

The Existence of Non-Minimal Solutions of the Yang-Mills-Higgs Equations Over R^3 with Arbitrary Positive Coupling Constant

L. M. Sibner^{1, *}, J. Talvacchia^{2, **}

¹ Department of Mathematics, Polytechnic University of New York, Brooklyn, NY 11201, USA.
Email: lsibner@photon.poly.edu

² Department of Mathematics, Swarthmore College, Swarthmore, PA 19081, USA.
Email: jtalvac1@cc.swarthmore.edu

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Abstract: This paper proves the existence of a non-trivial critical point of the $SU(2)$ Yang-Mills-Higgs functional on R^3 with arbitrary positive coupling constant. The critical point lies in the zero monopole class but has action bounded strictly away from zero.

1. Introduction

This paper establishes the existence of a non-globally-minimizing critical point with monopole number zero for the $SU(2)$ Yang-Mills-Higgs equations on R^3 with positive coupling constant λ . The Yang-Mills-Higgs equation on R^3 are a system of second order non-linear equations:

$$\begin{aligned} *D_A * F &= [D_A \phi, \phi] && \text{YMH}\lambda(1), \\ *D_A * D_A \phi &= \frac{\lambda}{2} \phi (|\phi|^2 - 1) && \text{YMH}\lambda(2). \end{aligned}$$

Here, the variables are A , a connection on a principal $SU(2)$ bundle and ϕ , a section of the vector bundle $E = su(2) \times R^3$ called the Higgs field. D_A is covariant differentiation and F is the curvature of the connection A , $F = dA + A \wedge A$.

These equations can be viewed as the variational equations of the action functional:

$$A(A, \phi) = \frac{1}{2} \|F_A\|_2^2 + \frac{1}{2} \|D_A \phi\|_2^2 + \frac{\lambda}{8} \| |\phi|^2 - 1 \|_2^2.$$

If we restrict to finite action solutions, then the Higgs field approaches an asymptotic limit. Namely we have

$$\lim_{|x| \rightarrow \infty} |\phi(x)| = 1,$$

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uniformly with $|x|$ (see [JT], p. 156). There is a topological invariant associated with the model, called the magnetic charge or monopole number,

$$k = \frac{1}{4\pi} \operatorname{tr} \int_{R^3} F \wedge D_A \phi.$$

This monopole number is an integer, as was shown by Groisser [G2]. This integer equals, in fact, the degree of the homotopy class of maps $S^2 \rightarrow S^2$ obtained by restricting the normalized Higgs field, $\frac{\phi}{|\phi|}$ to large spheres in R^3 .

A framework for analyzing the Yang-Mills-Higgs equations in terms of the Calculus of Variations was first established by C. Taubes ([T1, T2]). Observing that the Palais-Smale condition fails for these equations, Taubes developed a form of Ljusternik-Schnirelman theory of handle the analytic difficulties presented by the non-compactness and established the existence of a critical point with monopole number zero for the Yang-Mills-Higgs functional in the $\lambda = 0$ case that was non-minimizing even locally. In unpublished work, D. Groisser ([G1]) extended this result to the case λ sufficiently small. In this paper we show that there is a non-globally-minimizing solution with monopole number zero and λ an arbitrary positive constant. (*Remark.* It is possible that this solution is a local minimum.)

Theorem 1.1. *There exists a configuration c with monopole number zero that satisfies the Yang-Mills-Higgs equation (YMH λ 1) and (YMH λ 2) for arbitrary positive λ and such that c is not a global minimum of the action $\mathbf{A}(A, \phi)$.*

The proof draws heavily on the work of Taubes ([T1, T2]) and in many cases the proofs given are modifications of his arguments to show that the framework he established for $\lambda = 0$ holds in the more general case.

The outline of the paper is as follows: In Sect. 2 we give definitions and notation that will be useful in the main development. In Sect. 3 we establish the existence of a configuration that achieves the minimum action over all classes of configurations with non-zero monopole numbers. This result was previously unproved and interesting in its own right. In Sect. 4, we use this minimum to construct a non-trivial loop of configurations that generates the homotopy class for the min-max procedure. The key part of the section in the upper bound on the action that allows the min-max procedure to work. In Sect. 5, we execute the min-max argument and show that the critical point obtained must be non-globally-minimizing.

2. Definitions and Notation

As stated before, the variables for the $SU(2)$ Yang-Mills-Higgs equations are a connection A on the principal bundle $SU(2) \times R^3$ and a section ϕ of the vector bundle $su(2) \times R^3$. Thus A is a Lie Algebra valued 1-form and we write $A = A_i dx^i$, where $A_i(x)$ is a 2×2 traceless anti-hermitian matrix. Denote sections of a bundle by Γ . Denote compactly supported connections by Γ^c .

Definition 2.1. *A configuration, c , is an ordered pair $c = (A, \phi)$, where $A \in \Gamma(\mathcal{S} \otimes \mathcal{T}^*)$ and $\phi \in \Gamma(\mathcal{S})$ and for this problem $\mathcal{S} = su(2)$.*

The action $\mathbf{A}(c)$ is not finite for every possible configuration c , so we restrict ourselves to the subset of configurations that yield finite action and so that ϕ satisfies the boundary conditions described in Sect. 1.

Definition 2.2. Define the configuration space \mathbf{C} , by $\mathbf{C} = \{c = (A, \phi) \in \Gamma(\mathcal{S} \otimes \mathcal{T}^*) \times \Gamma(\mathcal{S}) \mid \mathbf{A}(c) < \infty \text{ and } \lim_{|x| \rightarrow \infty} |\phi(x)| = 1\}$.

The topology on \mathbf{C} is taken to be the intersection of the C^∞ topology with the weakest topology which renders continuous the functions:

1. $\check{A}(A, \phi) = \frac{1}{2} \|F_A\|_2^2 + \frac{1}{2} \|D_A \phi\|_2^2 : C \rightarrow R$.
2. $\|1 - |\phi|^2\|_2^2 : C \rightarrow R$.
3. $|\phi| : C \rightarrow \overline{C^0(R^3)}$, where $\overline{C^0(R^3)}$ is the set of continuous functions on R^3 with the topology induced by the sup norm.

Define the configuration space $\check{\mathbf{C}}$, by $\check{\mathbf{C}} = \{c = (A, \phi) \in \Gamma(\mathcal{S} \otimes \mathcal{T}^*) \times \Gamma(\mathcal{S}) \mid \check{A}(c) < \infty \text{ and } \lim_{|x| \rightarrow \infty} |\phi(x)| = 1\}$.

The topology on $\check{\mathbf{C}}$ is taken to be the same as the topology on \mathbf{C} save condition 2. Taubes [T1, T3] proved that $\pi_1(\check{\mathbf{C}}) \cong \pi_1(\text{Maps}(S^2, S^2)) \cong \pi_3(S^2)$. The inclusion $\mathbf{C} \hookrightarrow \check{\mathbf{C}}$ induces a homomorphism between $\pi_1(\mathbf{C})$ and $\pi_1(\check{\mathbf{C}})$.

The following operations will be used throughout the course of the paper and we establish notation for them here:

The Lie Algebra $su(2)$ as the vector space of 2×2 traceless anti-hermitian matrices has a positive definite inner product we denote by $(,)$. Thus if $\sigma^1, \sigma^2 \in su(2)$, then $(\sigma^1, \sigma^2) = -2 \text{ trace}(\sigma^1, \sigma^2)$.

We denote by $\bigwedge_p T^*$ the space of p -forms on R^3 , $p = 0, 1, 2, 3$. Take the usual Euclidean metric on T^* . This induces a positive inner product on $\bigwedge_p T^*$ via the Hodge star operator. Together these two metrics induce an inner product on $\mathcal{S} \otimes \bigwedge_p \mathcal{T}^*$, also denoted by $(,)$. The norm induced by this inner product will be denoted by $|\cdot|$. Thus for $\omega \in \mathcal{S} \otimes \bigwedge_p \mathcal{T}^*$, $|\omega| = (\omega, \omega)^{\frac{1}{2}}$.

