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Modules over $\mathfrak{U}_q(\mathfrak{sl}_2)$

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Abstract. The restricted quantum universal enveloping algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ decomposes in a canonical way into a direct sum of indecomposable left (or right) ideals. They are useful for determining the direct summands which occur in the tensor product of two simple $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules. The indecomposable finite-dimensional $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules are classified and located in the Auslander-Reiten quiver.

1. Introduction

One of the basic problems in the theory of quantum universal enveloping algebras is to decompose a tensor product of simple modules into a direct sum of indecomposable ones and hence to elucidate the structure of the corresponding fusion rule algebra. Although this problem is solved for $\mathfrak{U}_q(\mathfrak{sl}_2)$, it might still be interesting to derive the solution in a new way; at least in principle, the method used here can be generalised to higher rank quantum universal enveloping algebras. A distinguishing feature is that neither the quantum Casimir operator nor the R-matrix appears explicitly, nor occurs any tedious calculation whatever. Then, the finite-dimensional $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules are classified, partly because there seems to be some interest in that (see [Sm]). Still, at least the *result* should be known to the experts and also to some readers of [RT].

In Sect. 2 we set forth the algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ at $q = \exp(\pi i m/N)$.

The main issue of Sect. 3 is Theorem 3.7, which states how $\mathfrak{U}_q(\mathfrak{sl}_2)$ decomposes into a direct sum of indecomposable left ideals. In due course, several indecomposable $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules will emerge, among these the modules $\mathbf{P}_\ell$, which have the property that if

$$0 \to L \to E \to \mathbf{P}_\ell \to 0 \qquad \text{and} \qquad 0 \to \mathbf{P}_\ell \to F \to M \to 0$$

are short exact sequences of $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules, then $\mathbf{P}_\ell$ embeds as a direct summand into E and into F. The algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ exemplifies many useful concepts from algebra: the Jacobson radical, Loewy layers, the Cartan matrix, and so on. Furthermore, $\mathfrak{U}_q(\mathfrak{sl}_2)$ nicely illustrates the multiplicity relations pertaining to Frobenius algebras.

In Sect. 4 we solve the problem mentioned at the beginning of the introduction for $\mathfrak{U}_q(\mathfrak{sl}_2)$. It is easy to find out what the composition factors of the tensor product of two finite-dimensional $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules are. If one of the factors in the tensor product is a module $\mathbf{P}_\ell$, it likewise follows what the indecomposable direct summands of the tensor product are. These pieces of information suffice to solve the problem.

In Sect. 5 we leave the tensor category of finite-dimensional left $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules and peer at the abelian category ${}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\mathbf{mod}$ of finite-dimensional left $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules. Each module in ${}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\mathbf{mod}$ is in an essentially unique way a direct sum of indecomposable modules in ${}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\mathbf{mod}$ (Krull-Schmidt theorem). The classification of the latter modules uses a result of Kronecker's. In addition to the classification, we deduce the Auslander-Reiten quiver of $\mathfrak{U}_q(\mathfrak{sl}_2)$ in a nontechnical fashion.

For the reader's convenience, Sect. A recalls the fundamentals of Auslander-Reiten theory.

In Sect. B we find the example $\mathfrak{U}_q(\mathfrak{sl}_2)$ at $q = \exp(\pi i/4)$ worked out explicitly.

2. Summary of $\mathfrak{U}_q(\mathfrak{sl}_2)$

We fix an integer $N \geq 2$ and an integer m such that $2N, m$ are coprime. This then determines the primitive $2N^{\text{th}}$ root of 1,

$$q := \exp\left(\frac{\pi i m}{N}\right) . \tag{2.1}$$

$\mathfrak{U}_q(\mathfrak{sl}_2)$ is the associative $\mathbb{C}$-algebra[1] with 1 generated by K, E, F and subject to the relations[2]

$$\left.\begin{array}{ll} K^{2N} = 1\,, & E^N = 0 = F^N\,, \\[2ex] KEK^{-1} = q^2 E\,, & KFK^{-1} = q^{-2}F\,, \\[1ex] EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}\,. & \end{array}\right\} \tag{2.2}$$

This algebra is, in fact, a (quasi-triangular) Hopf algebra, whence the tensor product $L \otimes M := L \otimes_{\mathbb{C}} M$ of the two $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules L, M again acquires the structure of a $\mathfrak{U}_q(\mathfrak{sl}_2)$-module (and $L \otimes M \cong M \otimes L$ as $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules). The comultiplication will be recalled as we will need it, later.

Another piece of notation consists in the introduction of q-numbers: for $n \in \mathbb{Z}$ put

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} . \tag{2.3}$$

Since m is odd, definitions (2.1) and (2.3) show that

$$[n + N] = -[n] = [-n] . \tag{2.4}$$

As a convention, by ℓ we shall always denote an element in $\mathbb{Z}/2N\mathbb{Z}$, which means that $\ell + 2N = \ell$. Nonetheless, expressions like, for example, K^ℓ, $[\ell]$, or

[1] In order not to bother anyone with ring problems

[2] These relations define what is in fact a *restricted* quantum universal enveloping algebra. In some respect, it would be more natural to consider an infinite-dimensional algebra generated by K, K^{-1}, and divided powers of E and F. (Recall that for an algebraic group scheme the appropriate thing to study is its algebra of distributions, not its Lie algebra.) Even so, the algebra considered here is a bona fide object in the realm of finite-dimensional algebras

$\ell + 1 \equiv 0 \pmod N$ are well-defined. Another example is given by introducing the K-eigenvectors

$$\varphi_\ell := \sum_{h \in \mathbb{Z}/2N\mathbb{Z}} q^{-h\ell} K^h ,$$

such that

$$K\varphi_\ell = \varphi_\ell K = q^\ell \varphi_\ell , \qquad E\varphi_\ell = \varphi_{\ell+2} E , \qquad F\varphi_\ell = \varphi_{\ell-2} F . \tag{2.5}$$

The next lemma is well-known (cf., e. g., [Lu2]) and is basic for what follows.

Lemma 2.1. *The $2N^3$ vectors $F^a K^\ell E^b \in \mathfrak{U}_q(\mathfrak{sl}_2)$, where $a, b \in \{0, \dots, N-1\}$ and $\ell \in \mathbb{Z}/2N\mathbb{Z}$, constitute a vector space basis for $\mathfrak{U}_q(\mathfrak{sl}_2)$.*

Corollary 2.2. $\mathfrak{U}_q(\mathfrak{sl}_2) = \bigoplus_{\substack{\ell \in \mathbb{Z}/2N\mathbb{Z} \\ a,b \in \{0,\dots,N-1\}}} \mathbb{C} F^a \varphi_\ell E^b$ *as a vector space.*

3. A Canonical Decomposition of $\mathfrak{U}_q(\mathfrak{sl}_2)$ into PIMs

Our aim is to decompose $\mathfrak{U}_q(\mathfrak{sl}_2)$ into a direct sum of indecomposable left ideals. Such a decomposition is not unique. However, we shall introduce some favourable gradation of $\mathfrak{U}_q(\mathfrak{sl}_2)$ and then require a decomposition into homogeneous indecomposable left ideals. That decomposition will be shown to exist, and, furthermore, uniqueness will be saved.

The promised gradation is not far from the decomposition of the adjoint representation into K-eigenspaces and is given by

$$\mathfrak{U}_q(\mathfrak{sl}_2) = \bigoplus_{s=-(N-1)}^{N-1} \mathfrak{U}_q(\mathfrak{sl}_2)_s ,$$

where

$$\mathfrak{U}_q(\mathfrak{sl}_2)_s := \bigoplus_{\substack{a-b=s \\ \ell \in \mathbb{Z}/2N\mathbb{Z} \\ a,b \in \{0,\dots,N-1\}}} \mathbb{C} F^a \varphi_\ell E^b .$$

We say the elements in $\mathfrak{U}_q(\mathfrak{sl}_2)_s$ have *height* s. Note how K, E, F respect the gradation.

The Modules $\mathbf{P}_\ell$. The modules $\mathbf{P}_\ell$, whose definition or construction concerns us next, turn out to be certain maximal indecomposable left ideals in $\mathfrak{U}_q(\mathfrak{sl}_2)$. We shall construct these modules step by step, starting with a (left and right) K-eigenvector of highest height, viz. $F^{N-1}\varphi_\ell$.

Two special instances of the formula which calculates the commutators $[E^r, F^s]$ will be needed (cf. [Lu1]):

$$[F, E^s] = -[s]\, E^{s-1} \, \frac{K q^{-(1-s)} - K^{-1} q^{1-s}}{q - q^{-1}} , \tag{3.1}$$

$$E^{N-1} F^{N-1} = \sum_{h=0}^{N-1} (\text{nonzero coeff.})\, F^{N-1-h} E^{N-1-h} \prod_{j=0}^{h-1} \frac{K q^{-j} - K^{-1} q^j}{q - q^{-1}} . \tag{3.2}$$

Lemma 3.1. $FE^sF^{N-1}\varphi_\ell = -[s][\ell+s+1]\,E^{s-1}F^{N-1}\varphi_\ell$ *for* $s \in \mathbb{Z}_{>0}$.

Proof.

$$\begin{aligned} &FE^sF^{N-1}\varphi_\ell \\ &\overset{(2.2)}{=} [F,E^s]\,F^{N-1}\varphi_\ell \overset{\substack{(3.1)\\(2.5)}}{=} -[s]\,E^{s-1}\,\frac{Kq^{-(1-s)}-K^{-1}q^{1-s}}{q-q^{-1}}\,\varphi_{\ell+2}\,F^{N-1} \\ &\overset{\substack{(2.5)\\(2.3)}}{=} -[s][\ell+2-(1-s)]\,E^{s-1}\varphi_{\ell+2}\,F^{N-1} \overset{(2.5)}{=} -[s][\ell+s+1]\,E^{s-1}F^{N-1}\varphi_\ell\,. \end{aligned}$$

□

We define

$$\begin{aligned} \mathrm{mod}_N^+ &: \mathbb{Z} \longrightarrow \{1,\dots,N\}\,, \\ \mathrm{mod}_N^- &: \mathbb{Z} \longrightarrow \{0,\dots,N-1\} \end{aligned}$$

such that $\mathrm{mod}_N^\pm(h) \equiv h\,(\mathrm{mod}\,N)$. Recall that, by our convention, the functions $\mathrm{mod}_N^\pm$ are defined on $\mathbb{Z}/2N\mathbb{Z}$ as well.

Corollary 3.2. $\min\left\{s \in \mathbb{Z}_{>0} \;\middle|\; FE^sF^{N-1}\varphi_\ell = 0\right\} = \mathrm{mod}_N^+(-\ell-1)$,
$\max\left\{s \in \mathbb{Z}_{<N} \;\middle|\; FE^sF^{N-1}\varphi_\ell = 0\right\} = \mathrm{mod}_N^-(-\ell-1)$.

Thus, the N-dimensional left ideal $\mathfrak{U}_q(\mathfrak{sl}_2)F^{N-1}\varphi_\ell$ may be represented schematically as

$$\begin{array}{ll} \bullet & \alpha_\ell := F^{N-1}\varphi_\ell \\ \downarrow\uparrow & \\ \bullet & E\alpha_\ell \\ \downarrow\uparrow & \\ \bullet & E^2\alpha_\ell \\ \downarrow\uparrow & \\ \vdots & \\ \downarrow\uparrow & \\ \bullet & E^{\mathrm{mod}_N^+(-\ell-1)-1}\alpha_\ell =: \beta_\ell \\ \downarrow & \\ \bullet & E^{\mathrm{mod}_N^-(-\ell-1)}\alpha_\ell =: \tilde{\alpha}_\ell \\ \downarrow\uparrow & \\ \vdots & \\ \downarrow\uparrow & \\ \bullet & E^{N-1}\alpha_\ell =: \tilde{\beta}_\ell \end{array} \qquad (3.3)$$

Each dot stands for the 1-dimensional subspace spanned by the named vector. Upward (resp. downward) pointing arrows indicate a *nonzero* left-multiplication action by F (resp. by E). In the diagram pictured above, the single downward pointing arrow does not appear for $\ell+1 \equiv 0\,(\mathrm{mod}\,N)$, as then $\alpha_\ell = \tilde{\alpha}_\ell$ (or, equivalently, $\beta_\ell = \tilde{\beta}_\ell$); we put

$$\gamma_\ell := \alpha_\ell \qquad \text{for } \ell+1 \equiv 0\,(\mathrm{mod}\,N)\,. \qquad (3.4)$$

For $\ell+1 \not\equiv 0 \,(\mathrm{mod}\, N)$ we have $F^{N-1}\beta_\ell = 0$, and β_ℓ can be written as

$$\beta_\ell = F\gamma_\ell\,, \tag{3.5}$$

for a unique vector γ_ℓ of the form

$$\gamma_\ell = \sum_{\substack{a\in\{0,\ldots,N-2\}\\ h\in\mathbb{Z}/2N\mathbb{Z}\\ b\in\{0,\ldots,N-1\}}} c_{a,h,b}\, F^a \varphi_h\, E^b \quad \text{with } c_{a,h,b} \in \mathbb{C}\,.$$

Observe that γ_ℓ and $\tilde{\alpha}_\ell$ lie in the same homogeneous component of $\mathfrak{U}_q(\mathfrak{sl}_2)$ and have the same left and the same right K-eigenvalues; in particular,

$$K\gamma_\ell = q^{-\ell}\gamma_\ell\,, \tag{3.6}$$

which also holds for $\ell+1 \equiv 0\,(\mathrm{mod}\, N)$.

Lemma 3.3. $FE^s\gamma_\ell - E^s\beta_\ell = [s][\ell-s+1]\,E^{s-1}\gamma_\ell$ *for* $s \in \mathbb{Z}_{>0}$.

Proof.

$$\begin{aligned} FE^s\gamma_\ell - E^s\beta_\ell &\overset{(3.5)}{=} [F,E^s]\,\gamma_\ell \overset{(3.1)}{=} -[s]\,E^{s-1}\,\frac{Kq^{-(1-s)} - K^{-1}q^{1-s}}{q-q^{-1}}\,\gamma_\ell \\ &\overset{\substack{(3.6)\\(2.3)}}{=} -[s][-\ell-(1-s)]\,E^{s-1}\gamma_\ell \overset{(2.4)}{=} [s][\ell-s+1]\,E^{s-1}\gamma_\ell\,. \end{aligned}$$

[This calculation applies to all values of ℓ. For $\ell+1 \equiv 0\,(\mathrm{mod}\, N)$ simply notice that $E^s\beta_\ell = 0 = E^sF\gamma_\ell$; but then Lemma 3.3 says the same as Lemma 3.1.] □

Corollary 3.4. $F^{N-1}E^{N-1}\gamma_\ell$ *is a nonzero multiple of* $\tilde{\alpha}_\ell$. *In particular,* $E^s\gamma_\ell \neq 0$ *for* $s \in \{0,\ldots,N-1\}$.

Let us now illustrate the left ideal $\mathfrak{U}_q(\mathfrak{sl}_2)\gamma_\ell$ for $\ell+1 \not\equiv 0\,(\mathrm{mod}\, N)$:

$$\begin{array}{rcccl}
 & & & \bullet & \alpha_\ell \\
 & & & \downarrow\uparrow & \\
 & & & \vdots & \\
 & & & \downarrow\uparrow & \\
 & & & \bullet & \beta_\ell \\
 & & \nearrow & \downarrow & \\
\gamma_\ell & \bullet & & \bullet & \tilde{\alpha}_\ell \\
 & \downarrow\uparrow & \nearrow & \downarrow\uparrow & \\
 & \vdots & \vdots & \vdots & \\
 & \downarrow\uparrow & \nearrow & \downarrow\uparrow & \\
\delta_\ell := E^{\mathrm{mod}^-_N(\ell)}\gamma_\ell & \bullet & & \bullet & \tilde{\beta}_\ell \\
 & \downarrow & \nearrow & & \\
\tilde{\gamma}_\ell := E^{\mathrm{mod}^\pm_N(\ell+1)}\gamma_\ell & \bullet & & & \\
 & \downarrow\uparrow & & & \\
 & \vdots & & & \\
 & \downarrow\uparrow & & & \\
\tilde{\delta}_\ell := E^{N-1}\gamma_\ell & \bullet & & &
\end{array} \tag{3.7}$$

It is clear that for $s \in \{0, \ldots, \mathrm{mod}_N^-(\ell)\}$ the two vectors $E^s\gamma_\ell$ and $E^s\tilde{\alpha}_\ell$ are linearly independent.

Now we define $\mathbf{P}_\ell$: for $\ell \in \mathbb{Z}/2N\mathbb{Z}$ let [cf. (3.7), (3.3) and (3.4)]

$$\mathbf{P}_\ell := \mathfrak{U}_q(\mathfrak{sl}_2)\gamma_\ell \,.$$

The modules $\mathbf{P}_\ell$ and $\mathbf{P}_{\ell+N}$ yield the same skeleton-of-dots-and-arrows. That makes it reasonable to put

$$\overline{\mathbf{P}_\ell} := \mathbf{P}_{\ell+N} \,.$$

So we get an involution on the set of modules $\{\mathbf{P}_\ell \,|\, \ell \in \mathbb{Z}/2N\mathbb{Z}\}$. Another involution, $\ell \mapsto \bar{\ell}$, now written on the set of indices $\mathbb{Z}/2N\mathbb{Z}$ itself, is defined via the equation $\ell + \bar{\ell} = -2$.

