$$

Equivalently

$$R(x) = \frac{1-f(\alpha x)}{x} B , \tag{3.35}$$

using (3.32). Furthermore, the operator $W(x, y)$ defined in (3.10) is given by

$$e^{x^2 W(x,y)} = e^{-y(A+B)} e^{x^2 R(x)} e^{y(A+B)} \tag{3.36}$$

in the present case. Then, we obtain

$$W(x, y) = e^{-\alpha y} \frac{1-f(\alpha x)}{x} B , \tag{3.37}$$

using the commutation relation $|A+B, B| = [A, B] = \delta_A B = \alpha B$ and its consequent formula

$$e^{-y(A+B)} R(x) e^{y(A+B)} = e^{-yA} R(x) e^{yA} = e^{-y\delta_A} R(x) = e^{-\alpha y} R(x) . \tag{3.38}$$

Thus, we arrive at the representation

$$\begin{aligned} F_m(x) &= e^{x(A+B)} \exp\left[\sum_{j=1}^{n} (p_j x)^2 W(p_j x, (p_{j+1} + \cdots + p_n)x) \right] \\ &= e^{x(A+B)} \exp\left[\sum_{j=1}^{n} (p_j x)(1 - f(\alpha p_j x)) e^{-\alpha(p_{j+1} + \cdots + p_n)x} B \right] \end{aligned} \tag{3.39}$$

with the notation that $p_{n+1} + p_n \equiv 0$. Here we have used the commutability of $\{W(x, y)\}$. By expanding the second exponential factor in (3.39) with respect to x, we obtain

$$F_m(x) = e^{x(A+B)} \exp\left[\frac{\alpha}{2} \left(\sum_{j=1}^{n} p_j^2 \right) x^2 B + O(x^3) \right] \tag{3.40}$$

using (3.33). Therefore, our necessary condition is given by

$$\lim_{n \to \infty} \sum_{j=1}^{n} p_j^2 = 0 \quad \text{for } n = n(m) . \tag{3.41}$$

The present argument using the Lie algebra can be easily extended to the case $Q_s(x)$ to derive the necessary condition (3.30).

4. Some Applications

In the present section, we present some applications of our general theorem to the ordinary Trotter formula, the generalized Trotter-like formula, the unitary decomposition and the complex decomposition. The general scheme of higher-order decompositions is also discussed.

4.1. Trotter Formula. It is well known [7] that the Trotter decomposition

$$T_m(x) = (e^{\frac{x}{m}A} e^{\frac{x}{m}B})^m \tag{4.1}$$

converges to the exponential operator $e^{x(A+B)}$ in the infinite limit of m, namely

$$\lim_{m\to\infty} \| T_m(x) - e^{x(A+B)} \| = 0 \tag{4.2}$$

in a Banach space.

It is interesting that this convergence (4.2) is an immediate consequence of our general theorem. In fact, we have

$$T_m(x) = Q\left(\frac{x}{m}\right) Q\left(\frac{x}{m}\right) \cdots Q\left(\frac{x}{m}\right) \tag{4.3}$$

with

$$Q(x) = e^{xA} e^{xB} . \tag{4.4}$$

The parameters $\{p_{mj}\}$ in the theorem or more explicitly in Corollary 1 are given by

$$p_{m1} = p_{m2} = \cdots = p_{mm} = \frac{1}{m} \tag{4.5}$$

in the present case. Thus, we find that

$$\sum_{j=1}^{n(m)} |p_{mj}|^2 = \frac{m}{m^2} = \frac{1}{m} \to 0 \text{ as } m \to \infty . \tag{4.6}$$

Clearly, $\{|p_{mj} + \cdots + p_{mm}|\}$ are uniformly bounded for any m and j. Thus we obtain

$$\lim_{m\to\infty} Q_m(x) = e^{x(A+B)} \tag{4.7}$$

from our general Theorem in Sect. 2. In this case, the sum $\sum_j |p_{mj}|$ is finite (namely equal to unity) and consequently the proof of the convergence is quite easy as is well known [7].

