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Static Spherically Symmetric Solutions of the Einstein-Yang-Mills Equations

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Abstract: We study the global behaviour of static, spherically symmetric solutions of the Einstein-Yang-Mills equations with gauge group $SU(2)$. Our analysis results in three disjoint classes of solutions with a regular origin or a horizon. The 3-spaces (t = const.) of the first, generic class are compact and singular. The second class consists of an infinite family of globally regular, resp. black hole solutions. The third type is an oscillating solution, which although regular is not asymptotically flat.

1. Introduction

The interest in the study of solutions of the coupled Einstein-Yang-Mills (EYM) equations has recently received considerable impetus by the discovery of a class of nonsingular ("particlelike") [1] and nonabelian black hole solutions [2, 3]. The results of Bartnik and McKinnon obtained by numerical integration indicate the existence of a discrete family of globally regular, static, spherically symmetric solutions of a $SU(2)$ Yang-Mills field coupled to gravity. The members of this family can be characterized by the number n of zeros of the gauge potential W parametrizing the spherically symmetric ansatz. For each value of $n \geq 0$ there seems to be exactly one regular solution.

The first rigorous existence proof of a globally regular solution with one zero was given in [4]. More recently it has been extended to both globally regular [5] and black hole solutions [6] with an arbitrary number of zeros.

In the present paper we classify the global behaviour of solutions regular at the origin $r = 0$ or with a horizon at some r_h and find in both cases three different classes. The first class contains the generic solutions describing singular space-times of the "bag of gold" type [7] with compact 3-spaces. Next there are the globally regular, resp. black hole solutions with an arbitrary n. Finally there are oscillating

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solutions which may be considered as limits of regular solutions when the number of zeros goes to infinity. This type of solution has not been considered previously. Geometrically it describes a non-singular globally hyperbolic space-time without boundary, which is not asymptotically flat. The analysis of this solution provides interesting asymptotic statements for regular solutions with a large number of zeros. Based on this classification we present an existence proof of globally regular and black hole solutions for any n as well as of the oscillating solutions.

In Sect. 2 we derive the basic equations together with the relevant boundary conditions. In Sect. 3 we present numerical results for solutions with $n \leq 10$ and establish some remarkable empirical relations for their parameters. In Sect. 4 we prove the local existence and analyticity of solutions at the singular points $r = 0$, $r = \infty$, and $r = r_h$. In Sect. 5 we demonstrate some useful properties of the Yang-Mills equations in flat space. In Sect. 6 we analyze the global behaviour leading to the classification of solutions. In Sect. 7 we prove the local existence of oscillating solutions and derive their asymptotic behaviour. In Sect. 8 we prove the global existence of regular solutions for any n as well as of the oscillating solution. In Sect. 9 we prove the existence of black hole solutions for any n and $r_h > 0$ as well as of the oscillating solutions. Finally in Sect. 10 we explain the numerically found behaviour of solutions for large n ($n \gtrsim 4$).

2. Ansatz and Field Equations

We are interested in static, spherically symmetric solutions of the EYM equations. In this case the metric tensor of this space–time can be parametrized as [8]

$$ds^2 = e^{2\nu(R)}dt^2 - e^{2\lambda(R)}dR^2 - r^2(R)d\Omega^2 , \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. As long as $\frac{dr}{dR} \neq 0$ the function $r(R)$ can be chosen as coordinate. This is true in particular near $r = 0$ if the origin is a regular point and near $r = \infty$ if the space is asymptotically flat. It turns out to be convenient to express the line element in the form

$$ds^2 = A^2(r)\mu(r)dt^2 - \frac{dr^2}{\mu(r)} - r^2 d\Omega^2 . \tag{2}$$

The most general static, spherically symmetric ansatz for the $SU(2)$ Yang-Mills field W^a_μ can be written (in the Abelian gauge) as [9]

$$W^a_t = (0, 0, A_0) , \quad W^a_\theta = (\phi_1, \phi_2, 0) , \tag{3a}$$
$$W^a_r = (0, 0, A_1) , \quad W^a_\varphi = (-\phi_2 \sin\theta, \phi_1 \sin\theta, \cos\theta) . \tag{3b}$$

Clearly the above ansatz (3) is form invariant under gauge transformations around the third isoaxis, with A_0, A_1 transforming as $U(1)$ gauge fields and (ϕ_1, ϕ_2) as a doublet. The reduced EYM action can be explicitly written as $\left(\text{where a prime denotes } \frac{d}{dr}\right)$:

$$S = -\int dr A(r)\Big[\frac{1}{2G}(\mu + r\mu' - 1) + \frac{1}{g^2}\Big(-\frac{r^2}{2A^2}A_0'^2 + \mu[(\phi_1' + A_1\phi_2)^2 + (\phi_2' - A_1\phi_1)^2] - \frac{A_0^2}{\mu A^2}(\phi_1^2 + \phi_2^2) + \frac{(1 - \phi_1^2 - \phi_2^2)^2}{2r^2}\Big)\Big] ,$$

and G, resp. g denote the gravitational, resp. gauge coupling constant. By exploiting the $U(1)$ gauge invariance of the ansatz (3) one can set e. g. $\phi_2 = 0$. Then it follows from the field equations that $A_1(r) = 0$ (assuming that $\phi_1 \neq 0$). This way one is left with only two functions, A_0 and ϕ_1. It is known [10] that there are no regular asymptotically flat solutions, nor black holes with $A_0 \neq 0$ apart from the Reissner–Nordstrøm (RN) ones. Therefore we shall restrict ourselves to the case $A_0 \equiv 0$. The EYM action for this simplified ansatz yields (with $W \equiv \phi_1$):

$$S = -\int dr A(r)\Big[\frac{1}{2G}(\mu + r\mu' - 1) + \frac{1}{g^2}\Big(\mu W'^2 + V\Big)\Big] , \tag{4}$$

where

$$V = \frac{(1 - W^2)^2}{2r^2} . \tag{5}$$

Rescaling the variable $r \longrightarrow \dfrac{r\sqrt{G}}{g}$ and $S \rightarrow g\sqrt{G}S$ removes the dependence on G and g from S. The field equations derived from the resulting action are

$$(\mu W')' = \frac{W(W^2 - 1)}{r^2} - \frac{2\mu W'^3}{r} , \tag{6a}$$

$$\mu' = \frac{1}{r}\Big(1 - \mu - 2(\mu W'^2 + V)\Big) , \tag{6b}$$

$$A^{-1}A' = \frac{2W'^2}{r} . \tag{6c}$$

Note that Eqs. (6a b) are decoupled from Eq. (6c) for the metric function $A(r)$.

Sometimes it is convenient to express μ in terms of the "mass" function $m(r)$ through $\mu(r) = 1 - 2m(r)/r$. In the present context $m(r)$ represents the mass contained in a sphere with radius r. For asymptotically flat spacetimes it tends to the total mass M of the solution. The function m obeys the equation

$$m' = \mu W'^2 + \frac{(W^2 - 1)^2}{2r^2} , \tag{7}$$

and hence increases monotonously as long as $\mu \geq 0$.

The field equations (6) have obvious singular points at $r = 0$ and $r = \infty$ as well as for points r_0, where $\mu(r_0) = 0$. Regularity at $r = 0$ of the configuration desribed by Eqs. (2, 3) requires $\mu(r) = 1 + O(r^2)$, $W(r) = \pm 1 + O(r^2)$, and $A(r) = A(0) + O(r^2)$. Since $W \longrightarrow -W$ can be achieved by a gauge transformation we choose $W(0) = 1$. Similarly we assume $A(0) = 1$ since a rescaling of A corresponds to an irrelevant rescaling of the time coordinate. Inserting a power series expansion into Eq. (6) one finds

$$\begin{aligned} W(r) &= 1 - br^2 + O(r^4) , \\ \mu(r) &= 1 - 4b^2r^2 + O(r^4) , \\ A(r) &= 1 + 4b^2r^2 + O(r^4) , \end{aligned} \tag{8}$$

where b is an arbitrary parameter.

Similarly assuming a power series expansion in $\frac{1}{r}$ at $r = \infty$ for asymptotically flat solutions one finds $\lim_{r\to\infty} W(r) = \{\pm 1, 0\}$. It will be shown in Sect. 6 that $W(\infty) = 0$

cannot occur for globally regular solutions and therefore we ignore this case for the moment. For the remaining cases one finds

$$\begin{aligned} W(r) &= \pm\left(1 - \frac{c}{r} + O\left(\frac{1}{r^2}\right)\right), \\ \mu(r) &= 1 - \frac{2M}{r} + O\left(\frac{1}{r^4}\right), \\ A(r) &= A_\infty + O\left(\frac{1}{r^4}\right), \end{aligned} \tag{9}$$

where M is the mass of the solution in units of $gM_{\mathrm{Pl}} = \frac{g}{\sqrt{G}}$.

Solutions of Eq. (6) stay regular at a point r_h, where $\mu(r_h) = 0$ if

$$\begin{aligned} \mu(r_h + \rho) &= \mu'_h\,\rho + O(\rho^2)\,, \\ W(r_h + \rho) &= W_h + W'_h\,\rho + O(\rho^2)\,, \end{aligned} \tag{10}$$

with

$$\begin{aligned} \mu'_h &= \frac{1}{r_h}\left(1 - \frac{(W_h^2 - 1)^2}{r_h^2}\right), \\ W'_h &= \frac{W_h(W_h^2 - 1)}{\mu'_h r_h^2}\,. \end{aligned} \tag{11}$$

For r_h fixed there is one adjustable parameter W_h analogous to the parameter b at $r = 0$. If r_h and W_h are chosen such that $\mu'_h > 0$, the surface $r = r_h$ describes a regular event horizon. Asymptotically flat solutions with this behaviour are black holes and we will therefore refer to Eqs. (10, 11) as "black hole boundary conditions."

3. Numerical Results

In this chapter we give a short overview of some results obtained by numerical integration of Eq. (6). By using a suitably desingularized version of Eq. (6) (see Eq. (22)) we were able to start the numerical integration (an adaptive step size Runge-Kutta method) exactly at $r = 0$.

Our numerical analysis clearly indicates that the generic solution with the initial conditions (8) at $r = 0$ develops a singularity at some point $r_0(b)$, where μ tends to zero, $W(r_0)$ stays finite but $W'(r_0)$ diverges. If $b \ll 1$ then $W(r_0) < -1$ and $r_0 \gg 1$, on the other hand if $b \gg 1$ then $W(r_0) \approx 1$ and $r_0 \ll 1$. To obtain a regular solution this singularity has to be avoided by a suitable "tuning" of the parameter b. As the value of b increases from 0 to ≈ 0.24, the position of the singular point, $r_0(b)$, moves inwards up to a turning point, $r_t \approx 10$ and then starts to move out without any apparent limit as b increases further up to $b_1 \approx 0.45$. Fine tuning the parameter b allows to increase the region where W stays close to -1 before W starts to run away towards large negative, resp. positive values for $b < b_1$, resp. $b > b_1$, eventually leading to a zero of μ. The value b_1 corresponds to the globally regular solution with one zero of W found by Bartnik and McKinnon [1]. Then as b grows from b_1 to $b_2 \approx 0.65$ the same phenomenon takes place with a turning point $r_t \approx 70$, except that now W has two zeros and $W(r_0) > 1$. For b_2 one obtains another globally regular solution with two zeros of W. This behaviour of $r_0(b)$ repeats itself in the intervals

(b_n, b_{n+1}) with r_t rapidly increasing and b_n accumulating at $b_\infty \approx 0.706$. For $b > b_\infty$ the solution becomes singular already for $r_0 < 1$.

The minimum of the function $\mu_n(r)$ decreases rapidly (and moves towards $r \approx 1$) as n increases, making the numerical integration more and more difficult. The mass, M_n, starts at $M_1 \approx 0.83$ and converges rapidly to 1 with increasing n. Furthermore some of the zeros $r_{n,k}$ of $W_n(r)$ accumulate around $r = 1$, while the outermost zero at $r_{n,n}$ moves further and further out. For large n one can clearly distinguish three regions: the inner region $r \leq 1$, where $W_n(r)$ for increasing n approaches a non-trivial limit $W_\infty(r)$ with zeros accumulating at $r = 1$; the small-field region $1 \leq r \ll r_{n,n}$, where $W_n(r)$ is small and oscillates; and the asymptotic region $r \gg 1$, where $\mu_n(r) \approx 1$ and $(-1)^n W_n(r)$ is a universal (n independent) function of $\frac{r}{c_n}$, with c_n being the coefficient in the expansion (9). Since the zeros of W_n accumulate at $r = 1$ for $n \gg 1$ an essential singularity of the limiting solution, $W_\infty(r)$ is likely to occur. In Sects. 7 and 8 we shall prove the existence of such a limiting solution and analyze some of its properties. For $r > 1$ this limiting solution seems to be $W_\infty \equiv 0$, corresponding to the extremal RN black hole,

$$W(r) \equiv 0\,, \qquad A(r) \equiv 1\,, \qquad \mu(r) = (1 - \frac{1}{r})^2\,. \tag{12}$$

In the following we present some data of the numerical solutions and exhibit some of their remarkable properties for large values of n. The last entries in Tables 1 and 2 correspond to the limiting solution $(W_\infty(r),\ \mu_\infty(r))$.