- For $\ell + 1 \not\equiv 0 \,(\mathrm{mod}\, N)$ we set [cf. (3.7)]

$$\begin{aligned} \mathbf{V}_\ell^{\downarrow} &:= \mathfrak{U}_q(\mathfrak{sl}_2)\alpha_\ell \,, & \mathbf{V}_\ell^{\uparrow} &:= \mathfrak{U}_q(\mathfrak{sl}_2)\tilde{\delta}_\ell \,, \\ \Omega^1\mathbf{S}_\ell &:= \mathbf{V}_\ell^{\downarrow} + \mathbf{V}_\ell^{\uparrow} \,, & \mathbf{S}_\ell &:= \mathbf{P}_\ell/\Omega^1\mathbf{S}_\ell \end{aligned} \tag{3.8}$$

and get

$$\begin{aligned} &\mathbf{P}_\ell \supseteq \Omega^1\mathbf{S}_\ell \supseteq \mathbf{V}_\ell^{\downarrow} \supseteq \mathbf{V}_\ell^{\downarrow} \cap \mathbf{V}_\ell^{\uparrow} \supseteq 0 \,, \\ &\mathbf{P}_\ell \supseteq \Omega^1\mathbf{S}_\ell \supseteq \mathbf{V}_\ell^{\uparrow} \supseteq \mathbf{V}_\ell^{\downarrow} \cap \mathbf{V}_\ell^{\uparrow} \supseteq 0 \end{aligned}$$

as composition series built from homogeneous submodules, both series with composition factors isomorphic to $\mathbf{S}_\ell, \mathbf{S}_{\bar{\ell}}, \mathbf{S}_{\bar{\ell}}, \mathbf{S}_\ell$. The maximal semisimple submodule of $\mathbf{P}_\ell$ is the simple module $\mathbf{V}_\ell^{\downarrow} \cap \mathbf{V}_\ell^{\uparrow}$, hence $\mathbf{P}_\ell$ is indecomposable.

- For $\ell + 1 \equiv 0 \,(\mathrm{mod}\, N)$ the module $\mathbf{P}_\ell = \mathbf{P}_{\bar{\ell}}$ is a simple left $\mathfrak{U}_q(\mathfrak{sl}_2)$-module. These modules, $\mathrm{St} := \mathbf{P}_{N-1}$ and $\overline{\mathrm{St}}$ $(= \mathbf{P}_{2N-1})$, are termed the Steinberg modules. Just in order to have the correct notation, we put $\Omega^1\mathbf{S}_\ell := 0 \subseteq \mathbf{P}_\ell$ and $\mathbf{S}_\ell := \mathbf{P}_\ell/\Omega^1\mathbf{S}_\ell$.

The next proposition takes up some of the discussion above.

Proposition 3.5.

- *For* $\ell + 1 \not\equiv 0 \,(\mathrm{mod}\, N)$ *we have* [*cf.* (3.7)]

$$\mathbf{P}_\ell = \bigoplus_{h=0}^{N-1} \left(\mathbb{C}E^h\alpha_\ell \oplus \mathbb{C}E^h\gamma_\ell \right) \qquad \textit{as a vector space,}$$

and this $2N$*-dimensional left* $\mathfrak{U}_q(\mathfrak{sl}_2)$*-module is indecomposable.*

- *For* $\ell + 1 \equiv 0 \,(\mathrm{mod}\, N)$ *we have* [*cf.* (3.3)]

$$\mathbf{P}_\ell = \bigoplus_{h=0}^{N-1} \mathbb{C}E^h\alpha_\ell \qquad \textit{as a vector space,}$$

and this N*-dimensional left* $\mathfrak{U}_q(\mathfrak{sl}_2)$*-module is simple.*

The Decomposition.

Lemma 3.6. *Right multiplication by E^h defines an isomorphism $\mathbf{P}_\ell \xrightarrow{\cong} \mathbf{P}_\ell E^h$ of left $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules for $h \in \{0, \ldots, \mathrm{mod}_N^-(\ell)\}$.*

Proof. We must show that right multiplication by $E^{\mathrm{mod}_N^-(\ell)}$ maps $\mathbf{P}_\ell$ injectively onto $\mathbf{P}_\ell E^{\mathrm{mod}_N^-(\ell)}$. This follows from

$$E^{N-1}\alpha_\ell\, E^{\mathrm{mod}_N^-(\ell)} \neq 0 \qquad \text{and} \qquad E^{N-1}\gamma_\ell\, E^{\mathrm{mod}_N^-(\ell)} \neq 0\,.$$

The first of these two formulae implies the second, for $F^{N-1}E^{N-1}\gamma_\ell$ is a nonzero multiple of $\tilde{\alpha}_\ell = E^{\mathrm{mod}_N^-(-\ell-1)}\alpha_\ell$,

$$\begin{aligned}
&E^{N-1}\alpha_\ell\, E^{\mathrm{mod}_N^-(\ell)}\\
&\overset{(3.3)}{=} E^{N-1}F^{N-1}\varphi_\ell\, E^{\mathrm{mod}_N^-(\ell)}\\
&\overset{(3.2)}{=} \sum_{h=0}^{N-1}(\text{nonzero coeff.})\, F^{N-1-h}E^{N-1-h}\left(\prod_{j=0}^{h-1}\frac{Kq^{-j}-K^{-1}q^{j}}{q-q^{-1}}\right)\varphi_\ell\, E^{\mathrm{mod}_N^-(\ell)}\\
&\overset{(2.5)\,(2.3)}{=} \sum_{h=0}^{N-1}(\text{nonzero coeff.})\, F^{N-1-h}E^{N-1-h}\left(\prod_{j=0}^{h-1}[\ell-j]\right)\varphi_\ell\, E^{\mathrm{mod}_N^-(\ell)}\\
&\overset{(2.5)\,(2.2)\,(2.3)}{=} (\text{nonzero coeff.})\, F^{N-1-\mathrm{mod}_N^-(\ell)}E^{N-1}\varphi_{-\ell} \neq 0\,.
\end{aligned}$$

□

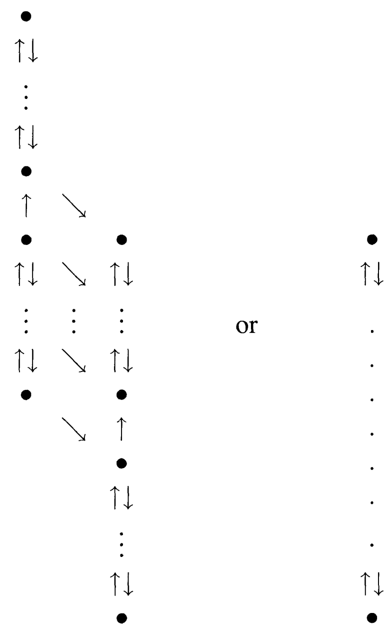
Remark [cf. (3.7)]. $\mathbf{P}_\ell E^{\mathrm{mod}_N^-(\ell)+1} = \mathfrak{U}_q(\mathfrak{sl}_2)\tilde{\delta}_{\bar{\ell}} \subseteq \mathbf{P}_{\bar{\ell}}$ for $\ell + 1 \not\equiv 0\,(\mathrm{mod}\, N)$.

Theorem 3.7. $\mathfrak{U}_q(\mathfrak{sl}_2) = \bigoplus_{\ell\in\mathbb{Z}/2N\mathbb{Z}}\ \bigoplus_{h=0}^{\mathrm{mod}_N^-(\ell)} \mathbf{P}_\ell E^h$ *as a left $\mathfrak{U}_q(\mathfrak{sl}_2)$-module is the unique decomposition of $\mathfrak{U}_q(\mathfrak{sl}_2)$ into a direct sum of homogeneous indecomposable left ideals.*

Proof. The sum $\sum_{h=0}^{\mathrm{mod}_N^-(\ell)} \mathbf{P}_\ell E^h$ $\left(\text{or even } \sum_{h=0}^{N-1} \mathbf{P}_\ell E^h\right)$ is in fact a direct sum because the summands lie in different right K-eigenspaces. Also, the outer sum is a direct one: $\mathbf{V}_\ell^\downarrow \cap \mathbf{V}_\ell^\uparrow \not\cong \mathbf{V}_{\ell'}^\downarrow \cap \mathbf{V}_{\ell'}^\uparrow$ for $\ell \neq \ell'$, and $\mathfrak{U}_q(\mathfrak{sl}_2)$ is a Frobenius algebra (see below). A dimension count now shows that we have completely exhausted $\mathfrak{U}_q(\mathfrak{sl}_2)$. This being established, we also have the stated uniqueness. □

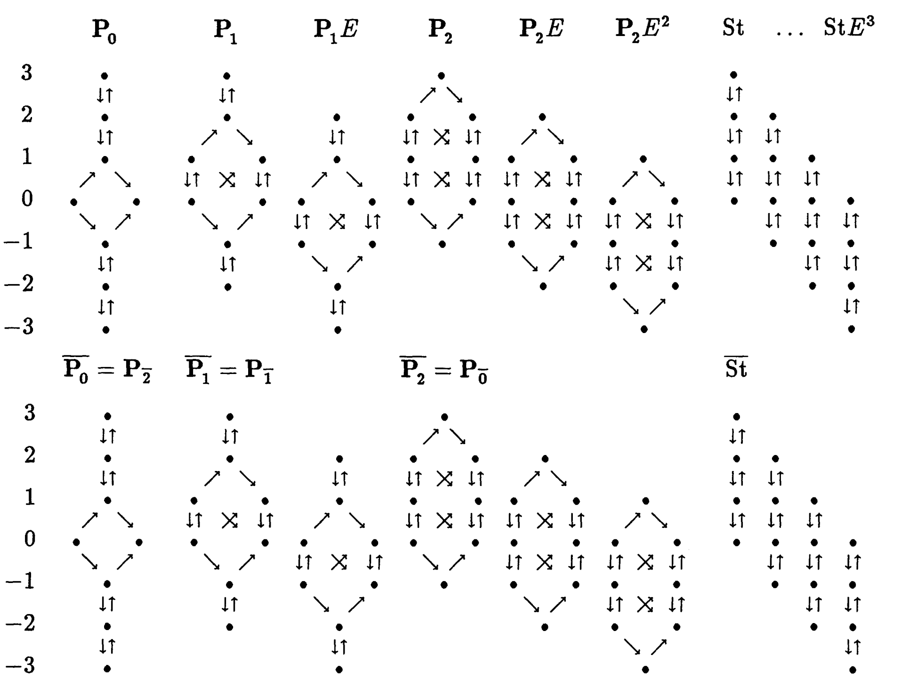
Thus, we have decomposed $\mathfrak{U}_q(\mathfrak{sl}_2)$ into a direct sum of indecomposable left ideals, each being isomorphic to $\mathbf{P}_\ell$ for some $\ell \in \mathbb{Z}/2N\mathbb{Z}$. A left ideal in $\mathfrak{U}_q(\mathfrak{sl}_2)$ which is isomorphic to $\mathbf{P}_\ell$ for some ℓ is called a *(left) principal indecomposable module* (PIM for short).

To construct the decomposition of $\mathfrak{U}_q(\mathfrak{sl}_2)$ into homogeneous indecomposable left ideals, we started with a vector of highest height, produced $\mathbf{P}_\ell$, and shifted $\mathbf{P}_\ell$ to $\mathbf{P}_\ell E^h$. Of course, we could have started with a vector of lowest height. We would then construct the left module which contains $E^{N-1}\varphi_{-\ell}$ and whose skeleton-of-dots-and-arrows looks like this:



Finally, we would shift such a module Q to QF^h for suitable values of h. By uniqueness, this procedure then yields the same decomposition of $\mathfrak{U}_q(\mathfrak{sl}_2)$ into PIMs. From now on, we shall draw the skeleton-of-dots-and-arrows in a more symmetric fashion.

Here is the diagram for $N = 4$ [see Sect. B, where the case $m = 1$ [cf. (2.1)] is worked out explicitly]:



Remark. It is not an accidental coincidence that each $\mathbf{P}_\ell$ occurs with multiplicity $\dim \mathbf{S}_\ell$ (see the remark about the "dimension formula" below).

Some Notions from Algebra. The decomposition of $\mathfrak{U}_q(\mathfrak{sl}_2)$ in Theorem 3.7 corresponds to a decomposition of $1 \in \mathfrak{U}_q(\mathfrak{sl}_2)$ into a sum of primitive orthogonal idempotents: $1 = \sum_j e_j$, where $e_j e_{j'} = \delta_{jj'} e_j$, and the adjective "primitive" means that the decomposition cannot be refined. Each e_j belongs to some $\mathbf{P}_\ell E^h$ (i.e., $\mathfrak{U}_q(\mathfrak{sl}_2)e_j = \mathbf{P}_\ell E^h$), and it turns out that e_j is a vector of height 0.

Theorem 3.7 together with the knowledge of the architecture of the PIMs yield several obvious corollaries. But first we review some terminology from algebra.

Let us dwell on the concept of projectivity. An object P in an abelian category $\mathcal{A}$ is termed *projective* if the functor $\mathrm{Hom}_{\mathcal{A}}(P, \ \ )$ from $\mathcal{A}$ to the category of abelian groups maps short exact sequences to short exact sequences. If we take for $\mathcal{A}$ the category ${}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\mathbf{mod}$ of finite-dimensional left $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules, the requirement for P to be projective is the following: for any epimorphism $M \xrightarrow{\pi} L$ in ${}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\mathbf{mod}$, the corresponding map $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P, M) \xrightarrow{\pi_*} \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P, L)$ of abelian groups (here, in fact, complex vector spaces) also has to be surjective (since the Hom-functor is always left exact, this is already sufficient). In other words: given a homomorphism $P \xrightarrow{\varphi} L$, there exists a homomorphism ψ making the diagram

$$\begin{tikzcd} & P \arrow[dl, "\psi"'] \arrow[d, "\varphi"] & \\ M \arrow[r, "\pi"] & L \arrow[r] & 0 \end{tikzcd}$$

commutative. In the special case where $L = P$ and $\varphi = \mathrm{id}_P$, ψ is a section of π, that is, $M = \ker \pi \oplus \mathrm{im}\,\psi \cong \ker \pi \oplus P$. In this situation, we may, in particular, choose M to be a finite-dimensional free $\mathfrak{U}_q(\mathfrak{sl}_2)$-module which projects onto P. Hence projective modules are direct summands of free modules. On the other hand, this property characterises the projective modules because for a direct summand of a free module it is clear how to construct a required lift. An *injective* module is a module which fulfils the concept dual to that of projectivity.

A *projective cover* of a module M is a projective module P which projects onto M (or, more precisely, the epimorphism $P \twoheadrightarrow M$) and is minimal with respect to this property. It is then unique up to isomorphism. For example, $\mathbf{P}_\ell$ is a projective cover of $\mathbf{S}_\ell$ or of $\mathbf{P}_\ell/\mathbf{V}_\ell^{\downarrow}$; when $\ell + 1 \not\equiv 0 \,(\mathrm{mod}\, N)$, the module $\Omega^1\mathbf{S}_\ell$ has $\mathbf{P}_{\bar{\ell}} \oplus \mathbf{P}_{\bar{\ell}}$ as a projective cover. An *injective hull* of a module is defined via the dual concept.

Recall that the nonsemisimplicity of a finite-dimensional algebra A over an algebraically closed field is measured by a two-sided ideal: the *(Jacobson) radical* $\mathbf{J}(A)$, which is always a nilpotent ideal. It is the minimal two-sided ideal such that the algebra $A/\mathbf{J}(A)$ is semisimple, i.e., $A/\mathbf{J}(A)$ is a direct sum of full matrix algebras.

Corollary 3.8. [*cf.* (3.8)]. $\mathbf{J} := \mathbf{J}(\mathfrak{U}_q(\mathfrak{sl}_2)) = \bigoplus_{\ell \in \mathbb{Z}/2N\mathbb{Z}} \ \bigoplus_{h=0}^{\mathrm{mod}_N^-(\ell)} \Omega^1 \mathbf{S}_\ell E^h$,

$$\mathfrak{U}_q(\mathfrak{sl}_2)/\mathbf{J} \cong \mathbf{J}^2, \qquad \mathbf{J}^3 = 0 .$$

If $P = Ae$ is a PIM, then $Ae/\mathbf{J}(A)e$ is a simple left A-module. $Ae \mapsto Ae/\mathbf{J}(A)e$ induces a bijection between the set of isomorphism classes of indecomposable projective modules and that of simple modules.

Corollary 3.9. *For each $h = 0, \ldots, N-2$ there are the two $(h+1)$-dimensional modules $\mathbf{S}_h$ and $\mathbf{S}_{h+N}$, and, together with* St *and* $\overline{\mathrm{St}}$, *these $2N$ modules provide a nonredundant list for the isomorphism classes of simple $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules.*

For a finite-dimensional module M one defines its *radical*, $\operatorname{rad} M$, as the intersection of the maximal submodules of M and its *socle*, $\operatorname{soc} M$, as the sum of the minimal (i. e., simple) submodules, which is then a semisimple submodule of M.

Corollary 3.10. *For $\ell + 1 \not\equiv 0 \,(\operatorname{mod} N)$ the* Loewy layers *of $\mathbf{P}_\ell$ look like*

$$\begin{array}{rclcll} \mathbf{P}_\ell/\operatorname{rad}\mathbf{P}_\ell & = & \mathbf{P}_\ell/\mathbf{J}\,\mathbf{P}_\ell & = & \mathbf{S}_\ell & \textit{(the head of } \mathbf{P}_\ell\textit{)}, \\ \operatorname{rad}\mathbf{P}_\ell/\operatorname{soc}\mathbf{P}_\ell & = & \mathbf{J}\,\mathbf{P}_\ell/\mathbf{J}^2\mathbf{P}_\ell & \cong & \mathbf{S}_{\bar\ell} \oplus \mathbf{S}_{\bar\ell} & \textit{(the heart of } \mathbf{P}_\ell\textit{)}, \\ \operatorname{soc}\mathbf{P}_\ell & = & \mathbf{J}^2\mathbf{P}_\ell & \cong & \mathbf{S}_\ell & \textit{(the socle of } \mathbf{P}_\ell\textit{)}. \end{array}$$

It is known (see [LS]) that finite-dimensional Hopf algebras over a field (or, more generally, over a principal ideal domain) are *Frobenius algebras*, which means that the two regular representations are equivalent. (These Frobenius algebras must not be mistaken for the Frobenius algebras showing up, for example, in [Ma].) The two regular representations being equivalent, in turn, implies *self-injectivity*, namely, that each finite-dimensional module is projective if and only if it is injective. Hence every short exact sequence of $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules

$$0 \to L \to E \to M \to 0$$

splits, that is, $E \cong L \oplus M$, if L or M is projective.

