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The Structure of Perturbative Quantum Gravity on a de Sitter Background

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Abstract: Classical gravitation on de Sitter space suffers from a linearization instability. One consequence is that the causal response to a spatially localized distribution of positive energy cannot be globally regular. We use this fact to show that no causal Green's function can give the correct linearized response to certain bilocalized distributions, even though these distributions obey the constraints of linearization stability. We avoid the problem by working on the open submanifold spanned by conformal coordinates. The retarded Green's function is first computed in a simple gauge, then the rest of the propagator is inferred by analyticity – up to the usual ambiguity about real, analytic and homogeneous terms. We show that the latter can be chosen so as to give a propagator which does not grow in any direction. The ghost propagator is also given and the interaction vertices are worked out.

1. Introduction

The study of graviton fluctuations on a de Sitter background is fascinating because infrared effects in quantum gravity may provide a mechanism through which an initially positive cosmological constant relaxes to zero. It is therefore frustrating that we lack a perturbative formalism which is even valid at tree order! Of course the vertices can be worked out with a bit of patience, and various solutions for the gauge fixed propagator have been reported [1–3]. The imaginary parts of these propagators ought to give Green's functions which describe how the classical theory responds to external stress energy. The trouble is that the linearized response inferred in this way is wrong, even for the trivial case of a freely falling point mass [3].

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We wish to emphasize that the propagators obtained in refs. [1–3] really do obey the differential equations they are claimed to satisfy. The problem is rather that solving the dynamical equations of a gauge fixed theory is not quite the same thing as enforcing the combination of constraints and dynamical equations which define the invariant theory. The constrained fields of the invariant theory depend upon the instantaneous matter sources whereas the fields of the gauge fixed theory respond to the past history of these sources through causal Green's functions. Usually this is a distinction without a difference because conserved sources must be present, in some form, at sufficiently early times to have causally affected the constrained field anywhere. This is not so in de Sitter space; owing to the exponential expansion of distances one can find conserved sources which *never* causally affect certain points. An unfortunate synergy between this fact and the background's linearization instability implies that there are sources for which no causal Green's function can give the correct linearized response over the full manifold. These facts are proved in Sects. 2 and 3, the discussion of which we shall here anticipate in order to better motivate our proposed solution.

We show in Sect. 3 that the spatial sections of the D-dimensional de Sitter manifold can be taken to be $(D-1)$-spheres. Consider a conserved source of positive energy density which is localized on these $(D-1)$-spheres. A consequence of the linearization instability (discussed in Sect. 2) is that the linearized response engendered by such a source cannot be regular throughout the $(D-1)$-sphere. This fact does not conflict with our perception that localized sources seem to be realizable – and that they can induce perfectly regular geometries – because the constraints of linearization stability can be easily satisfied by adding sources on the far side of the $(D-1)$-sphere. Owing to the causal structure of the background these compensating sources would not be observable in the vicinity of the first source until very late times.[1]

Consider now a linearization stable distribution which consists of localized sources on either side of the $(D-1)$-sphere. Provided the sources are not too singular a globally regular, linearized response can be found. Suppose that it is given by a causal Green's function. Since a null ray requires the lifetime of the universe to move even halfway around the $(D-1)$-sphere the linearized response to the full distribution must consist at early times of a nonzero region around each source, separated by an extensive region of zero response between the two sources. Since the field equations are linear, and are solved locally by zero, it follows that the response to each source must *separately* give a globally regular solution to the linearized equations with just one source. But no such solution exists by virtue of the linearization instability. We therefore conclude that no causal Green's function can give the correct response to the full distribution, despite the fact that it is linearization stable.

This is the great obstacle to any formulation of de Sitter perturbation theory which is simultaneously global, covariant and causal. Before describing our pro-

[1] The instability can also be evaded if one excites global, negative energy modes of the gravitational field itself. However, these global modes cannot represent the response from a causal Green's function since they fail to vanish outside the light cone of the sources. Another possibility is that the solution to a localized distribution exists but requires more than one coordinate patch for its expression. In this case the problem in obtaining a globally regular response on a single patch would come in attempting to extend the solution beyond the causal horizon of the source. No causal Green's function can reproduce the necessary response – regular within the causal horizon with a coordinate singularity beyond – because they vanish outside the lightcone of the source points.

posed solution it is important to mention that the problem is one of efficiency rather than principle. There is no question that a global formulation of perturbation theory can be worked out *canonically* by imposing the appropriate linearization constraints. These are superficially acausal but introduce no physical violation of causality. The only impediment to this approach is the finite duration of human interest in the result. It requires extraordinary efforts to apply canonical methods even to simple scalar field theories in flat space; accomplishing anything this way for quantum gravity on de Sitter space might require a very long time.

Since we are loath to forsake either covariance or manifest causality we shall instead abandon globalism. The danger in this approach is that information might tend to flow into or out of whatever subset of the full manifold we take as the physical arena. That such an embarrassment can be avoided derives from the same peculiar causal structure which has frustrated previous global formulations: distances in de Sitter space expand so rapidly that the future lightcone of even a point in the infinite past encompasses only a fraction of the $(D-1)$-sphere at any instant. The future lightcone under discussion has the topology of R^{D-1}. It achieves openness, despite the compact spatial sections, by escaping into the ever more inflated $(D-1)$-spheres of the future. Of course a lightcone is null, but we can obtain an open surface which is spacelike by simply evolving each point of the lightcone for a fixed proper time along parallel timelike geodesics. We call the subset of de Sitter space which is swept out in this way, "an open submanifold." Our solution to the previously described problem is to formulate gravitational perturbation theory on an open submanifold.

It is clearly convenient to consider the constant τ foliations of the initial lightcone to be surfaces of simultaneity. These spatial sections are also Cauchy surfaces; once initial value data is given on such a surface it completely determines the course of future evolution. This is why no information leaks into or out of open submanifolds at finite times. (We regard the initial lightcone as residing in the infinite past.) Though open submanifolds fail to cover all of de Sitter space they do have the property that observers at any finite time are in causal contact with all sufficiently ancient sources. Open submanifolds also happen to be free of the linearization instability.

The last two features mean that a causal Green's function can give the correct linearized response on an open submanifold. It is also possible to impose a coordinate system on this submanifold which is conformal to flat space:

$$ds^2 = \frac{1}{(Hu)^2}(-du^2 + \|d\vec{x}\|^2) . \tag{1.1}$$

In Sect. 4 we exploit these facts to solve for the retarded Green's function of the conformally rescaled graviton field, $\psi_{\mu\nu} \equiv (Hu)^2 h_{\mu\nu}$, in a natural gauge. Our analysis is everywhere in D dimensions but it is worth noting here the simplicity of this Green's function, $[{}_{\rho\sigma}G^{\alpha\beta}_{\text{ret}}](x, x')$, in the physically relevant case of $D=4$:

$$-\frac{H^2}{4\pi}\,\theta(u'-u)\left\{\frac{u'u}{\|\vec{x}'-\vec{x}\|}\,\delta(u'-u-\|\vec{x}'-\vec{x}\|)\,[2\delta_\rho{}^{(\alpha}\delta_\sigma{}^{\beta)} - \eta_{\rho\sigma}\eta^{\alpha\beta}]\right.$$

$$\left. + \theta(u'-u-\|\vec{x}'-\vec{x}\|)\,[2\bar{\delta}_\rho{}^{(\alpha}\bar{\delta}_\sigma{}^{\beta)} - 2\bar{\eta}_{\rho\sigma}\bar{\eta}^{\alpha\beta}]\right\} . \tag{1.2}$$

(A bar over a tensor suppresses its zero components while parentheses around tensor indices denote symmetrization. In four dimensions the Hubble constant is $H=\sqrt{\frac{1}{3}\Lambda}$.) As a check we show in Sect. 5 that the D-dimensional Green's function gives the correct linearized response to a freely falling point mass. The imaginary part of the propagator equals half the sum of the advanced and retarded Green's functions. In Sect. 6 we invoke analyticity to infer the real part up to the usual ambiguity about terms which are real, analytic and which solve the homogeneous equation. It is again worth noting the remarkable simplicity of our result for $D=4$. Up to the aforementioned ambiguity the four dimensional propagator $i[{}_{\rho\sigma}\Delta^{\alpha\beta}](x,x')$ is equal to:

$$\frac{H^2}{8\pi^2}\left\{\frac{2u'u}{(x-x')^2+i\varepsilon}\left[2\delta_\rho{}^{(\alpha}\delta_\sigma{}^{\beta)}-\eta_{\rho\sigma}\eta^{\alpha\beta}\right]\right.$$
$$\left.-\ln(H^2(x-x')^2+i\varepsilon)\left[2\bar{\delta}_\rho{}^{(\alpha}\bar{\delta}_\sigma{}^{\beta)}-2\bar{\eta}_{\rho\sigma}\bar{\eta}^{\alpha\beta}\right]\right\}. \tag{1.3}$$

What real, analytic and homogeneous terms should be added depends upon the choice of vacuum – a choice for which there is no clear criterion in a time dependent background such as this. However, we do show that real, analytic and homogeneous terms exist which make the resulting propagator remain finite as the separation between x and x' approaches either spacelike or timelike infinity, and which do not disrupt the correct flat space limit ($H\to 0$ and $u, u'\to\infty$ such that $Hu, Hu'\to 1$ and $u-u'$ remains finite) already possessed by expression (1.3). Section 6 concludes with a computation of the ghost propagator and the interaction vertices.

2. Linearization Instability

Linearization instability is a potential pathology of both classical and quantum perturbation theory. It arises when gauge invariance – or current conservation – is not enough to ensure that solutions to the linearized invariant field equations can be perturbatively corrected to give asymptotic series solutions to the full equations. The usual paradigm is electrodynamics with nondynamical geometry:

$$ds^2=-dt^2+r_0^2\,d\Omega_{D-1}^2\equiv g_{\mu\nu}\,dx^\mu\,dx^\nu \tag{2.1}$$

on the constant curvature manifold $S^{D-1}\times R$. It is not generally possible to solve Maxwell's equations:

$$\partial_\mu(g^{\mu\rho}\,g^{\nu\sigma}\,F_{\rho\sigma}\sqrt{-g})=J^\nu \tag{2.2}$$

on this manifold, even for current densities which obey $\partial_\nu J^\nu=0$. One also needs the total charge to vanish:

$$Q\equiv\int_{S^{D-1}} d^{D-1}x\,J^0=0\,. \tag{2.3}$$

(Note that the measure in (2.3) is the naive, constant one because the integrand is a density.) The mathematical reason for this is that integrating the derivative of a nonsingular function over a compact manifold gives zero:

$$\int_{S^{D-1}} d^{D-1}x\,\partial_\mu\,(g^{\mu\rho}g^{0\sigma}F_{\rho\sigma}\sqrt{-g}) = -\int_{S^{D-1}} d^{D-1}x\,\partial_i(g^{ij}F_{j0}\sqrt{-g}) = 0\,. \tag{2.4}$$

The physical reason is that the flux from any net charge would have nowhere to go on the $(D-1)$-sphere.