Table 1. Parameters of the globally regular solutions (W_n, μ_n)

n	b_n	M_n	c_n	$\mu_{\min}$
1	0.45371627277	0.82864698216	0.893382	0.2424
2	0.65172552552	0.97134549426	8.86389	0.03506
3	0.69704005033	0.99531647219	$5.89323 \cdot 10^1$	0.002974
4	0.70487847794	0.99923619279	$3.66335 \cdot 10^2$	0.0002530
5	0.70616866087	0.99987546806	$2.25189 \cdot 10^3$	$5.6090 \cdot 10^{-5}$
6	0.70637932997	0.99997969696	$1.38174 \cdot 10^4$	$3.3251 \cdot 10^{-5}$
7	0.70641368476	0.99999668991	$8.47562 \cdot 10^4$	$2.3565 \cdot 10^{-6}$
8	0.70641928597	0.99999946034	$5.19852 \cdot 10^5$	$1.8580 \cdot 10^{-7}$
9	0.70642019917	0.99999991201	$3.18867 \cdot 10^6$	$4.0140 \cdot 10^{-8}$
10	0.70642034805	0.99999998565	$1.95572 \cdot 10^7$	$2.3651 \cdot 10^{-8}$
∞	0.70642037705	1	–	0

From Tables 1 and 2 one can already deduce some features of the behaviour of the parameters b_n, M_n, c_n, and the location of the zeros $r_{n,k}$ of W_n for large n. Let us define

$$\Delta b_n = \frac{b_\infty - b_n}{b_\infty - b_{n+1}}\,, \quad \Delta M_n = \frac{1 - M_n}{1 - M_{n+1}}\,,$$
$$\Delta c_n = \frac{c_{n+1}}{c_n}\,, \qquad \Delta r_n = \frac{r_{n+1,n+1}}{r_{n,n}}\,, \tag{13}$$

and list these Δ's in Table 3.

Table 2. Location of the zeros $r_{n,k}$ of W_n

n	Zeros of W_n
1	1.54567
2	1.10010 3.69573
3	0.98983 1.66635 14.1647
4	0.96900 1.15086 3.67658 77.0184
5	0.96551 1.02679 1.66962 14.0019 462.179
6	0.96493 1.00369 1.15222 3.67610 76.0503 2824.58
7	0.96484 0.99982 1.02778 1.66970 13.9977 456.281 17314.8
8	0.96482 0.99919 1.00461 1.15225 3.67609 76.0251 2788.44 106194.
9	0.96482 0.99909 1.00073 1.02780 1.66971 13.9975 456.127 17093.2 651351.
10	0.96482 0.99907 1.00010 1.00464 1.15225 3.67609 76.0244 2787.50 104834. 3995185.
∞	0.96482 0.99907 0.99998 ...

Table 3. Quotients of parameters for consecutive solutions

n	Δb_n	ΔM_n	Δc_n	Δr_n
1	4.6203	5.9800	9.9217	2.3910
2	5.8308	6.1181	6.6486	3.8327
3	6.0836	6.1318	6.2162	5.4373
4	6.1255	6.1334	6.1471	6.0009
5	6.1324	6.1337	6.1359	6.1114
6	6.1335	6.1337	6.1340	6.1301
7	6.1336	6.1336	6.1335	6.1331
8	6.1335	6.1333	6.1338	6.1336
9	6.1325	6.1315	6.1333	6.1337

On the basis of Table 3 we establish the following empirical asymptotic formulae for b_n, M_n, and c_n:

$$b_n = b_\infty - 2.186 \cdot e^{-n\alpha}\,, \quad M_n = 1 - 1.081 \cdot e^{-n\alpha}\,, \quad c_n = 0.2595 \cdot e^{n\alpha}\,, \tag{14}$$

where $\alpha \approx 1.814$, resp. $e^\alpha \approx 6.134$. We should like to mention at this point that a somewhat different empirical mass-formula was given in Ref. [1]. These empirical formulae and the value $\alpha = \frac{\pi}{\sqrt{3}} \approx 1.8138$ will be explained in Sect. 10.

4. Local Existence

In Sect. 2 we have stated necessary conditions for solutions regular at the singular points $r = 0$, $r = \infty$, and $\mu = 0$. Here we shall prove the local existence and analyticity of such solutions. For $r = 0$ this has been already proven in ref. [4], but we feel that it is worth to present our proof since it is an application of the standard technique to linearize differential equations at a singular point [11, 12]. A number of general theorems allows to relate the asymptotic behaviour of solutions near the singular point to that of the linearized system. The regular solutions lie on the "stable manifold." Below we give a version of such a theorem taking into account some special properties of Eq. (6). It permits to parametrize the family of regular solutions by parameters determined at the singular point and yields the analyticity of the solutions in r and these parameters.

Proposition 1. *Consider a system of differential equations for $n + m$ functions $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$,*

$$t\frac{du_i}{dt} = t^{\mu_i} f_i(t, u, v) , \qquad t\frac{dv_i}{dt} = -\lambda_i v_i + t^{\nu_i} g_i(t, u, v) , \tag{15}$$

with constants $\lambda_i > 0$ and integers $\mu_i, \nu_i \geq 1$ and let $\mathscr{C}$ be an open subset of R^n such that the functions f and g are analytic in a neighbourhood of $t = 0$, $u = c$, $v = 0$ for all $c \in \mathscr{C}$. Then there exists an n-parameter family of solutions of the system (15) *such that*

$$u_i(t) = c_i + O(t^{\mu_i}) , \qquad v_i(t) = O(t^{\nu_i}) , \tag{16}$$

where $u_i(t)$ and $v_i(t)$ are defined for $c \in \mathscr{C}$, $|t| < t_0(c)$ and are analytic in t and c.

Proof. Following the standard procedure [11] p. 330ff, we convert the differential Eqs. (15) with boundary conditions (16) into integral equations and seek a solution through successive approximations. Starting with $u^{(0)} = c$, $v^{(0)} = 0$ we define

$$\begin{aligned} u_i^{(n+1)}(t) &= c_i + \int_0^t t'^{\mu_i - 1} f_i(t', u^{(n)}, v^{(n)})\, dt' , \\ v_i^{(n+1)}(t) &= t^{-\lambda_i} \int_0^t t'^{\lambda_i + \nu_i - 1} g_i(t', u^{(n)}, v^{(n)})\, dt' . \end{aligned} \tag{17}$$

Due to the assumed analyticity of f and g one can easily estimate these integrals and deduce that there exists a $t_0(c) > 0$ such that for $|t| < t_0(c)$ the $u^{(n)}$, $v^{(n)}$ converge to a solution and that this solution is analytic in c and t.

Remark. Prop. 1 implies in particular the standard result [11] of the analyticity of the stable manifold, i.e., that in a neighbourhood of the singular point $t = 0$ the v_i can be expressed as analytic functions of t and the u_i.

Proposition 2. *There exists a one-parameter family of local solutions of Eqs.* (6a, b) *near $r = 0$ analytic in b and r such that*

$$\begin{aligned} W(r) &= 1 - br^2 + O(r^4) , \\ \mu(r) &= 1 - 4b^2 r^2 + O(r^4) . \end{aligned} \tag{18}$$

Proof. Introducing

$$w_1 = \frac{1 - W}{r^2} , \quad w_2 = \frac{\mu W'}{r} , \quad w_3 = \frac{1 - \mu}{r^2} , \tag{19}$$

Eqs. (6a, b) can be written as

$$\begin{aligned} rw_1' &= -2w_1 - w_2 + r^2 h_1 , \qquad rw_2' = -2w_1 - w_2 + r^2 h_2 , \\ rw_3' &= -3w_3 + 4w_1^2 + 2w_2^2 + r^2 h_3 , \end{aligned} \tag{20}$$

where h_i are polynomials in w, r^2, and $\frac{1}{\mu}$. Next we substitute

$$w_1 = u_1 + v_1 , \quad w_2 = -2u_1 + v_1 , \quad w_3 = 2v_2 + 2(2u_1^2 - v_1^2) , \tag{21}$$

and find

$$ru_1' = r^2 f_1 , \qquad rv_i' = -3v_i + r^2 g_i , \tag{22}$$

where f_1 and g_i are polynomials in u, v, r^2, and $\frac{1}{\mu}$. According to Prop. 1 there exists a one-parameter family of solutions such that

$$u_1 = b + O(r^2) , \quad v_i = O(r^2) . \tag{23}$$

Proposition 3. *There exists a two-parameter family of local solutions of Eqs.* (6a, b) *near $r = \infty$ analytic in c, M, and $\frac{1}{r}$ such that*

$$\begin{aligned} W(r) &= \pm\left(1 - \frac{c}{r} + O\left(\frac{1}{r^2}\right)\right) , \\ \mu(r) &= 1 - \frac{2M}{r} + O\left(\frac{1}{r^4}\right) . \end{aligned} \tag{24}$$

These are the only solutions with $|W| \to 1$, $rW' \to 0$, and $\mu \to 1$ as $r \to \infty$.

Proof. Let us assume that $W \to 1$ as $r \to \infty$ (the case $W \to -1$ follows trivially from the symmetry of the field equations (6)) and introduce

$$s = \frac{1}{r} , \quad w_1 = r(1 - W) , \quad w_2 = r^2 \mu W' , \quad w_3 = r(1 - \mu) . \tag{25}$$

Then the field equations can be written as $\left(\text{with } f' = \frac{df}{ds}\right)$

$$sw_1' = -w_1 + w_2 + sh_1 , \quad sw_2' = 2w_1 - 2w_2 + sh_2 , \quad sw_3' = s^3 h_3 , \tag{26}$$

where h_i are polynomials in w, s, and $\frac{1}{\mu}$. Similarly to the case above when $r \to 0$ we substitute

$$w_1 = u_1 + v_1 , \quad w_2 = u_1 - 2v_1 , \quad w_3 = u_2 , \tag{27}$$

and find

$$su_1' = sf_1 , \qquad su_2' = s^3 f_2 , \qquad sv_1' = -3v_1 + sg_1 , \tag{28}$$

where f_i and g_1 are polynomials in u, v, s, and $\frac{1}{\mu}$. According to Prop. 1 there exists a two-parameter family of solutions such that

$$u_1 = c + O(s) , \quad u_2 = 2M + O(s^3) , \quad v_i = O(s) . \tag{29}$$

The nonlinear terms in Eq. (26) are such that $s^2 h_1$, $s^2 h_2$, and $s^4 h_3$ are polynomials in sw, s, and $\frac{1}{\mu}$ and vanish for $sw \to 0$. Uniqueness is implied by the fact that bounded solutions must lie on the stable manifold [11].

Proposition 4. *There exists a two-parameter family of local non-degenerate black hole solutions of Eqs.* (6a, b) *satisfying the boundary conditions* (10, 11) *defined for* $r_h > 0$, W_h *such that* $\mu'_h > 0$, *and* $|r - r_h| < \rho_0(r_h, W_h)$ *and analytic in* r_h, W_h, *and* r.

Proof. Introducing $\rho = r - r_h$ as new independent variable as well as

$$\begin{aligned} u_1(\rho) &= r\,, & u_2(\rho) &= W\,, \\ w_1(\rho) &= \frac{\mu}{\rho}\,, & w_2(\rho) &= \frac{\mu W'}{\rho}\,, \end{aligned} \tag{30}$$

we find $\left(\text{with } f' = \frac{df}{d\rho}\right)$

$$\begin{aligned} \rho u_1' &= \rho\,, \qquad \rho u_2' = \rho \frac{w_3}{w_1}\,, \\ \rho w_i' &= -w_i + F_i(u) + \rho h_i(u, w)\,, \end{aligned} \tag{31}$$

where h_i are polynomials in $\frac{1}{u_1}$, $\frac{1}{w_1}$, u, and w and

$$\begin{aligned} F_1(u) &= \frac{1}{u_1}\left(1 - \frac{(u_2^2 - 1)^2}{u_1^2}\right)\,, \\ F_2(u) &= \frac{u_2(u_2^2 - 1)}{u_1^2}\,. \end{aligned} \tag{32}$$

Next we substitute

$$w_i = v_i + F_i \tag{33}$$

and find

$$\rho v_i' = -v_i + \rho g_i(u, v)\,, \tag{34}$$

where g_i are polynomials in $\frac{1}{u_1}$, $\frac{1}{w_1}$, u, and v. According to Prop. 1 there exists a two-parameter family of solutions such that

$$u_1 = r_h + \rho\,, \quad u_2 = W_h + O(\rho)\,, \quad v_i = O(\rho)\,, \tag{35}$$

defined for $r_h > 0$, and W_h such that $\mu'_h > 0$.

Remark. The regularity of $A(r)$ can, in all these cases, be easily deduced integrating Eq. (6c) with an additional free parameter (initial value).

5. Flat Space Solutions

As already discussed in Sect. 2, for regular solutions μ tends to 1 at the singular points $r = 0$, resp. $r = \infty$. Hence we expect that in a neighbourhood of these points the YM-potential W is well described by the solutions of the flat-space YM-theory with the field equation

$$W'' = \frac{W(W^2 - 1)}{r^2} . \tag{36}$$

Therefore we first study the solutions of this comparatively simple second order equation. Although most of their qualitative features are known [13, 14] we recall and also prove them as they will be frequently needed throughout the paper.

Introducing the "time"-coordinate $\tau = \ln r$ we obtain the autonomous form

$$\ddot{W} - W(W^2 - 1) = \dot{W} , \tag{37}$$

that may be viewed as describing the motion of a particle in the potential $V = -(W^2 - 1)^2/4$ under the influence of a "negative friction" due to the term $\dot{W}$. Correspondingly the "energy" of the particle

$$E = \frac{1}{2}\dot{W}^2 - \frac{1}{4}(W^2 - 1)^2 \tag{38}$$

grows in time according to $\dot{E} = \dot{W}^2$.