Remark. The "dimension formula" for Frobenius algebras (for a more precise statement see [CR, (61.13) or (61.16)]) in our context reads

$$\dim \mathfrak{U}_q(\mathfrak{sl}_2) = \sum_{\ell \in \mathbb{Z}/2N\mathbb{Z}} \dim \mathbf{P}_\ell \cdot \dim(\mathbf{P}_\ell/\operatorname{rad}\mathbf{P}_\ell)\,.$$

In order not to rely upon the statement made above – that finite-dimensional Hopf algebras over a field are Frobenius algebras – we may argue as follows. Look at the vector $\Lambda := \tilde\alpha_0 = \tilde\beta_0 = E^{N-1}F^{N-1}\varphi_0$, which is a basis for the module $\operatorname{soc}\mathbf{P}_0$. Since $K \cdot \Lambda = \Lambda$, $E \cdot \Lambda = 0$, and $F \cdot \Lambda = 0$, it follows that $\operatorname{soc}\mathbf{P}_0$ realises the trivial representation, that is, the representation afforded by the counit or augmentation ε:

$$X \cdot \Lambda = \varepsilon(X)\Lambda \quad \text{for all } X \in \mathfrak{U}_q(\mathfrak{sl}_2)\,. \tag{3.9}$$

Equality (3.9) tells us that Λ is a *left integral* in $\mathfrak{U}_q(\mathfrak{sl}_2)$. The nonsingular bilinear form on the dual $\mathfrak{U}_q(\mathfrak{sl}_2)^*$ of the Hopf algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$

$$(p, q) \longmapsto pq(\Lambda)$$

provides an isomorphism $\mathfrak{U}_q(\mathfrak{sl}_2)^* \cong \mathfrak{U}_q(\mathfrak{sl}_2)$ as left $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules (actually, even as left Hopf modules), showing that $\mathfrak{U}_q(\mathfrak{sl}_2)$ is a Frobenius algebra. As $\dim(\mathbf{P}_0/\operatorname{rad}\mathbf{P}_0) = 1$, the space of left integrals is 1-dimensional – which follows from $\dim\operatorname{soc}\mathbf{P}_0 = 1$ because $\mathfrak{U}_q(\mathfrak{sl}_2)$ is a Frobenius algebra. The subspace $\mathbb{C}\Lambda$ is thus a two-sided ideal of $\mathfrak{U}_q(\mathfrak{sl}_2)$, and the equality $\Lambda \cdot K = \Lambda$ then shows that Λ is a *two-sided integral* in $\mathfrak{U}_q(\mathfrak{sl}_2)$, that is, (3.9) and

$$\Lambda \cdot X = \varepsilon(X)\Lambda \quad \text{for all } X \in \mathfrak{U}_q(\mathfrak{sl}_2) \tag{3.10}$$

hold. Equation (3.10) follows because there is only one further equivalence class of 1-dimensional representations of $\mathfrak{U}_q(\mathfrak{sl}_2)$, which has K acting by -1. Hence $\mathfrak{U}_q(\mathfrak{sl}_2)$

is a *unimodular* Hopf algebra. Actually, $\mathfrak{U}_q(\mathfrak{sl}_2)$ is an algebra of an even more special kind, which we shall show in Sect. 5.

Since $\mathfrak{U}_q(\mathfrak{sl}_2)$ is a Frobenius algebra, decomposing $\mathfrak{U}_q(\mathfrak{sl}_2)$ into *right* PIMs does not yield anything essentially new. But what we shall do is to give the decomposition of $\mathfrak{U}_q(\mathfrak{sl}_2)$ into indecomposable two-sided ideals. For this purpose, we have to combine PIMs into *blocks*. As we already know the composition factors of the PIMs, this is immediate (in fact, we needn't even know that and simply would apply the remark after Lemma 3.6):

$$\begin{aligned}
\mathbb{B}_\ell &:= \bigoplus_{h=0}^{\operatorname{mod}_{\overline{N}}(\ell)} \mathbf{P}_\ell E^h \oplus \bigoplus_{\bar{h}=0}^{\operatorname{mod}_{\overline{N}}(\bar{\ell})} \mathbf{P}_{\bar{\ell}} E^{\bar{h}} \qquad \text{for } \ell+1 \not\equiv 0 \,(\operatorname{mod} N)\,, \\
\mathbb{B}_{\mathrm{St}} &:= \bigoplus_{h=0}^{N-1} \mathrm{St} E^h\,, \\
\mathbb{B}_{\overline{\mathrm{St}}} &:= \bigoplus_{\bar{h}=0}^{N-1} \overline{\mathrm{St}} E^{\bar{h}}\,.
\end{aligned}$$

Corollary 3.11. $\mathfrak{U}_q(\mathfrak{sl}_2) = \left(\bigoplus_{\ell=0}^{N-2} \mathbb{B}_\ell\right) \oplus \mathbb{B}_{\mathrm{St}} \oplus \mathbb{B}_{\overline{\mathrm{St}}}$ *is the (unique) decomposition of the algebra* $\mathfrak{U}_q(\mathfrak{sl}_2)$ *into indecomposable two-sided ideals.*

Corollary 3.12. *The Cartan matrix of* $\mathfrak{U}_q(\mathfrak{sl}_2)$ *(i. e., the matrix whose* (ℓ, h)*-entry is* $\dim \operatorname{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\mathbf{P}_h, \mathbf{P}_\ell)$*, which is the same number as the multiplicity with which the simple module* $\mathbf{P}_h/\mathbf{J}\,\mathbf{P}_h$ *occurs in a composition series of* $\mathbf{P}_\ell$*) looks like*

$$C = \begin{pmatrix} \boxed{\begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix}} & & & & \mathbf{0} \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 2 & 2 \\ 2 & 2 \end{matrix}} & & \\ & \mathbf{0} & & \boxed{1} & \\ & & & & \boxed{1} \end{pmatrix},$$

where the row and column indices run over $0, \bar{0}, \ldots, N-2, \overline{N-2}, N-1, 2N-1$.

Remark. The Cartan matrix above has the form $C = {}^{t}DD$ for a "decomposition matrix"

$$D = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} & & & & \mathbf{0} \\ & \ddots & & & \\ & & \boxed{\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}} & & \\ & \mathbf{0} & & \boxed{1} & \\ & & & & \boxed{1} \end{pmatrix}.$$

It has some connexion with the composition series of the modules $\mathbf{V}_\ell^{\downarrow}$ and $\mathbf{V}_\ell^{\uparrow}$, which were defined in (3.8).

4. Tensor Products

Fact 1. *In a finite-dimensional associative $\mathbb{C}$-algebra, each finite-dimensional projective module is isomorphic to a direct sum of PIMs.*

[This follows immediately from the Krull-Schmidt theorem.]

Fact 2. *In a Hopf $\mathbb{C}$-algebra H, the tensor product of a finite-dimensional projective module P with an arbitrary module M is projective.*

[On $H \otimes M := H \otimes_{\mathbb{C}} M$ we have two tensor product structures: the usual one given by $g(h \otimes m) := \sum g_{(1)}h \otimes g_{(2)}m$ (here $\triangle(g) = \sum g_{(1)} \otimes g_{(2)}$ is a convenient notation for the comultiplication $\triangle$) and the tensor product structure as an induced module (where M is treated merely as a vector space), namely, $g(h \otimes m) := gh \otimes m$. The crucial point to note is that these two left H-modules are indeed isomorphic: the two morphisms given by $h \otimes m \overset{\alpha}{\longmapsto} \sum h_{(1)} \otimes S\Big(h_{(2)}\Big)m$ (where S is the antipode) and $h \otimes m \overset{\beta}{\longmapsto} \sum h_{(1)} \otimes h_{(2)}m$ are two mutually inverse isomorphisms between the two tensor product structures. Let us compute $\alpha \circ \beta$:

$$\begin{aligned}
\alpha \circ \beta\,(h \otimes m) &= \sum h_{(1)(1)} \otimes S\Big(h_{(1)(2)}\Big)h_{(2)}m \\
&= \sum h_{(1)} \otimes S\Big(h_{(2)(1)}\Big)h_{(2)(2)}m && \text{by coassociativity} \\
&= \sum h_{(1)} \otimes \varepsilon\Big(h_{(2)}\Big)m && \text{by the antipode axiom} \\
& && (\varepsilon \text{ is the counit}) \\
&= \sum h_{(1)}\,\varepsilon\Big(h_{(2)}\Big) \otimes m = h \otimes m && \text{by the counit axiom.}
\end{aligned}$$

A similar computation shows that $\beta \circ \alpha = \mathrm{id}$.

According to Fact 1, we may assume without loss of generality that P is a PIM. So we have to pass from H to a direct summand of H, and it is here that the structure of M as an H-module comes into play.]

Our aim is to decompose $\mathbf{P}_\ell \otimes \mathbf{S}_h$ into a direct sum of indecomposable modules. Facts 1 and 2 show that $\mathbf{P}_\ell \otimes \mathbf{S}_h \cong \bigoplus_{j \in \mathbb{Z}/2N\mathbb{Z}} \mathbf{P}_j^{\oplus a_j}$, where the multiplicities a_j remain to be determined. The computation of the a_j's is just a combinatorial exercise. To explain this last statement, we recall that the comultiplication is given by

$$\begin{aligned}
\triangle(K) &= K \otimes K\ , \\
\triangle(E) &= E \otimes 1 + K \otimes E\ , \\
\triangle(F) &= F \otimes K^{-1} + 1 \otimes F\ .
\end{aligned}$$

With the first formula, we can compute the dimensions of the left K-eigenspaces. This knowledge is not sufficient. But the last two formulae show that tensor product formation is compatible with the height structure. More precisely, we may decompose $\mathbf{P}_\ell \otimes \mathbf{S}_h$ as a vector space into a direct sum of subspaces [3],

$$\mathbf{P}_\ell \otimes \mathbf{S}_h = \bigoplus_s \Big(\mathbf{P}_\ell \otimes \mathbf{S}_h\Big)_s := \bigoplus_{t+u=s} (\mathbf{P}_\ell)_t \otimes (\mathbf{S}_h)_u\,,$$

[3] Since $\Omega^1\mathbf{S}_h$ is a homogeneous submodule of $\mathbf{P}_h$, $\mathbf{S}_h$ inherits the gradation from $\mathbf{P}_h$

such that E (resp. F) maps $\left(\mathbf{P}_\ell \otimes \mathbf{S}_h\right)_s$ into $\left(\mathbf{P}_\ell \otimes \mathbf{S}_h\right)_{s-1}$ $\left(\text{resp. into } \left(\mathbf{P}_\ell \otimes \mathbf{S}_h\right)_{s+1}\right)$ by left or right multiplication. A left K-eigenvalue of highest height determines a module isomorphic to a PIM which is a direct summand of $\mathbf{P}_\ell \otimes \mathbf{S}_h$. We then delete the corresponding K-eigenvalues and continue splitting off direct summands.

An example may clarify the procedure. Let us compute for $N = 5$ the tensor product $\mathbf{P}_2 \otimes \mathbf{S}_3$. The height-and-left-$K$-eigenvalue-structures of $\mathbf{P}_2$ and $\mathbf{S}_3$, respectively, may be pictured as [cf. (3.7), (2.5) and (3.6)]

$$\begin{array}{c} 4 \\ 6 \\ 8\ 8 \\ 0\ 0 \\ 2\ 2 \\ 4 \\ 6 \end{array} \qquad \text{and} \qquad \begin{array}{c} 7 \\ 9 \\ 1 \\ 3 \end{array} ,$$

where a number ℓ stands for a K-eigenvalue q^ℓ. Now we do the tensor product and get

$$\begin{array}{cccc}
\hline
1 & & & \\ \hline
3 & 3 & & \\ \hline
5\ 5 & 5 & 5 & \\ \hline
7\ 7 & 7\ 7 & 7 & 7 \\ \hline
9\ 9 & 9\ 9 & 9\ 9 & 9 \\ \hline
1 & 1\ 1 & 1\ 1 & 1\ 1 \\ \hline
3 & 3 & 3\ 3 & 3\ 3 \\ \hline
 & 5 & 5 & 5\ 5 \\ \hline
 & & 7 & 7 \\ \hline
 & & & 9 \\ \hline
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccccc}
\hline
1 & & & & \\ \hline
3 & 3 & & & \\ \hline
5 & 5 & 5 & 5 & \\ \hline
7 & 7 & 7\ 7 & 7\ 7 & \\ \hline
9 & 9\ 9 & 9\ 9 & 9\ 9 & \\ \hline
 & 1\ 1 & 1\ 1 & 1\ 1 & 1 \\ \hline
 & 3 & 3\ 3 & 3\ 3 & 3 \\ \hline
 & 5 & 5 & 5 & 5 \\ \hline
 & 7 & & & 7 \\ \hline
 & & & & 9 \\ \hline
\end{array} ,$$

from which we read off that

$$\mathbf{P}_2 \otimes \mathbf{S}_3 \cong \overline{\mathrm{St}} \oplus \mathbf{P}_1 \oplus \mathbf{P}_3 \oplus \mathbf{P}_3 \oplus \overline{\mathrm{St}} \; .$$

In order to compute $\mathbf{P}_\ell \otimes \mathbf{S}_h$ in general, we may restrict ourselves to the cases where the conditions

$$(\ell + 2N\mathbb{Z}) \cap \{0, \ldots, N-1\} \neq \varnothing \quad \text{and} \quad (h + 2N\mathbb{Z}) \cap \{0, \ldots, N-1\} \neq \varnothing$$

are both fulfilled. It is because the modules $\mathbf{P}_\ell$ and $\overline{\mathbf{P}_\ell} = \mathbf{P}_{\ell+N}$ (resp. $\mathbf{S}_h$ and $\overline{\mathbf{S}_h} := \mathbf{S}_{h+N}$) differ in their height-and-left-K-eigenvalue-structures by the replacements of each left K-eigenvalue $q^\jmath$ with $q^{\jmath+N} = -q^\jmath$ only that this reduction is possible. In other words,

$$\begin{aligned} \overline{\mathbf{P}_\ell} \otimes \overline{\mathbf{S}_h} &\cong \mathbf{P}_\ell \otimes \mathbf{S}_h \; , \\ \overline{\mathbf{P}_\ell} \otimes \mathbf{S}_h &\cong \mathbf{P}_\ell \otimes \overline{\mathbf{S}_h} \cong \overline{\mathbf{P}_\ell \otimes \mathbf{S}_h} \; , \end{aligned} \tag{4.1}$$

where the last expression is defined in the obvious way by extending the involution $\mathbf{P}_\ell \mapsto \overline{\mathbf{P}_\ell}$ to the set of isomorphism classes of (finite-dimensional projective) modules.

It is useful to introduce certain quantities: we define σ by

$$\{\sigma(\ell)\} := \Big((\ell + 2N\mathbb{Z}) \cup (\bar{\ell} + 2N\mathbb{Z})\Big) \cap \{0, \dots, N-1\} \qquad \text{for } \ell \neq -1$$

$$\left(\text{e.g., for } N = 5: \quad \begin{array}{c|ccccccccc} \ell & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \sigma(\ell) & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \end{array}\right)$$

and put [cf. (3.7), (3.4) and (3.3)]

$$\widehat{\mathbf{P}}_\ell := \bigoplus_{h=0}^{N-1} \Big(\mathbb{C}E^h \alpha_\ell \oplus \mathbb{C}E^h \gamma_\ell\Big) ,$$

that is, (cf. Proposition 3.5) $\widehat{\mathbf{P}}_\ell = \mathbf{P}_\ell$ for $\ell + 1 \not\equiv 0 \,(\mathrm{mod}\, N)$ but $\widehat{\mathbf{P}}_\ell \cong \mathbf{P}_\ell \oplus \mathbf{P}_\ell$ for $\ell + 1 \equiv 0 \,(\mathrm{mod}\, N)$. Each of the modules $\widehat{\mathbf{P}}_\ell$ is $2N$-dimensional.

Proposition 4.1. *For "$0 \le h \le \ell < N$" (i.e., $0 \le \mathrm{mod}^-_{2N}(h) \le \mathrm{mod}^-_{2N}(\ell) < N$) we have*

$$\widehat{\mathbf{P}}_\ell \otimes \mathbf{S}_h \cong \bigoplus_{j=0}^{h} \widehat{\mathbf{P}}_{\sigma(\ell - h + 2j)} .$$

For "$0 \le \ell < h < N$" (i.e., $0 \le \mathrm{mod}^-_{2N}(\ell) < \mathrm{mod}^-_{2N}(h) < N$) we have

$$\widehat{\mathbf{P}}_\ell \otimes \mathbf{S}_h \cong \widehat{\mathbf{P}}_h \otimes \mathbf{S}_\ell \,\oplus\, \overline{\widehat{\mathbf{P}}_{N-1} \otimes \mathbf{S}_{h-\ell-1}} .$$

Furthermore,

$$\widehat{\mathbf{P}}_\ell \otimes \mathbf{S}_h \cong \mathbf{S}_h \otimes \widehat{\mathbf{P}}_\ell .$$

Proof. The last part, that is, the commutativity property follows because $\widehat{\mathbf{P}}_\ell \otimes \mathbf{S}_h$ and $\mathbf{S}_h \otimes \widehat{\mathbf{P}}_\ell$ have the same height-and-left-K-eigenvalue-structures. (So we need not mention quasi-triangularity here.)