By the term, "modes," we mean solutions of the linearized field equations. A theory which is linearization unstable possesses modes which cannot be corrected to give full solutions and do not therefore represent true degrees of freedom. In the electrodynamical example these bogus modes reside in the charged matter sector but they would appear as well in the gauge sector of nonabelian gauge theories on this manifold. Whatever the model, the existence of unphysical modes means that naive perturbation theory is wrong. The cure is to expunge the unwanted modes by imposing linearization constraints which restore the integrability of perturbation theory. For our electrodynamical paradigm the necessary constraint is just that the total charge should vanish. Since Q is constant this need only be done on the initial value surface; naive perturbation theory can then be developed on the subset of linearized solutions which obey the constraint.

Linearization in stabilities arise in gravity on any background which possesses at least one Killing vector and a compact, spacelike Cauchy surface [4, 5]. To understand why we must first elucidate the nature of gravitational perturbation theory. Consider the action for Einstein's theory in D spacetime dimensions:

$$S[g] \equiv \frac{1}{\kappa^2}\int d^Dx[R-(D-2)\Lambda]\sqrt{-g}\,. \tag{2.5}$$

Our conventions are that the metric has spacelike signature, $\kappa^2 \equiv 16\pi G$, and $R^\alpha{}_{\beta\mu\nu} \equiv \Gamma^\alpha{}_{\nu\beta,\mu} + \Gamma^\alpha{}_{\mu\rho}\,\Gamma^\rho{}_{\nu\beta} - (\mu \leftrightarrow \nu)$. Perturbation theory is based upon the expansion:

$$g_{\mu\nu} \equiv \hat{g}_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{2.6}$$

where $\hat{g}_{\mu\nu}$ is an exact solution. Indices on the graviton field, $h_{\mu\nu}$, are raised and lowered using $\hat{g}_{\mu\nu}$. Following Abbott and Deser [6] we segregate the field equations into a linearized part and a graviton stress tensor density:

$$0 = \frac{\delta S[\hat{g}+\kappa h]}{\delta h_{\mu\nu}} = \mathscr{D}^{\mu\nu\rho\sigma}\,h_{\rho\sigma} - \kappa T^{\mu\nu}[h]\,, \tag{2.7a}$$

$$\begin{aligned}\mathscr{D}^{\mu\nu\rho\sigma}\,h_{\rho\sigma} \equiv{}& \frac{1}{2}(h^{\mu\nu;\rho}{}_{\rho} + h^{\rho;}{}_{\rho}{}^{\mu\nu} - h^{\rho\mu;\nu}{}_{\rho} - h^{\rho\nu;\mu}{}_{\rho})\sqrt{-\hat{g}}\\ &+ \Lambda h^{\mu\nu}\sqrt{-\hat{g}} - \frac{1}{2}\,\hat{g}^{\mu\nu}(h^\rho{}_\rho{}^{;\sigma}{}_\sigma - h^{\rho\sigma;}{}_{\rho\sigma} + \Lambda h^\rho{}_\rho)\sqrt{-\hat{g}}\,.\end{aligned} \tag{2.7b}$$

Note that the semicolon denotes covariant differentiation with respect to the background metric.

The perturbative field equations are obtained by substituting the expansion:

$$h_{\mu\nu} = \sum_{n=1}^{\infty} \kappa^{n-1} h^{(n)}_{\mu\nu} \tag{2.8}$$

into (2.7a) and then segregating powers of κ:

$$\mathscr{D}_{\mu\nu}^{\ \ \rho\sigma} h^{(1)}_{\rho\sigma} = 0$$

$$\mathscr{D}_{\mu\nu}^{\ \ \rho\sigma} h^{(2)}_{\rho\sigma} = T^{(2)}_{\mu\nu}[h^{(1)}]$$

$$\vdots$$

$$\mathscr{D}_{\mu\nu}^{\ \ \rho\sigma} h^{(n)}_{\rho\sigma} = T^{(n)}_{\mu\nu}[h^{(1)}, h^{(2)}, \ldots, h^{(n-1)}] \,. \tag{2.9}$$

Naive perturbation theory operates by systematically correcting any linearized solution, $\phi_{\mu\nu}$, to give an asymptotic series solution in the sense of (2.8). Provided each of the equations (2.9) is integrable the various terms in this expansion are:

$$h^{(1)}_{\mu\nu} = \phi_{\mu\nu}$$

$$h^{(2)}_{\mu\nu}[\phi] = \mathscr{D}^{-1}_{\mu\nu}{}^{\rho\sigma} T^{(2)}_{\rho\sigma}[\phi]$$

$$\vdots$$

$$h^{(n)}_{\mu\nu}[\phi] = \mathscr{D}^{-1}_{\mu\nu}{}^{\rho\sigma} T^{(n)}_{\rho\sigma}[\phi, h^{(2)}[\phi], \ldots, h^{(n-1)}[\phi]] \,. \tag{2.10}$$

The fundamental degrees of freedom in both classical and quantum perturbation theory are the initial value data which characterize all the linearized solutions for which this system is integrable.

Of course the kinetic operator is not generally invertible on account of gauge invariance. As a consequence of the vanishing of the following divergence:

$$\begin{aligned}(\mathscr{D}^{\mu\nu\rho\sigma} h_{\rho\sigma})_{;\nu} = {} & \frac{1}{2}(h^{\mu\rho;\sigma}{}_{\sigma\rho} - h^{\mu\rho;\sigma}{}_{\rho\sigma} + \Lambda h^{\mu\rho;}{}_{\rho})\sqrt{-\hat{g}} \\ & + \frac{1}{2}(h^{\rho}{}_{\rho}{}^{;\mu\sigma}{}_{\sigma} - h^{\rho}{}_{\rho}{}^{;\sigma}{}_{\sigma}{}^{\mu} - \Lambda h^{\rho}{}_{\rho}{}^{;\mu})\sqrt{-\hat{g}} \\ & + \frac{1}{2}(h^{\rho\sigma;}{}_{\rho\sigma}{}^{\mu} - h^{\rho\sigma;\mu}{}_{\rho\sigma} + \Lambda h^{\mu\rho;}{}_{\rho})\sqrt{-\hat{g}} \,,\end{aligned} \tag{2.11}$$

we see that $\mathscr{D}^{-1}_{\mu\nu}{}^{\rho\sigma}\, T_{\rho\sigma}$ cannot exist unless:

$$T^{\mu\nu}{}_{;\nu} \equiv T^{\mu\nu}{}_{,\nu} + \hat{\Gamma}^{\mu}{}_{\nu\rho}\, T^{\rho\nu} = 0 \,. \tag{2.12}$$

This and the requirement that the field equations contain no more than second derivatives is what imposes such strong constraints upon the allowed interactions [7].

Conservation in the sense of (2.12) does follow, at least on shell, for the graviton stress tensor. This is enough to ensure the integrability of (2.9) when the Cauchy surfaces are noncompact or when the background is free of isometries. Suppose, however, that the background possesses at least one Killing vector:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \tag{2.13}$$

and that the Cauchy surfaces are compact and spacelike. Consider one such surface C with timelike normal vector $c^{\mu}(x)$. Because C is compact we can partially integrate derivatives along it without producing surface terms. This and the fact that ξ_{μ} is a Killing vector suffice, after considerable manipulation, to show that for all fields $h_{\mu\nu}$ which are free of coordinate singularities:

$$\oint_C d^{D-1}x\, c_\mu \xi_\nu \mathscr{D}^{\mu\nu\rho\sigma} h_{\rho\sigma} = 0 \tag{2.14}$$

whether or not $h_{\mu\nu}$ is a linearized solution. An immediate consequence is that the kinetic operator has another zero eigenvector in addition to those implied by gauge invariance. If even one of the source terms on the right-hand side of (2.9) should have support in this direction then the associated linearized solution cannot be corrected to give a full solution which is free of coordinate singularities.[2] Perturbative integrability obviously requires:

$$Q^i = \oint_C d^{D-1}x\, c_\mu \xi^i_\nu T^{\mu\nu}[h] = 0 \tag{2.15}$$

for every Killing vector ξ^i_μ. It is straightforward to show that these constraints are time independent [6] so that once imposed on C they continue to be obeyed on foliations.

We conclude this section with a discussion of the relation between linearization instability and gauge fixing. When a linearization instability afflicts the invariant field equations it must occur as well when the gauge is fixed by imposing conditions on the field. However, if the gauge is fixed by adding terms to the action then one can obtain a set of field equations which are perturbatively integrable. This happens for our electrodynamical model in Feynman gauge:

$$\square A^\nu \equiv A^{\nu;\rho}{}_\rho \sqrt{-g} = J^\nu\,. \tag{2.16}$$

In addition to the two fictitious photons modes the solution set of the full gauge fixed theory – including the matter equations – contains a completely gauge invariant but nonneutral, and hence unphysical, sector. This sector must be suppressed by fiat before the Feynman gauge solutions can obey the invariant field equations.

The nonneutral sector of Feynman gauge electrodynamics can be eliminated by imposing (2.3) on any initial value surface – or upon the asymptotic states in quantum field theory. This fact ought to seem more surprising than it does because the two systems – (2.2) and (2.16) – are solved in very different ways. One of the D Maxwell's equations is a constraint which need only be enforced on the initial value surface; current conservation and the other $(D-1)$ equations conspire to preserve it under time evolution. This constraint equation is solved by adjusting the spatial variation of the gauge field at a given instant so that the gradient of the electric field gives the charge density at that instant. There is no causal relation between the source and the longitudinal electric field that it engenders; the induced electric field depends upon the charge density throughout the surface of simultaneity.

In contradistinction to (2.2) all D of the equations in Feynman gauge electrodynamics are dynamical. The general solution is given in terms of a retarded

[2] What saves flat space for $\Lambda = 0$ is the noncompactness of the usual Cauchy surface. If we effectively compactify the Cauchy surface by requiring that solutions fall off too rapidly – say $h_{ij} \sim r^{1-D}$ and $\dot{h}_{ij} \sim r^{-D}$ – then a similar linearization instability results. The same effective compactification is what causes the linearization instability of pure higher derivative gravity [8].

Green's function:

$$A_\mu(x) = \phi_\mu(x) + \int d^D x' [{}_\mu G^\nu_{\rm ret}](x, x') J_\nu(x') , \tag{2.17}$$

where ϕ_μ is a homogeneous solution and the Green's function obeys:

$$\Box [{}_\mu G^\nu_{\rm ret}](x, x') = \delta_\mu{}^\nu \delta^D(x - x') . \tag{2.18}$$

Of course the homogeneous term represents free electromagnetic radiation – and to get a solution of the invariant theory we want $\phi^{\mu;}{}_\mu = 0$ – but the feature which ought to shock is how the field depends upon the current density. In (2.17) we see that the field responds causally to the source; the effect at any point in space and time is built up by propagating forward the response to the source at each instant inside the past lightcone of that point.