4.2. Generalized Trotter-Like Formulas. If we know an s-th order decomposition $Q_s(x)$ (i.e., of index s), namely

$$e^{x\mathscr{H}} = Q_s(x) + O(|x|^{s+1}) , \tag{4.8}$$

then we can construct the following generalized Trotter-like formula [29, 30]

$$e^{x(A_1 + \cdots + A_q)} = \left[Q_s\left(\frac{x}{m}\right)\right]^m + O\left(\frac{|x|^{s+1}}{m^s}\right) . \tag{4.9}$$

The convergence of this formula can be also shown immediately from our general theorem, as in the case of the ordinary Trotter formula in Sub-sect. 4.1. In fact, the condition (2.5) of the main theorem is satisfied as follows:

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}|p_{mj}|^{s+1}=\lim_{m\to\infty}\frac{m}{m^{s+1}}=\lim_{m\to\infty}\frac{1}{m^s}=0 \tag{4.10}$$

for $s>0$. The case $s=1$ corresponds to the ordinary Trotter formula.

4.3. Fractal Decomposition. It is easy to confirm explicitly the condition (2.5) for fractal decompositions [1∼5, 22, 24, 25, 32]. For example, the symmetric decompositions $\{\hat{S}_{2m}(x)\}$ in (1.14) with (1.15) converge to $e^{x\mathscr{H}}$ with $\mathscr{H}=A_1+A_2+\cdots+A_q$ in the limit $m\to\infty$, because

$$\sum_{j=1}^{n(m)}|p_{mj}|^3\sim\left(\frac{1}{3^m}\right)^3\times 5^m=\left(\frac{5}{27}\right)^m\to 0\,, \tag{4.11}$$

as has already been discussed in Remark 4 in Sect. 2.

A more rigorous confirmation is given as follows:

$$\begin{aligned}\sum_{j=1}^{n(m)}|p_{mj}|^3&=\prod_{k=3}^{m}\left(\sum_{j=1}^{5}|p_{kj}|^3\right)\\&=\prod_{k=3}^{m}\frac{4+4^{3/(2k-1)}}{(4-4^{1/(2k-1)})^3}<\left(\frac{7}{8}\right)^{m-2}\,,\end{aligned} \tag{4.12}$$

since $4^{1/(2k-1)}\leqq 2^{2/5}<2$ and $4^{3/(2k-1)}<2^{6/5}<3$ for $k\geqq 3$. Thus we arrive at the desired result

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)}|p_{mj}|^3=0\,. \tag{4.13}$$

It is also easy to confirm that $\{|p_j+p_{j+1}+\cdots+p_n|\}$ are all bounded for any m and j. They are all smaller than unity. This confirmation yields the convergence of the fractal decomposition.

This kind of confirmation can be easily made for any other fractal decomposition as follows. In general, we consider the recursive scheme (1.10), namely

$$Q_{2m}(x)=Q_{2m-2}(t_{m,1}x)\ldots Q_{2m-2}(t_{m,r}x) \tag{4.14}$$

with $Q_2(x)=S(x)$ and with the parameters $\{t_{m,j}\}$ determined by (1.11). Then, the $2m^{\text{th}}$ order decomposition $Q_{2m}(x)$ is expressed as the product of the second-order symmetric decomposition $S(x)$, namely

$$Q_{2m}(x)=S(p_{m1}x)S(p_{m2}x)\ldots S(p_{mm}x)\,. \tag{4.15}$$

Here, the parameters $\{p_{mj}\}$ are given by the "direct" product of $\{t_{k,j}\}$ as $\{p_{2j}\}\equiv\{t_{2,j}\}$ and

$$\{p_{kj}\}=\{p_{k-1j}\}\otimes\{t_{k,j}\}\,, \tag{4.16}$$

similarly to the explicit example (1.15).

The fractal decompositions to be considered here satisfy the condition (1.20) and the condition that

$$1>\sum_{j=1}^{r}|t_{2,j}|^3\geqq\sum_{j=1}^{r}|t_{m,j}|^3\quad\text{for } m\geqq 3\,. \tag{4.17}$$

Therefore, we obtain

$$\sum_{j=1}^{n(m)} |p_{mj}|^3 = \prod_{k=2}^{m} \left(\sum_{j=1}^{r} |t_{k,j}|^3 \right)$$
$$\leqq \left(\sum_{j=1}^{r} |t_{2,j}|^3 \right)^{m-1} \to 0 \,. \tag{4.18}$$

This yields the convergence of the fractal decompositions $\{Q_{2m}(x)\}$ constructed by the recursive scheme using the parameters $\{t_{k,j}\}$ or $\{p_{kj}\}$ satisfying the conditions (1.20) and (4.17). Namely, we arrive at the uniform convergence

$$\lim_{m\to\infty} \| Q_{2m}(x) - \mathrm{e}^{x\mathscr{H}} \| = 0 \,, \tag{4.19}$$

in any compact region of x for the bounded operators $\{A_j\}$.