The global features of the solutions of Eq. (37) are best illustrated in a phase-space diagram displaying the integral curves of the vector field $X = (\dot{W}, W(W^2 - 1) + \dot{W})$ in the $(W, \dot{W})$-plane (Fig. 1). This vector field has three singular points at $(0, 0)$ and $(\pm 1, 0)$. Linearization at these points exhibits the former one as a focal point with the eigenvalues $\lambda = (1 \pm i\sqrt{3})/2$ and the latter as saddle points with the eigenvalues $\lambda = 2$ and $\lambda = -1$. Since all the eigenvalues have non-vanishing real parts the phase-flow of Eq. (37) is topologically equivalent to that of the linearized equations in a neighbourhood of the singular points (compare [12] p. 48). That implies in particular that from each saddle-point there emerge two separatrices describing the stable ($\lambda = -1$) and unstable ($\lambda = 2$) manifolds of the flow. Specializing the results of Props. 2 and 3 to $\mu \equiv 1$ it turns out to be possible to parametrize the stable and unstable manifolds in the form $W = \pm(1 - ce^{-\tau} + O(e^{-2\tau}))$ and $W = \pm(1 - be^{2\tau} + O(e^{4\tau}))$ respectively.

Closer analysis reveals that the stable manifold has a branch spiralling into the fixed point (0,0) and one running off to infinity. Both branches of the second separatrix run away to infinity. The asymptotic solution close to the fixed point (0,0) is of the form

$$W(\tau) = Ce^{\frac{1}{2}\tau} \sin\left(\frac{\sqrt{3}}{2}\tau + \theta\right) . \tag{39}$$

The global behaviour of the phase flow is described by

Proposition 5. i) *Trajectories of Eq.* (37) *staying bounded for* $\tau \to \infty$ *have to run into one of the saddle points along the separatrix corresponding to the stable direction* ($\lambda = -1$).

ii) *Trajectories staying bounded for* $\tau \to -\infty$ *have to run either into* $(0, 0)$ *or to one of the saddle points along the separatrix corresponding to the unstable* ($\lambda = 2$) *direction.*

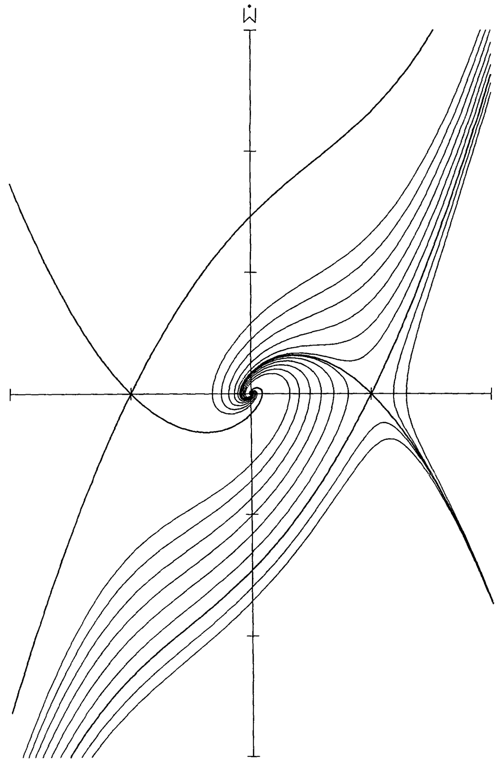

Fig. 1. Phase-space diagram for Eq. (37), showing the separatrices and some selected trajectories

iii) *All remaining trajectories run to infinity within finite time $\bar{\tau}$ with the asymptotic behaviour*

$$W(\tau) = \pm \frac{\sqrt{2}}{\bar{\tau} - \tau} + O(1) . \tag{40}$$

Proof. i) and ii) are a consequence of the theorem of Poincaré-Bendixson [12] p. 29, since there are no periodic orbits. For the proof of iii) we rewrite Eq. (37) as the phase-space equation

$$\frac{d\dot{W}}{dW} = \frac{W(W^2 - 1)}{\dot{W}} + 1 . \tag{41}$$

Without restriction we may assume $|W| > 1$. Putting $\dot{W} = (W^2 - 1)v$ and $x = \ln(W^2 - 1)$ we obtain from Eq. (41),

$$\frac{dv}{dx} = \frac{1}{v} - 2v + (1 + e^x)^{-\frac{1}{2}} . \tag{42}$$

From this form it is not difficult to see that $v^2 \to \frac{1}{2}$ for $x \to \infty$ and hence $\dot{W}^2 \to (W^2 - 1)^2$ for large W leading to the asymptotic behaviour claimed in iii).

According to Prop. 5 solutions regular at $r = 0$ have to start at one of the saddle points, say (1,0) and follow the separatrix corresponding to $\lambda = 2$. Close to the singular point W behaves as $W(r) = 1 - br^2 + O(r^4)$ as described in Prop. 2. For $b < 0$ the solution runs to infinity along the upper branch of the separatrix, for $b > 0$

the solution follows the lower branch. Numerical analysis shows that this curve cuts the line $W = -1$ at $\dot{W} \approx -2.327$. Without the use of numerics, using Eq. (37) it is straightforward to show that this curve stays above its tangent at the saddle point as long as $W \geq -1$. Once $W \leq -1$ the solution cannot turn back and tends monotonously to infinity within finite time $\bar{\tau}$. Numerically we found that W diverges for

$$r = |b|^{-\frac{1}{2}}\bar{r} \quad \text{with} \quad \bar{r} = e^{\bar{\tau}} \approx \begin{cases} 5.317 & \text{for } b > 0 \,, \\ 1.746 & \text{for } b < 0 \,. \end{cases} \tag{43}$$

The observation that solutions regular at $r = 0$ run off to infinity is in accordance with the known result that there are no non-trivial globally regular, static solutions of the pure YM theory [15].

Since the singular points of Eq. (37) are non-degenerate the flow of X is structurally stable [12], i.e., the phase flow in a bounded region changes continuously with a small perturbation. This will become relevant, when one takes into account the effects of the gravitational field leading to a perturbation of the form ($|\delta| \ll 1$)

$$\ddot{W} - W(W^2 - 1) = (1 + \delta)\dot{W} \,. \tag{44}$$

Such perturbations leave the fixed points invariant, but modify the τ-dependence of the solutions. The relevant properties of the solutions of the perturbed equation is expressed in the

Proposition 6. *Solutions of Eq.* (44) *with* $|\delta| \ll 1$ *have the properties:*
i) *Solutions staying bounded have to run into one of the singular points as in the unperturbed case.*
ii) *Solutions coming close enough to one of the saddle points* $(\pm 1, 0)$, *but missing it, leave the strip* $|W| \leq 1$.
iii) *Close to the saddle points* $(\pm 1, 0)$ *the solutions can be parametrized in the form*

$$W(\tau) = \pm(1 - c(\tau)e^{-\tau} - b(\tau)e^{2\tau}) \tag{45}$$

with slowly varying functions b and c.

Proof. i) and ii) are immediate consequences of the structural stability of the vector field X (compare [12] p. 40) and Prop. 5. The last property follows from the topological equivalence between the vector field corresponding to Eq. (44) and its linearization [12] p. 48.

6. Global Behaviour

Let us now turn to the analysis of the global behaviour of the solutions of the full EYM equations (6). In view of the physical interpretation we shall limit ourselves to solutions regular at $r = 0$ or with black hole boundary conditions at $r = r_h$. When we say that some property holds for all r in this context we always mean for $r \geq 0$, resp. for $r \geq r_h$.

Our analysis will reveal that in general the coordinate r is not suitable to describe the full global geometry of the solutions, because it becomes stationary at some point. This will manifest itself as a zero of μ and a singularity of W'. It is possible to remove this "coordinate singularity" by using a different independent variable. Before introducing this new variable we state some useful properties of the solutions as functions of r.

Proposition 7. *As long as $\mu(r) > 0$ the function $W(r)$ can have neither maxima in the regions $W > 1$ and $0 > W > -1$ nor minima for $W < -1$ and $0 < W < 1$. The only solutions with extrema of W at $W = 0, \pm 1$ at some regular point $r > 0$ are those with constant W.*

Proof. This follows immediately by putting $W' = 0$ in Eq. (6a).

Corollary. (i) *If $|W(r_0)| > 1$ and $W(r_0)W'(r_0) > 0$ for some r_0 then $|W(r)| > 1$ for all $r \geq r_0$.*
(ii) *If W has only a finite number of zeros and $|W(r)| \leq 1$ for all $r \geq r_0$ for some r_0 then $\lim_{r\to\infty} W(r)$ exists, since $W(r)$ will be monotonous for r large enough.*

Proposition 8. *If $\mu(r_0) < 1$ for some r_0 then $\mu(r) < 1$ for all $r \geq r_0$.*

Proof. Putting $\mu(r) = 1$ in Eq. (6b) one gets $\mu'(r) < 0$, showing that μ cannot cross 1 from below.

Proposition 9. *As long as $0 < \mu < 1$ the solutions are regular functions of r.*

Proof. Let us take some finite interval $I = [r_0, r_1)$ and assume that the solution is regular in I. We will show that the solution stays regular (i.e., finite) also at r_1. From $\mu(r_1) > 0$ we get

$$r_1 > 2m(r_1) \geq 2\int_{r_0}^{r_1} \mu W'^2\, dr\ . \tag{46}$$

According to the assumption $\mu_{\min} = \min(\mu : r \in \bar{I}) > 0$, hence

$$\int_{r_0}^{r_1} W'^2\, dr < \frac{r_1}{2\mu_{\min}}\ . \tag{47}$$

From this we conclude that $A(r_1)$ is finite. Furthermore using the Schwarz inequality it follows that $W(r_1)$ is finite. Integrating Eq. (6a) we also get the finiteness of $W'(r_1)$.

In order to desingularize the equations when $\mu(r) \to 0$ we shall go back to the more general parametrization (1) of the metric considering r as a function of a coordinate τ defined through $dr = rN d\tau$ with $N = \sqrt{\mu}$. Thus the metric takes the form

$$ds^2 = A^2N^2dt^2 - r^2(\tau)(d\tau^2 + d\Omega^2)\ . \tag{48}$$

In addition we introduce $U = NW'$ and

$$\kappa = \frac{1}{2N}(1 + N^2 + 2U^2 - (W^2-1)^2/r^2)\ . \tag{49}$$

The significance of the combination κ is that it stays finite when N tends to zero for growing τ in spite of its vanishing denominator and allows to desingularize Eq. (6) for $N \to 0$. Using κ the equations can be written in the form (with $\dot{} = d/d\tau$)

$$\dot{r} = rN\ , \tag{50a}$$

$$\dot{W} = rU\ , \tag{50b}$$

$$\dot{\kappa} = 1 + 2U^2 - \kappa^2\ , \tag{50c}$$

$$\dot{N} = (\kappa - N)N - 2U^2\ , \tag{50d}$$

$$\dot{U} = \frac{W(W^2-1)}{r} - (\kappa - N)U\ , \tag{50f}$$

$$(AN)^{\cdot} = (\kappa - N)AN\ . \tag{50g}$$

In the following we interpret κ as an additional dependent variable and Eq. (49) as a constraint on the initial data that is, of course, preserved by the differential Eq. (50).

It is now obvious that for finite τ these equations are no more singular for $N = 0$. The only possible singularities occur for $r = 0$ or when some of the dependent variables diverge.

Since $N \leq 1$ for the type of solutions considered, we have $\ln(r(\tau)/r(\tau_0)) \leq \tau - \tau_0$. For solutions regular at $r = 0$ the variable τ behaves like $\tau \sim \ln r$ for $r \to 0$, i.e., $\tau \to -\infty$, and

$$\kappa(r) = 1 + 2b^2 r^2 + O(r^4) . \tag{51a}$$

For black hole solutions we may choose $\tau = 0$ at the horizon; then $\tau \sim \sqrt{r - r_h}$ and

$$\kappa(\tau) = \frac{1}{\tau} + O(\tau) . \tag{51b}$$

Applying Prop. 1 to $u(\tau) = (r, W, \kappa N - U^2)$, and $v(\tau) = \left(U, \kappa - \frac{1}{\tau}\right)$ we see that r, W, N, U, and $\kappa - \frac{1}{\tau}$ are analytic functions of τ, r_h, and W_h for all τ sufficiently small, $r_h > 0$, and W_h including the case $r_h^2 \leq (W_h^2 - 1)^2$ where $W(r)$ is not analytic and the solution does not describe a regular horizon.

Important bounds on κ are expressed by the

Lemma 10. i) *If $\kappa(\tau_0) > 1$ for some τ_0 then $\kappa(\tau) > 1$ for all $\tau > \tau_0$.*
ii) *If $\kappa(\tau_0) + N(\tau_0) < 2$ for some τ_0 then $\kappa(\tau) + N(\tau) < 2$ for all $\tau > \tau_0$.*
iii) *If $\kappa(\tau_0) + N(\tau_0) > 2$ for some τ_0 then $\kappa(\tau) + N(\tau) < 2 + \epsilon$ for any $\epsilon > 0$ and large enough τ.*

Proof. i) Put $\xi = \frac{\kappa - 1}{\kappa + 1}$ with $1 > \xi(\tau_0) > 0$, then

$$\dot{\xi} = -2\xi + (1 - \xi)^2 U^2 \geq -2\xi , \tag{52}$$

and therefore $1 > \xi(\tau) \geq e^{-2(\tau - \tau_0)} \xi(\tau_0) > 0$.
ii) Put $\eta = \kappa + N - 2$, then

$$\dot{\eta} = -\eta - \frac{1}{4}\eta^2 - \frac{3}{4}(\kappa - N)^2 \leq -\eta , \tag{53}$$

and therefore $\eta(\tau) \leq e^{-(\tau - \tau_0)} \eta(\tau_0) < 0$.
iii) This follows from $\dot{\eta} < -\frac{1}{4}\eta^2$ for $\eta > 0$.