There is no contradiction between the two approaches because current conservation relates the instantaneous charge density – which produces the longitudinal electric field in the invariant formalism – to the past current density – which gives the same field in the Feynman gauge formalism. Suppose, for example, that the source consists of stationary opposite point charges on either side of the $(D-1)$-sphere. We would certainly obtain different results from the two methods if these charges appeared out of nowhere at a given instant and then disappeared some time later. Maxwell's equations imply that the longitudinal electric field must be everywhere nonzero within this interval whereas the Feynman gauge system would give zero field of any sort outside of the future lightcones of the creation events. The disagreement is forbidden because the spontaneous appearance of two separated charges would violate current conservation. If the two charges are around at some instant then they must have been around as well in the past for at least as long as it would require both to have emerged from the same point. Truly stationary charges must have been around forever and it is the accumulated, perfectly causal response to this past current density which gives rise to the instantaneous longitudinal electric field.

Consistency between the two formalisms is so well known as to constitute almost a physical cliché. Little remarked is the fact that this consistency depends crucially upon a property of the nondynamical geometry of the model, namely that any two points on a surface of simultaneity lie within the future lightcone of some point far enough back in the past. De Sitter space shares the $S^{D-1} \times R$ topology of our electrodynamical model, but its geometry is far different. In particular, we are about to see that distances expand so rapidly in de Sitter space that opposite and simultaneous points on the $(D-1)$-sphere never come into causal contact, while distant simultaneous points only reach causal contact at very late times.

3. Geometry and Coordinates in de Sitter Space

De Sitter space is the unique maximally symmetric solution to the vacuum Einstein equation with a positive cosmological constant. These facts mean that the curvature bears the following simple relation to the metric:

$$R^\alpha{}_{\beta\gamma\delta} = H^2 (\delta^\alpha{}_\gamma g_{\beta\delta} - \delta^\alpha{}_\delta g_{\beta\gamma}) , \tag{3.1}$$

where the D-dimensional Hubble constant is $H^2 \equiv \frac{1}{D-1} \Lambda$. (Note that although the de Sitter metric is to be the background for the perturbative discussions of succeeding sections we shall spare ourselves the notational bother of placing a "hat" over the symbol $g_{\mu\nu}$ throughout this section.) Substitution of (3.1) in the equation of geodesic deviation:

$$\frac{D^2 \Delta^\mu(\tau)}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}(\chi(\tau))\dot{\chi}^\nu(\tau)\Delta^\rho(\tau)\dot{\chi}^\sigma(\tau) \tag{3.2a}$$

$$= -H^2 \Delta^\mu g_{\rho\sigma}\dot{\chi}^\rho\dot{\chi}^\sigma + H^2\dot{\chi}^\mu g_{\rho\sigma}\dot{\chi}^\rho \Delta^\sigma \tag{3.2b}$$

reveals that the deviation $\Delta^\mu(\tau)$ between two nearby geodesics $\chi^\mu(\tau)$ expands exponentially provided the geodesics are initially parallel and timelike.

The most convenient way to describe D-dimensional de Sitter space is as the surface of constant length H^{-1} from the origin of $(D+1)$-dimensional Minkowski space. Note that both our de Sitter metric and the Minkowski metric are assumed to have spacelike signature. De Sitter coordinates are represented by lower case letters; the Minkowski coordinates are denoted by upper case letters. General de Sitter indices are Greek and general Minkowski indices are capital English letters from the first of the alphabet. Lower case English indices from the middle of the alphabet (i, j, etc.) run from 1 to $D-1$; capital English indices from the middle of the alphabet (I, J, etc.) run from 1 to D. The embedding is:

$$HX^A = E^A(x) , \tag{3.3a}$$

where the mappings are assumed to obey $\eta_{AB}\, E^A(x)\, E^B(x) \equiv E(x) \cdot E(x) = 1$. The inverse can be defined for X^A on the embedded surface; for general X^A it is defined by considering the mapping to be homogeneous of degree zero:

$$Hx^\mu = e^\mu(X) = e^\mu(\lambda X) . \tag{3.3b}$$

A consequence is that the associated Jacobians invert one another in the following sense:

$$\frac{1}{H^2}\left(\frac{\partial E^A}{\partial x^\mu}\right)\left(\frac{\partial e^\nu}{\partial X^A}\right) = \delta_\mu{}^\nu , \tag{3.4a}$$

$$\frac{1}{H^2}\left(\frac{\partial e^\mu}{\partial X^A}\right)\left(\frac{\partial E^B}{\partial x^\mu}\right) = \delta_A{}^B - E_A E^B . \tag{3.4b}$$

Note as well that $\frac{\partial E^A}{\partial x^\mu} E_A = 0 = \frac{\partial e^\mu}{\partial X^A} E^A$.

All geometrically meaningful quantities in a de Sitter coordinate system can be expressed in terms of the functions $E^A(x)$ and $e^\mu(X)$. For example, the de Sitter metric is:

$$g_{\mu\nu} = \frac{1}{H^2}\frac{\partial E^A}{\partial x^\mu}\frac{\partial E_A}{\partial x^\nu} , \quad g^{\mu\nu} = \frac{1}{H^2}\frac{\partial e^\mu}{\partial X^A}\frac{\partial e^\nu}{\partial X_A} . \tag{3.5}$$

The $\frac{1}{2} D(D+1)$ isometries of de Sitter space are obtained by subjecting the embedding coordinates to $O(1, D)$ transformations and then inverting:

$$x^{\mu'} = \frac{1}{H} e^\mu(\Omega^A{}_B X^B) . \tag{3.6a}$$

This gives an immediate and explicit expression for the Killing vectors:

$$\xi^{\mu}_{AB}(x)=\frac{1}{H^2}\frac{\partial e^{\mu}(X)}{\partial X^A}E_B(x)-\frac{1}{H^2}\frac{\partial e^{\mu}(X)}{\partial X^B}E_A(x)\,. \tag{3.6b}$$

when a geodesic connects two points x^{μ} and y^{μ} it is found by inverting the trivial geodesic between X^A and Y^A:

$$\chi^{\mu}(\tau)=\frac{1}{H}e^{\mu}[(1-\tau)X+\tau Y]\,. \tag{3.7}$$

(Another way is by inverting the "great circle" traced out by the $SO(1, D)$ rotation which carries X^A onto Y^A.) The distance $l(x, y)$ along such a geodesic obeys:

$$\sin^2\left[\frac{1}{2}H\ell(x, y)\right]=\frac{1}{2}-\frac{1}{2}E(x)\cdot E(y)\equiv 1-z(x, y)\,. \tag{3.8}$$

De Sitter space has the curious property that certain points cannot be connected by a geodesic. The "length" between such points is still defined by (3.8); for these cases the function $z(x, y)$ lies between zero and negative infinity.

De Sitter vector and tensor fields can be mapped onto the tangent space of the Minkowski embedding thusly:

$$V^A(x)\equiv\frac{1}{H}\frac{\partial E^A(x)}{\partial x^{\mu}}v^{\mu}(x)\,. \tag{3.9a}$$

Vector and tensor fields from the Minkowski embedding are mapped into the de Sitter tangent space using the inverse Jacobian:

$$v^{\mu}(x)\equiv\frac{1}{H}\frac{\partial e^{\mu}(X)}{\partial X^A}V^A(X)\,. \tag{3.9b}$$

Parallel transport from X^A to Y^A on the Minkowski embedding is accomplished by the $SO(1, D)$ transformation which carries X^A onto Y^A along the "great circle" connecting them:

$$\begin{aligned}V^A(Y)=&\,V^A(X)+2HX\cdot V(X)\,HY^A\\&-\frac{(HX+HY)\cdot V(X)}{1+HX\cdot HY}(HX^A+HY^A)\,.\end{aligned} \tag{3.10}$$

By inversion we infer the following simple form for the de Sitter parallel transport matrix:

$$H^2[{}_{\mu}g^{\nu}](x, y)=\frac{\partial E^A(x)}{\partial x^{\mu}}\frac{\partial e^{\nu}(Y)}{\partial Y^A}-\frac{1}{1+E(x)\cdot E(y)}\frac{\partial E^A(x)}{\partial x^{\mu}}E_A(y)\,E^B(x)\frac{\partial e^{\nu}(Y)}{\partial Y^B}\,. \tag{3.11}$$

Since all Minkowski indices are contracted this object is invariant under the isometries of de Sitter space, as is the metric and the geodesic length.

Expression (3.11) is known as a bitensor. Indices on the left belong to the tangent space of the first argument, indices on the right belong to the tangent space of the second argument. Other commonly used bitensors are the geodesic normals at x^μ and y^μ:

$$[{}_\mu n](x,y) \equiv \frac{\partial \ell(x,y)}{\partial x^\mu} = -\frac{1}{H}\csc[H\ell(x,y)]\,\frac{\partial E^A(x)}{\partial x^\mu}\,E_A(y)\,, \tag{3.12a}$$

$$[n_\mu](x,y) \equiv \frac{\partial \ell(x,y)}{\partial y^\mu} = -\frac{1}{H}\csc[H\ell(x,y)]\,E_A(x)\,\frac{\partial E^A(y)}{\partial y^\mu}\,, \tag{3.12b}$$

These bitensors are also de Sitter invariants. A theorem of Allen and Jacobson [9] asserts that any de Sitter invariant bitensor can be expressed using the length, the metric, the parallel transport matrix and the geodesic normals.

Since de Sitter space has the topology of $S^{D-1}\times R$ it is natural to use a coordinate system in which the spatial sections are S^{D-1}. These "closed coordinates" consist of a time t_1, $-\infty<t_1<\infty$, and $(D-1)$ angles obeying $0\leqq\alpha_i\leqq\pi$ for $i=1,\ldots,D-2$ and $0\leqq\alpha_{D-1}<2\pi$. The embedding is:

$$E^0 = \sinh(Ht_1)\,, \tag{3.13a}$$

$$E^I = \cosh(Ht_1)\sin(\alpha_1)\ldots\sin(\alpha_{D-I})\cos(\alpha_{D+1-I})\,, \tag{3.13b}$$

where we define $\alpha_D\equiv 0$. The unique inverse which is homogeneous of degree zero is:

$$e^0 = \operatorname{atanh}\left(\frac{X^0}{\sqrt{(X^1)^2+\cdots+(X^D)^2}}\right), \tag{3.14a}$$

$$e^i = \operatorname{atan}\left(\frac{\sqrt{(X^1)^2+\cdots+(X^{D-i})^2}}{X^{D+1-i}}\right). \tag{3.14b}$$

A trivial application of (3.5) gives the line element:

$$ds^2 = -dt_1^2 + \frac{1}{H^2}\cosh^2(Ht_1)\,d\Omega^2_{D-1}\,. \tag{3.15}$$

Although the factor of $\cosh^2(Ht_1)$ seems to indicate that de Sitter space contracts for $t_1<0$ this is a fiction of the coordinate system. We saw from expression (3.2) that the deviation between initially parallel, timelike and freely falling observers expands exponentially at all points in de Sitter space. The curves of constant α_i are indeed timelike geodesics but it can be shown using (3.11) that they are only parallel to one another at $t_1=0$. Before this time they draw together for no other reason than that they were initially converging at a tremendous rate. The uniform de Sitter expansion gradually slows their approach, and actually reverses it at $t_1=0$. Thereafter the exponential expansion of distances is manifest.