Generating functions in y represent the height-and-left-K-eigenvalue-structures η for the various modules under consideration in the following way, where we put $x := yq^2$ to abbreviate the notation:

$$\eta(\mathbf{S}_h) := q^{-h} + yq^{-h+2} + \ldots + y^h q^h = q^{-h}\,\frac{1 - x^{h+1}}{1 - x} \qquad \text{for "} 0 \le h < N \text{"},$$

$$\eta(\mathbf{S}_{\bar{h}}) := q^{-\bar{h}} + yq^{-\bar{h}+2} + \ldots + y^{\bar{h}-N} q^{\bar{h}} = q^{-\bar{h}}\,\frac{1 - x^{\bar{h}-N+1}}{1 - x} \qquad \text{for "} 0 \le h < N-1 \text{"},$$

$$\eta(\widehat{\mathbf{P}}_\ell) := (1 + y^N)\,\eta(\mathbf{S}_{\bar{\ell}}) + 2\,y^{\bar{\ell}-N+1}\eta(\mathbf{S}_\ell) \qquad \text{for "} 0 \le \ell < N-1 \text{"},$$

$$\eta(\widehat{\mathbf{P}}_{N-1}) := 2\,\eta(\mathbf{S}_{N-1}) ,$$

so that

$$\eta(\widehat{\mathbf{P}}_\ell) = (1 + y^N)\,q^{-\bar{\ell}}\,\frac{1 - x^{\bar{\ell}-N+1}}{1 - x} + 2\,y^{\bar{\ell}-N+1} q^{-\ell}\,\frac{1 - x^{\ell+1}}{1 - x} \qquad \text{for "} 0 \le \ell < N \text{"},$$

and finally

$$\eta(\widehat{\mathbf{P}}_{2N-1}) := -(1 + y^N)\,\eta(\mathbf{S}_{N-1}) .$$

Let "$0 \le h \le \ell < N$". To prove the first formula in the proposition we shall see that

$$\eta(\widehat{\mathbf{P}}_\ell)\,\eta(\mathbf{S}_h) = \sum_{j=0}^{h} y^{f_{\ell,h}(j)}\eta(\widehat{\mathbf{P}}_{\sigma(\ell-h+2j)})$$

for a suitable set of exponents $\{\, f_{\ell,h}(j) \mid j = 0, \dots, h \,\}$. We find

$$f_{\ell,h}(j) = \frac{1}{2}\Big(\sigma(\ell - h + 2j) - (\ell - h)\Big) .$$

Now let "$0 \le \ell < h < N$". Then we have

$$\eta(\widehat{\mathbf{P}}_\ell)\,\eta(\mathbf{S}_h) = y^{h-\ell}\eta(\widehat{\mathbf{P}}_h)\,\eta(\mathbf{S}_\ell) + \eta(\widehat{\mathbf{P}}_{2N-1})\,\eta(\mathbf{S}_{h-\ell-1}) ,$$

which proves the second formula in the proposition. □

Remark. Using Proposition 4.1, we can calculate $\widehat{\mathbf{P}}_\ell \otimes M$, where M is an arbitrary (finite-dimensional) $\mathfrak{U}_q(\mathfrak{sl}_2)$-module. In fact, up to isomorphism the tensor product remains the same if we replace M by, say, the direct sum of the composition factors of M.

In the same vein as Proposition 4.1 we prove the next lemma, which amounts to nothing more than the classical Clebsch-Gordan theorem.

Lemma 4.2. *For "$0 \le h \le \ell < N$" (i. e., $0 \le \operatorname{mod}^-_{2N}(h) \le \operatorname{mod}^-_{2N}(\ell) < N$) the two modules* [*cf.* (3.8)]

$$\mathbf{S}_\ell \otimes \mathbf{S}_h \quad \textit{and} \quad \bigoplus_{\substack{j=0 \\ \text{"}\ell-h+2j<N\text{"}}}^{h} \mathbf{S}_{\ell-h+2j} \;\oplus\; \bigoplus_{\substack{j=0 \\ \text{"}\ell-h+2j\ge N\text{"}}}^{h} \Omega^1\mathbf{S}_{\sigma(\ell-h+2j)}$$

have the same composition factors, that is, their classes are equal in $K(\mathfrak{U}_q(\mathfrak{sl}_2))$, the Grothendieck ring of $\mathfrak{U}_q(\mathfrak{sl}_2)$.

Corollary 4.3. *For "$1 \le \ell < N-1$" (i. e., $1 \le \operatorname{mod}^-_{2N}(\ell) < N-1$) we have*

$$\mathbf{S}_\ell \otimes \mathbf{S}_1 \cong \mathbf{S}_{\ell-1} \oplus \mathbf{S}_{\ell+1} ;$$

furthermore,

$$\mathbf{S}_0 \otimes \mathbf{S}_1 \cong \mathbf{S}_1 \quad \textit{and} \quad \mathbf{S}_{N-1} \otimes \mathbf{S}_1 \cong \mathbf{P}_{N-2} .$$

Proof. The first of the last two formulae is trivial, the second is contained in Proposition 4.1. By Lemma 4.2 we have

$$\operatorname{class}(\mathbf{S}_\ell \otimes \mathbf{S}_1) = \operatorname{class}(\mathbf{S}_{\ell-1} \oplus \mathbf{S}_{\ell+1}) \quad \text{in } K(\mathfrak{U}_q(\mathfrak{sl}_2)) .$$

As $\mathbf{S}_{\ell-1}$ and $\mathbf{S}_{\ell+1}$ belong to different blocks, the equation above implies that $\mathbf{S}_\ell \otimes \mathbf{S}_1 \cong \mathbf{S}_{\ell-1} \oplus \mathbf{S}_{\ell+1}$.

Alternatively, from $\mathbf{P}_\ell \twoheadrightarrow \mathbf{S}_\ell$ we get $\mathbf{P}_\ell \otimes \mathbf{S}_1 \twoheadrightarrow \mathbf{S}_\ell \otimes \mathbf{S}_1$, and now we employ Proposition 4.1. □

Proposition 4.4. *The indecomposable direct summands of $\mathbf{S}_\ell \otimes \mathbf{S}_h$ are either simple or projective.*

Proof. Without loss of generality, we consider the cases "$0 \le h \le \ell < N$" only. The standard proof then goes as follows (cf. [GK]): Corollary 4.3 shows that the modules $\mathbf{S}_\ell$ and $\mathbf{S}_h$ occur as direct summands of suitable tensor powers of $\mathbf{S}_1$ and that Proposition 4.4 holds for $h = 1$. Now we invoke Fact 2, and the general case follows. □

Theorem 4.5. *For "$0 \leq h \leq \ell < N$" (i. e., $0 \leq \mathrm{mod}_{2N}^{-}(h) \leq \mathrm{mod}_{2N}^{-}(\ell) < N$) we have*

$$\mathbf{S}_\ell \otimes \mathbf{S}_h \cong \bigoplus_{\substack{j=0 \\ \mu_{\ell,h}(j)=1}}^{h} \mathbf{S}_{\ell-h+2j} \;\oplus \bigoplus_{\substack{j=0 \\ \mu_{\ell,h}(j)=2 \\ \text{"}\ell-h+2j<N\text{"}}}^{h} \mathbf{P}_{\ell-h+2j} \,,$$

where for $j \in \{0,\dots,h\}$ we put

$$\mu_{\ell,h}(j) := \#\Big\{ \jmath \in \{0,\dots,h\} \;\Big|\; \sigma(\ell-h+2\jmath) = \sigma(\ell-h+2j) \Big\} \in \{1,2\} \,.$$

Proof. Put Proposition 4.1, Lemma 4.2, and Proposition 4.4 together. □

Now we can compute $\mathbf{S}_\ell \otimes \mathbf{S}_h \cong \mathbf{S}_h \otimes \mathbf{S}_\ell$ for all values of ℓ and h by means of applying formulae analogous to (4.1).

5. Finite-Dimensional $\mathfrak{U}_q(\mathfrak{sl}_2)$-Modules

This section deals with the *abelian* category ${}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\mathbf{mod}$ of finite-dimensional left $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules. Here we have to consider the representation theories for each block of $\mathfrak{U}_q(\mathfrak{sl}_2)$. The blocks $\mathbb{B}_{\mathrm{St}}$ and $\mathbb{B}_{\overline{\mathrm{St}}}$ – both isomorphic to $\mathrm{Mat}(N\times N,\mathbb{C})$ – deserve no further comment.

For almost the rest of this section we fix ℓ such that $\ell + 1 \not\equiv 0 \,(\mathrm{mod}\, N)$.

An Equivalent Category. We begin investigating the category ${}_{\mathbb{B}_\ell}\mathbf{mod}$ of finite-dimensional left $\mathbb{B}_\ell$-modules. Recall that $\mathbf{P}_\ell$ and $\mathbf{P}_{\bar\ell}$ represent the two types of PIMs for $\mathbb{B}_\ell = \mathbb{B}_{\bar\ell}$. Put

$$\mathcal{B} := \mathrm{End}_{\mathfrak{U}_q(\mathfrak{sl}_2)}\Big(\mathbf{P}_\ell \oplus \mathbf{P}_{\bar\ell}\Big) = \mathrm{End}_{\mathbb{B}_\ell}\Big(\mathbf{P}_\ell \oplus \mathbf{P}_{\bar\ell}\Big) \,.$$

The algebra $\mathcal{B}^{\mathrm{op}}$ is the *opposite algebra* (i. e., the multiplication is reversed) of $\mathcal{B}$. It is a standard fact – known as Morita theory – that the categories ${}_{\mathbb{B}_\ell}\mathbf{mod}$ and ${}_{\mathcal{B}^{\mathrm{op}}}\mathbf{mod}$ are equivalent. In fact, we may view $\mathbf{P}_\ell \oplus \mathbf{P}_{\bar\ell}$ as a $\mathbb{B}_\ell$-$\mathcal{B}^{\mathrm{op}}$-bimodule; the vector space $\mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_\ell \oplus \mathbf{P}_{\bar\ell}, \mathbb{B}_\ell)$ then becomes a $\mathcal{B}^{\mathrm{op}}$-$\mathbb{B}_\ell$-bimodule, and the functors

$$\begin{aligned} \mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_\ell \oplus \mathbf{P}_{\bar\ell}, \mathbb{B}_\ell)\otimes_{\mathbb{B}_\ell} \quad &: \quad {}_{\mathbb{B}_\ell}\mathbf{mod} \longrightarrow {}_{\mathcal{B}^{\mathrm{op}}}\mathbf{mod} \,, \\ (\mathbf{P}_\ell \oplus \mathbf{P}_{\bar\ell})\otimes_{\mathcal{B}^{\mathrm{op}}} \quad &: \quad {}_{\mathcal{B}^{\mathrm{op}}}\mathbf{mod} \longrightarrow {}_{\mathbb{B}_\ell}\mathbf{mod} \end{aligned}$$

provide an equivalence of abelian categories ${}_{\mathbb{B}_\ell}\mathbf{mod} \simeq {}_{\mathcal{B}^{\mathrm{op}}}\mathbf{mod}$. The algebra $\mathcal{B}^{\mathrm{op}}$ is the *basic algebra* of $\mathbb{B}_\ell$; every simple (left) $\mathcal{B}^{\mathrm{op}}$-module is 1-dimensional.

We shall describe the algebra $\mathcal{B}$ explicitly. In matrix notation $\mathcal{B}$ may be written as

$$\mathcal{B} = \begin{pmatrix} \mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_\ell, \mathbf{P}_\ell) & \mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_{\bar\ell}, \mathbf{P}_\ell) \\ \mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_\ell, \mathbf{P}_{\bar\ell}) & \mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_{\bar\ell}, \mathbf{P}_{\bar\ell}) \end{pmatrix} .$$

$\mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_\ell, \mathbf{P}_\ell)$ is generated as a vector space by the two $\mathbb{B}_\ell$-homomorphisms given by [see (3.7)] $\gamma_\ell \mapsto \gamma_\ell$ (i. e., $\mathrm{id}_{\mathbf{P}_\ell}$) and $\gamma_\ell \mapsto \tilde\alpha_\ell$, whereas $\mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_\ell, \mathbf{P}_{\bar\ell})$ is the vector space generated by the two $\mathbb{B}_\ell$-homomorphisms given by $\mathsf{e} : \gamma_\ell \mapsto \tilde\gamma_{\bar\ell}$ and

$\mathsf{f} : \gamma_\ell \mapsto \alpha_{\bar\ell}$. Similarly, we have $\mathrm{Hom}_{\mathbb{B}_\ell}(\mathbf{P}_{\bar\ell}, \mathbf{P}_\ell) = \mathbb{C}\bar{\mathsf{e}} \oplus \mathbb{C}\bar{\mathsf{f}}$, where $\bar{\mathsf{e}} : \gamma_{\bar\ell} \mapsto \tilde\gamma_\ell$ and $\bar{\mathsf{f}} : \gamma_{\bar\ell} \mapsto \alpha_\ell$. The algebra $\mathcal{B}$ is thus the 8-dimensional $\mathbb{C}$-algebra generated by

$$\bar{\mathsf{Z}}\mathsf{Z} := \begin{pmatrix} \mathrm{id}_{\mathbf{P}_\ell} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{Z}\bar{\mathsf{Z}} := \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_{\mathbf{P}_{\bar\ell}} \end{pmatrix},$$

$$\mathsf{E} := \begin{pmatrix} 0 & 0 \\ \mathsf{e} & 0 \end{pmatrix}, \qquad \bar{\mathsf{E}} := \begin{pmatrix} 0 & \bar{\mathsf{e}} \\ 0 & 0 \end{pmatrix},$$

$$\mathsf{F} := \begin{pmatrix} 0 & 0 \\ \mathsf{f} & 0 \end{pmatrix}, \qquad \bar{\mathsf{F}} := \begin{pmatrix} 0 & \bar{\mathsf{f}} \\ 0 & 0 \end{pmatrix}.$$

Besides the obvious relations among the elements of $\mathcal{B}$ defined above, the identities $\bar{\mathsf{F}}\mathsf{E} = \bar{\mathsf{E}}\mathsf{F}$ and $\mathsf{F}\bar{\mathsf{E}} = \mathsf{E}\bar{\mathsf{F}}$ follow from $\bar{\mathsf{f}}\mathsf{e} = \bar{\mathsf{e}}\mathsf{f}$ and $\mathsf{f}\bar{\mathsf{e}} = \mathsf{e}\bar{\mathsf{f}}$. Moreover, we have $\bar{\mathsf{E}}\mathsf{E} = \bar{\mathsf{F}}\mathsf{F} = \mathsf{E}\bar{\mathsf{E}} = \mathsf{F}\bar{\mathsf{F}} = 0$. Note that the three elements $\bar{\mathsf{Z}}\mathsf{Z}$, $\mathsf{E} + \bar{\mathsf{E}}$, and $\mathsf{F} + \bar{\mathsf{F}}$ already generate $\mathcal{B}$ as an algebra with 1.

Look at the inclusions for classes of finite-dimensional algebras

$$\{\text{self-injective algebras}\} \supseteq \{\text{Frobenius algebras}\} \supseteq \{\text{symmetric algebras}\}.$$

The first two types of algebras were defined in Sect. 3. A *symmetric algebra* A over the field k is an algebra which admits a linear form $t : A \to k$ such that no left (and no right) ideal of A different from the zero ideal lies in the kernel of t and such that, in addition, $t(xy) = t(yx)$ for every $x, y \in A$. The group algebras of finite groups over a field form a prominent subclass of the class of symmetric algebras. Certainly, $\mathfrak{U}_q(\mathfrak{sl}_2)$ is not a group algebra. But $\mathfrak{U}_q(\mathfrak{sl}_2)$ is a symmetric algebra. To see this we must show that each block of $\mathfrak{U}_q(\mathfrak{sl}_2)$ is a symmetric algebra, which is certainly true for the simple algebras $\mathbb{B}_{\mathrm{St}}$ and $\mathbb{B}_{\overline{\mathrm{St}}}$. Let us consider the generic block $\mathbb{B}_\ell$. Symmetry of algebras is not affected by Morita equivalence – like self-injectivity, but unlike the property of being a Frobenius algebra. The linear form $\mathcal{B}^{\mathrm{op}} \to \mathbb{C}$ defined by

$$\bar{\mathsf{Z}}\mathsf{Z}, \mathsf{Z}\bar{\mathsf{Z}}, \bar{\mathsf{F}}\mathsf{E}, \mathsf{F}\bar{\mathsf{E}} \longmapsto 1, \quad \mathsf{E}, \bar{\mathsf{E}}, \mathsf{F}, \bar{\mathsf{F}} \longmapsto 0$$

reveals that $\mathcal{B}^{\mathrm{op}}$ and thus $\mathbb{B}_\ell$ are symmetric algebras. Now the symmetry of the algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ again implies (cf. [Hu]) that the Hopf algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$ is unimodular, as already observed in Sect. 3.

Let

$$\mathcal{K} := \mathbb{C}[X, Y]/(X^2, Y^2)$$

be the *Kronecker algebra* over $\mathbb{C}$. By abuse of notation we shall write X instead of $X \bmod (X^2, Y^2)$ and Y instead of $Y \bmod (X^2, Y^2)$.

The algebra $\widehat{\mathcal{B}}$, generated by $\mathsf{Z}, \bar{\mathsf{Z}}, \mathsf{E}, \bar{\mathsf{E}}, \mathsf{F}, \bar{\mathsf{F}}$ (or just by $\mathsf{Z}, \bar{\mathsf{Z}}, \mathsf{E}, \mathsf{F}$), is defined by the requirement that

$$\mathsf{Z} \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \bar{\mathsf{Z}} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\mathsf{E} \longmapsto \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}, \qquad \bar{\mathsf{E}} \longmapsto \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix},$$

$$\mathsf{F} \longmapsto \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}, \qquad \bar{\mathsf{F}} \longmapsto \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}$$

define an isomorphism $\widehat{\mathcal{B}} \xrightarrow{\cong} \mathrm{Mat}(2\times 2, \mathcal{K})$; and we identify $\mathcal{B}$ as a subalgebra of $\widehat{\mathcal{B}}$. It might be instructive for the reader to draw diagrams for the PIMs of $\mathcal{B}$ and of $\widehat{\mathcal{B}}$.

We want to get rid of the "${}^{\mathrm{op}}$". This is achieved by means of the isomorphism $\widehat{\mathcal{B}} \xrightarrow{\cong} \widehat{\mathcal{B}}^{\mathrm{op}}$ given by

$$\mathsf{Z} \mapsto \bar{\mathsf{Z}}, \qquad \bar{\mathsf{Z}} \mapsto \mathsf{Z}, \qquad \mathsf{E} \mapsto \bar{\mathsf{E}}, \qquad \mathsf{F} \mapsto \bar{\mathsf{F}}.$$

It restricts to an isomorphism $\mathcal{B} \xrightarrow{\cong} \mathcal{B}^{\mathrm{op}}$.