As an example of non-uniform decompositions [4, 5], we consider here the following multiple tilde decomposition [5]

$$Q_m(x) = [Q_{m-1}(p_m x)]^\lambda [\tilde{Q}_{m-1}(q_m x)]^\mu [Q_{m-1}(p_m x)]^\lambda \tag{4.20}$$

with the tilde operator $\tilde{Q}(x)$ [4,5] defined by $\tilde{Q}(x) = Q^{-1}(-x)$. Here the parameters p_m and q_m are given [5] by

$$p_m = \frac{1}{2\lambda + (-1)^m \mu (2\lambda/\mu)^{1/m}} \text{ and } q_m = (-1)^m \left(\frac{2\lambda}{\mu}\right)^{1/m} p_m \,. \tag{4.21}$$

They have the following limits

$$\begin{cases} \lim\limits_{m\to\infty} p_{2m} = \lim\limits_{m\to\infty} q_{2m} = \dfrac{1}{2\lambda+\mu} \,, \\ \lim\limits_{m\to\infty} p_{2m-1} = -\lim\limits_{m\to\infty} q_{2m-1} = \dfrac{1}{2\lambda-\mu} \,. \end{cases} \tag{4.22}$$

Here $2\lambda - \mu > 1$. Thus, the fractal dimensionality [31] of this decomposition is given [5] by

$$D = \frac{2\log(2\lambda+\mu)}{\log[(2\lambda+\mu)(2\lambda-\mu)]} \,. \tag{4.23}$$

As was pointed out in the previous paper [5], we have $1 < D < 2$ and

$$\lim_{\lambda\to\infty} D = 1 \,. \tag{4.24}$$

The above decomposition is reduced to the ordinary Trotter formula in the limit $\lambda \to \infty$. The condition (2.5) in the decomposition (4.20) is reduced to

$$\sum_{j=1}^{n(m)} |p_{mj}|^2 = \prod_{k=2}^{m} \left(\sum_{j=1}^{r} |p_{kj}|^2 \right) = \prod_{k=2}^{m} (2\lambda |p_k|^2 + \mu |q_k|^2)$$
$$= \prod_{k=2}^{m} \frac{2\lambda + \mu(2\lambda/\mu)^{1/k}}{(2\lambda + (-1)^k \mu (2\lambda/\mu)^{1/k})^2} \sim \left(\frac{1}{2\lambda-\mu}\right)^m \to 0 \tag{4.25}$$

for $2\lambda - \mu > 1$. Thus, we obtain

$$\lim_{m\to\infty} \| Q_m(x) - \mathrm{e}^{x(A_1+\cdots+A_q)} \| = 0 \tag{4.26}$$

uniformly in any compact region for x for the bounded operators $\{A_j\}$.

4.4. Unitary Decomposition. In quantum mechanics, the unitary operator $\exp(it\mathscr{H}/\hbar)$ with the hermitian operator $\mathscr{H}$ plays an important role and it is often decomposed in the form (2.3) with $x=it/\hbar$. The condition (2.5) is the same for real fractal decomposition and for unitary decomposition. Of course, the proof of the convergence for the unitary decomposition is easily found [22] without using our general theorem, because $\|\exp(it\mathscr{H}/\hbar)\|=1$.

4.5. Complex Decomposition. We consider the following recursive scheme

$$Q_m(x)=Q_{m-1}(p_{m1}x)\ldots Q_{m-1}(p_{mr}x) \tag{4.27}$$

with $Q_2(x)=S(x)$,

$$\sum_{j=1}^{r} p_{mj}=1 \quad\text{and}\quad \sum_{j=1}^{r} p_{mj}^{m}=0\,. \tag{4.28}$$

In particular, we discuss the following complex solution [1 ~ 3, 22]

$$p_{m1}\equiv p_m=\frac{1}{1+e^{i\pi/m}}=1-p_{m2} \tag{4.29}$$

for $r=2$, or more generally we consider the situation [22] in which s parameters of $\{p_{mj}\}$ take the same value p_m and the remaining $(r-s)$ parameters are equal to p'_m. Then, from (4.28), we obtain

$$p_m=\frac{1}{s(1+a_m e^{i\pi/m})}\ \text{and}\ p'_m=\left(\frac{s}{r-s}\right)^{1/m}\frac{e^{i\pi/m}}{s(1+a_m e^{i\pi/m})} \tag{4.30}$$

with

$$a_m=\left(\frac{r-s}{s}\right)^{1-1/m}\,. \tag{4.31}$$

Clearly, these parameters approach the Trotter limit:

$$\lim_{m\to\infty} p_m=\lim_{m\to\infty} p'_m=\frac{1}{r}\,. \tag{4.32}$$

Now the condition (2.5) is confirmed as

$$\begin{aligned}\prod_{k=3}^{m}\sum_{j=1}^{r}|p_{kj}|^3&=\sum_{k=3}^{m}(s|p_k|^3+(r-s)|p'_k|^3)\\ &=\prod_{k=3}^{m}\left(\frac{1+[(r-s)/s]^{1-3/k}}{s^2(1+a_k)^3}\right)\left(\frac{1+a_k}{|1+a_k e^{i\pi/k}|}\right)^3\\ &\leqq\prod_{k=3}^{m}\left(\frac{1/s^2+a_k^3/(r-s)^2}{(1+a_k)^3}\cdot\frac{1}{\cos^3(\pi/2k)}\right)\\ &\leqq M^3\cdot\prod_{k=3}^{m}\frac{1+a_k^3}{(1+a_k)^3}\,,\end{aligned} \tag{4.33}$$

where we have [22]

$$M = \prod_{k=3}^{\infty} \frac{1}{\cos\left(\dfrac{\pi}{2k}\right)} \quad \text{is finite}\,, \tag{4.34}$$

even in the limit $m \to \infty$. On the other hand, we have

$$\lim_{k\to\infty} a_k = \frac{r-s}{s} > 0\,. \tag{4.35}$$

Then, we arrive at the result

$$\lim_{m\to\infty} \prod_{k=3}^{m} \sum_{j=1}^{n} |p_{kj}|^3 = 0\,, \tag{4.36}$$

because

$$\lim_{m\to\infty} \prod_{k=3}^{m} \frac{1+a_k^3}{(1+a_k)^3} = 0 \tag{4.37}$$

for $0 < a_k < \infty$.

It is also easy to confirm the condition (ii) in (1.20) in the above decomposition. Thus, we arrive at the uniform convergence

$$\lim_{m\to\infty} \| Q_m(x) - \mathrm{e}^{x(A_1 + \cdots + A_q)} \| = 0 \tag{4.38}$$

in any compact region of x for the operators $\{A_j\}$ in a Banach space.

5. Summary and Discussion

In the present paper, we have found a general theorem concerning the convergence of decompositions of exponential operators in a Banach space. The general theorem gives the proof of the convergence of the Trotter formula, generalized Trotter-like formulas, the real fractal decomposition, the unitary decomposition and the complex decomposition. Namely, our general theorem gives a unified proof of the two limits $m \to \infty$ and $s \to \infty$ for the approximants $\{[Q_s(x/m)]^m\}$. The first ordinary type of convergence is easy to prove. The second new type of convergence has been studied for the first time in the present paper.

Since the convergence of the fractal decomposition (1.11) is proved, a more accurate bound is obtained easily in the form

$$\| S_{2m}(x) - \mathrm{e}^{x\mathscr{H}} \| \leqq L_{2m} |x|^{2m+1} \tag{5.1}$$

for small $|x|$ with some approximate constant L_{2m}, using the recursive scheme (4.14).

The present theorem can be easily extended to the following non-uniform decomposition

$$F_m(x) = Q_s(p_{m1}x) \ldots Q_{s_n}(p_{mn}x) \tag{5.2}$$

under the condition that

$$\lim_{m\to\infty}\sum_{j=1}^{n(m)} |p_{mj}|^{1+\min(s_1,s_2,\ldots,s_n)}=0\ . \tag{5.3}$$

It should be remarked here that the condition (2.5) with an index s is stronger than that with an index s' larger than s. Namely, if $\lim\sum_j|p_{mj}|^{s+1}=0$, then $\lim\sum_j|p_{mj}|^{s'+1}=0$ for $s<s'$, as it should be.

The general theory to construct higher-order decompositions has been proposed by the present author [3, 5] to give equations to determine the parameters $\{p_{mj}\}$ using Kubo's symmetrization operation [28] and the time-ordering operation. The minimal number $r_{\min}$ of the products of decompositions is given in this theory in terms of the free Lie algebra and the Möbius function [3, 33].

It is rather difficult to determine the parameters of non-uniform decompositions such as (2.3) and (5.2), even if we use the general theory of decomposition [3], except for multiple tilde decompositions of the form (4.20).