An L_2 inner product on $\Gamma(\mathcal{S} \otimes \bigwedge_p \mathcal{T}^*)$ is defined in the usual way:

$$\langle \omega_1, \omega_2 \rangle_2 = \int d^2x (\omega_1, \omega_2)(x),$$

where $\omega_i \in \Gamma(\mathcal{S} \otimes \bigwedge_p \mathcal{T}^*)$.

We denote the norm induced by this inner product by $\|\cdot\|$. Thus

$$\|\omega\|_2 = \langle \omega, \omega \rangle_2^{\frac{1}{2}}.$$

The covariant derivative on sections of \mathcal{S} is denoted by $D_A : \Gamma(\mathcal{S}) \rightarrow \Gamma(\mathcal{S} \otimes \mathcal{T}^*)$ and is defined by $D_A \phi = d\phi + [A, \phi]$. We can extend the covariant derivative to p -forms in two ways. The first extension we also denote by D_A :

$$D_A : \Gamma\left[\mathcal{S} \otimes \bigwedge_p \mathcal{T}^*\right] \rightarrow \Gamma\left[\mathcal{S} \otimes \bigwedge_{p+1} \mathcal{T}^*\right],$$

$$D_A \omega = d\omega + [A, \omega].$$

The second extension is denoted by ∇_A .

$$\nabla_A : \Gamma \left[\mathcal{S} \otimes \bigwedge_p \mathcal{S}^* \right] \rightarrow \Gamma \left[\mathcal{S} \otimes \bigwedge_p \mathcal{S}^* \otimes \mathcal{S}^* \right],$$

$$\nabla_A \omega = \sum_{i=1}^3 \left(\frac{\partial \omega}{\partial x_i} + [A_i, \omega] \right) \otimes dx^i.$$

Definition 2.3. For $c \in C$, the gradient of \mathbf{A} at \mathbf{c} , denoted $\nabla \mathbf{A}_c$, is a linear functional on $\Gamma^c((\mathcal{S} \otimes \mathcal{S}^*) \oplus \mathcal{S})$ defined by

$$\nabla \mathbf{A}_c(\psi) = \frac{d}{ds} \mathbf{A}(c + s\psi)|_{s=0}.$$

If $c = (A, \phi)$ and $\psi = (\omega, \eta)$, a short computation yields that

$$\nabla \mathbf{A}_c(\psi) = \langle D_A \omega, F \rangle_2 + \langle [\omega, \phi], D_A \phi \rangle_2 + \langle D_A \eta, D_A \phi \rangle_2 + \frac{\lambda}{2} \langle (|\phi|^2 - 1)\phi, \eta \rangle_2.$$

Definition 2.4. A configuration $c \in C$ is a critical point of A if $\nabla \mathbf{A}_c(\cdot) \equiv 0$ on $\Gamma^c((\mathcal{S} \otimes \mathcal{S}^*) \oplus \mathcal{S})$.

Definition 2.5. Let $c = (A, \phi) \in C$. Define the Banach space H_c to be the completion of $\Gamma^c((\mathcal{S} \otimes \mathcal{S}^*) \oplus \mathcal{S})$ in the following norm:

$$\|\psi\|_c^2 = \|\nabla_A \psi\|_2^2 + \|[\phi, \psi]\|_2^2.$$

One can extend this norm to $\Gamma^c((\mathcal{S} \otimes \bigwedge_p \mathcal{S}^*) \otimes \mathcal{S})$ in the obvious way. We denote by $\|\cdot\|_{c^*}$, the standard norm induced by $\|\cdot\|_c$ on the dual space.

Definition 2.6. Define the Banach space $K(R^3)$ to be the completion of $C_0^\infty(R^3)$ in the norm $\|\nabla(\cdot)\|_2$.

Define the Banach space $K_A(\mathcal{S} \otimes \bigwedge_p \mathcal{S}^*)$ for $A \in \Gamma(\mathcal{S} \otimes \mathcal{S}^*)$ and $p = 1, 2, 3$ to be the completion of $\Gamma^c(\mathcal{S} \otimes \bigwedge_p \mathcal{S}^*)$ in the norm

$$\|\phi\|_{K_A}^2 = \|\nabla_A \phi\|_2^2.$$

Definition 2.7 (Uhlenbeck). Let $\{c_i = (A_i, \phi_i)\}_{i=1}^\infty \in C$. The sequence $\{c_i\}$ is said to converge strongly to $L_{2, \text{loc}}^1$ to $c = (A, \phi) \in C$ if the following is true:

1. There exists a uniform open cover of R^3 by balls $\{V_\alpha\}$ of radius $r > 0$.
2. There exists, for each i, α , gauge transformations $g_\alpha(i) \in L_2^1(V_\alpha; SU(2))$.
3. For each α , the sequence $\{g_\alpha(i)c_i\}$ converges strongly in $L_2^1(V_\alpha; (\mathcal{S} \otimes \mathcal{S}^*) \oplus \mathcal{S})$ to some (A_α, ϕ_α) .
4. For each α, β the sequence $\{g_{\alpha\beta}(i)\} = \{g_\alpha(i)g_\beta^{-1}(i)\}$ converges strongly in $L_2^2(V_\alpha \cap V_\beta; SU(2))$.
5. In each $V_\alpha \cap V_\beta$, $(A_\alpha, \phi_\alpha) = g_{\alpha\beta}(A_\beta, \phi_\beta)$.
6. For each α , there exists $h_\alpha \in L_2^2(V_\alpha; SU(2))$ such that $h_\alpha c = (A_\alpha, \phi_\alpha)$ in V_α .

3. The Existence of the Minimum Over Non-Zero Monopole Classes

In this section we establish that the inf of the energy functional over all configurations with non-zero monopole number exists and is positive and, moreover, that there exists a configuration that achieves this inf.

We denote by C_k , the set of configurations in C whose monopole number is k .

Theorem 3.1. *There exists $c_0 \in C$ such that*

1. $\mathbf{A}(c_0) = \inf_{\substack{k \neq 0 \\ c \in C_k}} \mathbf{A}(c) = \mathbf{A}_0 > 0,$
2. $c_0 \in C_{k_0}$ for some $k_0 \neq 0$.

Proof of Theorem 3.1. Since each C_k is non-empty, the inf A_0 exists. Moreover, A is bounded below by $4\pi|k|$ on C_k (see [JT], p. 103), so in fact we have $A_0 \geq 4\pi$.

The argument that there exists a configuration that achieves this inf is more subtle. The idea is to construct a bounded sequence of configurations converging to \mathbf{A}_0 and then apply a weak compactness theorem due to Uhlenbeck. The hard part of this is insuring that the limiting configuration has positive monopole number. We execute this argument in the following sequence of lemmas:

Lemma 3.2. *Given $(A, \phi) \in C$, there exists $\hat{\phi}$ such that*

1. $\hat{\phi} - \phi \in K_A,$
2. $\mathbf{A}(A, \hat{\phi}) \leq \mathbf{A}(A, \phi),$
3. $*D_A *D_A \hat{\phi} = \frac{\lambda}{2}(1 - |\hat{\phi}|^2)\phi.$

Proof. This result was originally proved by Groisser. (See [G1], Appendix A.)

Modifying ϕ by an element of K_A does not change the monopole number. Thus, what we obtain from this lemma is that given A and fixed asymptotics for ϕ , (YMH λ 2) can always be solved without increasing the energy. Thus, one can restrict attention to minimizing sequences of configurations that apriori solve (YMH λ 2).