The algebra $\mathrm{Mat}(2\times 2, \mathcal{K})$ has basic algebra $\mathcal{K}$, so that we get the relations between algebras

$$\mathbb{B}_\ell \ \sim\ \mathcal{B}^{\mathrm{op}} \ \sim\ \mathcal{B} \subseteq \widehat{\mathcal{B}} \cong \mathrm{Mat}(2\times 2, \mathcal{K}) \ \sim\ \mathcal{K},$$

where $\sim$ denotes equivalence of the respective left module categories. What remains to be done is to compare ${}_{\mathcal{B}}\mathbf{mod}$ with ${}_{\widehat{\mathcal{B}}}\mathbf{mod}$ and to describe ${}_{\mathcal{K}}\mathbf{mod}$.

Let M be a (finite-dimensional) left $\mathcal{B}$-module. We have $M = \bar{\mathsf{Z}}\mathsf{Z}M \oplus \mathsf{Z}\bar{\mathsf{Z}}M$, and we define the left $\widehat{\mathcal{B}}$-module

$$\widehat{M} := \bar{\mathsf{Z}}\mathsf{Z}M \oplus \mathsf{Z}\bar{\mathsf{Z}}M \oplus \bar{\mathsf{Z}}(\mathsf{Z}\bar{\mathsf{Z}}M) \oplus \mathsf{Z}(\bar{\mathsf{Z}}\mathsf{Z}M)$$

in the obvious way.

An indecomposable (finite-dimensional) $\widehat{\mathcal{B}}$-module L decomposes via restriction to $\mathcal{B}$ into a direct sum of two indecomposable $\mathcal{B}$-modules, $L|_{\mathcal{B}} = L_1 \oplus L_2$, where $\bar{\mathsf{Z}}\mathsf{Z}$ (E, F) acts on L_1 (resp. L_2) as $\mathsf{Z}\bar{\mathsf{Z}}$ $(\bar{\mathsf{E}}, \bar{\mathsf{F}})$ acts on L_2 (resp. L_1). In particular, each $\widehat{\mathcal{B}}$-module is isomorphic to $\widehat{M}$ for some $\mathcal{B}$-module M; and we have $\widehat{L|_{\mathcal{B}}} \cong L \oplus L$ as $\widehat{\mathcal{B}}$-modules. Thus $M \mapsto \widehat{M}$ yields

$$\left\{\begin{array}{l}\text{isomorphism classes}\\ \text{of indecomposable}\\ \text{(finite-dimensional)}\\ \text{left } \mathcal{B}\text{-modules}\end{array}\right\} \xrightarrow{2:1} \left\{\begin{array}{l}\text{isomorphism classes}\\ \text{of indecomposable}\\ \text{(finite-dimensional)}\\ \text{left } \widehat{\mathcal{B}}\text{-modules}\end{array}\right\}.$$

Finally, we recall the relevant facts concerning ${}_{\mathcal{K}}\mathbf{mod}$. First note that the radical of $\mathcal{K}$ is $\mathbf{J}(\mathcal{K}) = \mathbb{C}X \oplus \mathbb{C}Y \oplus \mathbb{C}XY$. Assume M is a finite-dimensional $\mathcal{K}$-module without projective summands. Then $\operatorname{soc}\mathcal{K} = \mathbf{J}(\mathcal{K})^2 = \mathbb{C}XY$ annihilates M, and $\mathbf{J}(\mathcal{K})M \subseteq \operatorname{soc} M$. Choose now a vector space basis for M whose first elements constitute a basis for $\operatorname{soc} M$. The matrices of X and Y, respectively, are exactly the matrices of the form

$$\begin{pmatrix} 0 & \widetilde{X} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \widetilde{Y} \\ 0 & 0 \end{pmatrix}. \tag{5.1}$$

We are thus left with classifying pairs of matrices (5.1) under simultaneous conjugation by invertible matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, that is, we may replace $\widetilde{X}$ by $A\widetilde{X}B$ and $\widetilde{Y}$ by $A\widetilde{Y}B$, where A and B are invertible matrices.

The problem of classifying such pairs of matrices was solved by Kronecker more than a century ago [Kr]. His result yields the following nonredundant, exhaustive list, which tells us the isomorphism classes of indecomposable finite-dimensional $\mathcal{K}$-modules.

- $\mathbb{C}$ (the simple $\mathcal{K}$-module)
- $\mathcal{K}$ (the PIM)
- For each $n = 1, 2, \ldots$ the $\mathcal{K}$-module corresponding to the $n\times(n+1)$-matrices

$$\widetilde{X} = \begin{pmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix} \quad , \quad \widetilde{Y} = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{pmatrix}$$

- For each $n = 1, 2, \ldots$ the $\mathcal{K}$-module corresponding to the $(n+1)\times n$-matrices

$$\widetilde{X} = \begin{pmatrix} 1 & & 0 \\ 0 & \ddots & \\ & \ddots & 1 \\ 0 & & 0 \end{pmatrix} \quad , \quad \widetilde{Y} = \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{pmatrix}$$

- For each $\lambda \in \mathbb{C} \cup \{\infty\}$, $n = 1, 2, \ldots$ the $\mathcal{K}$-module corresponding to

$$\begin{aligned} &\widetilde{X} = \mathbf{1}_n\,, && \widetilde{Y} = \mathbf{J}_n(\lambda) && \text{if } \lambda \in \mathbb{C}\,, \\ &\widetilde{X} = \mathbf{J}_n(0)\,, && \widetilde{Y} = \mathbf{1}_n && \text{if } \lambda = \infty\,, \end{aligned}$$

where $\mathbf{J}_n(\mu)$ denotes a Jordan block matrix of size n with eigenvalue μ.

Remark (see [Ka]). The modules listed under the first and the last three points correspond to the positive roots of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. The correspondence is one-to-one for the real roots, whereas with every positive imaginary root is associated a variety of modules parametrised by the sphere $\mathbb{P}^1(\mathbb{C})$.

Remark (cf. [Ба, HR]). The Kronecker algebra over k, $k[X,Y]/(X^2, Y^2)$, for k a field of characteristic 2 is isomorphic to the group algebra over k of the Klein four group $\langle x, y \mid x^2 = y^2 = 1, xy = yx\rangle$. An isomorphism is given by $X \mapsto 1 + x$, $Y \mapsto 1 + y$.

Remark. Of course, the theory developed further, going beyond what shall be used here. There is Ringel's classification for the dihedral 2-groups [Rin1], which uses part of [ГП, GP] in a new shape (functorial filtrations). You may also look at [BS]. More recently, there are [WW, BR, Er] to name but three items.

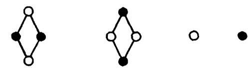
To introduce some notation, we give the list which classifies the isomorphism classes of indecomposable left $\mathbb{B}_\ell$-modules which parallels the list given above for the $\mathcal{K}$-modules.

- $\mathbf{S}_\ell$, $\mathbf{S}_{\bar{\ell}}$
- $\mathbf{P}_\ell$, $\mathbf{P}_{\bar{\ell}}$
- For each $n = 1, 2, \ldots$: $\Omega^n\mathbf{S}_\ell$, $\Omega^n\mathbf{S}_{\bar{\ell}}$
- For each $n = 1, 2, \ldots$: $\Omega^{-n}\mathbf{S}_\ell$, $\Omega^{-n}\mathbf{S}_{\bar{\ell}}$
- For each $\lambda \in \mathbb{C} \cup \{\infty\}$, $n = 1, 2, \ldots$: $\mathbf{M}^n_\ell(\lambda)$, $\mathbf{M}^n_{\bar{\ell}}(\lambda)$.

There are no new notations introduced under the first two points. Let us make a digression on the next two points and on the last point.

Digression on Ext. The modules $\Omega^1\mathbf{S}_\ell$ and $\Omega^1\mathbf{S}_{\bar\ell}$ have already been introduced [cf. (3.8)]. As the notation suggests, the modules $\Omega^{\pm n}\mathbf{S}_\ell$ and $\Omega^{\pm n}\mathbf{S}_{\bar\ell}$ are related to the loop space functor in algebraic topology. In fact, the projective modules play the role of the homotopically trivial objects.

It may be useful to depict the modules $\Omega^{\pm n}\mathbf{S}_\ell$, $\Omega^{\pm n}\mathbf{S}_{\bar\ell}$ by means of diagrams. Let us write

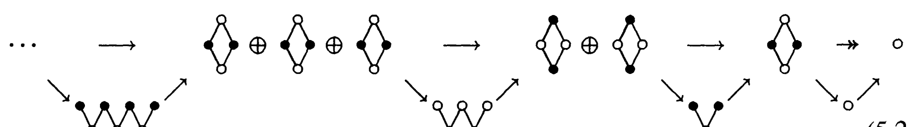


for the four modules $\mathbf{P}_\ell$, $\mathbf{P}_{\bar\ell}$, $\mathbf{S}_\ell$, $\mathbf{S}_{\bar\ell}$, respectively. If we compare these diagrams with those displayed earlier, we should note two things. Firstly, within a whole diagram, $\circ$ and $\bullet$ stand for $\mathrm{mod}_N^+(\ell+1)$-dimensional and $\mathrm{mod}_N^+(\bar\ell+1)$-dimensional vector subspaces, respectively. As we know, this is of minor importance. Secondly, whereas in the pictures in earlier sections the head was on the left- and the socle on the right-hand side, they are now at the top and bottom, respectively, such as we see it in Corollary 3.10.

Recall how a minimal projective resolution of a module M is constructed:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr, "\partial_3"] \arrow[dr] & & P_{\Omega^2 M} \arrow[rr, "\partial_2"] \arrow[dr] & & P_{\Omega^1 M} \arrow[rr, "\partial_1"] \arrow[dr] & & P_M \arrow[rr, "\partial_0"] \arrow[dr] & & M \arrow[r] & 0, \\
& \Omega^3 M \arrow[ur] & & \Omega^2 M \arrow[ur] & & \Omega^1 M \arrow[ur] & & \Omega^0 M = M \arrow[ur] & &
\end{tikzcd}$$

where P_X (or, more precisely, $P_X \twoheadrightarrow X$) denotes a projective cover for the module X and is then unique up to isomorphism; $\Omega^j M := \ker \partial_{j-1}$ for $j \in \mathbb{Z}_{>0}$; and the ∂'s are such that the diagram commutes. The horizontal sequence is thus an exact one. Here is "the" minimal projective resolution of $\mathbf{S}_\ell$ in diagrammatic form:

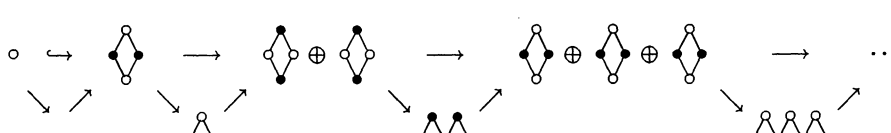


(5.2)

Interchanging black and white dots leads to "the" minimal projective resolution of $\mathbf{S}_{\bar\ell}$. This gives us the V -, W -, WV -, ... shaped modules $\Omega^n\mathbf{S}_\ell$ and $\Omega^n\mathbf{S}_{\bar\ell}$. To construct Λ -, M -, MΛ -, ... shaped modules, we use minimal injective resolutions of $\mathbf{S}_\ell$ and $\mathbf{S}_{\bar\ell}$. For a module M it reads

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & M \arrow[rr, "\partial^0"] \arrow[dr] & & I_M \arrow[rr, "\partial^1"] \arrow[dr] & & I_{\Omega^{-1}M} \arrow[rr, "\partial^2"] \arrow[dr] & & I_{\Omega^{-2}M} \arrow[rr, "\partial^3"] \arrow[dr] & & \cdots, \\
& M = \Omega^0 M \arrow[ur] & & \Omega^{-1} M \arrow[ur] & & \Omega^{-2} M \arrow[ur] & & \Omega^{-3} M \arrow[ur] & &
\end{tikzcd}$$

where I_X (or, more precisely, $X \hookrightarrow I_X$) denotes an injective hull for the module X and is then unique up to isomorphism; $\Omega^{-j}M := \operatorname{coker} \partial^{j-1}$ for $j \in \mathbb{Z}_{>0}$; and the ∂'s are such that the diagram commutes. The horizontal sequence is thus an exact one. Here is "the" minimal injective resolution of $\mathbf{S}_\ell$ in diagrammatic form:



(5.3)

And again, we may interchange black and white dots. We have thus seen the modules $\Omega^{-n}\mathbf{S}_\ell$ and $\Omega^{-n}\mathbf{S}_{\bar{\ell}}$, too.

Given two $\mathfrak{U}_q(\mathfrak{sl}_2)$-modules M and L, the $\mathbb{C}$-vector space $\mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, L)$ classifies extensions of M by L, that is, isomorphism classes of short exact sequences

$$0 \to L \to E \to M \to 0$$

with fixed end terms. The Ext functors may be calculated as

$$\begin{aligned}\mathrm{Ext}^n_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, L) &\cong \frac{\ker\Big(\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P_{\Omega^n M}, L) \xrightarrow{(\partial_{n+1})^*} \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P_{\Omega^{n+1} M}, L)\Big)}{\mathrm{im}\Big(\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P_{\Omega^{n-1} M}, L) \xrightarrow{(\partial_n)^*} \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P_{\Omega^n M}, L)\Big)} \\ &\cong \frac{\ker\Big(\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, I_{\Omega^{-n} L}) \xrightarrow{(\partial^{n+1})_*} \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, I_{\Omega^{-(n+1)} L})\Big)}{\mathrm{im}\Big(\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, I_{\Omega^{-(n-1)} L}) \xrightarrow{(\partial^n)_*} \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, I_{\Omega^{-n} L})\Big)} .\end{aligned}$$

The first formula calculates the right derived functors of the contravariant functor $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\ \ , L)$, and the second formula gives the right derived functors of the covariant functor $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M,\ \ )$.

If $L = S$ is a simple module, the differentials ∂^* are zero: if the composite

$$P_{\Omega^{n+1} M} \xrightarrow{\partial_{n+1}} P_{\Omega^n M} \longrightarrow S$$

were nonzero, $P_{\Omega^n M}$ would have a direct summand isomorphic to P_S lying in $\mathrm{im}\,\partial_{n+1}$. But then we could split off direct summands isomorphic to P_S from each of the modules $P_{\Omega^{n+1} M}$ and $P_{\Omega^n M}$ in the projective resolution of M we used; that contradicts minimality. Hence

$$\mathrm{Ext}^n_{\mathfrak{U}_q(\mathfrak{sl}_2)}(M, S) \cong \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(P_{\Omega^n M}, S) \cong \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\Omega^n M, S) .$$

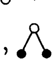
Similarly, for $M = S$ a simple module, the differentials ∂_* are zero: if the composite

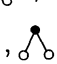
$$S \longrightarrow I_{\Omega^{-n} L} \xrightarrow{\partial^{n+1}} I_{\Omega^{-(n+1)} L}$$

were nonzero, $I_{\Omega^{-n} L}$ would have a direct summand isomorphic to I_S which injects into $I_{\Omega^{-(n+1)} L}$ by ∂^{n+1}. Again, this is impossible because of the minimality of the injective resolution used here. Hence

$$\mathrm{Ext}^n_{\mathfrak{U}_q(\mathfrak{sl}_2)}(S, L) \cong \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(S, I_{\Omega^{-n} L}) \cong \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(S, \Omega^{-n} L) .$$

Here are some sample computations in case $n = 1$:

$\mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\circ, \circ) \cong$ $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}($  $, \circ) = 0$; $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\circ,$  $) = 0$ (1)

$\mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\circ, \bullet) \cong$ $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}($  $, \bullet) \cong \mathbb{C}^2$; $\mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\circ,$  $) \cong \mathbb{C}^2$ (2)

$\mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\circ,$  $) \cong \mathrm{Hom}_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\circ, \circ) \cong \mathbb{C}$. (3)

(1) Here we only have the isomorphism class of the split extensions.

(2) An explicit isomorphism of $\mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\mathbf{V}^{\downarrow}_{\bar{\ell}} \cap \mathbf{V}^{\uparrow}_{\ell}, \mathbf{V}^{\downarrow}_{\bar{\ell}} \cap \mathbf{V}^{\uparrow}_{\bar{\ell}}) \left(\cong \mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\mathbf{S}_\ell, \mathbf{S}_{\bar{\ell}})\right)$ [cf. (3.8)] with $\mathbb{C}^2$ may be given as follows: the isomorphism class of the split extensions corresponds to $(0,0) \in \mathbb{C}^2$; for $(u,v) \in \mathbb{C}^2 - \{(0,0)\}$ the corresponding isomorphism class of short exact sequences is represented by [cf. (3.7)]

$$\begin{gathered} 0 \to \mathfrak{U}_q(\mathfrak{sl}_2)\tilde{\alpha}_{\bar{\ell}} \to \mathfrak{U}_q(\mathfrak{sl}_2)\big(u\tilde{\gamma}_{\bar{\ell}} + v\alpha_{\bar{\ell}}\big) \to \mathfrak{U}_q(\mathfrak{sl}_2)\tilde{\alpha}_\ell \to 0 \\ \tilde{\alpha}_{\bar{\ell}} \mapsto \tilde{\alpha}_{\bar{\ell}}\,, \qquad u\tilde{\gamma}_{\bar{\ell}} + v\alpha_{\bar{\ell}} \mapsto \tilde{\alpha}_\ell\,. \end{gathered}$$

The modules $\mathfrak{U}_q(\mathfrak{sl}_2)\big(u\tilde{\gamma}_{\bar{\ell}} + v\alpha_{\bar{\ell}}\big)$ and $\mathfrak{U}_q(\mathfrak{sl}_2)\big(u'\tilde{\gamma}_{\bar{\ell}} + v'\alpha_{\bar{\ell}}\big)$ are isomorphic if and only if (u,v) and $(u',v') \in \mathbb{C}^2 - \{(0,0)\}$ represent the same point λ in the complex projective line $\mathbb{P}^1(\mathbb{C})$. This gives us the family of modules $\mathbf{M}^1_\ell(\lambda)$.