Another misleading feature of these coordinates is the fact that the spatial sections are closed. No local observer who is constrained by causality can verify this closure. Consider, for example, a null geodesic which passes through the origin. By setting $\ell=0$ in Eq. (3.8) we see that any such curve must obey:

$$0 = \frac{1}{2} - \frac{1}{2}E^D(x) = \frac{1}{2} - \frac{1}{2}\cosh(Ht_1)\cos(\alpha_1)\,. \tag{3.16}$$

Hence even the lifetime of the universe from $t_1=-\infty$ to $t_1=+\infty$ suffices to carry a light ray only halfway around the spatial section.

Since the locus of points with $t_1=$constant is a compact, spacelike Cauchy surface (the normal vector is just $c^\mu=\delta_0{}^\mu$) we see that de Sitter space suffers from a linearization instability. However, the fact that no more than half the points on this Cauchy surface will ever come into causal contact with one another means that the constraints (2.15) have a peculiarly artificial form. This is most simply illustrated by introducing matter into the theory. Consider the action of a point particle of bare mass M whose worldline is $q^\mu(\tau)$:

$$S^{\mathrm{matter}}[q,g] = -M \int d\tau \sqrt{-g_{\alpha\beta}(q)\,\dot{q}^\alpha \dot{q}^\beta}\ . \tag{3.17}$$

If the particle is stationary at the spatial origin ($\alpha^i = 0$) then its worldline obeys $q^\mu(\tau) = \tau\, \delta_0{}^\mu$ and the associated stress tensor is:

$$\kappa T^{\mathrm{matter}}_{\mu\nu}(t_1, \vec{x}) = -\frac{\frac{1}{2}\kappa M\, \delta^{D-1}(\vec{x})}{\sqrt{1-\kappa h_{00}(t_1,\vec{x})}}\, \delta_\mu{}^0 \delta_\nu{}^0\ . \tag{3.18}$$

By virtue of the κ in the numerator this contributes to lowest order in the equation for $h^{(2)}_{\mu\nu}$ but the total stress tensor must still obey (2.15) if the metric is free of coordinate singularities. The ξ^μ_{IJ} Killing vectors possess no zero component and so vanish when contracted into (3.18). The remaining D vectors give:

$$\begin{aligned} c^\mu(t_1,\vec{x})\, \xi^\nu_{0I}(t_1,\vec{x})\, \kappa T^{\mathrm{matter}}_{\mu\nu}(t_1,\vec{x}) &= -\frac{\frac{1}{2}\kappa M \delta^{D-1}(\vec{x})}{\sqrt{1-\kappa h_{00}(t_1,\vec{x})}}\, \frac{X_I}{\sqrt{1+(HX^0)^2}} \\ &= -\frac{\frac{1}{2}\kappa M\, \delta^{D-1}(\vec{x})}{\sqrt{1-\kappa h_{00}(t_1,\vec{x})}}\, \frac{1}{H}\, \delta_I{}^D \cos(\alpha_1)\ . \end{aligned} \tag{3.19}$$

The nonvanishing contribution from $\xi\beta^\mu_{0D}$ does not integrate to zero. It could be cancelled by the global negative energy modes of the graviton stress tensor but in this case the response would be nonzero beyond the lightcone of the source. It follows that a single point mass cannot give a causal response which is free of coordinate singularities on the full manifold! This fact has been noted previously in three dimensions by Deser and Jackiw [10].

The phenomenon we have just described bears a superficial analogy to the electrodynamical model presented in Sect. 2 as the paradigm for a linearization instability. If the spatial sections are $(D-1)$-spheres of constant curvature then the net charge has to vanish. This must be so in order to prevent the contradiction which would occur when the flux from a net charge meets itself on the other side of the $(D-1)$-sphere. The zero charge condition is of course accomplished by having as many negative charges as there are positive ones. However, we have seen that de Sitter space is expanding so rapidly that the flux from a point mass can never encounter itself on the other side of the spatial section. Nor is the "zero energy" condition enforced by having negative mass sources. Owing to the oddness of $\cos(\alpha_1)$ under $\alpha_1 \to \alpha_1 + \pi$ we conclude from (3.19) that while a single mass is not allowed without a coordinate singularity there is no problem for a pair of identical masses on opposite sides of the $(D-1)$-sphere. This is very strange because we have seen that no observer can feel the effect of both masses before $t_1 = 0$, and the two masses *never* feel one another.

The rest of the argument was given in Sect. 1. If the response to such a dual source is regular beyond the source points at $\alpha_1 = 0$ and $\alpha_1 = \pi$ then it cannot come from a causal Green's function. Since we believe that it makes sense to consider the theory's response to an arbitrary (conserved) disturbance, and since we feel that this response must be causal we are powerfully motivated to consider restricting physics to a portion of the full manifold.

The minimal coordinate patch we might consider is the one which describes the submanifold within the causal horizon of an observer at the spatial origin. A convenient coordinatization is the static system given by a time t, $-\infty<t<\infty$, and a $(D-1)$-vector $\vec{r}$ of radius $r=\|\vec{r}\|\leqq 1/H$. The embedding is:

$$E^0=\sinh(Ht)\sqrt{1-H^2r^2}\ , \tag{3.20a}$$

$$E^i=Hr^i\ , \tag{3.20b}$$

$$E^D=\cosh(Ht)\sqrt{1-H^2r^2}\ . \tag{3.20c}$$

The homogeneous inverse is easily seen to be:

$$e^0=\operatorname{atanh}\left(\frac{X^0}{X^D}\right), \tag{3.21a}$$

$$e^i=\frac{X^i}{\sqrt{X\cdot X}}\ . \tag{3.21b}$$

A now familiar application of (3.5) yields the following line element:

$$ds^2=-(1-H^2r^2)\,dt^2+(1-H^2r^2)^{-1}\,dr^2+r^2\,d\Omega^2_{D-2}\ . \tag{3.22}$$

That the apparent problem at $Hr=1$ is only a coordinate singularity can be seen from the fact that all curvature invariants are finite at this point.

That the horizon is no barrier to outward propagation is shown by consideration of a null ray which intersects the origin. By setting $\ell=0$ in (3.8) we see that any such curve must obey:

$$Hr=\tanh(Ht)\ . \tag{3.23}$$

Though it would seem that the null ray cannot pass the horizon this is a figment of the coordinate system. In fact this geodesic agrees along its full extent with the geodesic defined in closed coordinates by (3.16); (3.23) only stops at $r=\frac{1}{H}$ because the static coordinate system runs out there. The time t is that of an observer in free fall at the spatial origin. Though it requires an infinite proper time for him to feel influences from beyond the horizon freely falling observers at all other points reach the horizon in a finite proper time. Since quantum field theory is dominated by the many rather than the few we must reject the static coordinate patch as too small a physical arena.

A much larger venue is provided by the "open" coordinate system. This consists of a time t_0, $-\infty<t_0<\infty$, and a $(D-1)$-vector $\vec{x}$ whose norm, $\|\vec{x}\|$, is unrestricted along the nonnegative real line. The embedding is:

$$E^0=\sinh(Ht_0)+\frac{1}{2}H^2\|\vec{x}\|^2\exp(Ht_0)\ , \tag{3.24a}$$

$$E^i=\exp(Ht_0)Hx^i\ , \tag{3.24b}$$

$$E^D=\cosh(Ht_0)-\frac{1}{2}H^2\|\vec{x}\|^2\exp(Ht_0)\ . \tag{3.24c}$$

The homogeneous inverse is:

$$e^0 = \ln\left[\frac{X^0+X^D}{\sqrt{X\cdot X}}\right], \tag{3.25a}$$

$$e^i = \frac{X^i}{X^0+X^D} . \tag{3.25b}$$

Another application of (3.5) gives the line element:

$$ds^2 = -dt_0^2 + \exp(2Ht_0)\,\|d\vec{x}\|^2 . \tag{3.26}$$

Although the lines of constant $\vec{x}$ are timelike geodesics it can be shown using (3.11) that they are not parallel to one another on any surface of constant t_0. However, since they are at all times drawing apart from one another the fact that their divergence increases exponentially with t_0 makes the uniform exponential expansion of distances manifest in these coordinates.

It is easy to see that in these coordinates the distance function obeys:

$$1-z(x,x') = -\sinh^2\left[\frac{1}{2}H(t_0-t_0')\right] + \frac{1}{4}H^2\,\|\vec{x}-\vec{x}\|^2 \exp[H(t_0+t_0')] . \tag{3.27}$$

Setting $\vec{x}'=0$ and $\ell(x,x')=0$ gives the equation for a null ray which passes through the spatial origin at t_0':

$$H\,\|\vec{x}\| = |\exp(-Ht_0') - \exp(-Ht_0)| . \tag{3.28}$$

By varying t_0 and t_0' we see that such a light ray can just cross the open coordinate patch in the lifetime of the universe.

Since $E^0+E^D = \exp(Ht_0) > 0$ these coordinates do *not* cover de Sitter space. They do, however, have the property that the locus of points with $t_0 = \text{const.}$ is a Cauchy surface, so it is valid to restrict physics to this submanifold. It is the $t_0 > 0$ segment of this space which would be joined to an open Friedmann–Robertson–Walker universe to describe the transition to an inflationary phase.

It is a trivial consequence of (3.1) that the de Sitter Weyl tensor vanishes. Any metric with this property is locally conformal to the Lorentz metric. In fact the entire open coordinate patch is conformal to flat space. The conformal time is just:

$$u \equiv \frac{1}{H}\exp(-Ht_0) . \tag{3.29}$$

Note the curious inversion; the infinite past corresponds to $u \to +\infty$ while the infinite future is at $u=0$. The conformal embedding is:

$$E^0 = \frac{1+H^2(\|\vec{x}\|^2-u^2)}{2Hu} , \tag{3.30a}$$

$$E^i = \frac{x^i}{u} , \tag{3.30b}$$

$$E^D = \frac{1-H^2(\|\vec{x}\|^2-u^2)}{2Hu} . \tag{3.30c}$$

Its unique homogeneous inverse is:

$$e^0 = \frac{\sqrt{X \cdot X}}{X^0 + X^D}, \tag{3.31a}$$

$$e^i = \frac{X^i}{X^0 + X^D}. \tag{3.31b}$$

Applying (3.5) results in the claimed conformal line element:

$$ds^2 = \Omega^2(-du^2 + d\vec{x} \cdot d\vec{x}), \tag{3.32a}$$

$$\Omega = \frac{1}{Hu} = E^0 + E^D. \tag{3.32b}$$

From (3.8) we infer a relation which will be of importance in the next section:

$$\sin^2\left[\frac{1}{2} H\ell(x, x')\right] = \frac{1}{4}\Omega(x)\Omega(x')H^2(x-x')^2 = 1 - z(x, x'), \tag{3.33}$$

where by x^2 we mean the Lorentz inner product, $x^2 \equiv -u^2 + \|\vec{x}\|^2$.