Lemma 3.3. *It is possible to choose a sequence of configurations $\{c_i\}$ with positive monopole number that satisfies $*D_{A_i} *D_{A_i} \phi_i = -\frac{\lambda}{2}(1 - |\phi_i|^2)\phi_i$ and such that*

1. $\lim_{i \rightarrow \infty} \mathbf{A}(c_i) = \mathbf{A}_0,$
2. $\mathbf{A}(c_i) \geq \mathbf{A}(c_{i+1}),$
3. $\lim_{i \rightarrow \infty} \|\nabla \mathbf{A}_i\|_{c_i^*} \rightarrow 0.$

Proof. Chose a sequence of configurations $\{c_i\}$ each with positive monopole number such that $\mathbf{A}(c_i) \rightarrow \mathbf{A}_0$. By a straightforward generalization of a proof of Taubes (see [T1], Sect. 6, or [G1], Sect. 2), one can show that $\|\nabla \mathbf{A}(c_i)\|_{c_i^*} \rightarrow 0$ or one could perturb a configuration c_i with $\mathbf{A}(c_i) = \mathbf{A}_0 + \varepsilon$ to a configuration \tilde{c}_i with $\mathbf{A}(\tilde{c}_i) < \mathbf{A}_0$, contradicting that \mathbf{A}_0 is the inf of the action. By Lemma 3.2, there exists a sequence $\{\hat{c}_i\}$ such that $\mathbf{A}(\hat{c}_i) \rightarrow \mathbf{A}_0$, $\|\nabla \mathbf{A}(\hat{c}_i)\|_{\hat{c}_i^*} \rightarrow 0$ and \hat{c}_i satisfies

$*D_{A_i} *D_{A_i} \hat{\phi}_i = \frac{-\lambda}{2}(1 - |\hat{\phi}_i|^2)\hat{\phi}_i.$ Now choose a monotone decreasing subsequence.

This lemma establishes the existence of a minimizing sequence that apriori satisfies the second Yang-Mills-Higgs equation. What is left to show is that the limit of this sequences of configurations is in fact a configuration with non-zero monopole number that solves both Yang-Mills-Higgs equations.

Definition 3.4. For $a \in R^3$, denote by $T_a c_i(x) = c_i(x - a)$ the configuration c_i translated by a .

Note that $\mathbf{A}(T_a c_i) = \mathbf{A}(c_i)$, $\|\nabla \mathbf{A}_{T_a c}\|_* = \|\nabla \mathbf{A}_c\|_*$ and if c_i satisfies $*D_{A_i} *D_{A_i} \phi_i = -\frac{\lambda}{2}(1 - |\phi_i|^2)\phi_i$ then so does $T_a c_i$. Thus the convergence properties of c_i will carry to a translated sequence.

Proposition 3.5. Let $\{c_i\}$ be a sequence of configurations and B a positive constant such that

$$\begin{aligned} \mathbf{A}(c_i) &\leq B, \\ *D_{A_i} *D_{A_i} \phi_i &= -\frac{\lambda}{2}(1 - |\phi_i|^2)\phi_i, \end{aligned}$$

and each c_i has monopole number $k_i > 0$. Assume $c_i \rightarrow c$ strongly in $L^1_{2, \text{loc}}$. Then there exists a subsequence, also denoted $\{c_i\}$, and a sequence of points $\{x_i\} \in R^3$ such that the translated sequence $\{T_{x_i} c_i\}$ converges strongly in $L^1_{2, \text{loc}}$ to a configuration c' with positive monopole number.

In order to prove this proposition, we first need a lemma concerning the zeroes of the Higgs field ϕ .

Lemma 3.6. Let $B > 0$. Let c be a configuration such that

$$\begin{aligned} \mathbf{A}(c) &\leq B, \\ *D_A *D_A \phi &= -\frac{\lambda}{2}(1 - |\phi|^2)\phi. \end{aligned}$$

Let $z(\phi) = \{x \in R^3 \mid |\phi(x)| = 0\}$. Then there exists a number $N = N(B)$ independent of ϕ such that the set $z(\phi)$ is contained in the union of N disjoint open balls of radius 1.

Proof of Lemma 3.6. For a configuration c satisfying the hypothesis, there exists a constant c_1 such that the Holder condition

$$|\phi(x) - \phi(y)| \leq c_1 |x - y|^{\frac{1}{2}} \quad \forall x, y \in R^3$$

is satisfied. (See [JT], p. 186–189.) Hence there exists a constant $c_2 > 0$ such that if $\phi(x) = 0$ and $|y - x| < c_2$, then $|\phi(y)| < \frac{1}{2}$. Let $\mu = \min(c_2, \frac{1}{3})$. Then for any $x \in z(\phi)$, we have

$$\int_{|y-x| \leq \mu} (1 - |\phi|)^6 dy \geq \frac{4\pi}{3} \mu^3 \frac{1}{64}.$$

However, for any $c \in C$, the L^6 norm of $1 - |\phi|$ has a bound depending only on the action:

$$\int_{R^3} (1 - |\phi|)^6 \leq c_3 \mathbf{A}(c)^3 \leq c_3 B^3$$

for some constant c_3 independent of the configuration c (see Corollary 4.13 of [T1]).

Now suppose $\{x_i\}_1^n$ is a subset of $z(\phi)$ with the property that for $i \neq j$, the open ball of radius 1 centered at x_i does not contain x_j . Then by the triangle inequality, the balls of radius $\frac{1}{3}$ centered at the x_i must be mutually disjoint, and since $\mu \leq \frac{1}{3}$, the same must be true of the balls of radius μ with these centers. Hence

$$c_3 B^3 \geq \int_{R^3} (1 - |\phi|)^6 \geq \sum_{i=1}^n \int_{|y-x_i| \leq \mu} (1 - |\phi|)^6 dy \geq n \frac{4\pi}{3} \mu^3 \frac{1}{64},$$

from which an upper bound N on n follows.

Finally, suppose that $z(\phi)$ cannot be covered by disjoint union of N open balls of radius 1. Then given any subset $\{x_i\}_1^N$ as above, there exists $x_{N+1} \in z(\phi)$ such that the distance from x_{N+1} to any other x_i is at least 1. Hence the ball of radius $\frac{1}{3}$ centered at x_{N+1} cannot intersect the ball of radius $\frac{1}{3}$ centered at any other x_i , so we have a collection of $N + 1$ points such that the balls of radius $\frac{1}{3}$ centered at the $\{x_i\}$ are mutually disjoint. From this contradiction the result follows.

Proof of Proposition 3.5. Use Lemma 3.6 to decompose R^3 (for each i) into a finite union of balls of radius one and their complement so that the zeroes of ϕ_i are contained in the union of these balls. Denote these balls by B_j^i , where $j = 1, \dots, n_i$ and their bounding spheres by S_j^i . Now n_i is a bounded infinite sequence and hence has a convergent subsequence. Since n_i is integer valued, this convergent subsequence must be a constant sequence. Denote this constant by n . Restrict our convergent sequence c_i to a subsequence (also denoted by c_i) such that R^3 decomposes into n balls containing the zeroes of ϕ_i and their complement.

Since there are no zeroes on the boundary of each ball, the winding number of k_j^i of ϕ_i on S_j^i is well defined. We claim $k_i = \sum_{j=1}^n k_j^i$. The argument is as follows:

Let $\hat{\phi}_i = \frac{\phi_i}{|\phi_i|}$.

Let $\omega = \frac{\sin \chi d\chi d\theta}{4\pi}$ be the volume form on S^2 .

By definition, $k_j^i = \int_{S_j^i} (\hat{\phi}_i)^* \omega$.