(3) $\mathrm{Ext}^1_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\mathbf{S}_\ell, \Omega^1\mathbf{S}_\ell)$ is spanned by the class of $0 \to \Omega^1\mathbf{S}_\ell \to \mathbf{P}_\ell \to \mathbf{S}_\ell \to 0$.

The notation for each pair of modules $\mathbf{M}^n_\ell(\lambda)$ and $\mathbf{M}^n_{\bar{\ell}}(\lambda)$ for $n \geq 2$ has not been fixed yet. We do this now by declaring that $\mathbf{M}^n_\ell(\lambda)$ is the module whose head is isomorphic to $\mathbf{S}^{\oplus n}_\ell$ (rather than $\mathbf{S}^{\oplus n}_{\bar{\ell}}$), which then conforms with the example (2) above.

Remark. A word about not necessarily finite-dimensional $\mathbb{B}_\ell$-modules: surely, having in mind the diagrams (5.2) or (5.3) for the modules $\Omega^{\pm n}\mathbf{S}_\ell$, we now can construct $\mathbb{B}_\ell$-modules whose number of composition factors has any prescribed cardinality.

Having the minimal resolution of the trivial module $\mathbf{S}_0$ [(5.2) for $\ell = 0$] we may consider the cohomology ring

$$H^\bullet(\mathfrak{U}_q(\mathfrak{sl}_2)) := \bigoplus_{n=0}^{\infty} \mathrm{Ext}^n_{\mathfrak{U}_q(\mathfrak{sl}_2)}(\mathbf{S}_0, \mathbf{S}_0)\,.$$

We may also splice (5.2) and (5.3) to get a complete resolution of $\mathbf{S}_0$ and consider something like the Tate cohomology. To fill in the details might be an interesting exercise for the reader.

The Auslander-Reiten Quiver. An appealing way of organising the set of isomorphism classes of indecomposable finite-dimensional $\mathbb{B}_\ell$-modules is to locate them in the Auslander-Reiten quiver (cf. Sect. A). We shall thus construct the Auslander-Reiten quiver of $\mathbb{B}_\ell$.

Theorem 5.1. *The components of the Auslander-Reiten quiver of $\mathbb{B}_\ell$ containing the PIMs have the form*

$$\Gamma_\ell := \qquad \begin{array}{c} [\mathbf{P}_\ell] \\ \nearrow \qquad \searrow \\ \ldots \rightrightarrows [\Omega^2\mathbf{S}_{\bar{\ell}}] \rightrightarrows [\Omega^1\mathbf{S}_\ell] \rightrightarrows [\mathbf{S}_{\bar{\ell}}] \rightrightarrows [\Omega^{-1}\mathbf{S}_\ell] \rightrightarrows [\Omega^{-2}\mathbf{S}_{\bar{\ell}}] \rightrightarrows \ldots \end{array}$$

$$(\Gamma_{\bar{\ell}} =) \qquad \begin{array}{c} [\mathbf{P}_{\bar{\ell}}] \\ \nearrow \qquad \searrow \\ \ldots \rightrightarrows [\Omega^2\mathbf{S}_\ell] \rightrightarrows [\Omega^1\mathbf{S}_{\bar{\ell}}] \rightrightarrows [\mathbf{S}_\ell] \rightrightarrows [\Omega^{-1}\mathbf{S}_{\bar{\ell}}] \rightrightarrows [\Omega^{-2}\mathbf{S}_\ell] \rightrightarrows \ldots \end{array}$$

The translation τ shifts the stable part of these quiver components two places to the left.

Proof. The Auslander-Reiten sequence from Proposition A.6 for $P = \mathbf{P}_\ell$ reads (cf. Corollary 3.10)

$$0 \longrightarrow \Omega^1\mathbf{S}_\ell \longrightarrow \mathbf{S}_{\bar\ell} \oplus \mathbf{S}_{\bar\ell} \oplus \mathbf{P}_\ell \longrightarrow \Omega^{-1}\mathbf{S}_\ell \longrightarrow 0\ .$$

The Auslander-Reiten sequence with cokernel term $\mathbf{S}_{\bar\ell}$ has $D\mathrm{Tr}\,\mathbf{S}_{\bar\ell} \cong \Omega^2\mathbf{S}_{\bar\ell}$ as kernel term. The last isomorphism follows by the remark after the proof of Theorem A.2. Propositions A.4 and A.5 help us concoct an Auslander-Reiten sequence having the form

$$0 \longrightarrow \Omega^2\mathbf{S}_{\bar\ell} \longrightarrow \Omega^1\mathbf{S}_\ell \oplus \Omega^1\mathbf{S}_\ell \oplus E' \longrightarrow \mathbf{S}_{\bar\ell} \longrightarrow 0\,.$$

Counting dimensions (or composition factors), we see that $E' = 0$. Induction shows that the "left half" of the component of the quiver $\Gamma_{\mathbb{B}_\ell}$ containing the vertex $[\mathbf{P}_\ell]$ is as given in the theorem. Similarly, starting with

$$0 \longrightarrow \mathbf{S}_{\bar\ell} \longrightarrow \Omega^{-1}\mathbf{S}_\ell \oplus \Omega^{-1}\mathbf{S}_\ell \oplus E'' \longrightarrow \mathrm{Tr}D\mathbf{S}_{\bar\ell} = \Omega^{-2}\mathbf{S}_{\bar\ell} \longrightarrow 0\ ,$$

we obtain the remaining part of the quiver component. We also get the component containing $[\mathbf{P}_{\bar\ell}]$.

An alternative is suggested in [AR, pp. 15–16]. □

In order to derive the components of the Auslander-Reiten quiver containing the vertices $[\mathbf{M}_\ell^n(\lambda)]$ or $[\mathbf{M}_{\bar\ell}^n(\lambda)]$, we shall calculate within the algebra $\mathcal{B}$, which is isomorphic to the basic algebra $\mathcal{B}^{\mathrm{op}}$ of $\mathbb{B}_\ell$. So let $\mathcal{M}_\ell^n(\lambda)$ [resp. $\mathcal{M}_{\bar\ell}^n(\lambda), \mathcal{P}_\ell, \mathcal{P}_{\bar\ell}$] be $\mathcal{B}$-modules (determined up to isomorphism) which correspond to the $\mathbb{B}_\ell$-modules $\mathbf{M}_\ell^n(\lambda)$ [resp. $\mathbf{M}_{\bar\ell}^n(\lambda), \mathbf{P}_\ell, \mathbf{P}_{\bar\ell}$].

We compute a minimal projective presentation for $\mathcal{M}_\ell^n(\lambda)$, $\lambda \in \mathbb{C}$. With respect to a suitable basis for that $2n$-dimensional $\mathcal{B}$-module we have

$$\bar{\mathsf{Z}}\mathsf{Z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathsf{E}+\bar{\mathsf{E}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{F}+\bar{\mathsf{F}} = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \qquad \text{on } \mathcal{M}_\ell^n(\lambda),$$

where, here and later, $1 := \mathbf{1}_n$, and $J := \mathbf{J}_n(\lambda)$ is an $n\times n$ Jordan block matrix with eigenvalue λ. We may choose a basis for $\mathcal{P}_\ell^{\oplus n}$ such that

$$\bar{\mathsf{Z}}\mathsf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{E}+\bar{\mathsf{E}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{F}+\bar{\mathsf{F}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{on } \mathcal{P}_\ell^{\oplus n}.$$

The matrix $\begin{pmatrix} 0 & 1 & J & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is then the matrix of a $\mathcal{B}$-homomorphism $\mathcal{P}_\ell^{\oplus n} \twoheadrightarrow \mathcal{M}_\ell^n(\lambda)$, which is already a projective cover. Its kernel, $\Omega^1\mathcal{M}_\ell^n(\lambda)$, embeds into $\mathcal{P}_\ell^{\oplus n}$ by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -J \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, so that

$$\bar{\mathsf{Z}}\mathsf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{E}+\bar{\mathsf{E}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{F}+\bar{\mathsf{F}} = \begin{pmatrix} 0 & -J \\ 0 & 0 \end{pmatrix} \qquad \text{on } \Omega^1\mathcal{M}_\ell^n(\lambda).$$

Since $\mathbf{J}_n(-\lambda)$ is the Jordan canonical form of $-J = -\mathbf{J}_n(\lambda)$, we obtain the isomorphism $\Omega^1\mathcal{M}_{\bar{\ell}}^n(\lambda) \cong \mathcal{M}_{\bar{\ell}}^n(-\lambda)$. At the same time, we get $\Omega^1\mathcal{M}_{\bar{\ell}}^n(-\lambda) \cong \mathcal{M}_{\ell}^n(\lambda)$ because $\mathbb{B}_\ell = \mathbb{B}_{\bar{\ell}}$. In particular, we have $\Omega^2\mathcal{M}_{\ell}^n(\lambda) \cong \mathcal{M}_{\ell}^n(\lambda)$.

But let us directly compute a projective cover of $\Omega^1\mathcal{M}_{\ell}^n(\lambda)$. We may choose a basis for $\mathcal{P}_{\bar{\ell}}^{\oplus n}$ such that

$$\bar{\mathsf{Z}}\mathsf{Z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{E}+\bar{\mathsf{E}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{F}+\bar{\mathsf{F}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{on } \mathcal{P}_{\bar{\ell}}^{\oplus n}.$$

The matrix $\left(\begin{smallmatrix} 0 & 1-J & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$ describes the projective cover $\mathcal{P}_{\bar{\ell}}^{\oplus n} \twoheadrightarrow \Omega^1\mathcal{M}_{\ell}^n(\lambda)$.

To sum up, we have the minimal projective presentation of $\mathcal{M}_{\ell}^n(\lambda)$

$$\mathcal{P}_{\bar{\ell}}^{\oplus n} \xrightarrow{\begin{pmatrix} 0 & 1 & -J & 0 \\ 0 & 0 & 0 & -J \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \mathcal{P}_{\ell}^{\oplus n} \xrightarrow{\begin{pmatrix} 0 & 1 & J & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \mathcal{M}_{\ell}^n(\lambda)\,.$$

Now we look closer at the middle term of the Auslander-Reiten sequence with cokernel term $\mathcal{M}_{\ell}^n(\lambda)$. In order to do so, we simply use the pullback diagram which we considered while proving Theorem A.2 – the existence of Auslander-Reiten sequences. The same bases for the modules $\mathcal{M}_{\ell}^n(\lambda), \mathcal{P}_{\ell}^{\oplus n}, \mathcal{P}_{\bar{\ell}}^{\oplus n}$ as above are used. The pullback diagram is (cf. the remark after the proof of Theorem A.2)

$$\begin{tikzcd} \mathcal{P}_{\bar{\ell}}^{\oplus n} \arrow[r, "\begin{pmatrix} 0 & 1 & -J & 0 \\ 0 & 0 & 0 & -J \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}"] & \mathcal{P}_{\ell}^{\oplus n} \\ E \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{pullback}"] & \mathcal{M}_{\ell}^n(\lambda) \arrow[u] \end{tikzcd} \qquad \theta_{\mathcal{M}_{\ell}^n(\lambda)} = \begin{pmatrix} 0 & A \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It suffices to know that the morphism $\theta_{\mathcal{M}_{\ell}^n(\lambda)}$ has a matrix of the form as indicated in the diagram above. This form is dictated by the fact that $\theta_{\mathcal{M}_{\ell}^n(\lambda)}$ is a $\mathcal{B}$-homomorphism. The pullback E is the submodule of the direct product of $\mathcal{P}_{\bar{\ell}}^{\oplus n}$ with $\mathcal{M}_{\ell}^n(\lambda)$ consisting of those elements that have the same image in $\mathcal{P}_{\ell}^{\oplus n}$. The result reads, w. r. t. a suitable basis,

$$\bar{\mathsf{Z}}\mathsf{Z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{E} + \bar{\mathsf{E}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathsf{F} + \bar{\mathsf{F}} = \begin{pmatrix} 0 & 0 & J & A \\ 0 & 0 & 0 & J \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{on } E.$$

Since the geometric multiplicity of the eigenvalue of the matrix $\begin{pmatrix} J & A \\ 0 & J \end{pmatrix}$ can not exceed two, we see that one of the following alternatives holds:

$$\begin{aligned} &E \cong \mathcal{M}_\ell^{2n}(\lambda), \\ &E \cong \mathcal{M}_\ell^{n-k}(\lambda) \oplus \mathcal{M}_\ell^{n+k}(\lambda) \text{ for some } k,\ 0 \le k < n. \end{aligned} \tag{5.4}$$

Note that A is not the zero matrix, otherwise $\iota_{\mathcal{M}_\ell^n(\lambda)}^{(\mathcal{M}_\ell^n(\lambda))} \circ \pi_{\mathcal{M}_\ell^n(\lambda)}^{(\mathcal{M}_\ell^n(\lambda))}(\mathrm{id}_{\mathcal{M}_\ell^n(\lambda)})$ would be zero, which is absurd. Thus, we get an Auslander-Reiten sequence

$$0 \to D\mathrm{Tr}\,\mathcal{M}_\ell^1(\lambda) \cong \Omega^2\mathcal{M}_\ell^1(\lambda) \cong \mathcal{M}_\ell^1(\lambda) \to \mathcal{M}_\ell^2(\lambda) \to \mathcal{M}_\ell^1(\lambda) \to 0.$$

Using Propositions A.4 and A.5, we know that the Auslander-Reiten sequence with cokernel term $\mathcal{M}_\ell^2(\lambda)$ looks like

$$0 \to \mathcal{M}_\ell^2(\lambda) \to \mathcal{M}_\ell^1(\lambda) \oplus E' \to \mathcal{M}_\ell^2(\lambda) \to 0.$$

By (5.4) we have $E' \cong \mathcal{M}_\ell^3(\lambda)$. We then inductively get Auslander-Reiten sequences of the form

$$0 \to \mathcal{M}_\ell^n(\lambda) \to \mathcal{M}_\ell^{n-1}(\lambda) \oplus \mathcal{M}_\ell^{n+1}(\lambda) \to \mathcal{M}_\ell^n(\lambda) \to 0$$

for $n \ge 2$.

Finally, I assert that the calculation above goes through for $\lambda = \infty$ mutatis mutandis.

Let us formulate the result again using the language of $\mathbb{B}_\ell$-modules.

Theorem 5.2. *For each $\lambda \in \mathbb{P}^1(\mathbb{C})$ we have two* 1-tubes *as components of the Auslander-Reiten quiver of $\mathbb{B}_\ell$, viz.*

$$\Gamma_{\ell,\lambda} := \quad [\mathbf{M}_\ell^1(\lambda)] \rightleftarrows [\mathbf{M}_\ell^2(\lambda)] \rightleftarrows [\mathbf{M}_\ell^3(\lambda)] \rightleftarrows \ldots$$

and

$$(\Gamma_{\bar{\ell},\lambda} =) \quad [\mathbf{M}_{\bar{\ell}}^1(\lambda)] \rightleftarrows [\mathbf{M}_{\bar{\ell}}^2(\lambda)] \rightleftarrows [\mathbf{M}_{\bar{\ell}}^3(\lambda)] \rightleftarrows \ldots.$$

The translation τ act as the identity.

Returning to the whole algebra $\mathfrak{U}_q(\mathfrak{sl}_2)$, that is, letting vary ℓ, we summarise the main issue of the present section in the theorem below.

Theorem 5.3. *(cf. Theorems 5.1 and 5.2).*
The Auslander-Reiten quiver of $\mathfrak{U}_q(\mathfrak{sl}_2)$ is

$$\Gamma_{\mathfrak{U}_q(\mathfrak{sl}_2)} = \left(\dot{\bigcup_{\ell+1 \not\equiv 0 \,(\mathrm{mod}\, N)}} \left(\{\Gamma_\ell\} \,\dot{\cup}\, \dot{\bigcup_{\lambda \in \mathbb{P}^1(\mathbb{C})}} \{\Gamma_{\ell,\lambda}\} \right) \right) \dot{\cup} \{[\mathrm{St}]\} \,\dot{\cup}\, \{[\bar{\mathrm{St}}]\}.$$

A. Addendum: Auslander-Reiten Theory

The main statement of Auslander-Reiten theory is that the Auslander-Reiten quiver of a category with Auslander-Reiten sequences is *locally finite*.

An attempt at giving the necessary background in Auslander-Reiten theory (also known as the theory of almost split sequences) is the content of this section. The existence proof for Auslander-Reiten sequences is adapted from [Ga2].

Let Λ be a finite-dimensional associative k-algebra with 1, where k is an algebraically closed field.

Fun Λ is the category of all contravariant additive $\Big[$i.e., $M, N \in {}_\Lambda\mathbf{mod}$, $f, g \in \mathrm{Hom}_\Lambda(N, M)$, $F \in \mathbf{Fun}\,\Lambda \Longrightarrow F(f+g) = F(f) + F(g) \in \mathrm{Hom}\Big(F(M), F(N)\Big)\Big]$ functors from ${}_\Lambda\mathbf{mod}$ to **Ab**, the abelian category of abelian groups. Morphisms in **Fun** Λ are natural transformations, as usual. For $F \xrightarrow{\alpha} G$ to be a morphism in **Fun** Λ, the requirement is that for each $M \in {}_\Lambda\mathbf{mod}$ there is $F(M) \xrightarrow{\alpha_M} G(M)$ in **Ab** such that if $N \xrightarrow{h} M$ is a morphism in ${}_\Lambda\mathbf{mod}$, then the diagram

$$\begin{array}{ccc} F(N) & \xrightarrow{\alpha_N} & G(N) \\ {\scriptstyle F(h)}\uparrow & & \uparrow{\scriptstyle G(h)} \\ F(M) & \xrightarrow{\alpha_M} & G(M) \end{array}$$

be commutative. Each $F \in \mathbf{Fun}\,\Lambda$ takes values in ${}_k\mathbf{Mod}$, the category of (not necessarily finite-dimensional) k-vector spaces. **Fun** Λ is an abelian category. Here is the way how exactness is measured in **Fun** Λ: $F' \xrightarrow{\alpha} F \xrightarrow{\beta} F''$ is exact in **Fun** Λ if and only if $F'(M) \xrightarrow{\alpha_M} F(M) \xrightarrow{\beta_M} F''(M)$ is exact in **Ab** for every $M \in {}_\Lambda\mathbf{mod}$.