In view of their behaviour near $r = 0$ we see that for regular solutions one has $1 \leq \kappa < 2$ for all τ and for black holes $1 < \kappa < 2 + \epsilon$ for sufficiently large τ (as long as $N \geq 0$).

Proposition 11. *If $|W(\tau)| > 1$ for some τ then N has a zero for some finite value τ_0, resp. r_0, with finite $W(\tau_0)$ and $U(\tau_0)$.*

Proof. Put $T = (W^2 - 1)/r$, then

$$\dot{T} = 2WU - NT , \qquad \dot{U} = WT - (\kappa - N)U , \tag{54}$$

and thus for any $\epsilon > 0$

$$(\ln|TU|)^{\cdot} \geq 2\sqrt{2}|W| - \kappa > 2\sqrt{2} - 2 - \epsilon \tag{55}$$

for large enough τ, using Lemma 10 and $WU \geq 0$ for $|W| \geq 1$ (compare Prop. 7).

It follows that we can find some τ_1 such that $|TU| \geq \frac{1}{\sqrt{2}}$ for $\tau \geq \tau_1$ and hence $T^2 + 2U^2 \geq 2$. The field equation (50d) for N can be written as

$$\dot{N} = \frac{1}{2}\left(1 - N^2 - T^2 - 2U^2\right) \tag{56}$$

and thus $\dot{N} \leq -\frac{1}{2}$ for $\tau \geq \tau_1$. In view of this inequality we conclude that N must reach the value 0 for some finite value τ_0. From the definition of τ we get $\tau > \ln r$, which implies that $r_0 = r(\tau_0)$ is finite as well.

Since κ is bounded according to the preceding lemma it follows from Eq. (50c) that U^2 is integrable and hence W is bounded. Finally U is bounded due to Eq. (49).

Proposition 12. If $|W| < 1$ and $W > 0$ for all $r > 0$ then N has a zero for some finite value τ_0, resp. r_0.

Proof. Assume $N(\tau) > 0$ for all τ. From Eq. (50) we get

$$(ANU)^{\cdot} = W(W^2 - 1)AN/r \tag{57}$$

and hence

$$U(\tau) = \frac{1}{AN}\int_{-\infty}^{\tau} \frac{W(W^2-1)AN}{r} d\tau' \leq 0\,, \tag{58}$$

implying that W is monotonously decreasing and therefore $\lim_{\tau\to\infty} W(\tau)$ exists. Furthermore

$$\begin{aligned}(NU)(\tau_0) - (NU)(\tau) &= \int_{\tau_0}^{\tau} \left(2U^3 - \frac{W(W^2-1)N}{r}\right) d\tau' \\ &\leq -\int_{\tau_0}^{\tau} \frac{W(W^2-1)N}{r} d\tau' < \infty\,.\end{aligned} \tag{59}$$

Hence there is some constant c such that

$$\int_{\tau_0}^{\tau} rNU^2 d\tau' \leq -c\int_{\tau_0}^{\tau} \dot{W} d\tau' \leq c \tag{60}$$

implying the boundedness of m and consequently $\lim_{\tau\to\infty} N(\tau) = 1$. From Eq. (50) we get

$$\ln(A(\tau)/A(\tau_0)) = \int_{\tau_0}^{\tau} \frac{2U^2}{N} d\tau' < \infty \tag{61}$$

and thus $\lim_{\tau\to\infty} A(\tau)$ exists. Using again Eq. (58) we see that $\lim_{\tau\to\infty} U(\tau)$ exists and is zero, because W has a limit. This implies

$$\int_{-\infty}^{\infty} \frac{W(W^2-1)AN}{r} d\tau' = 0 \tag{62}$$

contradicting the assumptions.

From Props. (11) and (12) we learn that for solutions extending to $r \to \infty$ the function W must remain between ± 1 and have at least one zero. Later we shall demonstrate that for any such solution W can only have a finite number of zeros and necessarily tends to ± 1 for $r \to \infty$.

Assume we start with either regular boundary conditions (8) for $r = 0$, or with black hole boundary conditions (10, 11) for a horizon at $r = r_h$, or just with any regular initial data with $r > 0$, $N > 0$, $\kappa \geq 1$ and integrate Eq. (50) towards increasing τ. Then there are three possible cases:

i) $N(\tau)$ has a zero at $\tau = \tau_0$.
ii) $N(\tau) > 0$ for all τ and $r(\tau)$ tends to infinity.
iii) $N(\tau) > 0$ for all τ and $r(\tau)$ remains bounded.

In the following we will study each of these cases in more detail.

Case i. $N(\tau)$ has a zero at $\tau = \tau_0$. Let r_0, W_0, U_0, κ_0 denote the values of r, W, U, κ at $\tau = \tau_0$. Case **i** is the generic one in the sense that a sufficiently small change in the initial data does not change the type of the solution but only leads to slightly changed values of τ_0, r_0, W_0, U_0, and κ_0.

First we observe that W_0 and U_0 cannot both be zero since then $N(\tau)$ would be identically zero. If $U_0 \neq 0$ then

$$\begin{aligned} W(\tau_0 + \sigma) &= W_0 + r_0 U_0 \sigma + O(\sigma^2) , \\ N(\tau_0 + \sigma) &= -2U_0^2 \sigma + O(\sigma^2) , \\ r(\tau_0 + \sigma) &= r_0 - r_0 U_0^2 \sigma^2 + O(\sigma^3) , \end{aligned} \tag{63}$$

whereas for $U_0 = 0$ and $W_0 \neq 0$

$$\begin{aligned} W(\tau_0 + \sigma) &= W_0 + \frac{W_0(W_0^2 - 1)}{2r_0} \sigma^2 + O(\sigma^3) , \\ N(\tau_0 + \sigma) &= -\frac{2W_0^2(W_0^2 - 1)^2}{3r_0^4} \sigma^3 + O(\sigma^4) , \\ r(\tau_0 + \sigma) &= r_0 - \frac{W_0^2(W_0^2 - 1)^2}{6r_0^3} \sigma^4 + O(\sigma^5) . \end{aligned} \tag{64}$$

In both cases $r(\tau)$ has a maximum and $N(\tau)$ changes from positive to negative values at $\tau = \tau_0$.

Since κ_0 is finite due to Lemma 10 the constraint (49) implies

$$(W_0^2 - 1)^2 = r_0^2(1 + 2U_0^2) \tag{65}$$

and hence either

$$|W_0| < 1 , \quad r_0 = \frac{1 - W_0^2}{\sqrt{1 + 2U_0^2}} < 1 , \tag{65a}$$

or

$$|W_0| > 1 , \quad r_0 = \frac{W_0^2 - 1}{\sqrt{1 + 2U_0^2}} . \tag{65b}$$

The sets of (r_0, W_0) values satisfying these conditions are depicted as regions A and B of the (r, W) half plane in Fig. 2. The complementary region C contains the allowed (r_h, W_h) values for black hole boundary conditions (compare Eq. (11)).

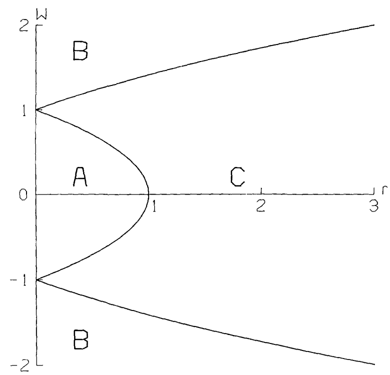

Fig. 2. Regions of the (r, W) half plane:
A and B contain the (r_0, W_0) values for case **i** where $N(\tau)$ has a zero,
C contains the allowed (r_h, W_h) values for black hole boundary conditions

It follows from Eq. (50) that $\dot{r} = rN$ is negative and monotonously decreasing for $\tau > \tau_0$. Hence r will decrease and become zero within a finite τ-interval provided the other quantities remain bounded.

The "energy" (38) is monotonously decreasing for $\tau \geq \tau_0$ since

$$\dot{E} = -(\kappa - 2N)\dot{W}^2 \, . \tag{66}$$

and has the value $-\frac{r_0^2}{4}$ at τ_0. Therefore $|W_0| > 1$ ($|W_0| < 1$) implies $|W| > 1$ ($|W| < 1$) for all $\tau \geq \tau_0$.

In order to desingularize Eq. (50) for $r \to 0$ we introduce yet another dependent variable

$$\lambda = \frac{W(W^2 - 1)}{r} + NU \, , \tag{67}$$

and replace Eqs. (50d–f) by

$$(rN)^{\cdot} = \kappa rN - 2rU^2 \, , \tag{68a}$$

$$\dot{U} = \lambda - \kappa U \, , \tag{68b}$$

$$\dot{\lambda} = (3W^2 - 2U^2 - 1)U \, , \tag{68c}$$

$$(rAN)^{\cdot} = \kappa rAN \, , \tag{68d}$$

again preserving the constraint (67). It is now obvious that Eqs. (50a–c, 68) are no more singular for $r = 0$.

Proposition 13. *For solutions of Eqs.* (49, 50) *with a zero of* $N(\tau)$ *at* $\tau = \tau_0$ *there exists a* $\tau_1 > \tau_0$ *such that* $r(\tau_1) = 0$ *and* W, U, rN, κ, λ, *and* rAN *remain finite for* $\tau \leq \tau_1$, *with*

$$U(\tau_1) = \pm W(\tau_1) \, , \quad rN(\tau_1) = \pm(1 - W^2(\tau_1)) \, , \tag{69}$$

where the upper sign applies for $|W| > 1$ *and the lower one for* $|W| < 1$.

Proof. As already mentioned $\dot{r}$ is negative and monotonously decreasing for $\tau > \tau_0$ and hence will become zero within a finite τ-interval provided the other functions remain bounded.

In order to demonstrate the boundedness of W we consider an interval $\bar{\tau} \leq \tau \leq \tau_1$ with $\bar{\tau} > \tau_0$ such that $r \geq 0$ and $rN \leq 0$. In this interval r^ϵ and $|rN|^{-\epsilon}$ are bounded and monotonous for all $\epsilon > 0$ and therefore their derivatives are absolutely integrable, i.e.,

$$\int r^\epsilon |N| d\tau < \infty\,, \qquad \int \frac{1}{r^{1-\epsilon}} d\tau < \infty\,, \qquad \int \frac{U^2}{r^\epsilon |N|^{1+\epsilon}} d\tau < \infty\,. \tag{70}$$

Since either $|W| < 1$ or $|W| > 1$ in the whole interval we need only consider the case $|W| > 1$. Using the Schwarz inequality it follows from (70) that $\int |U||N|^{-\epsilon} d\tau$ and hence $\int |\dot{W}||rN|^{-\epsilon} d\tau$ is finite for $0 < \epsilon < 1$. Integrating by parts we find

$$\frac{W}{|rN|^\epsilon}(\tau) + \epsilon \int_{\bar{\tau}}^{\tau} \frac{W}{|rN|^\epsilon}(\kappa - \frac{2U^2}{N}) d\tau' = \frac{W}{|rN|^\epsilon}(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} \frac{\dot{W}}{|rN|^\epsilon} d\tau'\,. \tag{71}$$

Since the r.h.s. is bounded and both terms on the l.h.s. have the same sign $W|rN|^{-\epsilon}$ is bounded. The constraint (49) implies $|rN| \leq |W^2 - 1|$ and hence W must be bounded.

Now we know that W and hence rN is bounded. Next we use the differential equation

$$(r^\epsilon U)^{\cdot} = \frac{W(W^2-1)}{r^{1-\epsilon}} - (\kappa - (1+\epsilon)N) r^\epsilon U\,. \tag{72}$$

A linear differential equation of the form $\dot{x} = a - bx$ has the explicit solution

$$x(\tau) = e^{-c(\tau)} \Big(x(\bar{\tau}) + \int_{\bar{\tau}}^{\tau} a(\tau') e^{c(\tau')} d\tau' \Big) \qquad \text{with} \qquad c(\tau) = \int_{\bar{\tau}}^{\tau} b(\tau') d\tau'\,, \tag{73}$$

that remains bounded if $|a|$ is integrable and $b \geq 0$. Thus $r^\epsilon U$ is bounded for $\epsilon > 0$ and consequently $|U^n|$ is integrable for any $n > 0$. Finally it follows from the differential Eqs. (50c, 68b–d) that κ, λ, rAN, and U are bounded and $rAN > 0$.

The relations (69) are a consequence of the constraints (49) and (67).

It follows from the form Eq. (48) of the metric that the $t =$ const. hypersurfaces are compact in this case. Expressing the 3-metric in isotropic coordinates yields a conformal factor that vanishes at the point corresponding to the zero of r at τ_1. Hence the geometry is singular at this point ("bag of gold" [7]).

Case ii. $N(\tau) > 0$ for all τ and $r(\tau)$ tends to infinity. This case describes the discrete family of asymptotically flat solutions found by Bartnik and McKinnon [1] as well as their black hole counterparts [2, 3].