Let S^i be a sphere large enough to that all the S_j^i lie in its interior. Then

$$k_i = \int_{S^i} (\hat{\phi}_i)^* \omega.$$

If Ω denotes the region inside S^i and the outside of the S_j^i , then by Stokes theorem

$$\int_{S^i} (\hat{\phi}_i)^* \omega - \sum_{j=1}^n \int_{S_j^i} (\hat{\phi}_i)^* \omega = \int_{\Omega} d((\hat{\phi}_i)^* \omega) = 0.$$

Thus

$$k_i = \sum_{j=1}^n k_j^i. \tag{3.1}$$

We now construct a translated sequence of configurations with positive monopole number whose limit configuration has the desired properties. Denote the center of the ball B_j^i by x_j^i . To begin, consider B_1^i and construct the translated sequence $T_{x_1^i}(c_i)$. Examine now the behavior of the balls B_j^i , $j \neq 1$ as we translated B_1^i back to the origin. To do this, consider $\lim_{i \rightarrow \infty} d(x_1^i, x_j^i) = a_{1j}$, $j \neq 1$ where d denotes distance. For each j there are two choices, $a_{1j} = \infty$ or a_{1j} is finite.

Categorize the indices j according to whether a_{1j} is finite or infinite. Denote the set of indices j such that a_{1j} is finite by $\alpha_{11}, \dots, \alpha_{1h_1}$ and denote the other indices by $\beta_{11}, \dots, \beta_{1k_1}$. Restrict $T_{x_1^i}(c_i)$ to a subsequence that converges in $L_{2, \text{loc}}^1$ on a bounded set containing $B_1^i \cup B_{\alpha_{11}}^i \cup \dots \cup B_{\alpha_{1h_1}}^i$. Rename this sequence $T_{x_{l_1}^i}(c_i)$. We now turn to indices $\beta_{11}, \dots, \beta_{1k_1}$, marking the balls that went off to infinity as B_1^i was translated back to the origin. Consider $B_{\beta_{11}}^i$ which we rename $B_{l_2}^i$. Construct the translated sequence $T_{x_{l_2}^i}(c_i)$ (restricting to the subset of indices such that $T_{x_{l_2}^i}(c_i)$ converges) and examine the behavior of the remaining balls $B_{\beta_{l_2}}^i, \dots, B_{\beta_{1k_1}}^i$ as we translate $B_{l_2}^i$ back to the origin by considering $d(x_{l_2}^i, x_{\beta_{1j}}^i) = a_{l_2j}$, $j \neq 1$. Again categorize the indices j according to whether a_{l_2j} is finite or infinite. Denote the indices such that a_{l_2j} is finite by $\alpha_{21}, \dots, \alpha_{2h_2}$ and the remaining indices by $\beta_{21}, \dots, \beta_{2l_2}$. Restrict $T_{x_{l_2}^i}(c_i)$ to a subsequence (also denoted $T_{x_{l_2}^i}(c_i)$) that converges on a bounded set containing

$$B_{l_2}^i \cup B_{\alpha_{21}}^i \cup \dots \cup B_{\alpha_{2h_2}}^i .$$

Repeat this process now on the infinite indices $\beta_{21} \dots \beta_{2k}$ until all indices are used up; i.e., until we have each ball B_j^i , $j = 1, \dots, n$ either translated back to the origin or remaining a finite distance from a ball translated back to the origin. We end up with m sequences, $T_{x_{l_1}^i}(c_i), \dots, T_{x_{l_m}^i}(c_i)$ such that

$$\begin{aligned} T_{x_{l_1}^i}(c_i) \text{ converges on } B(1) \supset B_{l_1}^i \cup B_{\alpha_{11}}^i \cup \dots \cup B_{\alpha_{1h_1}}^i , \\ T_{x_{l_2}^i}(c_i) \text{ converges on } B(2) \supset B_{l_2}^i \cup B_{\alpha_{21}}^i \cup \dots \cup B_{\alpha_{2h_2}}^i \\ \vdots \\ T_{x_{l_m}^i}(c_i) \text{ converges on } B(m) \supset B_{l_m}^i \cup B_{\alpha_{m1}}^i \cup \dots \cup B_{\alpha_{mh_m}}^i \end{aligned}$$

and

$$(B_{l_1}^i \cup B_{\alpha_{11}}^i \cup \dots \cup B_{\alpha_{1h_1}}^i) \cup \dots \cup (B_{l_m}^i \cup B_{\alpha_{m1}}^i \cup \dots \cup B_{\alpha_{mh_m}}^i) = \bigcup_{j=1}^n B_j^i .$$

By Eq. (3.1), at least one of the translated sequences must have the property that the winding number of each configuration on a sphere S_R of sufficiently large radius is positive for i sufficiently large. Thus, the monopole number of the limit configuration of that translated sequence must be positive by the strong convergence of the integral for monopole number on S_R . (All zeroes of the Higgs fields not going off to infinity as i increases are contained in S_R).

Proposition 3.7. *Let $\{c_i\}$ be a sequence of configurations with positive monopole number and B a positive constant that satisfy*

1. $\mathbf{A}(c_i) \leq B$.
2. $\lim_{i \rightarrow \infty} \|\nabla \mathbf{A}_i\|_{c_i^*} \rightarrow 0$.
3. $*D_A *D_A \phi = -\frac{\lambda}{2}(|\phi|^2 - 1)\phi$.

Then there exists a sequence of points $\{x_i\} \in R^3$ such that a subsequence of the translated sequence $\{T_{x_i}(c_i)\}$ converges strongly in $L^1_{2,loc}$ to a configuration $c = (A, \phi) \in C$ with positive monopole number, where c is a solution of the Yang-Mills-Higgs equations (YMH λ 1), (YMH λ 2) and $\mathbf{A}(c) = \mathbf{A}_0$.

Proof. That there exists a subsequence that converges to a configuration c that satisfies (YMH λ 1) and (YMH λ 2) follows from an application of the Uhlenbeck weak compactness theorem as in Proposition 5.6 of [T1]. That there exists a translated subsequence so that the limit configuration still satisfies the equations and has positive monopole number follows from Proposition 3.5.

4. Construction of the Trial Loop

In Sect. 3, we proved the existence of a configuration that minimizes the action over all classes of configurations with non-zero monopole number. Call this minimizing configuration c_0 and denote its action by $\mathbf{A}_0 = \mathbf{A}(c_0)$. We wish to use this configuration to construct a non-trivial loop of configurations that acts as a generator of a non-trivial homotopy class of $\pi_1(C)$ on which we will apply min-max procedures. An important consequence of the construction will be an upper bound on the action that will insure convergence in the min-max process.

At first glance, the success of such an endeavor may seem hopeless as the existence of c_0 is known only abstractly. However, much of the asymptotic behavior of c_0 at infinity is known due to work by Jaffe and Taubes [JT] and Dostoglou [D]. Decay estimates on the curvature, the Higgs field, and the covariant derivative of the Higgs field are known. It is also known that any finite action $SU(2)$ monopole tends to a $U(1)$ Yang-Mills connection on the sphere at infinity (see [D]). This turns out to be enough to obtain the necessary upper bounds in the action. For convenience, we gather together here the decay estimates used. For the remainder of this section, we will let (r, χ, θ) denote spherical co-ordinates on R^3 and $\hat{i}, \hat{j}, \hat{k}$ denote the following basis for $su(2)$:

$$\hat{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

For an $su(2)$ valued function ψ , we can define a decomposition of ψ with respect to an element ϕ in $su(2)$ into what we will call longitudinal and transverse components:

$$\begin{aligned} \psi &= \psi_L + \psi_T, \\ \psi_L &= \phi(\phi, \psi)|\phi|^{-2}. \end{aligned}$$

Lemma 4.1 (Taubes). *Let (A, ϕ) be a smooth finite action solution to the Yang-Mills-Higgs equations with $\lambda \geq 0$. Define $m_L = \min(\sqrt{\lambda}, 2)$ and $m_T = 1$. Then given $\varepsilon > 0$, there exists a constant K such that $\forall x \in R^3$,*

1. $0 \leq 1 - |\phi(x)| \leq Ke^{-(1-\varepsilon)m_L|x|}$,
2. $|D_A\phi|_L(x) \leq Ke^{-(1-\varepsilon)m_L|x|}$,
3. $|D_A\phi|_T(x) \leq Ke^{-(1-\varepsilon)m_T|x|}$,
4. $|F_T(x)| \leq Ke^{-(1-\varepsilon)m_T|x|}$,
5. $|F_L(x)| \leq \frac{K}{|x|^2}$.