4. The Retarded Green's Function

The great advantage of conformal coordinates is that the background metric, $\hat{g}_{\mu\nu}$, is so simply related to the Lorentz metric. To better exploit this simplicity we conformally rescale the full metric (background plus perturbation) as follows:

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \kappa h_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu}, \tag{4.1a}$$

and introduce the "pseudo-graviton field:"

$$\psi_{\mu\nu} \equiv \frac{1}{\kappa}(\tilde{g}_{\mu\nu} - \eta_{\mu\nu}). \tag{4.1b}$$

Indices on $\psi_{\mu\nu}$ are raised and lowered with the Lorentz metric.

The conformal rescaling allows us to reexpress the invariant Lagrangian:

$$\begin{aligned} \mathscr{L}_{\text{inv}} &= \frac{1}{\kappa^2}\sqrt{-g}\,[R - (D-2)(D-1)H^2] \\ &= \frac{1}{\kappa^2}\sqrt{-\tilde{g}}\,[\tilde{R}\Omega^{D-2} - 2(D-1)\Omega^{D-3}\Omega^{;\mu}{}_{\mu} \\ &\quad - (D-4)(D-1)\tilde{g}^{\mu\nu}\Omega_{,\mu}\Omega_{,\nu}\Omega^{D-4} - (D-2)(D-1)H^2\Omega^D], \end{aligned} \tag{4.2}$$

where $\tilde{R}$ is the Ricci scalar formed from $\tilde{g}_{\mu\nu}$ and $\Omega^{;\mu}{}_{\mu} \equiv \tilde{g}^{\mu\nu}(\Omega_{,\mu\nu} - \tilde{\Gamma}^{\rho}{}_{\mu\nu}\Omega_{,\rho})$ is the covariant derivative with respect to the rescaled metric. After many tedious rearrangements we extract a presumably irrelevant surface term:

$$\begin{aligned} \mathscr{S}^{\nu}{}_{,\nu} \equiv -\frac{1}{\kappa^2}\Big[&(\sqrt{-\tilde{g}})_{,\mu}\,\tilde{g}^{\mu\nu}\,\Omega^{D-2} + (\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\,\Omega^{D-2})_{,\mu} \\ &+ \left(\frac{D-4}{D-2}\right)\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}(\Omega^{D-2})_{,\mu} - \kappa\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\rho}\,\psi_{\rho}{}^{\nu}(\Omega^{D-2})_{,\mu}\Big]_{,\nu}, \end{aligned} \tag{4.3a}$$

and the following volume term:

$$\mathscr{L}_{\text{inv}}-\mathscr{S}^{\nu}{}_{,\nu}=-\frac{1}{2}(D-2)\sqrt{-\tilde{g}}\,\tilde{g}^{\rho\sigma}\,\tilde{g}^{\mu\nu}\,\psi_{\rho\sigma,\mu}\,\psi_{\nu}{}^{\alpha}\,\Omega_{,\alpha}\,\Omega^{D-3}$$

$$+\sqrt{-\tilde{g}}\,\tilde{g}^{\alpha\beta}\,\tilde{g}^{\rho\sigma}\,\tilde{g}^{\mu\nu}\left[\frac{1}{2}\psi_{\alpha\rho,\mu}\,\psi_{\nu\sigma,\beta}-\frac{1}{2}\psi_{\alpha\beta,\rho}\,\psi_{\sigma\mu,\nu}\right.$$

$$\left.+\frac{1}{4}\psi_{\alpha\beta,\rho}\,\psi_{\mu\nu,\sigma}-\frac{1}{4}\psi_{\alpha\rho,\mu}\,\psi_{\beta\sigma,\nu}\right]\Omega^{D-2}\,. \tag{4.3b}$$

The quadratic part is:

$$\mathscr{L}^{2}_{\text{inv}}=\left[\frac{1}{2}\psi^{\rho\sigma,\mu}\,\psi_{\mu\sigma,\rho}-\frac{1}{2}\psi^{\mu\nu}{}_{,\mu}\,\psi_{,\nu}+\frac{1}{4}\psi^{,\mu}\,\psi_{,\mu}\right.$$

$$\left.-\frac{1}{4}\psi^{\rho\sigma,\mu}\,\psi_{\rho\sigma,\mu}-\left(\frac{D-2}{2}\right)\psi^{\mu\nu}\,\psi_{,\mu}\,\phi_{,\nu}\right]\Omega^{D-2}\,, \tag{4.4}$$

where $\phi\equiv\ln(\Omega)$.

It is convenient to scale away multiplicative conformal factors by making the substitution:

$$\psi_{\mu\nu}=\Omega^{1-\frac{D}{2}}\,\chi_{\mu\nu} \tag{4.5}$$

in expression (4.4):

$$\mathscr{L}^{2}_{\text{inv}}=\frac{1}{2}\chi^{\rho\sigma,\mu}\chi_{\mu\sigma,\rho}-\frac{1}{2}\chi^{\mu\nu}{}_{,\mu}\chi_{,\nu}+\frac{1}{4}\chi^{,\mu}\chi_{,\mu}-\frac{1}{4}\chi^{\rho\sigma,\mu}\chi_{\rho\sigma,\mu}$$

$$+\left(\frac{D-2}{2}\right)\left\{-\chi^{\mu\rho,\sigma}\chi_{\rho\sigma}\phi_{,\mu}-\frac{1}{2}\chi^{\mu\nu}\chi_{,\mu}\,\phi_{,\nu}+\frac{1}{2}\chi\chi^{\mu\nu}{}_{,\mu}\phi_{,\nu}\right.$$

$$\left.+\frac{1}{4}\left(-\chi^{2}+\chi^{\rho\sigma}\chi_{\rho\sigma}\right)^{,\mu}\phi_{,\mu}\right\}$$

$$+\left(\frac{D-2}{2}\right)^{2}\left\{\frac{1}{2}\chi^{\mu\rho}\chi^{\nu}{}_{\rho}\phi_{,\mu}\phi_{,\nu}+\frac{1}{2}\chi\chi^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right.$$

$$\left.+\frac{1}{4}\chi^{2}\phi^{,\mu}\phi_{,\mu}-\frac{1}{4}\chi^{\rho\sigma}\chi_{\rho\sigma}\phi^{,\mu}\phi_{,\mu}\right\}. \tag{4.6}$$

The simplest gauge fixing functional seems to be $-\frac{1}{2}F_{\mu}F_{\nu}\eta^{\mu\nu}$, where:

$$F_{\mu}=\chi^{\nu}{}_{\mu,\nu}-\frac{1}{2}\chi_{,\mu}+\left(\frac{D-2}{2}\right)\chi^{\nu}{}_{\mu}\phi_{,\nu}+\left(\frac{D-2}{4}\right)\chi\phi_{,\mu}\,. \tag{4.7}$$

With this and some partial integrations the gauge fixed, quadratic Lagrangian reduces to the following simple form:

$$\mathscr{L}^{2}_{\text{GF}}=-\frac{1}{8}\chi\left[\partial^{2}+\left(\frac{D-2}{2}\right)\left(\frac{D}{2}\right)\frac{1}{u^{2}}\right]\chi+\frac{1}{4}\chi^{\rho\sigma}\left[\partial^{2}+\left(\frac{D-2}{2}\right)\left(\frac{D}{2}\right)\frac{1}{u^{2}}\right]\chi_{\rho\sigma}$$

$$-\left(\frac{D-2}{2}\right)\frac{1}{u^{2}}\chi^{0\rho}\chi_{0\rho}\,. \tag{4.8}$$

The gauge fixed kinetic operator is therefore:

$$\mathscr{D}_{\mu\nu}^{\rho\sigma} \equiv \left[\frac{1}{2}\delta_\mu^{(\rho}\,\delta_\nu^{\sigma)} - \frac{1}{4}\eta_{\mu\nu}\,\eta^{\rho\sigma}\right]\left[\partial^2 + \left(\frac{D-2}{2}\right)\left(\frac{D}{2}\right)\frac{1}{u^2}\right]$$
$$-\frac{D-2}{u^2}\,\delta_{(\mu}{}^0\,\delta_{\nu)}{}^{(\rho}\,\delta_0{}^{\sigma)}\,, \tag{4.9}$$

where parenthesized indices are symmetrized.

There are three sorts of homogeneous solutions:

$$\mathscr{D}_{\mu\nu}^{\rho\sigma}\,\varepsilon_{\rho\sigma} = \left[\partial^2 + \left(\frac{D-2}{2}\right)\left(\frac{D}{2}\right)\frac{1}{u^2}\right]\left(\frac{1}{2}\varepsilon_{\mu\nu} - \frac{1}{4}\varepsilon\,\eta_{\mu\nu}\right)$$
$$-\frac{D-2}{u^2}\,\delta_{(\mu}{}^0\,\varepsilon_{\nu)0} = 0\,. \tag{4.10}$$

If the polarization is purely spatial then it is annihilated by the operator:

$$\mathscr{D}_A \equiv \left[\partial^2 + \left(\frac{D-2}{2}\right)\left(\frac{D}{2}\right)\frac{1}{u^2}\right]. \tag{4.11a}$$

If the polarization is mixed time and space – that is, only the ε_{0i} components are nonzero – then it is annihilated by the operator:

$$\mathscr{D}_B \equiv \left[\partial^2 + \left(\frac{D-4}{2}\right)\left(\frac{D-2}{2}\right)\frac{1}{u^2}\right]. \tag{4.11b}$$

And there is a single homogeneous solution of the form $\varepsilon_{\mu\nu} = \left[\delta_\mu{}^0\,\delta_\nu{}^0 + \frac{1}{D-2}\,\eta_{\mu\nu}\right]$ $\bar{\varepsilon}$, where $\bar{\varepsilon}$ is annihilated by the operator:

$$\mathscr{D}_C \equiv \left[\partial^2 + \left(\frac{D-6}{2}\right)\left(\frac{D-4}{2}\right)\frac{1}{u^2}\right]. \tag{4.11c}$$

Note that the distinction between $\mathscr{D}_B$ and $\mathscr{D}_C$ disappears for $D=4$.