Proposition 14. *If $N(\tau) > 0$ for all τ and $\lim_{\tau \to \infty} r(\tau) = \infty$ then the solution is a member of the two-parameter family described in Prop. 3 with $N \to 1$ and $W \to \pm 1$.*

Proof. The strategy is to reduce the problem to the pure YM Eq. (37) by first showing that N and κ tend to one for $\tau \to \infty$.

First we show that U tends to zero for $\tau \to \infty$. We distinguish the two cases that W has only a finite number of zeros, resp. W has zeros for arbitrarily large r. In the first case we know from Prop. 7 that U cannot change sign for sufficiently large τ. Continuing as in the proof of Prop. 12 we conclude that m is bounded and $N \to 1$ for $\tau \to \infty$. Since in addition W tends to some limit for $\tau \to \infty$ we get $\lim_{\tau \to \infty} U(\tau) = 0$.

In the second case we can find a sequence $\tau_k \to \infty$ with $r_k = r(\tau_k) \to \infty$ and $U(\tau_k) = 0$. Integrating the equation for NU we get

$$|NU(\tau)| = |\frac{1}{A(\tau)} \int_{\tau_k}^{\tau} \frac{W(W^2-1)AN}{r} d\tau'| \le |\int_{r_k}^{r(\tau)} \frac{dr'}{r'^2}| \le \frac{1}{r_k} . \tag{74}$$

In order to show that this implies $\lim_{\tau\to\infty} U(\tau) = 0$ we require a positive lower bound for N. This follows from

$$2\kappa N - N^2 - 1 + \frac{(W^2-1)^2}{r^2} = 2U^2 \ge 0 \tag{75}$$

and Lemma 10, yielding su 14

$$4N - 3N^2 - 1 + \epsilon \ge 0 \tag{76}$$

for any ϵ and large enough τ, thus $N > \frac{2}{9}$.

Next we consider the Eq. (52) for $\xi = \frac{\kappa-1}{\kappa+1}$ and see that ξ tends to zero for $\tau \to \infty$, i.e., $\lim_{\tau\to\infty} \kappa(\tau) = 1$. From the Eq. (49) we derive $\lim_{\tau\to\infty} N(\tau) = 1$.

In the limit $N = \kappa = 1$ the Eqs. (50b, e) for W and U take the form of Eq. (37), which we have already discussed. Putting $\delta(\tau) = 2N - \kappa - 1$ we may rewrite Eqs. (50b, e) in the form (44) considering $\delta(\tau)$ as a small externally given perturbation of the vector field X. Proposition 6 implies that the solution has to tend to one of the saddle points. Since δ tends to zero for $\tau \to \infty$ it finally follows from Prop. 3 that the solution is a member of the two-parameter family described there.

Remark. This proposition implies in particular that there are no solutions with $W \to 0$ for $r \to \infty$, providing a simple proof of a theorem by Galtsov and Ershov [16, 10].

Case iii. $N(\tau) > 0$ for all τ and $r(\tau)$ remains bounded. We shall prove that r, W, U, N, and κ have a limit for $\tau \to \infty$.

Proposition 15. *If $N(\tau) > 0$ for all τ and $r(\tau)$ remains bounded then $r \to 1$, $W \to 0$, $U \to 0$, $N \to 0$, and $\kappa \to 1$ for $\tau \to \infty$. Furthermore W has infinitely many zeros.*

Proof. We first observe that r is monotonous and bounded and therefore has a limit r_0. Similarly m has a limit since it is monotonous and $N > 0$ implies $m < \frac{r}{2}$. Consequently N has a limit that must vanish since otherwise r could not be bounded. Next we use the constraint (49) to write the "energy" (38) in the form

$$E = -\frac{r^2}{4}(1 + N^2 - 2\kappa N) \tag{77}$$

and observe that E is monotonously decreasing for large τ (such that $N < \frac{1}{2}$) and tends to $-\frac{r_0^2}{4}$. Therefore $\dot{E}$ (Eq. (66)) and hence U^2 is integrable and $\lim_{\tau\to\infty} \kappa = 1$.

In the limit $N = 0$, $\kappa = 1$ the Eqs. (50b, e) lead to

$$\ddot{W} - W(W^2 - 1) = -\dot{W} , \tag{78}$$

i.e., the "time-reversed" form of Eq. (37). Putting $\delta(\tau) = \kappa - 2N - 1$ and replacing τ by $-\tau$ we may again rewrite Eqs. (50b, e) in the form (44). It follows from Prop. 11 that $|W| \le 1$ for all τ. Prop. 6 implies that the solution has to tend to one of the critical points. Since $W = \pm 1$ is incompatible with $E < 0$ this can only be the focal point $W = U = 0$ and therefore $r_0 = 1$ (compare Eq. (65)).

To show the existence of infinitely many zeros we consider the "phase function"

$$\varphi \equiv \arctan \frac{\sqrt{3}W}{W + 2\dot{W}} , \tag{79}$$

satisfying the equation

$$\dot{\varphi} = \frac{\sqrt{3}}{2}\left(1 + \frac{(\kappa - 1 - 2N)W\dot{W} - W^4}{W^2 + W\dot{W} + \dot{W}^2}\right) . \tag{80}$$

The Schwarz inequality allows to estimate

$$|\dot{\varphi} - \frac{\sqrt{3}}{2}| \leq \frac{\sqrt{3}}{2}|\kappa - 1 - 2N| + \frac{2}{\sqrt{3}}W^2 . \tag{81}$$

The r.h.s. tends to zero and hence φ diverges for $\tau \to \infty$.

We can formulate the main result of this chapter in

Theorem 16. *Any solution of Eq.* (50) *regular at either $r = 0$ or at a horizon $r = r_h$ belongs to one of the three classes, whose properties are described in Props. 13, 14, and 15.*

Case **iii** corresponds to a new type of solutions, which may be considered as the limit of case **i** when $\tau_0 \to \infty$ and $(W_0, U_0) \to (0, 0)$. Since these solutions have infinitely many zeros we call them oscillating. They will be further studied in the following chapter.

7. Oscillating Solutions

We shall first estimate the asymptotic behaviour for $\tau \to \infty$ of the oscillating solutions. Then we shall establish the local existence of these solutions at $\tau = \infty$ ($r = 1$) with the previously found asymptotic behaviour.

In order to estimate the behaviour of $W, U, \ldots$ for large τ we use an approximate form of Eq. (50) (with all non-leading terms neglected) and ignore, for the moment, the constraint (49). From Eqs. (50b, e) we get

$$\frac{W}{r}(\tau) = C_1 e^{-\frac{1}{2}\tau} \sin\left(\frac{\sqrt{3}}{2}\tau + \theta\right) , \tag{82a}$$

$$U(\tau) = C_1 e^{-\frac{1}{2}\tau} \sin\left(\frac{\sqrt{3}}{2}\tau + \frac{2\pi}{3} + \theta\right) , \tag{82b}$$

with some constants C_1 and θ. Next we use Eqs. (50d, a) to get

$$N(\tau) = C_2 e^{\tau} + \frac{1}{2}C_1^2 e^{-\tau}\left(1 - \frac{2}{\sqrt{7}}\cos(\sqrt{3}\tau + \phi + 2\theta)\right) , \tag{82c}$$

$$\ln r(\tau) = C_3 + C_2 e^{\tau} - \frac{1}{2}C_1^2 e^{-\tau}\left(1 - \frac{1}{\sqrt{7}}\cos(\sqrt{3}\tau + \phi + \frac{\pi}{3} + 2\theta)\right) , \tag{82d}$$

where $e^{i\phi} = \dfrac{1-3\sqrt{3}i}{2\sqrt{7}}$ and C_2, C_3 are again constants. Finally we use Eqs. (50c, f) to get

$$\kappa(\tau) = 1 + C_1^2 e^{-\tau}\Big(1 + \frac{1}{2}\cos(\sqrt{3}\tau + 2\theta)\Big) + C_4 e^{-2\tau} \ , \tag{82f}$$

$$(AN)(\tau) = C_5 e^{\tau} \ , \tag{82g}$$

with two more constants C_4 and C_5, where C_4 could be neglected at this point since it is multiplied by $e^{-2\tau}$ but will be needed later on.

Clearly we must choose $C_2 = 0$ to achieve that N tends to zero for $\tau \to \infty$ and $C_3 = 0$ in order to approximately satisfy the constraint (49). Note, however, that the constraint is also (approximately) satisfied for finite τ with $C_2 \neq 0$ as long as $C_2 e^{\tau}$ is of the same order as $C_1^2 e^{-\tau}$. If $C_2 > 0$ then N remains positive for $r > 1$, whereas N has a zero at some finite τ and $r < 1$ if $C_2 < 0$.

Proposition 17. i) *There exists a five-parameter family of local solutions of Eq.* (50) *such that $N(\tau) > 0$ for all τ and $r(\tau)$ remains bounded. They are defined for sufficiently large τ and can be parametrized in the form* (82) *with the constant parameters θ and C_i replaced by functions of τ. For $\tau \to \infty$ the functions $\theta(\tau)$, $C_1(\tau)$, $C_3(\tau)$, and $C_5(\tau)$ have limits, whereas $e^{2\tau}C_2(\tau)$ and $e^{-\tau}C_4(\tau)$ tend to zero.*
ii) *Among these solutions there is a four-parameter family satisfying the constraint* (49).

Proof. i) Inserting the ansatz (82) into Eq. (50) we obtain differential equations for the functions $\theta(\tau)$ and $C_i(\tau)$:

$$\dot\theta = e^{-\tau} f_0 \ , \tag{83a}$$

$$\dot C_1 = e^{-\tau} C_1 f_1 \ , \tag{83b}$$

$$(e^{2\tau}C_2)^{\cdot} = 2e^{2\tau}C_2 + e^{-\tau} f_2 \ , \tag{83c}$$

$$\dot C_3 = e^{-2\tau} f_3 \ , \tag{83d}$$

$$(e^{-\tau}C_4)^{\cdot} = -e^{-\tau}C_4 + e^{-\tau} f_4 \ , \tag{83f}$$

$$\dot C_5 = e^{-\tau} C_5 f_5 \ , \tag{83g}$$

where the f_i are homogeneous polynomials of degree 1 or 2 in C_1^2, $e^{\tau}(\kappa - 1)$, $e^{\tau}N$, and $e^{\tau}W^2$ with bounded coefficients, i.e., they are uniformly bounded if C_1, $e^{2\tau}C_2$, C_3, $e^{-\tau}C_4$, C_5 are bounded and τ is sufficiently large. Using again the textbook result [11] p. 330 we conclude that there exists a five-parameter family of solutions of Eq. (83) such that θ, C_1, C_3, and C_5 have a limit whereas $e^{2\tau}C_2$ and $e^{-\tau}C_4$ tend to zero for $\tau \to \infty$. These solutions are characterized by arbitrary initial values of θ, C_1, C_3, C_4, and C_5 for some sufficiently large τ; the initial value of C_2 is determined by the other initial data.
ii) Inserting the ansatz (82) into the constraint (49) we find that C_3 must vanish at least as $e^{-2\tau}$. Therefore we can choose arbitrary initial values for θ, C_1, C_4, and C_5 and satisfy the constraint by a suitable choice of the initial value for C_3.

It is instructive to describe the geometry corresponding to the oscillating solution. The asymptotic line element takes the form

$$ds^2 = C_5^2 e^{2\tau} dt^2 - d\tau^2 - d\Omega^2 \ . \tag{84}$$

Introducing the new coordinate $\sigma = C_5^{-1} e^{-\tau}$ the above line element (84) becomes:

$$ds^2 = \frac{dt^2 - d\sigma^2}{\sigma^2} - d\Omega^2 \tag{85}$$

which is the metric of the direct product of a two-pseudosphere and a two-sphere.

The t =const. hypersurfaces of the solution are non-compact. Asymptotically they become cylinders ($R^1 \times S^2$) exactly as those of the extremal RN solution near the horizon. Since W and $\dot{W}$ tend to zero for $\tau \to \infty$ the asymptotic form of the Yang-Mills field is also that of the extremal RN solution. Observe, however, that for the oscillating solution A diverges for $\tau \to \infty$ while $A = 1$ for the RN metric.

8. Globally Regular Solutions

In Sect. 4 we have stated the existence of a one-parameter family of local solutions of Eq. (6) regular near $r = 0$ and parametrized by $b = -\frac{1}{2}W''(0)$ (compare Prop. 2).

In Sect. 6 we have introduced an equivalent system of differential Eqs. (50), where the parameter b is determined by $\lim_{\tau \to -\infty} \frac{U}{r} = -2b$. Moreover we have shown that there are only three possible cases for the global behaviour of these solutions:

i) $N(\tau_0) = 0$ for some τ_0; $W(r)$ has a singularity at $r = r_0$ due to the choice of r as coordinate.

ii) $N > 0$ for all τ and $r \to \infty$ for $\tau \to \infty$. These are the globally regular solutions.

iii) $N > 0$ for all τ and $r \to 1$ for $\tau \to \infty$; $W(r)$ oscillates (has infinitely many zeros).