Proof. See [JT], p. 109.

Let (A, ϕ) be as in Lemma 4.1. Choose a radial gauge ($A_r = 0$). Let $i_R : S^2 \rightarrow R^3$ be the family of embeddings that sends the point (χ, θ) on the sphere to (R, χ, θ) on R^3 . Define $A_R = i_R^*(A)$.

Theorem 4.2 (Dostoglou). *A finite energy monopole becomes a pure $U(1)$ Yang-Mills connection at infinity. In analytic terms,*

$$(i) \quad \langle F_{A_R}, \phi \rangle = k_0 dS^2 + \omega(R, \cdot),$$

where k_0 is the monopole number, dS^2 is the surface element of the unit sphere, and ω is a real valued two form on R^3 with $|\omega| \leq \text{const. } r^{-3}$.

(ii) *In the radial gauge defined above, the connections A_R on the trivial bundle over S^2 converge pointwise to a connection \hat{A}_∞ , from which it follows that A has a limit A_∞ defined via $i_R^* A_\infty = \hat{A}_\infty$, and $A = A_\infty + B$, where $|B| \leq \frac{\varepsilon(r)}{r}$ with $\lim_{r \rightarrow \infty} \varepsilon(r) = 0$.*

Proof. See [D].

Now let $c_0 = (A_0, \phi_0)$ be the minimal configuration of Theorem 3.1. Using the above results, we obtain in a singular string gauge, where the string singularity is on the negative z axis.

Corollary 4.3. *The minimum configuration c_0 satisfies*

$$\begin{aligned} F_{A_0^g} &= k_0 \sin \chi \, d\chi \, d\theta \, \hat{i} + \omega_{\chi\theta} \, d\chi \, d\theta + (F)_T \\ &:= F_\infty + \omega_{\chi\theta} \, d\chi \, d\theta + (F)_T, \end{aligned} \tag{4.3a}$$

where $|\omega_{\chi\theta}| \leq \frac{b_1}{r}$ and $|(F)_T| \leq b_2 e^{-b_3 r}$ for some constants b_1, b_2 and $b_3 > 0$,

$$A_0^g = k_0(1 - \cos \chi) \, d\theta \, \hat{i} + B_0, \tag{4.3b}$$

where $(B_0)_r = 0$ and in any half-space, $z \geq c > 0$, $|B_0| \leq \frac{\varepsilon(r)}{r}$, and $|dB_0| \leq \frac{\varepsilon(r)}{r^2}$,

$$\int_{\frac{R}{2} \leq z \leq R} (|B_0|^2 + |dB_0|^2) d^3x \leq \frac{\varepsilon(R)}{R}, \tag{4.3c}$$

$$\phi_0^g = \hat{\phi} \hat{i} + \mu, \tag{4.3d}$$

where $\hat{\phi}$ denotes the component of ϕ_0^g in the \hat{i} direction, μ has terms involving \hat{j} and \hat{k} only and $|\mu| \leq b_4 e^{-b_5 r}$ for some constants b_4 and $b_5 > 0$.

Proof. (4.3a) follows immediately from Lemma 4.2(i).

To prove (4.3b and 4.3c), note that the connection defined by A_∞ restricted to S^2 is gauge equivalent on $S^2 - \text{south pole}$ to $k_0(1 - \cos \chi) d\theta \hat{i}$. Call this gauge information g . Note that g is independent of r . Extend g radially to a gauge transformation on all of R^3 . Then on any half space $t \geq c > 0$

$$\begin{aligned} A_0^g &= gA_0g^{-1} + gdg^{-1} \\ &= gA_\infty g^{-1} + gdg^{-1} + gBg^{-1} \end{aligned}$$

and g is smooth. Let $A_\infty^g = gA_\infty g^{-1} + gdg^{-1} = k_0(1 - \cos \chi) d\theta \hat{i}$. Let $B_0 = gBg^{-1}$. Since $B_0 = gBg^{-1}$, we have $|B_0| = |B| \leq \frac{\varepsilon(R)}{R}$.

To estimate dB_0 , recall that

$$\begin{aligned} F_{A_0^g} &= dA_0^g + A_0^g \wedge A_0^g \\ &= dA_\infty^g + dB_0 + B_0 \wedge B_0. \end{aligned}$$

Now comparing terms with the expression for $F_{A_0^g}$ given by (4.3a), we see that

$$|dB_0| \leq \frac{\varepsilon(R)}{R^2}.$$

Equation (4.3d) follows immediately from Lemma 4.1.

Lemma 4.4. (A_0^g, ϕ_0^g) is gauge equivalent to a smooth configuration (A_0^s, ϕ_0^s) on R^3 .

Proof. A_0^g has a string singularity on the half-line $\chi = \pi$. Since k_0 is an integer, the string can be inverted by a smooth gauge transformation. Therefore, the only apparent singularity is a point singularity at the origin. The Higgs action is finite and hence the results of [SS] imply that there is a gauge in which the configuration is smooth in a neighborhood of the origin.

We are now ready to define the loop of configurations. The definition is analogous to the definition in [SSU] in which the conformal invariance of the reflection in spheres in H^3 was exploited. Here in R^3 , we choose a plane $z = R$ far away from the origin in which to reflect.

Choose a cutoff function

$$\beta(x) = \begin{cases} 1, & \text{for } z \leq \frac{R}{2} \\ 0, & \text{for } z \geq \frac{3R}{4} \end{cases}.$$

Now define

$$\begin{aligned} \phi_R &= \hat{\phi} \hat{i} + \beta \mu, \\ A_R &= k_0(1 - \cos \chi) d\theta \hat{i} + \beta B_0. \end{aligned}$$

(Note μ and B_0 are defined as in Corollary 4.3 and that in a neighborhood of the plane $z = R$, the configuration (A_R, ϕ_R) is entirely in the \hat{i} direction.)

Next, let $\sigma_R : R^3 \rightarrow R^3$ denote the reflection in the plane $z = R$. In Cartesian coordinates, $\sigma_R(x, y, z) = (x, y, 2R - z)$.

The loop of configurations is now defined via the following gluing procedure:

$$\begin{aligned} \phi_\gamma &= \begin{cases} \phi_R, & \text{for } z \leq R; \\ \sigma_R^* \phi_R, & \text{for } z \geq R \end{cases} \\ A_\gamma &= \begin{cases} A_R & \text{for } z \leq R; \\ e^{-i\gamma} \sigma_R^* A_R e^{i\gamma} & \text{for } z \geq R \end{cases} \end{aligned}$$

where $0 \leq \gamma \leq 2\pi$.

Proposition 4.5. *The loop of configurations $c_\gamma = (A_\gamma^s, \phi_\gamma^s)$, $0 \leq \gamma \leq 2\pi$, defines a continuous map of $S^1 \rightarrow C_0$ which is homotopically nontrivial.*

Proof. The continuity of c_γ is obvious. (Note that the gauge transformations do not change any norms.)

To see that the monopole number of c_γ is zero, note that by Stokes theorem and the Bianchi identity

$$\frac{1}{4\pi} \int_{R^3} \text{tr } F_{A_\gamma} \wedge D_{A_\gamma} \phi_\gamma = \lim_{\varrho \rightarrow \infty} \int_{S_\varrho} \langle F_{A_\gamma}, \phi_\gamma \rangle,$$

where S_ϱ is the sphere of radius ϱ centered at $(0, 0, R)$.