It might be expected that each of these operators plays a role in the graviton Green's function and this is the case. The retarded Green's function is defined by the equation:

$$\mathscr{D}_{\mu\nu}^{\rho\sigma}\,[{}_{\rho\sigma}\mathscr{G}^{\alpha\beta}_{\text{ret}}](x,x') = \delta_\mu{}^{(\alpha}\,\delta_\nu{}^{\beta)}\,\delta^D(x-x') \tag{4.12}$$

and the boundary condition that it should vanish for $u' < u$. From general tensor analysis we can assume a solution of the form:

$$[{}_{\rho\sigma}\mathscr{G}^{\alpha\beta}_{\text{ret}}](x,x') = a(x,x')\,2\delta_\rho{}^{(\alpha}\,\delta_\sigma{}^{\beta)} + b(x,x')\,4\delta_{(\rho}{}^0\,\delta_{\sigma)}{}^{(\alpha}\,\delta_0{}^{\beta)} + c(x,x')\,\eta_{\rho\sigma}\,\eta^{\alpha\beta}$$
$$+ d(x,x')\,[\eta_{\rho\sigma}\,\delta_0{}^\alpha\,\delta_0{}^\beta + \delta_\rho{}^0\,\delta_\sigma{}^0\,\eta^{\alpha\beta}]$$
$$+ e(x,x')\,\delta_\rho{}^0\,\delta_\sigma{}^0\,\delta_0{}^\alpha\,\delta_0{}^\beta\,. \tag{4.13}$$

One now acts $\mathscr{D}_{\mu\nu}^{\rho\sigma}$ on this ansatz and segregates the terms proportional to each distinct tensor factor. The result is six scalar differential equations:

$$\delta_\mu{}^{(\alpha}\,\delta_\nu{}^{\beta)}\!:\ \mathscr{D}_A\,a(x,x') = \delta^D(x-x')\,, \tag{4.14a}$$

$$4\delta_{(\mu}{}^{0}\,\delta_{\nu)}{}^{(\alpha}\,\delta_{0}{}^{\beta)}:\ -\left(\frac{D-2}{2}\right)\frac{1}{u^2}\,a(x,x')+\frac{1}{2}\,\mathscr{D}_B\,b(x,x')=0\,, \tag{4.14b}$$

$$\eta_{\mu\nu}\,\eta^{\alpha\beta}:\ -\frac{1}{2}\,\mathscr{D}_A\left[a(x,x')+\left(\frac{D-2}{2}\right)c(x,x')-\frac{1}{2}\,d(x,x')\right]=0\,, \tag{4.14c}$$

$$\delta_{\mu}{}^{0}\,\delta_{\nu}{}^{0}\,\eta^{\alpha\beta}:\ \frac{D-2}{u^2}\,c(x,x')+\frac{1}{2}\left[\partial^2+\left(\frac{D-8}{2}\right)\left(\frac{D-2}{2}\right)\frac{1}{u^2}\right]d(x,x')=0\,, \tag{4.14d}$$

$$\eta_{\mu\nu}\,\delta_{0}{}^{\alpha}\,\delta_{0}{}^{\beta}:\ \mathscr{D}_A\left[b(x,x')-\left(\frac{D-2}{4}\right)d(x,x')+\frac{1}{4}\,e(x,x')\right]=0\,, \tag{4.14e}$$

$$\delta_{\mu}{}^{0}\,\delta_{\nu}{}^{0}\,\delta_{0}{}^{\alpha}\,\delta_{0}{}^{\beta}:\ -2\,\frac{D-2}{u^2}\,b(x,x')+\frac{D-2}{u^2}\,d(x,x')$$
$$+\frac{1}{2}\left[\partial^2+\left(\frac{D-8}{2}\right)\left(\frac{D-2}{2}\right)\frac{1}{u^2}\right]e(x,x')=0\,. \tag{4.14f}$$

To solve (4.14b) we make the substitution:

$$b(x,x')\equiv -a(x,x')+\bar{b}(x,x')\,, \tag{4.15a}$$

and use (4.14a) to conclude:

$$\mathscr{D}_B\,\bar{b}(x,x')=\delta^D(x-x')\,. \tag{4.15b}$$

Note that (4.14c) and the retarded boundary condition imply:

$$d(x,x')=2a(x,x')+(D-2)\,c(x,x')\,. \tag{4.16}$$

Now make the substitution:

$$c(x,x')\equiv -\frac{2}{D-3}\,a(x,x')+\frac{2}{(D-3)(D-2)}\,\bar{c}(x,x') \tag{4.17a}$$

in (4.14d). Using (4.14a) and (4.16) we conclude:

$$\mathscr{D}_C\,\bar{c}(x,x')=\delta^D(x-x')\,. \tag{4.17b}$$

It follows from (4.16) and (4.17) that:

$$d(x,x')=-\frac{2}{D-3}\,a(x,x')+\frac{2}{D-3}\,\bar{c}\,(x,x')\,. \tag{4.18}$$

Relation (4.14e) and the retarded boundary condition give:

$$e(x,x')=-4b(x,x')+(D-2)d(x,x') \tag{4.19a}$$

$$=2\left(\frac{D-4}{D-3}\right)a(x,x')+4\bar{b}(x,x')+2\left(\frac{D-2}{D-3}\right)\bar{c}(x,x')\,. \tag{4.19b}$$

Upon substitution of (4.15a), (4.18) and (4.19b) it can be seen that (4.14f) is obeyed as well. Note that the apparent singularity of (4.17a) and (4.18) in three dimensions is avoided because $a(x,x')=\bar{c}(x,x')$ for $D=3$.

The simplest way to solve (4.14a), (4.15b) and (4.17b) is to first obtain ordinary differential equations in the temporal variable u by Fourier transforming on $\vec{x}$. The three homogeneous equations are then reducible to Bessel's equation of order $\nu=\frac{D-1}{2}$, $\frac{D-3}{2}$ and $\frac{D-5}{2}$ respectively. The jump condition at $u=u'$ gives solutions proportional to $-N_\nu(ku')\,J_\nu(ku)+J_\nu(ku')\,N_\nu(ku)$, where $k=\|\vec{k}\|$ and $\vec{k}$ is the Fourier conjugate to $\vec{x}$. The angular parts of the inverse Fourier transform give another Bessel function and so we obtain the position space solution:

$$g_\nu(x,x';D)=\frac{\theta(\Delta u)\sqrt{uu'}}{4[2\pi\,\Delta x]^{(\frac{D-3}{2})}}\int_0^\infty dk k^{(\frac{D-1}{2})}J_{(\frac{D-3}{2})}(k\Delta x)$$
$$\times\{-N_\nu(ku')J_\nu(ku)+J_\nu(ku')N_\nu(ku)\}\;, \tag{4.20}$$

where $\Delta u\equiv u'-u$ and $\Delta x\equiv\|\vec{x}'-\vec{x}\|$.

It is not simple to evaluate the integral when D is odd and ν is integral but the result for even D and half integral ν is a series in derivatives of the lightcone theta function, $\theta(\Delta u-\Delta x)$. If $D=2d$ and $\nu=d-\frac{1}{2}-n$, then for $0\leqq n\leqq d-1$ we obtain:

$$g_{d-\frac{1}{2}-n}(x,x';2d)=-\frac{\theta(\Delta u)}{2[4\pi uu']^{d-1}}\sum_{k=n}^{d-1}\frac{(2d-n-k-2)!}{(d-k-1)!\,(k-n)!}$$
$$\times\left(-2\frac{uu'}{\Delta x}\frac{\partial}{\partial\Delta x}\right)^k\theta\kappa(\Delta u-\Delta x)\;. \tag{4.21a}$$

Except for the initial factor of $(uu')^{1-d}$ these functions are de Sitter invariant. To see this first note from (3.33) that $\theta(\Delta u)\,\theta(\Delta u-\Delta x)=\theta(\Delta u)\,\theta(z(x,x')-1)$. Now use (3.33) again to convert the Δx derivatives in (4.21a) to z derivatives. The result is:

$$g_{d-\frac{1}{2}-n}(x,x';2d)=-\frac{\theta(\Delta u)}{2[4\pi uu']^{d-1}}\sum_{k=n}^{d-1}\frac{(2d-n-k-2)!}{(d-k-1)!(k-n)!}$$
$$\times\delta^{(k-1)}[z(x,x')-1]\;, \tag{4.21b}$$

where we define $\delta^{(-1)}(z-1)\equiv\theta(z-1)$. The function $a(x,x')$ corresponds to the case of $n=0$. For $d\geqq 3$ the functions $\bar{b}(x,x')$ and $\bar{c}(x,x')$ are associated with $n=1$ and $n=2$ respectively; for $d=2$ they are both given by $n=1$. Note that only $a(x,x')$ contains a theta function.

Let us introduce the notation that a bar above a Lorentz metric or a Kronecker delta symbol means that the zero (i.e., u) component is projected out:

$$\bar{\eta}_{\mu\nu}\equiv\eta_{\mu\nu}+\delta_\mu{}^0\delta_\nu{}^0\;. \tag{4.22}$$

The tensor structure of our Green's function is most revealingly expressed by segregating it into terms proportional to the three scalar functions:

$$[{}_{\rho\sigma}\mathscr{G}^{\alpha\beta}_{\rm ret}](x,x')=a(x,x')\,[{}_{\rho\sigma}T^{\alpha\beta}_A]+\bar{b}(x,x')\,[{}_{\rho\sigma}T^{\alpha\beta}_B]+\bar{c}(x,x')\,[{}_{\rho\sigma}T^{\alpha\beta}_C]\;, \tag{4.23a}$$

$$[{}_{\rho\sigma}T^{\alpha\beta}_A]\equiv[2\bar{\delta}_\rho{}^{(\alpha}\,\bar{\delta}_\sigma{}^{\beta)}-\frac{2}{D-3}\bar{\eta}_{\rho\sigma}\,\bar{\eta}^{\alpha\beta}]\;, \tag{4.23b}$$

$$[{}_{\rho\sigma}T^{\alpha\beta}_B]\equiv 4\delta_{(\rho}{}^0\,\bar{\delta}_{\sigma)}{}^{(\alpha}\,\delta_0{}^{\beta)}\;, \tag{4.23c}$$

$$[{}_{\rho\sigma}T^{\alpha\beta}_C]\equiv\frac{2}{(D-3)(D-2)}[\eta_{\rho\sigma}+(D-2)\delta^0{}_\rho\delta^0{}_\sigma]\,[\eta^{\alpha\beta}+(D-2)\delta_0{}^\alpha\,\delta_0{}^\beta]\;. \tag{4.23d}$$

Now recall from (4.1) and (4.5) that the pseudo-graviton field and the graviton field relate to $\chi_{\mu\nu}$ thusly:

$$h_{\mu\nu}(x)=\Omega^2(x)\psi_{\mu\nu}(x)=\Omega^{3-\frac{D}{2}}(x)\,\chi_{\mu\nu}(x)\,, \tag{4.24a}$$

$$h^{\mu\nu}(x)=\Omega^{-2}(x)\,\psi^{\mu\nu}(x)=\Omega^{-1-\frac{D}{2}}(x)\,\chi^{\mu\nu}(x)\,. \tag{4.24b}$$

It follows that the retarded Green's function for the pseudo-graviton field is:

$$[{}_{\rho\sigma}G^{\alpha\beta}_{\rm ret}](x,x')=\Omega^{1-\frac{D}{2}}(x)\,\Omega^{1-\frac{D}{2}}(x')\,[{}_{\rho\sigma}\mathcal{G}^{\alpha\beta}_{\rm ret}](x,x') \tag{4.25a}$$

$$\begin{aligned}&\equiv 2\pi\theta(\Delta u)\,\{G_A(z-1)[{}_{\rho\sigma}T^{\alpha\beta}_{A}]\\ &+G_B(z-1)\,[{}_{\rho\sigma}T^{\alpha\beta}_{B}]\\ &+G_C(z-1)\,[{}_{\rho\sigma}T^{\alpha\beta}_{C}]\}\,,\end{aligned} \tag{4.25b}$$

where $z=z(x,x')$ is the length function defined in relation (3.8). Note that the functions G_A, G_B and G_C are de Sitter invariants. For example, when the dimension is even ($D=2d$) we have:

$$G_A(z-1)=-\frac{1}{4\pi}\left[\frac{H^2}{4\pi}\right]^{d-1}\sum_{k=0}^{d-1}\frac{(2d-k-2)!}{(d-k-1)!\,k!}\,\delta^{(k-1)}[z(x,x')-1]\,. \tag{4.26}$$

The functions G_B and G_C follow similarly from (4.21b).