It is now natural to investigate which of these three cases occurs for which values of b. It is convenient to further partition the set of b values according to the number of zeros of W as follows: For case **i** we denote by $\mathbf{Sing}_n$, $n = 0, 1, 2, \ldots$ the set of all b's such that $|W_0| > 1$ and W has n zeros and by $\mathbf{Sing}_\infty$ the set containing all b's such that $|W_0| < 1$ irrespective of the number of zeros of W. These are all open sets since W cannot have double zeros due to Prop. 7. For case **ii** we denote by $\mathbf{Reg}_n$, $n = 0, 1, 2, \ldots$ the set containing all b's such that W has n zeros (Bartnik-McKinnon type solutions). Finally we denote by **Osc** the set containing all b's for case **iii**.

For $b < 0$ the function W is monotonously increasing and Prop. 11 implies that $N(\tau_0) = 0$ for some τ_0, i.e., all negative numbers are contained in $\mathbf{Sing}_0$. For $b = 0$ we get the trivial solution with $W \equiv N \equiv 1$ that might be considered as the zeroth member of the family of globally regular solutions, i.e., $0 \in \mathbf{Reg}_0$. In the following we assume $b > 0$. First we shall show that if b is either sufficiently small or sufficiently large then $N(\tau)$ vanishes at some τ_0 and W is monotonously decreasing. For $b \ll 1$ we find $W_0 \ll -1$ and $r_0 \gg 1$, whereas for $b \gg 1$ we find $1 - W_0 \ll 1$ and $r_0 \ll 1$. In other words all sufficiently large positive b's are in $\mathbf{Sing}_\infty$ and all sufficiently small positive b's are in $\mathbf{Sing}_1$. Next we shall analyze how the behaviour of the solution can change with varying b and conclude that all the sets introduced above are nonempty. In particular there must be (at least) one $b_n \in \mathbf{Reg}_n$ for each nonnegative integer n and (at least) one $b_\infty \in \mathbf{Osc}$.

We start with the simplest case $|b| \ll 1$:

Proposition 18. *If $b > 0$, resp. $b < 0$ is sufficiently small then W is monotonously decreasing, resp. increasing and $N(\tau_0) = 0$ for some τ_0. The values of r, W, and U at τ_0 diverge for $b \to 0$.*

Proof. Substituting the rescaled variables $r \longrightarrow |b|^{-\frac{1}{2}}r$ and $U \longrightarrow |b|^{\frac{1}{2}}U$ into Eq. (50) we find that (50c, d) are replaced by the explicitly b-dependent equations

$$\dot{\kappa} = 1 + 2|b|U^2 - \kappa^2 , \tag{86a}$$

$$\dot{N} = (\kappa - N)N - 2|b|U^2 , \tag{86b}$$

and the initial condition $\lim\limits_{\tau\to-\infty} \frac{U}{r} = \mp 2$ becomes b-independent. When $b = 0$ then $\kappa \equiv N \equiv 1$ and therefore W satisfies the pure YM Eq. (37). According to Prop. 5 and the subsequent discussion the solution starts from the saddle point $W = 1$, $\dot{W} = 0$, follows the unstable manifold and eventually diverges at the finite value $r = \bar{r}$ (compare Eqs. (40, 43)).

Small values of b lead to a perturbation of this solution that is small as long as $\int_{-\infty}^{\tau} |b|U^2 d\tau' \ll 1$. Nevertheless $|b|^{-\frac{1}{2}}r$ (i.e., r before rescaling), $|W|$, and $|b|^{\frac{1}{2}}|U|$ reach values $\gg 1$ choosing, e.g., $\tau \approx \bar{\tau} - |b|^{-\gamma}$ with $\frac{1}{4} < \gamma < \frac{1}{3}$. Then Prop. 11 implies a zero of N.

Remark. A similar argument has been used in Ref. 4 to show that $W(r)$ monotonously decreases to -1 for sufficiently small values of b.

Supplementing the proof of Prop. 18 by a somewhat more quantitative analysis one finds

$$r_0 \approx \bar{r}\,|b|^{-\frac{1}{2}} , \quad |W_0| \approx 2^{-\frac{1}{6}}\, 3^{\frac{1}{3}}\, r_0^{\frac{2}{3}} , \quad |U_0| \approx 2^{-\frac{5}{6}}\, 3^{\frac{2}{3}}\, r_0^{\frac{1}{3}} , \quad \kappa_0 \approx 2 , \tag{87}$$

in good agreement with numerical results.

Next consider the other extreme case $b \gg 1$:

Proposition 19. *If b is sufficiently large then W is monotonously decreasing and $N(\tau_0) = 0$ for some τ_0. The values of the other variables at τ_0 are*

$$r_0 = \frac{\bar{r}}{b} + O\left(\frac{1}{b^2}\right) , \tag{88a}$$

$$W_0 = 1 - \sqrt{1 + 2\bar{U}^2}\frac{\bar{r}}{2b} + O\left(\frac{1}{b^2}\right) , \tag{88b}$$

$$U_0 = \bar{U} + O\left(\frac{1}{b}\right) , \tag{88c}$$

$$\kappa_0 = \bar{\kappa} + O\left(\frac{1}{b}\right) , \tag{88d}$$

with some constants $\bar{r} > 0$, $\bar{U} < 0$, and $1 < \bar{\kappa} < 2$.

Proof. Introducing $T = 2\frac{W-1}{r}$ and substituting the rescaled variable $r \longrightarrow \frac{r}{b}$ into Eq. (50) we find that Eqs. (50b, e) are replaced by the explicitly b-dependent equations

$$\dot{T} = 2U - NT , \tag{89a}$$

$$\dot{U} = \left(1 + \frac{3rT}{4b} + \frac{r^2T^2}{8b^2}\right)T - (\kappa - N)U \tag{89b}$$

and the initial condition $\lim\limits_{\tau\to-\infty} \frac{U}{r} = -2$ becomes b-independent.

For $\frac{1}{b} = 0$ these equations are the same as Eq. (54) with W replaced by 1. Hence the argument used in the proof of Prop. 11 applies and we can conclude that there is a τ_0 such that $N(\tau_0) = 0$. The values of r, T, U, and κ at τ_0 are finite and satisfy the relation $T^2 = 1 + 2U^2$ due to the constraint (49).

Small values of $\frac{1}{b}$ lead to a small perturbation of this solution.

Remark. A quite different argument has been used in Ref. 4 to show that $W(r)$ monotonously decreases but remains positive as long as $\mu > 0$ for all $b \geq 1$. Due to Prop. 12 this also implies that $N(\tau)$ has a zero whereas r, W, and U remain finite.

From numerical integration we find $\bar{r} \approx 0.47$, $\bar{U} \approx -0.93$, and $\bar{\kappa} \approx 1.53$.

Next we assume the set $\mathbf{Reg}_n$ is nonempty for some n, choose a $b_n \in \mathbf{Reg}_n$, and analyze the behaviour of solutions for b near b_n.

Lemma 20. *Given $b_n \in \mathbf{Reg}_n$ for any n then all b values in a sufficiently small neighbourhood of b_n are in either $\mathbf{Reg}_n$, or $\mathbf{Sing}_n$, or $\mathbf{Sing}_{n+1}$.*

Proof. For $n = 0$ this result and $b_0 = 0$ follow from Props. 11, 12, and 18.

According to Prop. 14 the solution for $b = b_n$, $n > 0$ is continuous in r for all r with $(W, rU, N, \kappa) \to ((-1)^n, 0, 1, 1)$ for $r \to \infty$. Props. 2 and 9 imply that the solutions are continuous in b and r. For any $\epsilon > 0$ we can therefore choose $\eta > 0$ and $r_2 > r_1 \gg 1$ such that for all $b \in [b_n - \eta, b_n + \eta]$ W has n zeros for $r < r_1$ and W, rU, N, and κ differ from their asymptotic values by less than ϵ for $r_1 \leq r \leq r_2$. Moreover $1 - N$ and $\kappa - 1$ remain small as long as $|W| \ll \sqrt{r}$ and $|rU| \ll \sqrt{r}$. Hence $(W, \dot{W})$ is near the fixed point $((-1)^n, 0)$ for $r_1 \leq r \leq r_2$ and satisfies the perturbed Yang-Mills equation (44) with $|\delta| \ll 1$. According to Prop. 6 there are three possibilities for $r > r_1$, b fixed, and ϵ sufficiently small:

a) the solution hits the fixed point as $r \to \infty$,

b) $(-1)^n W$ reaches the value 1 for some $\tilde{r}$ and exceeds that value for $r > \tilde{r}$,

c) $(-1)^n W$ has a maximum for some $\tilde{r}$ and turns back towards -1.

In the last two cases the solution closely follows the unstable manifold of Eq. (37) for $r > \tilde{r}$ as long as $|W| \ll \sqrt{r}$. In particular W is monotonous for $r > \tilde{r}$. Hence all points in a sufficiently small neighbourhood of b_n are contained in either $\mathbf{Reg}_n$ (case **a**), or $\mathbf{Sing}_n$ (case **b**), or $\mathbf{Sing}_{n+1}$ (case **c**).

For b near b_n we can parametrize the solution in the form

$$
\begin{aligned}
W(r) &= (-1)^n \left(1 - \bar{b}^{(0)} r^2 - \bar{c}^{(0)} r^{-1}\right) , \\
U(r) &= (-1)^n \left(-2\bar{b}^{(0)} r + \bar{c}^{(0)} r^{-2}\right) , \\
N(r) &= 1 - \bar{M}^{(0)} r^{-1} ,
\end{aligned}
\tag{90}
$$

where $\bar{b}^{(0)}$, $\bar{c}^{(0)}$, and $\bar{M}^{(0)}$ are analytic functions of b and r that are approximately r independent as long as W, rU, N, and κ are close to their asymptotic values (compare Prop. 6). We can choose $\eta > 0$ and $r_1 \gg 1$ such that $\bar{c}^{(0)}(r_1) > 0$ and $|\bar{b}^{(0)}(r_1)| \ll \bar{c}^{(0)}(r_1) r_2^{-3}$ for $|b - b_n| \leq \eta$. For case **a** discussed above we define $\tilde{b} = 0$, for case **b** we define $\tilde{b} = \bar{b}^{(0)}(\tilde{r}) < 0$, and for case **c** we define $\tilde{b} = \bar{b}^{(0)}(\tilde{r}) > 0$. For case **a** and **b** and $r > \tilde{r}$ the solutions are close to those for $|b| \ll 1$ discussed in Prop. 18 with $\tilde{b}$ playing the rôle of b.

Our numerical results indicate that $\tilde{b}$ is strictly increasing with b. This "transversality property" would imply that the number of zeros of W increases from n for

$b \le b_n$ to $n+1$ for $b > b_n$. Since we were not able to prove transversality we have to use a different argument leading to a weaker result.

For $b = b_n$ the functions $\bar{b}^{(0)}$, $\bar{c}^{(0)}$, and $\bar{M}^{(0)}$ have a power series expansion at $r = \infty$

$$\bar{b}^{(0)} = O(r^{-4}) , \qquad \bar{c}^{(0)} = c_n + O(r^{-1}) , \qquad \bar{M}^{(0)} = M_n + O(r^{-1}) , \tag{91}$$

i.e., this solution is the member of the family discussed in Prop. 3 with $(c, M) = (c_n, M_n)$. This two-parameter family of solutions can again be written in the form (90) with $\bar{b}^{(\infty)}$, $\bar{c}^{(\infty)}$, and $\bar{M}^{(\infty)}$ functions of c, M, and r. For (c, M) close to (c_n, M_n) and r fixed and sufficiently large the relation $\bar{b}^{(\infty)} = f(\bar{c}^{(\infty)}, \bar{M}^{(\infty)})$ defines an analytic function f, describing the stable manifold (compare the remark after Prop. 1). For b sufficiently close to b_n the difference

$$\Delta(b) = \bar{b}^{(0)}(b) - f(\bar{c}^{(0)}(b), \bar{M}^{(0)}(b)) \tag{92}$$

(for the same value of r) is analytic in b and vanishes for $b = b_n$.

Lemma 21. *Given the function $\Delta(b)$ defined above then there exists a neighbourhood of $b = b_n$ that contains no zeros of Δ except $b = b_n$.*

Proof. Assume there are other zeros of Δ in every neighbourhood of b_n, i.e., the zeros of Δ accumulate at b_n then Δ must identically vanish in the largest interval where the function $\Delta(b)$ remains analytic. An endpoint of that interval is, however, again in $\mathbf{Reg}_n$ and hence Δ is analytic in a neighbourhood of that endpoint. Consequently $\Delta(b)$ must vanish for all finite b in contradiction to Prop. 18 and 19.

Combining the results of Lemma 20 and 21 we obtain:

Proposition 22. *Given $b_n \in \mathbf{Reg}_n$ for any n then all $b < b_n$ (and similarly all $b > b_n$) sufficiently close to b_n are either in $\mathbf{Sing}_n$ or in $\mathbf{Sing}_{n+1}$.*

Similarly we assume the set **Osc** is nonempty, choose a $b_\infty \in \mathbf{Osc}$, and analyze the behaviour of solutions for b near b_∞.

Lemma 23. *Given $b_\infty \in \mathbf{Osc}$ and some n_0 then all b values in a sufficiently small neighbourhood of b_∞ are in either **Osc**, or $\mathbf{Sing}_\infty$, or $\bigcup_{n \ge n_0} (\mathbf{Reg}_n \cup \mathbf{Sing}_n)$.*

Proof. Since W has infinitely many zeros for $b = b_\infty$ and the solutions are continuous in b and τ all solutions for b sufficiently close to b_∞ must have at least n_0 zeros of W.