Reflections in the plane $z = R$ sends the surface area element dS^2 to $-dS^2$. Thus on the lower hemisphere of S_ϱ we have

$$\langle F_{A_\gamma}, \phi_\gamma \rangle = k_0 dS^2 + \omega,$$

where $|\omega| \leq \frac{1}{\varrho^3}$. On the upper hemisphere,

$$\begin{aligned} \langle F_{A_\gamma}, \phi_\gamma \rangle &= \langle d\sigma_R^* A_R + \sigma_R^* A_R \wedge \sigma_R^* A_R, \sigma_R^* \phi_R \rangle \\ &= \langle \sigma_R^* dA_R + \sigma_R^* (A_R \wedge A_R), \sigma_R^* \phi_R \rangle \\ &= \sigma_R^* \langle dA_R + A_R \wedge A_R, \phi_R \rangle \\ &= \sigma_R^* (k_0 dS^2) + \sigma_R^* (\omega) \\ &= -k_0 dS^2 + \omega', \end{aligned}$$

where $|\omega'| \leq \frac{1}{\varrho^3}$. Thus $\lim_{\varrho \rightarrow \infty} \int_{S_\varrho} \langle F_{A_\gamma}, \phi_\gamma \rangle = 0$.

To see that c_γ is homotopically non-trivial, we take the loop out of the string gauge. Note that A_γ has a string singularity on the half-line $\chi = \pi, z \leq 0$, and on its reflected image, $\chi = 0, z \geq 2R$. For $z \leq R$,

$$A_\gamma = k_0(1 - \cos \chi) d\theta \hat{i} + \beta B_0$$

while for $z \geq R$,

$$A_\gamma = k_0(1 - \cos(\sigma_R^*(\chi))) d\theta \hat{i} + \sigma_R^*(\beta B_0).$$

The gauge transformation that removes the string for $z \leq R$ is

$$u(\chi, \theta) = \begin{pmatrix} \cos \frac{\chi}{2} & i \sin \frac{\chi}{2} e^{-k_0 \theta} \\ i \sin \frac{\chi}{2} e^{i k_0 \theta} & \cos \frac{\chi}{2} \end{pmatrix}.$$

The gauge transformation removing the string for $z \geq R$ and twisted by an angle γ , $0 \leq \gamma \leq 2\pi$, may be obtained from u by replacing χ by $\sigma_R^*(\chi)$ and θ by $\theta + \frac{\gamma}{k_0}$. In matrix form

$$u_\gamma(\chi, \theta) = \begin{pmatrix} \cos\left(\frac{\sigma_R^*(\chi)}{2}\right) & i \sin\left(\frac{\sigma_R^*(\chi)}{2}\right) e^{-i(k_0\theta + \gamma)} \\ i \sin\left(\frac{\sigma_R^*(\chi)}{2}\right) e^{i(k_0\theta + \gamma)} & \cos\left(\frac{\sigma_R^*(\chi)}{2}\right) \end{pmatrix}.$$

To get from one gauge to the other, we note there exists a unique ν with $|\nu| < \pi$ such that $u_\gamma \circ u^{-1} = e^\nu$. Let

$$\lambda(z) = \begin{cases} 0 & \text{for } z \leq \frac{R}{2} \\ 1 & \text{for } z \geq \frac{3R}{2} \end{cases}.$$

Then the global gauge transformation $\omega_\gamma = e^{\lambda\nu}$ removes both string singularities. We now look at the Higgs field ϕ in this smooth gauge. For $z \leq \frac{R}{2}$ and $x, y \rightarrow \infty$,

$$\phi_\gamma^1 = \begin{pmatrix} i \cos \chi & -\sin \chi e^{-ik_0\theta} \\ \sin \chi e^{ik_0\theta} & -i \cos \chi \end{pmatrix}.$$

For $z \geq \frac{3R}{2}$ and $z, y \rightarrow \infty$,

$$\phi_\gamma^2 = \begin{pmatrix} i \cos \sigma_R^*(\chi) & -\sin \sigma_R^*(\chi) e^{-i(k_0\theta + \gamma)} \\ \sin \sigma_R^*(\chi) e^{i(k_0\theta + \gamma)} & i \cos \sigma_R^*(\chi) \end{pmatrix}.$$

In between, $\phi_\gamma^3 = u^{-1} e^{-\lambda\nu} \phi_\gamma e^{\lambda\nu} u$, $0 \leq \lambda \leq 1$.

We now show that $\phi_\gamma(\chi, \theta)$ represents a loop on the space $\text{Maps}(S^2, S^2)$ to which an extension of the classical theorem of Hopf can be applied. We take a model for S^2 consisting of a cylinder $\left\{ (x, y, z) \mid x^2 + y^2 = 1, \frac{-R}{2} \leq z \leq \frac{5R}{2} \right\}$ union the two hemispheres $H^+ = \left\{ (x, y, z) \mid x^2 + y^2 + \left(z - \frac{5R}{2}\right)^2 = 1, z \geq \frac{5R}{2} \right\}$ and $H^- = \left\{ (x, y, z) \mid x^2 + y^2 + \left(z + \frac{R}{2}\right)^2 = 1, z \leq \frac{-R}{2} \right\}$. We denote this non-standard sphere by $[S^2]$. The boundary of R^3 can be defined as a limit of directed rays based at $(0, 0, R)$. The loop ϕ_γ maps S^1 into $\text{Maps}([S^2], S^2)$. Let

$$f : S^1 \times [S^2] \rightarrow S^2$$

be defined by $f(\gamma, \chi, \theta, z) = \phi_\gamma(\chi, \theta, z)$.

Now let p and q be two points on the unit circle in S^2 having only (j, k) components. Then the inverse images, $f^{-1}(p)$ and $f^{-1}(q)$ are contained in the cylindrical part of the preimage, $S^1 \times S^1 \times \{z = R\}$. This is easy to see from the definition of the Higgs field and the fact that at $z = R$, $\chi = \frac{\pi}{2}$ so the diagonal components of ϕ_γ vanish there. The map f is transverse at these two points whose

preimages lie on a torus in $S^1 \times [S^2]$ and can be represented, for $0 \leq \gamma \leq 2\pi$, $0 \leq l \leq k_0$, as

$$f^{-1}(p) = \left\{ \left(\gamma, e^{i\left(\frac{2\pi l}{k_0}\right)} \right) \right\}$$

and

$$f^{-1}(q) = \left\{ \left(\gamma, e^{i\left(\frac{2\pi l}{k_0} + \gamma + \varepsilon\right)} \right) \right\}$$

for some ε , $0 < \varepsilon < \frac{2\pi}{k_0}$. The geometric picture of each of the two preimages is that of k_0 distinct circles that wrap around the torus once and each circle in $f^{-1}(p)$ links with each circle in $f^{-1}(q)$ all the same linking number ± 1 . From this picture, one sees that the linking number of this map, is $\pm k_0^2 \neq 0$. By an extension of Hopf's classical theorem (see [H1], chapter 14), since the linking number is non-zero, the map represents a non-trivial element of $\pi_1(\text{Maps}(S^2, S^2))$.

We now show that the energy of c_γ is less than twice the minimum energy over non-zero monopole classes.

Define

$$\mathbf{A}_R(c) = \frac{1}{2} \int_{z \leq R} d^3x \left\{ |F_A|^2 + |D_A \phi|^2 + \frac{\lambda}{4} (1 - |\phi|^2)^2 \right\}.$$

Proposition 4.6. $\mathbf{A}(c_\gamma) < 2\mathbf{A}_0$.

Proof. This will follow once we prove

- (i) $\mathbf{A}(c_\gamma) = 2\mathbf{A}_R(c_\gamma)$,
- (ii) $\mathbf{A}_R(c_0^g) \leq \mathbf{A}_0 - \frac{K}{R} + O\left(\frac{1}{R^2}\right)$,
- (iii) $\mathbf{A}_R(c_\gamma) \leq \mathbf{A}_R(c_0^g) + \frac{\varepsilon(R)}{R}$.

with R sufficiently large. Here K is a positive constant and $\varepsilon(R) \rightarrow 0$ as $R \rightarrow \infty$.