For $N, M \in {}_\Lambda\mathbf{mod}$ one puts $(N, M) := \mathrm{Hom}_\Lambda(N, M)$ in order to abbreviate the notation. With this notation we have $(\ \ , M) \in \mathbf{Fun}\,\Lambda$ for every $M \in {}_\Lambda\mathbf{mod}$.

The Yoneda lemma tells us that $\varphi \mapsto \varphi_M(\mathrm{id}_M)$ provides an isomorphism

$$\mathrm{Hom}_{\mathbf{Fun}\,\Lambda}\Big((\ \ , M), F\Big) \xrightarrow{\cong} F(M)\ ,$$

which is functorial in $M \in {}_\Lambda\mathbf{mod}$ and $F \in \mathbf{Fun}\,\Lambda$. Using the Yoneda lemma we see

$$\mathrm{Hom}_{\mathbf{Fun}\,\Lambda}\Big((\ \ , M),\ \ \Big) : \mathbf{Fun}\,\Lambda \to \mathbf{Ab}$$

is an exact functor, i.e., that $(\ \ , M)$ is a projective object in **Fun** Λ. A *finitely generated* object in **Fun** Λ is, by definition, an epimorphic image of some $(\ \ , M)$ with $M \in {}_\Lambda\mathbf{mod}$. Every finitely generated projective object in **Fun** Λ is isomorphic to $(\ \ , M)$ for some $M \in {}_\Lambda\mathbf{mod}$.

The radical $\mathrm{rad}\,F$ of $F \in \mathbf{Fun}\,\Lambda$ is the intersection of all maximal subfunctors of F. Here is a description of $\mathrm{rad}(\ \ , M)$ for $M \in {}_\Lambda\mathbf{mod}$: fix a decomposition $M = \bigoplus_{j=1}^{m} M_j$ with the M_j's indecomposable. Let $N \in {}_\Lambda\mathbf{mod}$, and decompose $N = \bigoplus_{i=1}^{n} N_i$ with the N_i's indecomposable. Any homomorphism $f \in (N, M)$ determines a matrix (f_{ji}), where $f_{ji} \in (N_i, M_j)$.

Lemma A.1. $f \in \mathrm{rad}(N, M) := \mathrm{rad}(\ \ , M)(N)$ *if and only if none of the* f_{ji}*'s is invertible.*

Proof. Put

$$E_M(N) := \left\{ f \in (N, M) \,\middle|\, \text{none of the } f_{ji}\text{'s is invertible} \right\} .$$

Hence if $N_i \ncong M_j$, then $f_{ji} \in (N_i, M_j)$ without any further restriction. But for $N_i \cong M_j$ we have $(N_i, M_j) \cong \mathrm{End}_\Lambda M_j$, which is a local ring (because M_j is indecomposable), and $\left\{ f_{ji} \in (N_i, M_j) \,\middle|\, f_{ji} \text{ is not invertible} \right\}$ corresponds to the maximal ideal of $\mathrm{End}_\Lambda M_j$. For $(N' \xrightarrow{g} N) \in (N', N)$ it follows that $f \in E_M(N)$ implies $f \circ g \in E_M(N')$. We thus see that $E_M \subseteq (\ \ , M)$ – and $E_M \subsetneqq (\ \ , M)$ unless $M = 0$. By construction $E_M \cong \bigoplus_{j=1}^{m} E_{M_j}$, so that it remains to be seen $E_M = \mathrm{rad}(\ \ , M)$ if $M \in {}_\Lambda\mathbf{mod}$ is indecomposable. So let M be indecomposable, and suppose $F \subsetneqq (\ \ , M)$. We shall show that $F \subseteq E_M$. Since the members of $\mathbf{Fun}\,\Lambda$ are additive by definition, it suffices to show that $F(N) \subseteq E_M(N)$ for every indecomposable $N \in {}_\Lambda\mathbf{mod}$. Let N be such, and let $n \in F(N) \subseteq (N, M)$. The element n corresponds, by the Yoneda lemma, to the natural transformation $(\ \ , N) \xrightarrow{\nu} F \subsetneqq (\ \ , M)$ with $\nu_N(\mathrm{id}_N) = n = (N, n)(\mathrm{id}_N)$, which shows that $n : N \to M$ is not invertible, whence $n \in E_M(N)$. $\square$

If $M \in {}_\Lambda\mathbf{mod}$ is indecomposable, the functor $S_M := (\ \ , M)/\mathrm{rad}(\ \ , M)$ is a simple object in $\mathbf{Fun}\,\Lambda$. Denote by $\pi^{(M)}$ the natural projection $(\ \ , M) \xrightarrow{\pi^{(M)}} S_M$.

Theorem A.2. *Let $M \in {}_\Lambda\mathbf{mod}$ be indecomposable. The simple functor S_M has a minimal projective resolution in $\mathbf{Fun}\,\Lambda$ of the form*

$$0 \longrightarrow (\ \ , M_2) \longrightarrow (\ \ , M_1) \longrightarrow (\ \ , M) \longrightarrow S_M \longrightarrow 0 .$$

Moreover, $M_2 = 0$ if M is projective, whereas for a nonprojective M, the module M_2 is indecomposable.

Proof. We first treat the case where M is a projective module. We then have $\mathrm{rad}(\ \ , M) = (\ \ , \mathrm{rad}\, M)$, considered as subfunctors of $(\ \ , M)$ ["$\supseteq$" is surely true; for "$\subseteq$" note that $N \xrightarrow{f} M$ not split epi implies $\mathrm{im}\, f \subseteq \mathrm{rad}\, M$]. Hence we have the minimal projective resolution

$$0 \longrightarrow (\ \ , \mathrm{rad}\, M) \longrightarrow (\ \ , M) \longrightarrow S_M \longrightarrow 0 .$$

Suppose now that M is not projective. There are several duality functors $\mathrm{Hom}_k(\ \ , k)$, namely, from ${}_k\mathbf{mod}$ to ${}_k\mathbf{mod}$, or from ${}_\Lambda\mathbf{mod}$ to $\mathbf{mod}_\Lambda$ (= the category of finite-dimensional right Λ-modules) and back, or also from the subcategory of $\mathbf{Fun}\,\Lambda^{\mathrm{op}}$ consisting of covariant additive functors ${}_\Lambda\mathbf{mod} \to {}_k\mathbf{mod}$ to the subcategory of $\mathbf{Fun}\,\Lambda$ consisting of contravariant additive functors ${}_\Lambda\mathbf{mod} \to {}_k\mathbf{mod}$ and back. We shall, by abuse of notation, denote all of these different functors by D and shall identify D^2 with the corresponding identity functor.

We define $\iota^{(M)} \in \mathrm{Hom}_{\mathbf{Fun}\,\Lambda}\Big(S_M, D(M,\ \ )\Big)$ by the requirement that $D\iota^{(M)}$ correspond to $\mathrm{id}_k \in DS_M(M) = Dk$ via the Yoneda isomorphism (for covariant functors). Clearly, $D\iota^{(M)}$ is an epimorphism, whence $\iota^{(M)}$ is a monomorphism.

We begin to construct the required resolution by deriving a (minimal) projective presentation of $D(M,\ \ )$ in $\mathbf{Fun}\,\Lambda$. Let

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a minimal projective presentation of M in ${}_\Lambda\mathbf{mod}$. For every $N \in {}_\Lambda\mathbf{mod}$ the sequence $0 \longrightarrow (M,N) \xrightarrow{(p_0,N)} (P_0,N) \xrightarrow{(p_1,N)} (P_1,N)$ is exact, and therefore $D(P_1,N) \xrightarrow{D(p_1,N)} D(P_0,N) \xrightarrow{D(p_0,N)} D(M,N) \longrightarrow 0$ is exact, which means that

$$D(P_1,\ \ ) \xrightarrow{D(p_1,\ \ )} D(P_0,\ \ ) \xrightarrow{D(p_0,\ \ )} D(M,\ \ ) \longrightarrow 0$$

is exact in $\mathbf{Fun}\,\Lambda$.

For the functor $\mathrm{Hom}_\Lambda(\ \ ,\Lambda) : {}_\Lambda\mathbf{mod} \to \mathbf{mod}_\Lambda$ we use the notation $P \mapsto \check{P}$ on objects and $f \mapsto {}^{\mathrm{t}}f$ on morphisms (cf. [DM]). The functorial morphism in $P, N \in {}_\Lambda\mathbf{mod}$

$$\begin{aligned} \check{P} \otimes_\Lambda N &\longrightarrow (P,N) \\ f \otimes n &\longmapsto \Big(p \mapsto f(p)\,n\Big) \end{aligned}$$

is an isomorphism under the assumption that P be projective. The adjunction isomorphism

$$\mathrm{Hom}_k(\check{P} \otimes_\Lambda N, k) \xrightarrow{\cong} \mathrm{Hom}_\Lambda\Big(N, \mathrm{Hom}_k(\check{P}, k)\Big)$$

gives us the functorial isomorphism $D(\check{P} \otimes_\Lambda\ \ ) \xrightarrow{\cong} (\ \ , D\check{P})$. Composition yields

$$\alpha^{(P)} : D(P,\ \ ) \to D(\check{P} \otimes_\Lambda\ \ ) \to (\ \ , D\check{P}),$$

which is functorial in P and is an isomorphism if P is projective.

We may construct the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & D(P_1,\ \ ) \arrow[r, "D(p_1,\ \ )"] \arrow[d, "\alpha^{(P_1)}"', "\cong"] \arrow[dr, phantom, "(1)"] & D(P_0,\ \ ) \arrow[r, "D(p_0,\ \ )"] \arrow[d, "\alpha^{(P_0)}"', "\cong"] & D(M,\ \ ) \arrow[r] & 0 \\
0 \arrow[r, "(2)"] & (\ \ ,\ker D{}^{\mathrm{t}}p_1) \arrow[r] & (\ \ , D\check{P}_1) \arrow[r, "(\ \ ,D{}^{\mathrm{t}}p_1)"] & (\ \ , D\check{P}_0) \arrow[dr, phantom, "(3)"] & & \\
0 \arrow[r] & (\ \ ,\ker D{}^{\mathrm{t}}p_1) \arrow[r] \arrow[u, equal] & (\ \ , D\check{P}_1 \times_{D\check{P}_0} M) \arrow[r] \arrow[u] \arrow[ur, phantom, "\text{(4) pullback}"] & (\ \ ,M) \arrow[r, "\pi^{(M)}"] \arrow[u, "(\ \ ,\theta_M)"'] & S_M \arrow[r] \arrow[uu, hook, "\iota^{(M)}"'] & 0
\end{tikzcd}$$

$$(5)$$

(1) Naturality of $\alpha^{(P)}$ in P.

(2) $\mathrm{Hom}_\Lambda(N,\ \ )$ is left exact for each $N \in {}_\Lambda\mathbf{mod}$.

(3) $D(p_0,\ \ ) \circ \Big(\alpha^{(P_0)}\Big)^{-1}$ is an epimorphism and $(\ \ ,M)$ is projective imply that $\iota^{(M)} \circ \pi^{(M)}$ factors through $(\ \ , D\check{P}_0)$, giving $M \xrightarrow{\theta_M} D\check{P}_0$ by the Yoneda lemma.

(4) The pullback in $\mathbf{Fun}\,\Lambda$ stems from the pullback in ${}_\Lambda\mathbf{mod}$. Firstly, let $N \in {}_\Lambda\mathbf{mod}$, and apply the functor $(N,\quad)$ to a pullback in ${}_\Lambda\mathbf{mod}$. The functor $(N,\quad)$ is a right adjoint functor (right adjoint to $N \otimes_k \quad$), so preserves arbitrary inverse limits and, in particular, pullbacks. Secondly, having the pullbacks from the first part for each $N \in {}_\Lambda\mathbf{mod}$, one simply checks that the universal property holds in order to get a pullback in $\mathbf{Fun}\,\Lambda$.
(5) Exactness at $(\quad, M)$ follows by a diagram chase – use that $\iota^{(M)}$ is a monomorphism for showing that it is a complex and the pullback for showing that we have $\ker \pi^{(M)} \subseteq \mathrm{im}\Big((\quad, D\check{P}_1 \times_{D\check{P}_0} M) \to (\quad, M)\Big)$.

It remains to be seen that the projective resolution got in the last row is minimal. To this end we show that $\ker D{}^t\!p_1$ is indecomposable.

We shall recall the operation of *transposing* a module (cf. [AB]). To a finite-dimensional left (resp. right) Λ-module M without nonzero projective direct summands one associates a finite-dimensional right (resp. left) Λ-module $\mathrm{Tr}\,M$ without nonzero projective direct summands in such a way that $\mathrm{Tr}\,\mathrm{Tr}\,M \cong M$ and $\mathrm{Tr}(M_1 \oplus M_2) \cong \mathrm{Tr}\,M_1 \oplus \mathrm{Tr}\,M_2$. It is constructed in the following way: from the minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \longrightarrow M \longrightarrow 0$ we chose above, one forms $\mathrm{Tr}\,M := \mathrm{coker}\,{}^t\!p_1$, so that from $\check{P}_0 \xrightarrow{{}^t\!p_1} \check{P}_1 \longrightarrow \mathrm{Tr}\,M \longrightarrow 0$ we get another exact sequence, $0 \longrightarrow D\mathrm{Tr}\,M \longrightarrow D\check{P}_1 \xrightarrow{D{}^t\!p_1} D\check{P}_0$, so that $D\mathrm{Tr}\,M = \ker D{}^t\!p_1$, which is thus indecomposable ($\mathrm{Tr}D\ker D{}^t\!p_1 \cong M$; note that $\mathrm{Tr}\,\mathrm{Tr}\,M \cong M$ holds because projective modules are reflexive). $\square$

Remark. For a symmetric algebra, the modules DM and $\check{M}$ are isomorphic; and, in particular, using the minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \longrightarrow M \longrightarrow 0$, we get

$$D\mathrm{Tr}\,M = \ker D{}^t\!p_1 \cong \ker p_1 = \Omega^2 M\,,$$

the second syzygy module of M.

Definition. *Let $M \in {}_\Lambda\mathbf{mod}$ be an indecomposable module which is not projective, and let*

$$0 \longrightarrow (\quad, N) \xrightarrow{(\ ,n)} (\quad, E) \xrightarrow{(\ ,m)} (\quad, M) \longrightarrow S_M \longrightarrow 0$$

be a minimal projective resolution of S_M in $\mathbf{Fun}\,\Lambda$. *Then*

$$0 \longrightarrow N \xrightarrow{n} E \xrightarrow{m} M \longrightarrow 0$$

is a short exact sequence and is termed an Auslander-Reiten sequence.

Exactness follows by inserting Λ. Since M is not projective, we have $S_M(\Lambda) = 0$ indeed.

Remark. The proof of existence of Auslander-Reiten sequences showed that the kernel term is $N \cong D\mathrm{Tr}\,M$ and hence that the cokernel term is $M \cong \mathrm{Tr}DN$.

Remark. Unicity of Auslander-Reiten sequences up to equivalence (of short exact sequences if we fix the end terms) follows from the unicity of projective resolutions.

Proposition A.3. *Let $M \in {}_\Lambda\mathbf{mod}$ be an indecomposable module which is not projective, and let*

$$0 \longrightarrow N \xrightarrow{n} E \xrightarrow{m} M \longrightarrow 0$$

be a short exact sequence which does not split. Then the following two statements are equivalent.

(i) *The sequence above is an Auslander-Reiten sequence.*

(ii) *N is indecomposable, and every morphism $x \in \operatorname{rad}(X, M)$ [i. e., $x : X \to M$ is not split epi] factorises via m (X running through $X \in {}_\Lambda\mathbf{mod}$).*

Proof. A morphism $X \to M$ which factorises via m is of course not split epi, that is, $\operatorname{rad}(X, M) \supseteq \operatorname{im}(X, m)$. The other inclusion, $\operatorname{rad}(X, M) \subseteq \operatorname{im}(X, m)$, is equivalent to each $x \in \operatorname{rad}(X, M)$ factoring via m.
(i) $\Longrightarrow$ (ii) We already know that N is indecomposable. Furthermore, we have $\operatorname{rad}(\ \ , M) = \ker \pi^{(M)} = \operatorname{im}(\ \ , m)$, which yields the assertion.
(ii) $\Longrightarrow$ (i) $\operatorname{rad}(\ \ , M) = \operatorname{im}(\ \ , m)$, by assumption. As $\ker \pi^{(M)} = \operatorname{rad}(\ \ , M)$, the sequence $0 \longrightarrow (\ \ , N) \xrightarrow{(\ \ ,n)} (\ \ , E) \xrightarrow{(\ \ ,m)} (\ \ , M) \longrightarrow S_M \longrightarrow 0$ is exact at $(\ \ , M)$, too. The minimality of this projective resolution follows because N is indecomposable. $\square$

Remark. The criterion given in the last proposition is what is usually taken for being the definition of Auslander-Reiten sequences. It explains the term "almost split sequence".

Remark. There is an analogous criterion for a short exact sequence

$$0 \longrightarrow N \xrightarrow{n} E \xrightarrow{m} M \longrightarrow 0 \tag{A.1}$$

to be an Auslander-Reiten sequence, dual to that of the previous proposition. Namely, (A.1) shall not split, the modules N and M shall be indecomposable, and every $y \in \operatorname{rad}(N, Y)$ [i.e., $y : N \to Y$ is not split mono] shall factorise via n (Y running through $Y \in {}_\Lambda\mathbf{mod}$).