For b sufficiently close to b_∞, $r \approx 1$, and $|N| \ll 1$ we can parametrize the solution in the form (82a–d). In order to eliminate C_1 we rewrite the solution in terms of

$$\bar{\tau} = \tau - \ln C_1^2 , \quad \bar{\theta}^{(0)} = \theta + \frac{\sqrt{3}}{2} \ln C_1^2 , \quad \bar{C}_2^{(0)} = C_1^2 C_2 , \quad \text{and} \quad C_3^{(0)} = C_3 . \tag{93}$$

The variable $\bar{\tau}$ is monotonously increasing with τ for $\bar{\tau} \gg 1$, $|N| \ll 1$, and $|\kappa - 1| \ll 1$ (compare Eq. (83b)) and hence we can consider $\bar{\theta}^{(0)}$, $\bar{C}_2^{(0)}$, and $C_3^{(0)}$ as functions of $\bar{\tau}$ defined for $\bar{\tau}$ larger than some $\bar{\tau}_1$. From the constraint (49) it follows that $C_3^{(0)}$ is negligible compared to $e^{\bar{\tau}} \bar{C}_2^{(0)}$.

There are again three possibilities for $\bar{\tau} > \bar{\tau}_1$ and b fixed:

a) $e^{\bar{\tau}} \bar{C}_2^{(0)}$ tends to zero for $\bar{\tau} \to \infty$, i.e., $b \in \mathbf{Osc}$,

b) r reaches the value 1 for some $\tilde{\tau}$ and exceeds that value for $\bar{\tau} > \tilde{\tau}$, i.e., b is in some $\mathbf{Reg}_n$ or $\mathbf{Sing}_n$,

c) N has a zero for some $\tilde{\tau}$ and $r < 1$, i.e., $b \in \mathbf{Sing}_\infty$.
For case **a** we define $\tilde{C}_2 = 0$, for case **b** we define $\tilde{C}_2 = \bar{C}_2^{(0)}(\bar{\tau}) > 0$, and for case **c** we define $\tilde{C}_2 = \bar{C}_2^{(0)}(\bar{\tau}) < 0$.

Again the numerical results indicate that $\tilde{C}_2$ is strictly decreasing with b. This transversality would imply that $b \in \mathbf{Sing}_\infty$ for $b > b_\infty$ and that b is in some $\mathbf{Reg}_n$ or $\mathbf{Sing}_n$ for $b < b_\infty$.

The solution for $b = b_\infty$ is a member of the four-parameter family discussed in Prop. 17. If this family of solutions is expressed in terms of $\bar{\theta}^{(\infty)}(\bar{\tau})$, $\bar{C}_2^{(\infty)}(\bar{\tau})$, and $C_3^{(\infty)}(\bar{\tau})$ as above then the relation $\bar{C}_2^{(\infty)} = f(\bar{\theta}^{(\infty)}, C_3^{(\infty)})$ for $\bar{\tau}$ fixed and sufficiently large defines again an analytic function f. For b sufficiently close to b_∞ the difference

$$\Delta(b) = \bar{C}_2^{(0)}(b) - f(\bar{\theta}(b), C_3^{(0)}(b)) \tag{94}$$

is again analytic in b and vanishes for $b = b_\infty$.

Lemma 24. *Given the function $\Delta(b)$ defined above then there exists a neighbourhood of $b = b_\infty$ that contains no zeros of Δ except $b = b_\infty$.*

Proof. The proof is completely analogous to that of Lemma 21.

Combining the results of Lemma 20 and 21 we obtain:

Proposition 25. *Given $b_\infty \in$ **Osc** and some n_0 then all $b < b_\infty$ (and similarly all $b > b_\infty$) sufficiently close to b_∞ are either in $\mathbf{Sing}_\infty$ or in $\bigcup_{n \geq n_0} (\mathbf{Reg}_n \cup \mathbf{Sing}_n)$.*

Combining the results of this chapter we obtain:

Theorem 26 (Existence of globally regular solutions). i) *The sets $\mathbf{Reg}_n$ and **Osc** are all nonempty, i.e., for each $n = 0, 1, 2, \ldots$ there exists a globally regular solution with n zeros of W for at least one $b_n \in \mathbf{Reg}_n$ and there exists an oscillating solution with $N > 0$ for all τ and $r \to 1$ for $\tau \to \infty$ for at least one $b_\infty \in$ **Osc**.*
ii) *The union $\bigcup_{n \geq 0} \mathbf{Reg}_n$ has accumulation points that are contained in **Osc**, i.e., there exists at least one sequence of globally regular solutions and one oscillating solution W_∞ such that $W_n(r) \to W_\infty(r)$ for $r < 1$ and $W_n(r) \to 0$ for $r \geq 1$ for $n \to \infty$.*

Proof. i) It follows from Prop. 11 that all $b < 0$ are in $\mathbf{Sing}_0$ and from Prop. 19 that all $b \gg 1$ are in $\mathbf{Sing}_\infty$. As a consequence of Props. 22 and 25 there must be at least one $b_n \in \mathbf{Reg}_n$ for each n and at least one $b_\infty \in$ **Osc** as b varies from negative to large positive values.
ii) Each of the sets $\mathbf{Reg}_n$ and **Osc** consists of a finite number of points but there must be accumulation points in the union $\bigcup_{n \geq 0} \mathbf{Reg}_n$ since the b_n are all bounded. Moreover two values b_n and b_{n+1} are separated by (at least) one open set contained in $\mathbf{Sing}_{n+1}$ and hence these accumulation points must be in **Osc**.

Remark. Transversality would imply that these solutions are unique, i.e., there is exactly one b_n for each n and exactly one b_∞ with $b_n < b_{n+1}$ and $b_n \to b_\infty$ as $n \to \infty$.

9. Black Hole Solutions

In Sect. 4 we have stated the existence of a two-parameter family of local non-degenerate black hole solutions of Eq. (6) parametrized by (r_h, W_h) with $r_h > 0$ and $(W_h^2 - 1)^2 < r_h^2$ (compare Prop. 4). For $W_h = 0$ and $r_h > 1$ these are the abelian RN black holes that tend to the extremal RN solution for $r_h \to 1$. Hence we can restrict the initial values to the domain

$$\begin{aligned} \sqrt{1-r_h} < W_h < \sqrt{1+r_h} \text{ for } 0 < r_h < 1 \,, \\ 0 < W_h < \sqrt{1+r_h} \text{ for } r_h \geq 1 \,. \end{aligned} \tag{95}$$

As discussed earlier (after Eq. (51b)) W and N are analytic functions of τ, r_h, and W_h. Near the horizon $r_h = r(0)$ we find the expansion

$$\begin{aligned} W(\tau) &= W_h\Big(1 + \frac{(W_h^2-1)\tau^2}{4}\Big) + O(\tau^4) \,, \\ N(\tau) &= \Big(1 - \frac{(W_h^2-1)^2}{r_h^2}\Big)\Big(\frac{\tau}{2} + O(\tau^3)\Big) - \frac{W_h^2\tau^3}{4} + O(\tau^5) \,. \end{aligned} \tag{96}$$

In order to analyze the global behaviour of these solutions we proceed in a similar way as we did for the regular solutions in the preceding chapter. First we partition the domain of initial data (95) into sets $\mathbf{Reg}_n$ (regular black hole solutions), $\mathbf{Sing}_n$, $\mathbf{Sing}_\infty$, and $\mathbf{Osc}$ as before. Next we define for each r_h and n the sets $\mathbf{Reg}_n^{r_h} = \{W_h : (r_h, W_h) \in \mathbf{Reg}_n\}$ and similarly $\mathbf{Sing}_n^{r_h}$, and for each $r_h < 1$ the sets $\mathbf{Sing}_\infty^{r_h}$, and $\mathbf{Osc}^{r_h}$. We shall conclude that for each n and r_h there is at least one $W_{hn} \in \mathbf{Reg}_n^{r_h}$ and for each $r_h < 1$ there is at least one $W_{h\infty} \in \mathbf{Osc}^{r_h}$.

For $W_h > 1$ the function W is monotonously increasing and Prop. 11 implies that $N(\tau_0) = 0$ for some τ_0, i.e., the values $W_h > 1$ are contained in $\mathbf{Sing}_0^{r_h}$ for all r_h. For $W_h = 1$ we get the trivial solution with $W \equiv 1$ and the Schwarzschild metric, i.e., $0 \in \mathbf{Reg}_0^{r_h}$ for all r_h.

The case $W_h \approx 1$ for any r_h is analogous to $|b| \ll 1$ for solutions regular at $r = 0$.

Proposition 27. *Given $r_h > 0$ and $W_h < 1$, resp. $W_h > 1$ such that $|1 - W_h| \ll \min(r_h, 1)$ then W is monotonously decreasing, resp. increasing and $N(\tau_0) = 0$ for some τ_0, i.e., all $W_h < 1$ sufficiently close to 1 are in $\mathbf{Sing}_1^{r_h}$. The values of r, W, and U at τ_0 diverge for $W_h \to 1$.*

Proof. The solution depends continuously on W_h and therefore W, U, N, and κ remain close to their values for the Schwarzschild solution with the same r_h until $r \gg \max(r_h, 1)$ and $1 - N \ll 1$. Then we can proceed as in the proof of Lemma 20 with the only difference that the extremum of W is at $\tilde{r} = r_h$.

The case $r_h < 1$ and W_h close to $\sqrt{1-r_h}$ is analogous to that of $b \gg 1$.

Proposition 28. *Given $0 < r_h < 1$ and W_h such that $0 < \dfrac{W_h - \sqrt{1-r_h}}{r_h} \ll 1$ then W is monotonously decreasing and $N(\tau_0) = 0$ for some $\tau_0 \ll 1$ with $0 < W(\tau_0) < W_h$, i.e., W_h values sufficiently close to $\sqrt{1-r_h}$ are in $\mathbf{Sing}_\infty$.*

Proof. W and N are analytic functions of τ with the expansion (96). Hence $N(\tau_0) = 0$ for $\tau_0 \approx \sqrt{\frac{8(W_h - \sqrt{1-r_h})}{r_h W_h}}$ and $W(\tau_0) \approx 2\sqrt{1-r_h} - W_h$.

Next consider the remaining extreme case $r_h \geq 1$ and $W_h \ll 1$:

Proposition 29. *Given* $r_h \geq 1$ *and some* n_0 *then all sufficiently small* W_h *are in* $\bigcup_{n \geq n_0} (\mathbf{Reg}_n^{r_h} \cup \mathbf{Sing}_n^{r_h})$.

Proof. The solution depends continuously on W_h and therefore W, U, N, and κ remain close to their values for the RN solution with the same r_h until $r \gg 1$ and $1 - N \ll 1$. Then with W and rU still small the solution is close to the form (39) with at least n_0 zeros provided W_h is small enough.

Remark. For $r_h \to \infty$ and W_h fixed (but not necessary small) W and N tend to universal functions of $\frac{r}{r_h}$.

The remaining argument is completely analogous to the one used for the regular solutions. Therefore we just formulate the results without proof.

Lemma 30. *Given* $(r_h, W_h)_n \in \mathbf{Reg}_n$ *for any* n *then all initial data sufficiently close to* $(r_h, W_h)_n$ *are in either* $\mathbf{Reg}_n$, *or* $\mathbf{Sing}_n$, *or* $\mathbf{Sing}_{n+1}$.

Proposition 31. *Given* $(r_h, W_{hn}) \in \mathbf{Reg}_n^{r_h}$ *for any* n *and* r_h *then all* $W_h > W_{hn}$ *(and similarly all* $W_h < W_{hn}$*) sufficiently close to* W_{hn} *are either in* $\mathbf{Sing}_n^{r_h}$ *or in* $\mathbf{Sing}_{n+1}^{r_h}$.

Lemma 32. *Given* $(r_h, W_h)_\infty \in \mathbf{Osc}$ *and some* n_0 *then all initial data sufficiently close to* $(r_h, W_h)_\infty$ *are in either* $\mathbf{Osc}$, *or* $\mathbf{Sing}_\infty$, *or* $\bigcup_{n \geq n_0} (\mathbf{Reg}_n \cup \mathbf{Sing}_n)$.

Proposition 33. *Given* $(r_h, W_{h\infty}) \in \mathbf{Osc}^{r_h}$ *for any* $r_h < 1$ *and some* n_0 *then all* $W_h > W_{h\infty}$ *(and similarly all* $W_h < W_{h\infty}$*) sufficiently close to* $W_{h\infty}$ *are either in* $\mathbf{Sing}_\infty^{r_h}$ *or in* $\bigcup_{n \geq n_0} (\mathbf{Reg}_n^{r_h} \cup \mathbf{Sing}_n^{r_h})$.