The fractal decomposition has always some negative parameters [1 ~ 4] among $\{p_{mj}\}$, but it can be applied even to irreversible processes such as nonlinear diffusion processes described by the Fokker–Planck operator. Even if the operators $\{A_j\}$ in (1.7) are not bounded, the present higher-order decomposition (1.7) can be defined in a restricted space such as the L^2 space in which operands or functions are restricted so that the relevant operators may be well defined [34].

The present higher-order decomposition may be called the "exponential perturbation expansion," which preserves the symmetry of the original exponential operator such as the unitarity in quantum mechanics and symplectic property in Hamiltonian dynamics. Our theorem can also be applied to the inner derivation $\delta_{\mathscr{H}}\equiv\mathscr{H}^{\times}$.

The present theorem gives the foundation of the fractal path integral [1 ~ 5] based on the fractal decompositions (4.13) and (4.20).

It will also an interesting problem in the future to study explicitly the magnitude of the correction term $(F_m(x)-\mathrm{e}^{x\mathscr{H}})$ in the whole range of x.

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Appendix: On Lemma 1

Here we discuss the condition (3.2) more explicitly.

A-1) If $Q_1(x)=Q(x)=\mathrm{e}^{xA_1}\ldots\mathrm{e}^{xA_q}$, then we have

$$\begin{aligned}\|\mathrm{e}^{-x\mathscr{H}}Q(x)-\mathbb{1}\| &= \|\mathscr{P}_1(\mathrm{e}^{-x\mathscr{H}}Q(x)-\mathbb{1})\|\\ &= \|\mathscr{P}_1(\mathrm{e}^{-x\mathscr{H}}Q(x))\| \leqq \mathscr{P}_1(\mathrm{e}^{|x|\,\|\mathscr{H}\|+\|\mathscr{H}\|_q})\\ &= \mathrm{e}^{|x|(\|\mathscr{H}\|+\|\mathscr{H}\|_q)}-(1+|x|(\|\mathscr{H}\|+\|\mathscr{H}\|_q))\ .\end{aligned} \tag{A.1}$$

Here $\mathscr{P}_1$ denotes the projection operator [7] defined by

$$\mathscr{P}_n f(x)=\sum_{k=n+1}^{\infty}\frac{f^{(k)}(0)}{k!}x^k\ . \tag{A.2}$$

The notation $\| \mathscr{H} \|_q$ is also defined by

$$\| \mathscr{H} \|_q \equiv \sum_{j=1}^{q} \| A_j \| . \tag{A.3}$$

Thus, the region of x for the inequality (3.2), namely

$$\| \mathrm{e}^{-x\mathscr{H}} Q(x) - \mathbb{1} \| < 1 \tag{A.4}$$

to hold is, at least, given by $|x| < x_1$, where x_1 is given by the solution of the equation

$$\mathrm{e}^{y_1} = 2 + y_1 ; \; y_1 = x_1 (\| \mathscr{H} \| + \| \mathscr{H} \|_q) . \tag{A.5}$$

Clearly we have

$$x_1 > x_0 \equiv \frac{1}{2} (\log 2) / (\| \mathscr{H} \| + \| \mathscr{H} \|_q) , \tag{A.6}$$

A-2) For $Q_2(x) = S(x)$, we have

$$\mathrm{e}^{y_2} = 1 + (1 + y_2 + \frac{1}{2} y_2^2) , \tag{A.7}$$

and consequently we obtain $y_2 > y_1 > 0$.

A-3) In general, for $Q_s(x)$, we obtain

$$\begin{aligned} \| \mathrm{e}^{-x\mathscr{H}} Q_s(x) - \mathbb{1} \| &= \| \mathscr{P}_s (\mathrm{e}^{-x\mathscr{H}} Q_s(x) - \mathbb{1}) \| \\ &= \| \mathscr{P}_s (\mathrm{e}^{-x\mathscr{H}} Q_s(x)) \| \\ &= \| \mathscr{P}_s (\mathrm{e}^{-x\mathscr{H}} \mathrm{e}^{x\tau_{11} A_1} \mathrm{e}^{x\tau_{12} A_2} \mathrm{e}^{x\tau_{21} A_1} \dots) \| \\ &\leqq \mathscr{P}_s (\mathrm{e}^{y(x)}) ; \; y(x) = x \left(\| \mathscr{H} \| + \sum_{k_1 j} |\tau_{kj}| \, \| A_j \| \right) . \end{aligned} \tag{A.8}$$

Thus, we have

$$\mathrm{e}^{y_s} = 1 + \sum_{k=0}^{s} \frac{1}{k!} y_s^k . \tag{A.9}$$

Then, we have $y_s > y_{s-1} > \cdots y_2 > y_1 > 0$.

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