Property (i) follows from the construction. To prove (ii) note that

$$\mathbf{A}_R(c_0^g) \leq \mathbf{A}_0 - \frac{1}{2} \int_{z \geq R} |F_{A_0^g}|^2 d^3x.$$

We compute the integral of $|F_\infty|^2$. In spherical coordinates,

$$F_\infty \wedge *F_\infty = \frac{k_0 \sin \chi \, dr \, d\chi \, d\theta}{r^2} = \frac{k_0}{r^4} d(\text{Vol}).$$

In cylindrical coordinates, $x = \varrho \cos t$, $y = \varrho \sin t$, $z = z$, $\varrho = x^2 + y^2$,

$$F_\infty \wedge *F_\infty = k_0 \left(\frac{\varrho \, d\varrho}{(\varrho^2 + z^2)} dt \, dz \right).$$

Therefore,

$$\int_{z \geq R} |F_\infty|^2 d^3x = k_0 \int_r \int_0^{2\pi} \int_0^\infty \frac{\varrho \, d\varrho}{(\varrho^2 + z^2)} dt \, dz = \frac{\pi k_0}{R}.$$

Since $F_{A_0^g} = F_\infty + \omega + F_T$, we have

$$\left(\int_{z \geq R} |F_{A_0^g}|^2 d^3x \right)^{\frac{1}{2}} \geq \left(\frac{\pi k_0}{R} \right)^{\frac{1}{2}} - \frac{\text{const}}{R^{\frac{3}{2}}}$$

and

$$\mathbf{A}_R(c_0^g) \leq \mathbf{A}_0 - \frac{\pi k_0}{2R} + O\left(\frac{1}{R^2}\right),$$

proving (ii).

To prove (iii), we estimate the three quantities appearing in the Higgs functional, noting that c_γ agrees with c_0^g for $z \leq \frac{R}{2}$. First,

$$\phi_R = \phi_0^g + (\beta - 1)\mu$$

for $\frac{R}{2} \leq z \leq R$ with $|\mu| \leq e^{-mR}$ and m a positive constant. Therefore

$$\int_{z \leq R} (1 - |\phi_R|^2)^2 d^3x \leq \int_{z \leq R} (1 - |\phi_0^g|^2)^2 d^3x + O(e^{-mR}).$$

Secondly, using orthogonality and the fact that $\beta \leq 1$,

$$\begin{aligned} \int_{z \leq R} |D_{A_R} \phi_R|^2 d^3x &\leq \int_{z \leq R} |D_{A_R}(\hat{\phi}^i)|^2 d^3x + O(e^{-mR}) \\ &\leq \int_{z \leq R} |D_{A_0}(\hat{\phi}^i)|^2 d^3x + O(e^{-mR}) \\ &\leq \int_{z \leq R} |D_{A_0}(\phi_0)|^2 d^3x + O(e^{-mR}). \end{aligned}$$

Finally, comparing curvature terms,

$$F_{A_R} - F_{A_0} = (\beta - 1)(dB_0 - k_0(1 - \cos \chi) d\theta \hat{i} \wedge B_0) + (d\beta)B_0 + (\beta^2 - 1)B_0 \wedge B_0$$

for $\frac{R}{2} \leq z \leq R$. Using the decay estimates of Corollary 4.3,

$$\int_{\frac{R}{2} \leq z \leq R} |F_{A_R} - F_{A_0^g}|^2 d^3x \leq \frac{\text{const } \varepsilon(R)}{R}.$$

Computations made previously show that

$$\int_{\frac{R}{2} \leq z \leq R} |F_{A_0}|^2 d^3x \leq \frac{\pi k_0}{R} + O\left(\frac{1}{R^2}\right).$$

Using this estimate and the fact that $F_{A_0^g}$ and F_{A_R} agree for $z \leq \frac{R}{2}$, we obtain

$$\int_{z \leq R} |F_{A_R}|^2 d^3x \leq \int_{z \leq R} |F_{A_0^g}|^2 d^3x + \frac{\varepsilon(R)}{R},$$

which proves (iii) and hence completes the proof of Proposition 4.6.

5. Min-Max Theory on C

In this section, we execute, the min-max argument on the non-trivial homotopy class generated by the mapping $c_\gamma : S^1 \rightarrow C$ as defined in Sect. 4. The argument follows the framework established by Taubes in the $\lambda = 0$ case. Groisser, [G1], observed that the main body of the Taubes min-max procedure generalizes to the case $\lambda > 0$ if one can find an appropriate loop with which to begin the process. Taubes' original proof [T1, 2] and Groisser's generalization of them [G1] kept close track of the base point of the loop, in order to insure that the maximum action achieved on a loop in a given homotopy class was bounded away from zero. However, Taubes later discovered [T3] that this conclusion was true even on the level of free homotopy, which greatly simplifies our work here. The details of the final proof of this section, insuring that the limit configuration in the min-max process does not jump monopole classes are included as this approach is different from [T1] and overcomes an obstacle to the full $\lambda > 0$ extension of the min-max process in [G1].

Let Ω be the homotopy class of $\pi_1(C)$ generated by the loop constructed in Sect. 4. Denote the elements of this homotopy class by $c(\cdot)$. To each mapping $c(\cdot)$ in this homotopy class, we can associate a configuration $\bar{c} = c(y_0)$, where

$$\mathbf{A}(c(y_0)) = \sup_{y \in S^1} \mathbf{A}(c(y)).$$

We call \bar{c} the configuration associated to $c(\cdot)$. As $\mathbf{A}(c(y))$ is continuous, we are assured that an associated configuration exists. In the case that the sup is achieved at more than one configuration, there is no complication in what follows in picking any one to be the associated configuration. Note that each associated configuration has monopole number zero.

We begin with Taubes' observation that the infimum of the action of the associated configuration in a nontrivial homotopy class is bounded away from zero.

Definition 5.1. $a_\infty := \inf_{c \in \Omega} \mathbf{A}(\bar{c})$

Proposition 5.2. *If Ω is a nontrivial homotopy class, then $a_\infty > 0$.*

Proof. Suppose $a_\infty = 0$. We will obtain a contradiction using a lemma of Taubes.

Let $c(\cdot) \in \Omega$, and suppose $\mathbf{A}(\bar{c}) = \varepsilon$. Let $\check{C} = \{c = (A, \phi) \mid \check{\mathbf{A}}(c) < \infty \text{ and } \lim_{x \rightarrow \infty} |\phi| = 1\}$ (cf. Definition 2.2), and define the topology on \check{C} just as the topology on C , save no restriction on $\|1 - |\phi|^2\|^2$. Since $c(\cdot)$ is continuous as a map from S^1 to C , it is clearly continuous as a map from S^1 to \check{C} , and $\sup_{y \in S^1} \check{\mathbf{A}}(c(y)) \leq \varepsilon$. Let

$\check{\Omega} \in \pi_1(\check{C})$ be the homotopy class of $c(\cdot)$ in the larger space \check{C} , and let \check{a}_∞ be defined as in Definition 5.1, but using $\check{\Omega}$ and $\check{\mathbf{A}}$. The above shows that $\check{a}_\infty \leq a_\infty$. Hence $a_\infty = 0$ implies $\check{a}_\infty = 0$. But Corollary E.1.2 of [T3] implies that if $\check{a}_\infty = 0$, then

any $c(\cdot) \in \check{\Omega}$ is homotopic in \check{C} to a map from S^1 to the space \mathcal{M}_0 of configurations which $\check{A} = 0$. However from [T1, Sect. 3], $\pi_1(\check{C}) \cong \pi_1(\text{Maps}(S^2, S^2)) \cong \pi_3(S^2)$, and the isomorphism is given by the map from S^1 to $\text{Maps}(S^2, S^2)$ obtained by restricting $\phi/|\phi|$ to a large 2-sphere. Thus any map from S^1 to \mathcal{M}_0 represents to zero element of $\pi_1(\text{Maps}(S^2, S^2))$.