Combining the results of this chapter we obtain:

Theorem 34 (Existence of black hole solutions). i) *The sets* $\mathbf{Reg}_n^{r_h}$ *and* $\mathbf{Osc}^{r_h}$ *are all nonempty, i.e., for each* $n = 0, 1, 2, \ldots$ *and* r_h *there exists a regular black hole solution with* n *zeros of* W *for at least one* $W_{hn} \in \mathbf{Reg}_n^{r_h}$ *and for each* $r_h < 1$ *there exists an oscillating solution* W_∞ *with* $N > 0$ *for all* $\tau > 0$ *and* $r \to 1$ *for* $\tau \to \infty$ *for at least one* $W_{h\infty} \in \mathbf{Osc}^{r_h}$.
ii) *The union* $\bigcup_{n \geq 0} \mathbf{Reg}_n^{r_h}$ *has accumulation points that are contained in* $\mathbf{Osc}^{r_h}$ *for* $r_h < 1$, *resp. accumulates at* $W_h = 0$ *for* $r_h \geq 1$, *i.e., for each* $r_h < 1$ *there exists at least one sequence of black hole solutions and one limiting solution such that* $W_n(r) \to W_\infty(r)$ *for* $r < 1$ *and* $W_n(r) \to 0$ *for* $r \geq 1$ *as* $n \to \infty$ *and for each* $r_h \geq 1$ *there exists at least one sequence of black hole solutions such that* $W_h \to 0$ *and* $W_n(r) \to 0$ *as* $n \to \infty$.
iii) *The connected components of the sets* $\mathbf{Reg}_n$ *and* $\mathbf{Osc}$ *are either isolated points or analytic curves in the* (r_h, W_h) *plane.*

Proof. i) and ii) exactly as the proof of Theorem 26.
iii) Props. 4 and 3, resp. 17 state the local existence of families of solutions at the singular points $r = r_h$ and $r = \infty$, resp. $\tau = \infty$. The values of these solutions in a neighbourhood of the singular points define analytic manifolds that can be extended at least as long as $N > 0$. The intersections of the analytic manifold defined near $r = r_h$ with that defined near $r = \infty$, resp. $\tau = \infty$ are again analytic manifolds. The (r, W) values of these intersections at $r = r_h$ are the sets $\bigcup_{n \geq 0} \mathbf{Reg}_n$, resp. $\mathbf{Osc}$.

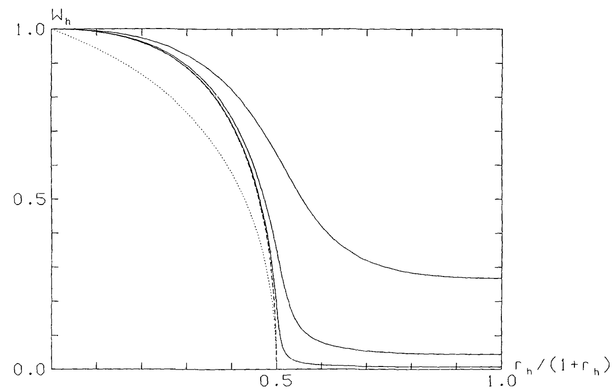

Fig. 3. Initial data for black hole solutions:
$W_{hn}(r_h)$ for $n = 1, 2, 3$ (solid lines), $W_{h\infty}(r_h)$ (dashed line), and the curve $r_h = 1 - W_h^2$ (dotted line)

Remark. One can also define a transversality property for black holes with r_h fixed with similar consequences as for regular solutions.

Figure 3 shows the initial data $W_{hn}(r_h)$ for $n = 1, 2, 3$ and $W_{h\infty}(r_h)$ for black hole solution obtained by numerical integration.

10. Large n Behaviour

In Sect. 3 we observed a characteristic n-dependence of the parameters of the globally regular solutions (Eq. (14)). Already for $n \gtrsim 4$ a typical scaling factor e^{α} with $\alpha \approx 1.8138$ appears in the comparison of the n^{th} with the $(n+1)^{\text{st}}$ solution. Whereas b_n and M_n seem to converge exponentially fast to finite limits b_∞, resp. $M_\infty = 1$, the parameters c_n increase with n as $e^{n\alpha}$.

In the following we shall explain this asymptotic behaviour for large n. This explanation is based on a characteristic behaviour of the solutions with many zeros of W that is already rather pronounced for $n = 4$. As described in Sect. 3 the numerical results suggest to introduce three regions: The inner region (I) where the solutions are close to the limiting solution with $b = b_\infty$. The small field region (II) where W is small and $N \approx 1 - \frac{1}{r}$, the extremal RN solution. And finally the asymptotic region (III) where $N \approx 1$ and $(-1)^n W_n$ is a universal function of $\frac{r}{c_n}$. The latter is nothing but the separatrix of the flat space YM Eq. (37) running from $(W, \dot{W}) = (0, 0)$ to $(1, 0)$, as described in Sect. 5. Since W_n is very small in region II it is well approximated by a solution of the linearized YM equation in the RN background. The boundary conditions for this linear problem are provided by the universal solutions in regions I and III. This will eventually explain the discrete spectrum of solutions.

For $b \approx b_\infty$ and $r \approx 1$ the solution can be expressed in the form (82) with slowly varying coefficient functions $C_i(b, \tau)$ and $\theta(b, \tau)$. The solution for $b = b_\infty$ is characterized by the limiting values (for $\tau \to \infty$) $C_1(b_\infty)$, $\theta(b_\infty)$, and $C_2(b_\infty) = C_3(b_\infty) = 0$. For $b \lesssim b_\infty$ the function $C_2(b, \tau)$ must be positive otherwise $r(\tau)$ would stay smaller than one. Clearly as b tends to b_∞ the value $r(\tau) = 1$ will be reached for larger and larger values of τ. Introducing the modified variable $\bar{\tau} = \tau - \ln C_1^2$ used before we can parametrize the solution in a neighbourhood of $r = 1$ by

$$\ln r_{\mathrm{I}}(\bar\tau) = C_3 + \bar C_2 e^{\bar\tau} - \frac{1}{2}\bar C_2^2 e^{2\bar\tau} - \frac{1}{2}e^{-\bar\tau}\left(1 - \frac{1}{\sqrt{7}}\cos\left(\sqrt{3}\bar\tau + \phi + \frac{\pi}{3} + 2\bar\theta\right)\right), \quad (97a)$$

$$N_{\mathrm{I}}(\bar\tau) = \bar C_2 e^{\bar\tau} - \bar C_2^2 e^{2\bar\tau} + \frac{1}{2}e^{-\bar\tau}\left(1 - \frac{2}{\sqrt{7}}\cos(\sqrt{3}\bar\tau + \phi + 2\bar\theta)\right), \quad (97b)$$

$$W_{\mathrm{I}}(\bar\tau) = e^{-\frac{1}{2}\bar\tau}\sin\left(\frac{\sqrt{3}}{2}\bar\tau + \bar\theta\right), \quad (97c)$$

$$U_{\mathrm{I}}(\bar\tau) = e^{-\frac{1}{2}\bar\tau}\sin\left(\frac{\sqrt{3}}{2}\bar\tau + \frac{2\pi}{3} + \bar\theta\right). \quad (97d)$$

These expressions differ from those of Eq. (82) by the non-leading terms proportional to $\bar C_2^2$. With these terms using the constraint (49) one can show that $C_3 \ll e^{-\bar\tau}$. The phase $\bar\theta$ is defined only up to multiples of 2π. We use this freedom to adjust $\bar\theta$ such that $\frac{\sqrt{3}}{2}\bar\tau + \bar\theta = m\pi$ at the m^{th} zero of W. The functions $\bar\theta$ and $\bar C_2$ are approximately τ-independent as long as $e^{-\bar\tau} \ll \bar C_2 e^{\bar\tau} \ll 1$. It is important that the validity of this condition extends into the region II with $r > 1$ where the solution can be well approximated by the solution of the linearized YM equation in the RN background

$$r_{\mathrm{II}}(\bar\tau) = 1 + \bar C_2 e^{\bar\tau}, \quad (98a)$$

$$N_{\mathrm{II}}(\bar\tau) = \frac{\bar C_2 e^{\bar\tau}}{1 + \bar C_2 e^{\bar\tau}}, \quad (98b)$$

$$W_{\mathrm{II}}(\bar\tau) = e^{-\frac{1}{2}\bar\tau}\sin\left(\frac{\sqrt{3}}{2}\bar\tau + \bar\theta\right) + \bar C_2 e^{\frac{1}{2}\bar\tau}\sin\left(\frac{\sqrt{3}}{2}\bar\tau + \frac{\pi}{3} + \bar\theta\right), \quad (98c)$$

with the choice of parameters such that the two forms of the solution match in the overlap of regions I and II. In region II the position of the m^{th} zero of W shifts from $\frac{\sqrt{3}}{2}\bar\tau + \bar\theta = m\pi$ for $\bar C_2 e^{\bar\tau} \ll 1$ to $\frac{\sqrt{3}}{2}\bar\tau + \frac{\pi}{3} + \bar\theta = m\pi$ for $\bar C_2 e^{\bar\tau} \gg 1$.

Finally the form (98 of the solution has to be joined to the one obtained in region III. There $\bar C_2 e^{\bar\tau} \gg 1$, i.e., $1 - N \ll 1$ and hence W approximately solves the flat space YM equation with boundary condition $W(\tau) \to \pm 1$. In the region where W is small this solution can be approximated by

$$r_{\mathrm{III}}(\hat\tau) = c e^{\hat\tau}, \quad (99a)$$

$$W_{\mathrm{III}}(\hat\tau) = \pm\hat C_1 e^{\frac{1}{2}\hat\tau}\sin\left(\frac{\sqrt{3}}{2}\hat\tau + \frac{\pi}{3} + \hat\theta\right), \quad (99b)$$

with the normalization $W_{\mathrm{III}} \to \pm(1 - e^{-\hat\tau})$. Again the phase $\hat\theta$ is adjusted such that $\frac{\sqrt{3}}{2}\hat\tau + \frac{\pi}{3} + \hat\theta = -m\pi$ at the last but m^{th} zero of W. Matching this form with the one valid in region II we obtain the conditions

$$\bar C_{2,n} e^{\bar\tau} = c_n e^{\hat\tau}, \quad (100a)$$

$$\bar C_{2,n} e^{\frac{1}{2}\bar\tau} = \hat C_1 e^{\frac{1}{2}\hat\tau}, \quad (100b)$$

$$\frac{\sqrt{3}}{2}\bar\tau + \bar\theta = \frac{\sqrt{3}}{2}\hat\tau + \hat\theta + n\pi, \quad (100c)$$

where n is the total number of zeros of W. Eliminating $\bar{\tau}$ and $\hat{\tau}$ we obtain the "quantization condition"

$$\bar{C}_{2,n} = \hat{C}_1 e^{\frac{1}{\sqrt{3}}(\bar{\theta}-\hat{\theta}-n\pi)} \equiv \bar{C}_{2,0} e^{-n\frac{\pi}{\sqrt{3}}} , \tag{101a}$$

and the relation

$$c_n = \bar{C}_{2,n}^{-1} \hat{C}_1^2 \equiv c_0 e^{n\frac{\pi}{\sqrt{3}}} . \tag{101b}$$

In order to estimate the mass M_n we consider the function

$$\bar{m} = m + \frac{1-W^4}{2r} - NUW = \frac{r}{2}(1-N^2) + \frac{1-W^4}{2r} - NUW , \tag{102}$$

satisfying the equation

$$\dot{\bar{m}} = 2(WU^3 - W^3U) . \tag{103}$$

Hence we can express the mass in the form

$$M = \bar{m}(\infty) = \bar{m}(\tau_1) + \int_{\tau_1}^{\infty} 2(WU^3 - W^3U)d\tau , \tag{104}$$

and choose a τ_1 at the end of region I where $C_3 \ll e^{-\bar{\tau}} \ll \bar{C}_2 e^{\bar{\tau}} \ll 1$. Neglecting terms much smaller than $\bar{C}_2$ we obtain

$$\bar{m}(\tau_1) = 1 - \frac{N^2 - \ln^2 r}{2} - NUW = 1 - \bar{C}_2 e^{\bar{\tau}}(N - \ln r + UW) = 1 - \frac{3}{4}\bar{C}_2 . \tag{105}$$

Integrating WU^3 yields a negligible result in regions II and III. In order to integrate W^3U we choose a τ_2 in the overlap of regions II and III where $\bar{C}_2 e^{\bar{\tau}} \gg 1$ and find that the contribution from region II is again negligible; in region III we obtain

$$-2\int_{\tau_2}^{\infty} W^3U d\tau = \left(\frac{W^2(1-W^2)}{r} - WU + rU^2\right)\Big|_{\tau_2}^{\infty} = -\frac{3}{4c}\hat{C}_1^2 = -\frac{3}{4}\bar{C}_2 . \tag{106}$$

Combining these results we find

$$M_n = 1 - \frac{3}{2}\bar{C}_{2,n} = 1 - \frac{3}{2}\bar{C}_{2,0} e^{-n\frac{\pi}{\sqrt{3}}} . \tag{107}$$

Assuming transversality $\bar{C}_2(b)$ is approximately proportional to $b_\infty - b$ whereas the b-dependence of $\bar{\theta}$ can be neglected. Numerical integration of the oscillating solution and its variation with respect to b yields

$$\bar{\theta} \approx 1.22457 \qquad \text{and} \qquad \bar{C}_2 \approx 0.329739 \cdot (b_\infty - b) , \tag{108}$$

numerical integration of Eq. (37) yields

$$\hat{\theta} \approx 0.339811 \qquad \text{and} \qquad \hat{C}_1 \approx 0.432478 . \tag{109}$$

Together with Eqs. (101, 107) we obtain the asymptotic expressions

$$b_n = b_\infty - 2.18595 \cdot e^{-n\alpha} , \quad M_n = 1 - 1.08119 \cdot e^{-n\alpha} , \quad c_n = 0.259489 \cdot e^{n\alpha} , \tag{110}$$

with $\alpha = \frac{\pi}{\sqrt{3}} \approx 1.81380$ and $e^\alpha \approx 6.13371$ in excellent agreement with the empirical formulae (14). Similar formulae could be derived for black hole solutions with $r_h < 1$ fixed and b_n replaced by $W_{hn}(r_h)$.

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