From (4.24) we see that the retarded Green's function for the graviton field is:

$$[{}_{\rho\sigma}H^{\alpha\beta}_{\rm ret}](x,x')=\Omega^2(x)\Omega^{-2}(x')\,[{}_{\rho\sigma}G^{\alpha\beta}_{\rm ret}](x,x') \tag{4.27a}$$

$$\begin{aligned}&\equiv 2\pi\theta(\Delta u)\,\{G_A(z-1)\,[{}_{\rho\sigma}\tau^{\alpha\beta}_{A}](x,x')\\ &+G_B(z-1)\,[{}_{\rho\sigma}\tau^{\alpha\beta}_{B}](x,x')\\ &+G_C(z-1)\,[{}_{\rho\sigma}\tau^{\alpha\beta}_{C}](x,x')\}\,.\end{aligned} \tag{4.27b}$$

The bitensor functions $[{}_{\rho\sigma}\tau^{\alpha\beta}_{A}](x,x')\equiv\Omega^2(x)\Omega^{-2}(x')\,[{}_{\rho\sigma}T^{\alpha\beta}_{A}]$, etc., are not de Sitter invariants. Of course $\Omega^2(x)\eta_{\rho\sigma}=\hat{g}_{\rho\sigma}(x)$ and $\Omega^{-2}(x')\eta^{\alpha\beta}=\hat{g}^{\alpha\beta}(x')$, the problem comes with $\Omega(x)\Omega^{-1}(x')\delta_\rho{}^\alpha$. Using expressions (3.11) and (3.12) one can show that:

$$\begin{aligned}\frac{\Omega(x)}{\Omega(x')}\,\delta_\mu{}^\nu=&[{}_\mu\hat{g}^\nu](x,x')+2\sqrt{z(1-z)}\,\{\omega_\mu(x)\,[\hat{n}^\nu](x,x')+[{}_\mu\hat{n}](x,x')\omega^\nu(x')\}\\ &+2(1-z)\,\{[{}_\mu\hat{n}](x,x')\,[\hat{n}^\nu](x,x')-\omega_\mu(x)\,\omega^\nu(x')\}\,,\end{aligned} \tag{4.28a}$$

where:

$$\omega_\mu(x)\equiv\Omega(x)\,\delta_\mu{}^0\,. \tag{4.28b}$$

Note that $\omega_\mu(x)$ is *not* a vector in the same sense that $[\hat{n}_\mu](x,y)$ is. The latter is defined invariantly by (3.12b) in any coordinate system whereas $\omega_\mu(x)$ is defined by (4.28b) in conformal coordinates, and in other systems by transforming it like a vector. Recall that *anything* is covariant if it is defined in a particular coordinate system.

This breaking of de Sitter invariance derives from the gauge fixing functional (4.7). Had we used a de Sitter invariant gauge we would have obtained an invariant graviton retarded Green's function. An example of such a gauge fixing functional is

$-\frac{1}{2}\sqrt{-\hat{g}}\hat{g}^{\mu\nu} F_\mu F_\nu$ where:

$$F_\mu = h^\rho{}_{\mu;\rho} - \frac{1}{2} h^\rho{}_{\rho;\mu} . \tag{4.29}$$

We will not trouble to work out the associated Green's function.

Although a de Sitter invariant Green's function could be obtained we do not choose to do so for three reasons. First, full de Sitter invariance is not observable on the open submanifold because some de Sitter transformations carry points on this coordinate patch off of it. Second, we will see in Sect. 6 that the propagator cannot be de Sitter invariant. Although unexpected there is precedent for this result in the previous work of Allen and Folacci [11] for the massless minimally coupled scalar field on a de Sitter background. A similar result for gravity becomes well nigh inevitable when one notes that the scalar kinetic operator of Allen and Folacci is the same as the kinetic operator for purely spatial polarizations of the pseudo-graviton field in our gauge.

The final reason we prefer a noninvariant gauge is that it permits a simple expression for the Green's function. Since the coordinate patch and the propagator must in any case break de Sitter invariance it seems worthwhile to introduce a bit more noninvariance in order to simplify the tensor algebra. Of course to those not familiar with quantum gravity our relation (4.27), with its ancillary definitions in previous expressions, probably seems formidably complex. However, it is actually quite simple for a graviton Green's function, and less complex than the invariant expressions which have been obtained previously. The reason for this seems to be that the conformal flatness of de Sitter space is a more powerful organizing principle than de Sitter invariance.

This point is forcefully illustrated by consideration of the Maxwell Green's function on four dimensional de Sitter space. Since free electromagnetism is conformally invariant for $D=4$ we can find the Green's function very simply by first conformally rescaling the metric and then adding the Feynman gauge fixing functional:

$$\frac{1}{2}(\eta^{\mu\nu} A_{\mu,\nu})^2 . \tag{4.30}$$

The result is:

$$[{}_\mu G_\nu](x, x') = \frac{\theta(\Delta u)}{4\pi} \eta_{\mu\nu} \delta[(x-x')^2] \tag{4.31a}$$

$$= \frac{H^2}{16\pi} \Omega(x)\,\Omega(x')\,\eta_{\mu\nu} \delta[z(x, x') - 1] . \tag{4.31b}$$

Because this is the flat space Green's function it is as simple to study $D=4$ electrodynamics on de Sitter space as it is in flat space. But from (4.28) we see that $\Omega(x)\,\Omega(x')\,\eta_{\mu\nu}$ is not de Sitter invariant. To obtain an invariant Green's function we would need an invariant gauge fixing functional such as:

$$-\frac{1}{2}\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\hat{g}^{\rho\sigma} A_{\mu,\nu} A_{\rho;\sigma} = -\frac{1}{2}[\eta^{\mu\nu}(A_{\mu,\nu} + 2A_\mu \phi_{,\nu})]^2 , \tag{4.32}$$

where $\phi \equiv \ln(\Omega)$. The resulting Green's function (for which take the imaginary part of the 1-form result recently obtained by Folacci [12]) is invariant but it is not simple. It contains a term proportional to $[{}_{\mu}\hat{g}_{\mu}]$ and another proportional to $[{}_{\mu}\hat{n}]$ $[\hat{n}_{\nu}]$. The invariant scalar proportionality functions are in each case more complicated than $\delta(z-1)$; what is worse, they are different. From (4.28) we see that it cannot be otherwise. It would be silly to use the second gauge for most calculations. The more efficient course is to use the first gauge condition and cheerfully accept the implied commitment to conformal coordinates.

Gravitation is not as straightforward as electrodynamics because Einstein's theory is not conformally invariant in four dimensions, yet the simplification effected in conformal coordinates with (4.7) is even greater. The invariant treatment requires five fundamental tensors – as opposed to our three – and the scalar proportionality functions are so complicated that their specification in a general gauge requires several pages of definitions [2]. By far the most condensed invariant expression is the one obtained by Antoniadis and Mottola [3] in four dimensions with a choice of gauge where only the spin two and spin zero parts of the graviton are nonzero. The divergent response that they report for a freely falling point mass is due to the infinite integration over u' of the theta function term in their spin zero Green's function. Our function G_A possesses a theta function but the reader can see from the tensor structure of expression (4.23b) that this term fails to couple with the stress tensor (3.18) of a freely failing point mass. Of course just achieving finiteness does not guarantee that this response is correct. We turn now to proving that it is.

5. Response to a Freely Falling Point Mass

From (3.17) and (4.1) we see that the action for a static point particle can be written as:

$$S^{\text{matter}}[q,g] \equiv -M \int d\tau \sqrt{-g_{\alpha\beta}(q(\tau))\, \dot{q}^{\alpha}(\tau)\, \dot{q}^{\beta}(\tau)}\,, \tag{5.1a}$$

$$\underset{q^{\mu}=\delta_0{}^{\mu}\tau}{\longrightarrow} -M \int d\tau\, \Omega(q) \sqrt{1-\kappa\psi_{00}(q)}\,. \tag{5.1b}$$

It follows that the linearized pseudo-graviton stress tensor is:

$$-\frac{\delta S^{\text{matter}}}{\delta\psi^{\alpha\beta}(x)_{|\psi=0}} = -\frac{1}{2}\kappa M \delta_{\alpha}{}^{0}\delta_{\beta}{}^{0}\Omega(x)\delta^{D-1}(\vec{x})\,. \tag{5.2}$$

From our solution (4.25) we see that the induced pseudo-graviton field must be:

$$\kappa\psi_{\rho\sigma}(x) = \int d^D x' [{}_{\rho\sigma}G^{\alpha\beta}_{\text{ret}}](x,x') \left\{-\frac{1}{2}\kappa^2 M \delta_{\alpha}{}^{0}\delta_{\beta}{}^{0}\Omega(x')\,\delta^{D-1}(\vec{x}')\right\}, \tag{5.3a}$$

$$= \left[2\delta_{\rho}{}^{0}\,\delta_{\sigma}{}^{0} + \frac{2}{D-2}\eta_{\rho\sigma}\right] \frac{(D-2)}{(D-3)^2} \frac{G_D M}{(\Omega\|\vec{x}\|)^{D-3}}\,, \tag{5.3b}$$

where we define the D-dimensional Newton constant:

$$G_D \equiv 2\frac{(D-3)^2}{(D-2)} \frac{\Gamma\left(\frac{D-3}{2}\right)}{\pi^{\frac{D-3}{2}}} G = \frac{1}{4}\left(\frac{D-3}{D-2}\right) \frac{\Gamma\left(\frac{D-1}{2}\right)}{\pi^{\frac{D-1}{2}}} \kappa^2 \tag{5.4}$$

to make the force law simple in the Newtonian limit. Note that of the three basis tensors in $[{}_{\rho\sigma}G^{\alpha\beta}_{\mathrm{ret}}](x, x')$ only $[{}_{\rho\sigma}T^{\alpha\beta}_{C}]$ couples to the stress energy of this source. Had the tensor $[{}_{\rho\sigma}T^{\alpha\beta}_{A}]$ not decoupled there would have been a logarithmic divergence from the infinite integration of the theta function over u'. This is essentially what happens with the spin zero part of the Green's function reported by Antoniadis and Mottola [3].