Therefore if $a_\infty = 0$, it follows that the map $S^1 \rightarrow \text{Maps}(S^2, S^2)$ obtained from $c(\cdot)$ by restricting $\phi/|\phi|$ to a large 2-sphere is homotopically trivial. But the construction in Sect. 4 guarantees that this is not the case.

Definition 5.3. We will call a sequence $\{c_i = (A_i, \phi_i)\}$ a good sequence if it satisfies the following four conditions:

1. $\lim_{i \rightarrow \infty} \mathbf{A}(c_i) = a_\infty$,
2. $\mathbf{A}(c_i) \geq \mathbf{A}(c_{i+1})$,
3. $\lim_{i \rightarrow \infty} \|\nabla \mathbf{A}_{c_i}\|_{c_i^*} \rightarrow 0$,
4. $\|\phi_i\|_\infty$ and $\|D_{A_i} * D_{A_i} \phi_i\|_2$ are bounded independent of i .

Proposition 5.4. There exists a sequence of loops $\{c_i(\cdot)\} \in \Omega$ for which the sequence $\{\bar{c}_i\}$ is a good sequence.

Proof. See Lemma 6.9 of [G1].

Proposition 5.5. Let $\{c_i(\cdot)\}$ be a sequence of loops for which $\{\bar{c}_i\}$ is a good sequence. Then there exists a sequence of points $\{x_i\}$ with the following properties:

1. The sequence $\{T_{x_i} \bar{c}_i\}$ has a subsequence that converges strongly in $L^1_{2, \text{loc}}$ to $c = (A, \phi) \in C$.
2. (A, ϕ) satisfies the Yang-Mills-Higgs equations (YMH λ 1) and (YMH λ 2).
3. $\mathbf{A}(A, \phi) > 0$.

Proof. See Proposition 6.2 and Theorem 6.1 of [G1].

Finally, we show that the limit configuration has monopole number zero. Note that since the limiting configuration in Proposition 5.5 has action bounded strictly away from zero, the fact that it lies in C_0 will imply that it is not a global minimum.

Proposition 5.6. Let $\{c_i\} \in C_0$ be a good sequence of configurations that converges strongly in $L^1_{2, \text{loc}}$ to a solution $c \in C$ of (YMH λ 1) and (YMH λ 2). If $\mathbf{A}(c_i) < 2\mathbf{A}_0$, then $c \in C_0$.

Proof. Recall that the monopole number is given by

$$k = \frac{1}{4\pi} \langle D_A \phi, * F_A \rangle_2.$$

We first remark that if there exists a ball B such that for all i , all of the zeroes of the Higgs field ϕ_i , remain inside the ball, then the theorem follows immediately from the strong convergence of the above integral on B . Thus in what follows, we may assume that for any R , there exists i_0 such that for all $i > i_0$, a zero of ϕ_i lies outside of B_R , the ball of radius R centered at the origin.

Now assume that the limiting configuration c is not in C_0 , but in C_k , $k \neq 0$ with $\mathbf{A}(c) = E \geq \mathbf{A}_0$. By the strong convergence of $\{c_i\}$ to c on B_R , the integral giving the monopole number must be close to k inside the ball and $-k$ outside the ball for i sufficiently large. If one chooses R large enough, most of the energy, say $E - \varepsilon$, will be inside the ball. By our initial remarks, for i large enough, ϕ_i has a zero at x_i lying outside $B_R \equiv B_0$.

Consider now the translated sequence $\{T_{x_i} c_i\}$. Without loss of generality we may assume that $\{T_{x_i} c_i\}$ converges to a configuration \tilde{c} with non-zero monopole number. We argue this as follows. Recall from Lemma 3.6 that there exists a number N , independent of i , such that the zero set of ϕ_i lies in B_0 union a collection of disjoint open unit balls $B_1^{(i)}, \dots, B_N^{(i)}$. Let k_j^i denote the winding number of $\phi/|\phi|$ on the boundary of $B_j^{(i)}$. Then for all i , $\sum_{j=1}^N k_j^i = -k \neq 0$. Suppose there were no subsequences $\{i', j'\}$ having both the following properties: (i) the center of $B_{j'}^{(i')}$ goes off to infinity as $i \rightarrow \infty$ and (ii) $k_{j'}^{i'} \neq 0$. Then there would exist R' such that all balls B_j^i for which $k_j^i \neq 0$ lie inside a fixed ball of radius R' . Since the monopole number of ϕ_i is the sum of the winding numbers on these non-wandering balls, our remark above implies that $c \in C_0$, contradicting the hypothesis. Hence we can assume, after some relabeling of subsequences, that there is a subsequence $\{c_i\}$ of our original sequence with the property that the center of $B_1^{(i)}$ goes to infinity as $i \rightarrow \infty$ and for which $k_1^i \neq 0, \forall i$.

Now let x_i be the center of $B_1^{(i)}$, and consider the sequence $\{T_{x_i} c_i\}$. By taking a further subsequence and increasing the radius of $B_1^{(i)}$, if necessary, to a number R_1 , we may assume that for any R' , there exists i' such that for $i > i'$, the only zeroes of $\{T_{x_i} \phi_i\}$ lying inside a ball of radius R' centered at the origin lie inside the ball B_1 of radius R_1 centered at the origin (otherwise we would obtain a contradiction as above). Since the condition defining a good sequence are translation invariant, $\{T_{x_i} c_i\}$ is again a good sequence. Hence by Proposition 5.5 there exists a subsequence (relabelled $\{T_{x_i} c_i\}$) that converges strongly in $L^2_{2, \text{loc}}$ to some configuration \tilde{c} . By the strong $L^2_{2, \text{loc}}$ convergence, the winding number of ϕ_i on the boundary of B_1 stabilizes as $i \rightarrow \infty$, and it cannot stabilize to zero since $k_1^i \neq 0, \forall i$. Furthermore, given ε_1 there exists R' such that the monopole number of \tilde{c} is given to within ε by an integral over $B_{R'}(0)$. For sufficiently large i , this ball will contain no zeroes of any $T_{x_i} \phi_i$ outside B_1 and hence the winding number of $\tilde{\phi}$ on the boundary of B_1 will equal the winding number of ϕ_i on the boundary of $B_{R'}(0)$. By taking R' larger still, we can also insure that the monopole number integrand, integrated over $B_{R'}(0)$, gives the winding number of $\tilde{\phi}$ on the boundary of $B_{R'}(0)$ to within ε_1 . Thus to within $2\varepsilon_1$, the limit of the monopole number of \tilde{c} equals the winding number of $\tilde{\phi}$ on the boundary of B_1 , which is non-zero. Since ε_1 was arbitrary, we conclude that the monopole number of \tilde{c} is non-zero.

Thus, we can construct balls $B_1^{(i)}(x_i)$ of radius R' , where R' is such that for i large enough the sequence $\{T_{x_i}(c_i)\}$ has energy $E - \varepsilon' \geq \mathbf{A}_0 - \varepsilon'$ on $T_{x_i}(B_1^{(i)}(x_i))$. For i large enough, $B_1^{(i)}(x_i)$ and B_0 are disjoint. Restrict $\{c_i\}$ to a subsequence so that this is so. Then for i large enough, $\mathbf{A}(c_i) \geq 2\mathbf{A}_0 - \varepsilon - \varepsilon'$. Since ε and ε' are arbitrary, this contradicts the fact that $\mathbf{A}(c_i)$ is strictly less than $2\mathbf{A}_0$.

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