Definition. *Let $X, Y \in {}_\Lambda\mathbf{mod}$. One calls $f \in (X, Y)$ an* irreducible morphism *if f is neither split mono nor split epi and if, in addition, for every factorisation through any $Z \in {}_\Lambda\mathbf{mod}$,*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y, \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

either g is split mono, or h is split epi.

Remark. By considering the factorisation $X \to \operatorname{im} f \hookrightarrow Y$ of f (or, alternatively, $X \twoheadrightarrow X/\ker f \to Y$), we see that an irreducible morphism is either mono or epi.

Remark. Let $X, Y \in {}_\Lambda\mathbf{mod}$ be indecomposable modules. The set of irreducible morphisms from X to Y is

$$\operatorname{rad}(X, Y) - \operatorname{rad}^2(X, Y) \ .$$

This follows from

$$\operatorname{rad}^2(X, Y) = \sum_{Z \in {}_\Lambda\mathbf{mod}} \underbrace{\operatorname{rad}(Z, Y)}_{\text{“not split epi”}} \circ \underbrace{\operatorname{rad}(X, Z)}_{\text{“not split mono”}} \ .$$

Definition. *Let $X, Y \in {}_\Lambda\mathbf{mod}$ be indecomposable modules. The* module of irreducible morphisms *from X to Y is defined as*

$$\mathrm{Irr}(X,Y) := \mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y) \quad .$$

The following two propositions connect the notion of an Auslander-Reiten sequence to the notion of irreducible morphisms. For proofs see [Rin2], for example.

Proposition A.4. *Let $0 \longrightarrow N \xrightarrow{n} E \xrightarrow{m} M \longrightarrow 0$ be an Auslander-Reiten sequence. Assume that the indecomposable modules $P, I \in {}_\Lambda\mathbf{mod}$ are projective and injective, respectively. Finally, let $0 \neq X \in {}_\Lambda\mathbf{mod}$. Then the irreducible morphisms*

$$\left\{\begin{array}{c} X \to M \\ X \to P \\ N \to X \\ I \to X \end{array}\right\} \text{ are exactly } \left\{\begin{array}{l} X \xrightarrow{\cong} E_X \hookrightarrow E_X \oplus E' = E \xrightarrow{m} M \\ X \xrightarrow{\cong} E_X \hookrightarrow E_X \oplus E' = \mathrm{rad}\, P \hookrightarrow P \\ N \xrightarrow{n} E = E_X \oplus E' \twoheadrightarrow E_X \xrightarrow{\cong} X \\ I \twoheadrightarrow I/\mathrm{soc}\, I = E_X \oplus E' \twoheadrightarrow E_X \xrightarrow{\cong} X \end{array}\right\}, \text{ where in}$$

these compositions $\hookrightarrow$ and $\twoheadrightarrow$ denote the respective natural inclusions and natural projections.

Proposition A.5.

(i) *Let*

$$0 \longrightarrow N \xrightarrow{\begin{pmatrix} n_1 \\ \vdots \\ n_d \\ n' \end{pmatrix}} Z^{\oplus d} \oplus E' \xrightarrow{(m_1 \ldots m_d \ m')} M \longrightarrow 0$$

be an Auslander-Reiten sequence with $Z \in {}_\Lambda\mathbf{mod}$ indecomposable and such that no direct summand of E' is isomorphic to Z. Then we have the equality $\dim_k \mathrm{Irr}(Z, M) = \dim_k \mathrm{Irr}(N, Z) = d$.

(ii) *Let $P \in {}_\Lambda\mathbf{mod}$ be an indecomposable projective module, and suppose*

$$0 \longrightarrow Z^{\oplus d} \oplus E' \xrightarrow{(m_1 \ldots m_d \ m')} P \twoheadrightarrow P/\mathrm{rad}\, P \longrightarrow 0$$

is a short exact sequence with $Z \in {}_\Lambda\mathbf{mod}$ indecomposable and that no direct summand of E' is isomorphic to Z. Then we have $\dim_k \mathrm{Irr}(Z, P) = d$.

(iii) *Let $I \in {}_\Lambda\mathbf{mod}$ be an indecomposable injective module, and suppose*

$$0 \longrightarrow \mathrm{soc}\, I \hookrightarrow I \xrightarrow{\begin{pmatrix} n_1 \\ \vdots \\ n_d \\ n' \end{pmatrix}} Z^{\oplus d} \oplus E' \longrightarrow 0$$

is a short exact sequence with $Z \in {}_\Lambda\mathbf{mod}$ indecomposable and that no direct summand of E' is isomorphic to Z. Then we have $\dim_k \mathrm{Irr}(I, Z) = d$.

Here, the residue classes (modulo the respective radical squares) of $n_1, \ldots, n_d$ or $m_1, \ldots, m_d$ form vector space bases for the respective modules of irreducible morphisms.

The next proposition describes the Auslander-Reiten sequence whose middle term contains the indecomposable module P as a direct summand, provided the module P is both projective and injective.

Proposition A.6. *Let $P \in {}_{\Lambda}\mathbf{mod}$ be an indecomposable module which is both projective and injective. Then there is an Auslander-Reiten sequence*

$$0 \longrightarrow \operatorname{rad} P \longrightarrow \operatorname{rad} P/\operatorname{soc} P \oplus P \longrightarrow P/\operatorname{soc} P \longrightarrow 0\,.$$

Definition. *The* Auslander-Reiten quiver Γ_{Λ} *of the algebra Λ is the quiver whose set of vertices is the set of isomorphism classes of all the indecomposable modules $X \in {}_{\Lambda}\mathbf{mod}$. The vertex corresponding to X is denoted by $[X]$. There is a* $\dim_k \operatorname{Irr}(X,Y)$*-fold arrow from $[X]$ to $[Y]$.*

If one deletes the vertices (and the arrows which begin or end there) $[I]$, $[D\mathrm{Tr}\, I]$, $[(D\mathrm{Tr})^2 I], \ldots$, where I is injective (and indecomposable), and the sequence stops if a projective module occurs, and one also deletes the vertices $[P]$, $[\mathrm{Tr}DP]$, $[(\mathrm{Tr}D)^2 P], \ldots$, where P is projective (and indecomposable), and the sequence stops if an injective module shows up, one obtains the largest subquiver of Γ_{Λ} for which $D\mathrm{Tr}$ induces an automorphism τ, the translation; this subquiver is termed the stable quiver. It is a quiver satisfying the axioms of a *stable representation quiver*. Such quivers were studied by Riedtmann [Ri], while she classified the representation-finite self-injective algebras over algebraically closed fields.

B. Appendix: $\mathfrak{U}_q(\mathfrak{sl}_2)$ at $q = \exp(\pi i/4)$

The following table displays the modules $\mathbf{P}_{\ell}$.

$\mathbf{P}_0$

$$\begin{array}{rcl}
 & & F^3\varphi_0 \\
 & & \downarrow\uparrow \\
 & & -\sqrt{2}F^2\varphi_0 + F^3\varphi_2 E \\
 & & \downarrow\uparrow \\
 & & 2F\varphi_0 - \sqrt{2}F^2\varphi_2 E + F^3\varphi_4 E^2 \\
 & \nearrow & \downarrow \\
2\varphi_0 - \sqrt{2}F\varphi_2 E + F^2\varphi_4 E^2 & & F^3\varphi_6 E^3 \\
\downarrow & \nearrow & \\
F^2\varphi_6 E^3 & & \\
\downarrow\uparrow & & \\
-\sqrt{2}F\varphi_6 E^3 & & \\
\downarrow\uparrow & & \\
2\varphi_6 E^3 & &
\end{array}$$

$\mathbf{P}_1$

$$\begin{array}{rcl}
 & & F^3\varphi_1 \\
 & & \downarrow\uparrow \\
 & & -F^2\varphi_1 + F^3\varphi_3 E \\
 & \nearrow & \downarrow \\
-F\varphi_1 + F^2\varphi_3 E & & F^3\varphi_5 E^2 \\
\downarrow\uparrow & \nearrow & \downarrow\uparrow \\
-\varphi_1 + F\varphi_3 E + F^2\varphi_5 E^2 & & F^2\varphi_5 E^2 + F^3\varphi_7 E^3 \\
\downarrow & \nearrow & \\
F\varphi_5 E^2 + F^2\varphi_7 E^3 & & \\
\downarrow\uparrow & & \\
-\varphi_5 E^2 - F\varphi_7 E^3 & &
\end{array}$$

$\mathbf{P}_2$

$$\begin{array}{rcl}
 & & F^3\varphi_2 \\
 & \nearrow & \downarrow \\
F^2\varphi_2 & & F^3\varphi_4 E \\
\downarrow\uparrow & \nearrow & \downarrow\uparrow \\
\sqrt{2}F\varphi_2 + F^2\varphi_4 E & & \sqrt{2}F^2\varphi_4 E + F^3\varphi_6 E^2 \\
\downarrow\uparrow & \nearrow & \downarrow\uparrow \\
2\varphi_2 + 2\sqrt{2}F\varphi_4 E + F^2\varphi_6 E^2 & & 2F\varphi_4 E + \sqrt{2}F^2\varphi_6 E^2 + F^3\varphi_0 E^3 \\
\downarrow & \nearrow & \\
2\varphi_4 E + \sqrt{2}F\varphi_6 E^2 + F^2\varphi_0 E^3 & &
\end{array}$$

$\mathbf{P}_3 = \mathrm{St}$

$$\begin{array}{l}
F^3\varphi_3 \\
\downarrow\uparrow \\
F^2\varphi_3 + F^3\varphi_5 E \\
\downarrow\uparrow \\
2F\varphi_3 + 2F^2\varphi_5 E + F^3\varphi_7 E^2 \\
\downarrow\uparrow \\
2\varphi_3 + 2F\varphi_5 E + F^2\varphi_7 E^2 + F^3\varphi_1 E^3
\end{array}$$

$$\mathbf{P}_4 = \overline{\mathbf{P}_0} = \mathbf{P}_{\bar{2}}$$

$$\begin{array}{ccl}
 & & F^3\varphi_4 \\
 & & \downarrow\uparrow \\
 & & \sqrt{2}F^2\varphi_4 + F^3\varphi_6E \\
 & & \downarrow\uparrow \\
 & & 2F\varphi_4 + \sqrt{2}F^2\varphi_6E + F^3\varphi_0E^2 \\
 & \nearrow & \downarrow \\
2\varphi_4 + \sqrt{2}F\varphi_6E + F^2\varphi_0E^2 & & F^3\varphi_2E^3 \\
\downarrow & \nearrow & \\
F^2\varphi_2E^3 & & \\
\downarrow\uparrow & & \\
\sqrt{2}F\varphi_2E^3 & & \\
\downarrow\uparrow & & \\
2\varphi_2E^3 & &
\end{array}$$

$$\mathbf{P}_5 = \overline{\mathbf{P}_1} = \mathbf{P}_{\bar{1}}$$

$$\begin{array}{ccl}
 & & F^3\varphi_5 \\
 & & \downarrow\uparrow \\
 & & F^2\varphi_5 + F^3\varphi_7E \\
 & \nearrow & \downarrow \\
F\varphi_5 + F^2\varphi_7E & & F^3\varphi_1E^2 \\
\downarrow\uparrow & \nearrow & \downarrow\uparrow \\
-\varphi_5 - F\varphi_7E + F^2\varphi_1E^2 & & -F^2\varphi_1E^2 + F^3\varphi_3E^3 \\
\downarrow & \nearrow & \\
-F\varphi_1E^2 + F^2\varphi_3E^3 & & \\
\downarrow\uparrow & & \\
-\varphi_1E^2 + F\varphi_3E^3 & &
\end{array}$$

$$\mathbf{P}_6 = \overline{\mathbf{P}_2} = \mathbf{P}_{\bar{0}}$$

$$\begin{array}{ccl}
 & & F^3\varphi_6 \\
 & \nearrow & \downarrow \\
F^2\varphi_6 & & F^3\varphi_0E \\
\downarrow\uparrow & \nearrow & \downarrow\uparrow \\
-\sqrt{2}F\varphi_6 + F^2\varphi_0E & & -\sqrt{2}F^2\varphi_0E + F^3\varphi_2E^2 \\
\downarrow\uparrow & \nearrow & \downarrow\uparrow \\
2\varphi_6 - 2\sqrt{2}F\varphi_0E + F^2\varphi_2E^2 & & 2F\varphi_0E - \sqrt{2}F^2\varphi_2E^2 + F^3\varphi_4E^3 \\
\downarrow & \nearrow & \\
2\varphi_0E - \sqrt{2}F\varphi_2E^2 + F^2\varphi_4E^3 & &
\end{array}$$

$$\mathbf{P}_7 = \overline{\mathrm{St}}$$

$$\begin{array}{l}
F^3\varphi_7 \\
\downarrow\uparrow \\
-F^2\varphi_7 + F^3\varphi_1E \\
\downarrow\uparrow \\
2F\varphi_7 - 2F^2\varphi_1E + F^3\varphi_3E^2 \\
\downarrow\uparrow \\
-2\varphi_7 + 2F\varphi_1E - F^2\varphi_3E^2 + F^3\varphi_5E^3
\end{array}$$

$\frac{1}{16}\,e_\ell^h$ are the primitive orthogonal idempotents such that $\mathfrak{U}_q(\mathfrak{sl}_2)e_\ell^h = \mathbf{P}_\ell E^h$.

$$e_0^0 = 2\varphi_0 - \sqrt{2}F\varphi_2E + F^2\varphi_4E^2 + \sqrt{2}F^3\varphi_6E^3$$

$$e_1^0 = 2\varphi_1 - 2F\varphi_3E - 2F^2\varphi_5E^2 \qquad e_1^1 = 2F\varphi_1E - 2F^2\varphi_3E^2$$

$$e_2^0 = 2\varphi_2 - F^2\varphi_6E^2 - \sqrt{2}F^3\varphi_0E^3 \qquad e_2^1 = \sqrt{2}F\varphi_2E - F^2\varphi_4E^2 - \sqrt{2}F^3\varphi_6E^3 \qquad e_2^2 = F^2\varphi_2E^2 - \sqrt{2}F^3\varphi_4E^3$$

$$e_3^0 = 2\varphi_3 + 2F\varphi_5E + F^2\varphi_7E^2 + F^3\varphi_1E^3 \qquad e_3^1 = 2F\varphi_3E + 2F^2\varphi_5E^2 + F^3\varphi_7E^3 \qquad e_3^2 = F^2\varphi_3E^2 + F^3\varphi_5E^3$$
$$e_3^3 = F^3\varphi_3E^3$$

$$e_4^0 = 2\varphi_4 + \sqrt{2}F\varphi_6E + F^2\varphi_0E^2 - \sqrt{2}F^3\varphi_2E^3$$

$$e_5^0 = 2\varphi_5 + 2F\varphi_7E - 2F^2\varphi_1E^2 \qquad e_5^1 = -2F\varphi_5E - 2F^2\varphi_7E^2$$

$$e_6^0 = 2\varphi_6 - F^2\varphi_2E^2 + \sqrt{2}F^3\varphi_4E^3 \qquad e_6^1 = -\sqrt{2}F\varphi_6E - F^2\varphi_0E^2 + \sqrt{2}F^3\varphi_2E^3 \qquad e_6^2 = F^2\varphi_6E^2 + \sqrt{2}F^3\varphi_0E^3$$

$$e_7^0 = 2\varphi_7 - 2F\varphi_1E + F^2\varphi_3E^2 - F^3\varphi_5E^3 \qquad e_7^1 = -2F\varphi_7E + 2F^2\varphi_1E^2 - F^3\varphi_3E^3 \qquad e_7^2 = F^2\varphi_7E^2 - F^3\varphi_1E^3$$
$$e_7^3 = -F^3\varphi_7E^3$$

Here are the block idempotents ($\mathfrak{U}_q(\mathfrak{sl}_2)b_x = b_x\mathfrak{U}_q(\mathfrak{sl}_2) = \mathbb{B}_x$):

$$\begin{aligned}
b_0 &= \frac{1}{16}\Big(2\varphi_0 + 2\varphi_6 - \sqrt{2}F\varphi_2E - \sqrt{2}F\varphi_6E - F^2\varphi_0E^2 - F^2\varphi_2E^2 + F^2\varphi_4E^2 + F^2\varphi_6E^2 \\
&\qquad + \sqrt{2}F^3\varphi_0E^3 + \sqrt{2}F^3\varphi_2E^3 + \sqrt{2}F^3\varphi_4E^3 + \sqrt{2}F^3\varphi_6E^3\Big) \\
b_1 &= \frac{1}{8}\Big(\varphi_1 + \varphi_5 + F\varphi_1E - F\varphi_3E - F\varphi_5E + F\varphi_7E - F^2\varphi_1E^2 - F^2\varphi_3E^2 - F^2\varphi_5E^2 - F^2\varphi_7E^2\Big) \\
b_2 &= \frac{1}{16}\Big(2\varphi_2 + 2\varphi_4 + \sqrt{2}F\varphi_2E + \sqrt{2}F\varphi_6E + F^2\varphi_0E^2 + F^2\varphi_2E^2 - F^2\varphi_4E^2 - F^2\varphi_6E^2 \\
&\qquad - \sqrt{2}F^3\varphi_0E^3 - \sqrt{2}F^3\varphi_2E^3 - \sqrt{2}F^3\varphi_4E^3 - \sqrt{2}F^3\varphi_6E^3\Big) \\
b_{\mathrm{St}} &= \frac{1}{16}\Big(2\varphi_3 + 2F\varphi_3E + 2F\varphi_5E + F^2\varphi_3E^2 + 2F^2\varphi_5E^2 + F^2\varphi_7E^2 + F^3\varphi_1E^3 + F^3\varphi_3E^3 + F^3\varphi_5E^3 + F^3\varphi_7E^3\Big) \\
b_{\overline{\mathrm{St}}} &= \frac{1}{16}\Big(2\varphi_7 - 2F\varphi_1E - 2F\varphi_7E + 2F^2\varphi_1E^2 + F^2\varphi_3E^2 + F^2\varphi_7E^2 - F^3\varphi_1E^3 - F^3\varphi_3E^3 - F^3\varphi_5E^3 - F^3\varphi_7E^3\Big)
\end{aligned}$$

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