Expression (5.3) is certainly the linearized solution to the gauge fixed equations of motion; we shall now prove that it also satisfies the invariant field equations, $R_{\mu\nu}=(D-1)H^2 g_{\mu\nu}$, to linearized order. It is simplest to do this by first computing the conformally rescaled curvature tensors and then inverting the Weyl transformation. Since the conformal metric is $\tilde{g}_{\mu\nu}=\eta_{\mu\nu}+\kappa\psi_{\mu\nu}$ we see that the conformally rescaled Riemann tensor can be expanded as follows:

$$\tilde{R}_{\alpha\beta\gamma\delta}=-\frac{1}{2}\,\kappa(\psi_{\alpha\gamma,\beta\delta}-\psi_{\delta\alpha,\gamma\beta}+\psi_{\beta\delta,\alpha\gamma}-\psi_{\gamma\beta,\delta\alpha})+O(\kappa^2)\,. \tag{5.5}$$

However, it is well to note that in this problem we are really doing perturbation theory in the small parameter:

$$\varepsilon(u, \vec{x})\equiv\frac{G_D M}{(\Omega\,\|\vec{x}\,\|)^{D-3}}\,. \tag{5.6}$$

Substituting (5.3) into (5.5) gives:

$$\tilde{R}_{0i0j}=\left[\delta_{ij}-(D-1)\frac{x_i x_j}{\|\vec{x}\,\|^2}-\left(\frac{D-4}{D-3}\right)\frac{\|\vec{x}\,\|^2}{u^2}\,\delta_{ij}\right]\frac{\varepsilon}{\|\vec{x}\,\|^2}+O(\varepsilon^2)\,, \tag{5.7a}$$

$$\tilde{R}_{0ijk}=[x_j\,\delta_{ik}-x_k\,\delta_{ij}]\,\frac{\varepsilon}{u\,\|\vec{x}\,\|^2}+O(\varepsilon^2)\,, \tag{5.7b}$$

$$\begin{aligned}\tilde{R}_{ijk\ell}=&-\left(\frac{D-1}{D-3}\right)[\delta_{ik}\,x_j\,x_\ell-\delta_{\ell i}\,x_k\,x_j+\delta_{j\ell}\,x_i\,x_k-\delta_{kj}\,x_\ell\,x_i]\,\frac{\varepsilon}{\|\vec{x}\,\|^4}\\ &+\frac{2}{D-3}\,[\delta_{ik}\,\delta_{j\ell}-\delta_{i\ell}\,\delta_{jk}]\,\frac{\varepsilon}{\|\vec{x}\,\|^2}+O(\varepsilon^2)\,.\end{aligned} \tag{5.7c}$$

Simple contraction produces the conformally rescaled Ricci tensor:

$$\tilde{R}_{00}=-\frac{(D-4)(D-1)}{(D-3)}\,\frac{\varepsilon}{u^2}+O(\varepsilon^2)\,, \tag{5.8a}$$

$$\tilde{R}_{0i}=(D-2)\,\frac{\varepsilon}{u\,\|\vec{x}\,\|^2}\,x_i+O(\varepsilon^2)\,, \tag{5.8b}$$

$$\tilde{R}_{ij}=\left(\frac{D-4}{D-3}\right)\frac{\varepsilon}{u^2}\,\delta_{ij}+O(\varepsilon^2)\,, \tag{5.8c}$$

and the conformally rescaled Ricci scalar:

$$\tilde{R}=2\,\frac{(D-4)(D-1)}{(D-3)}\,\frac{\varepsilon}{u^2}+O(\varepsilon^2)\,. \tag{5.9}$$

If $g_{\mu\nu}=\Omega^2\,\tilde{g}_{\mu\nu}$ then the Ricci tensor formed from $g_{\mu\nu}$ is:

$$R_{\mu\nu}=\tilde{R}_{\mu\nu}+(D-2)\,[\phi_{,\mu}\,\phi_{,\nu}-\phi_{:\mu\nu}]-\tilde{g}_{\mu\nu}[(D-2)\phi_{,\rho}\,\phi_{,\sigma}+\phi_{:\rho\sigma}]\,\tilde{g}^{\rho\sigma}\,, \tag{5.10}$$

where we remind the reader that $\phi\equiv\ln(\Omega)$ and that a colon denotes covariant differentiation with respect to the conformally rescaled metric $\tilde{g}_{\mu\nu}$. Expression (5.10) is valid for any conformal factor; exploiting a special property of ours – namely, $\phi_{,\mu}\,\phi_{,\nu}=\phi_{,\mu\nu}$ – allows us to write:

$$R_{\mu\nu}=\tilde{R}_{\mu\nu}+(D-2)\,\tilde{\Gamma}^{\rho}{}_{\mu\nu}\,\phi_{,\rho}+\tilde{g}_{\mu\nu}[-(D-1)\phi_{,\rho\sigma}+\tilde{\Gamma}^{\lambda}{}_{\rho\sigma}\,\phi_{,\lambda}]\,\tilde{g}^{\rho\sigma}\,. \tag{5.11}$$

At this stage the expression is still correct to all orders in ε. Recognizing now that $\phi_{,\rho}=-\frac{1}{u}\,\delta_{\rho}{}^{0}$ and substituting our linearized solution (5.3) we find:

$$\tilde{\Gamma}^{\rho}{}_{00}\,\phi_{,\rho}=\frac{\varepsilon}{u^2}+O(\varepsilon^2)\,, \tag{5.12a}$$

$$\tilde{\Gamma}^{\rho}{}_{0i}\,\phi_{,\rho}=-\frac{\varepsilon}{u\,\|\vec{x}\,\|^2}\,x_i+O(\varepsilon^2)\,, \tag{5.12b}$$

$$\tilde{\Gamma}^{\rho}{}_{ij}\,\phi_{,\rho}-=\frac{1}{D-3}\frac{\varepsilon}{u^2}\,\delta_{ij}+O(\varepsilon^2)\,. \tag{5.12c}$$

Combining this with (5.8) gives the necessary result:

$$\tilde{R}_{\mu\nu}+(D-2)\,\tilde{\Gamma}^{\rho}{}_{\mu\nu}\,\phi_{,\rho}=-\frac{2}{D-3}\frac{\varepsilon}{u^2}\,\tilde{g}_{\mu\nu}+O(\varepsilon^2)\,. \tag{5.13}$$

Substitution into (5.11), a couple of simple traces, and the recognition that $\Omega=\frac{1}{Hu}$ implies:

$$\begin{aligned} R_{\mu\nu}=&\left[-\frac{2}{D-3}\frac{\varepsilon}{u^2}+(D-1)\frac{1}{u^2}+2\left(\frac{D-1}{D-3}\right)\frac{\varepsilon}{u^2}\right.\\ &\left.-2\left(\frac{D-2}{D-3}\right)\frac{\varepsilon}{u^2}+O(\varepsilon^2)\right]\tilde{g}_{\mu\nu} &\text{(5.14a)}\\ =&\left[(D-1)\frac{1}{u^2}+O(\varepsilon^2)\right]\tilde{g}_{\mu\nu} &\text{(5.14b)}\\ =&(D-1)\,H^2 g_{\mu\nu}+O(\varepsilon^2)\,. &\text{(5.14c)}\end{aligned}$$

It follows that the linearized response indeed obeys the invariant field equations as well as the gauge fixed ones.

Contact with an exact solution can be made by evaluating the Weyl tensor:

$$\tilde{C}_{0i0j}=\left[\delta_{ij}-(D-1)\frac{x_i\,x_j}{\|\vec{x}\,\|^2}\right]\frac{\varepsilon}{\|\vec{x}\,\|^2}+O(\varepsilon^2)\,, \tag{5.15a}$$

$$\tilde{C}_{0ijk}=O(\varepsilon^2)\,, \tag{5.15b}$$

$$\tilde{C}_{ijk\ell} = -\left(\frac{D-1}{D-3}\right)[\delta_{ik}\, x_j\, x_\ell - \delta_{\ell i}\, x_k\, x_j + \delta_{j\ell}\, x_i\, x_k - \delta_{kj}\, x_\ell\, x_i]\,\frac{\varepsilon}{\|\vec{x}\|^4}$$
$$+\frac{2}{D-3}[\delta_{ik}\,\delta_{j\ell} - \delta_{i\ell}\,\delta_{jk}]\,\frac{\varepsilon}{\|\vec{x}\|^2} + O(\varepsilon^2)\,. \tag{5.15c}$$

This is most usefully expressed by raising the first index – whereupon the conformal rescaling ceases to matter – and representing the spatial indices in a spherical coordinate basis:

$$S^i_x \equiv \frac{x^i}{\|\vec{x}\|}\,, \tag{5.16a}$$

$$S^i_{\alpha_j} \equiv \frac{\partial x^i}{\partial \alpha_j}\,. \tag{5.16b}$$

Note that these basis vectors obey the following orthogonality relations:

$$S^i_x S^j_x \delta_{ij} = 1\,, \tag{5.17a}$$

$$S^i_x S^j_{\alpha_k} \delta_{ij} = 0\,, \tag{5.17b}$$

$$S^i_{\alpha_k} S^j_{\alpha_\ell} \delta_{ij} = \delta_{k\ell}\,\|\vec{x}\|^2 \sin^2(\alpha_1)\sin^2(\alpha_2)\ldots\sin^2(\alpha_{k-1}) \equiv \|\vec{x}\|^2 \Omega_{k\ell}\,. \tag{5.17c}$$

Remembering to include a minus sign in raising u indices we see that to lowest order:

$$C^u{}_{xux} = -S^i_x\, S^j_x\, \tilde{C}_{0i0j} + O(\varepsilon^2) = (D-2)\frac{\varepsilon}{\|\vec{x}\|^2} + O(\varepsilon^2)\,, \tag{5.18a}$$

$$C^u{}_{\alpha_i u \alpha_j} = -S^k_{\alpha_i}\, S^\ell_{\alpha_j}\, \tilde{C}_{0k0\ell} + O(\varepsilon^2) = -\varepsilon\Omega_{ij} + O(\varepsilon^2)\,, \tag{5.18b}$$

$$C^x{}_{\alpha_i x \alpha_j} = S^k_x\, S^\ell_{\alpha_i}\, S^m_x\, S^n_{\alpha_j}\, \tilde{C}_{k\ell mn} + O\,(\varepsilon^2) = -\varepsilon\Omega_{ij} + O(\varepsilon^2)\,, \tag{5.18c}$$

$$C^{\alpha_i}{}_{\alpha_j\alpha_k\alpha_\ell} = \frac{1}{\|\vec{x}\|^2}\,\Omega^{im}\, S^n_{\alpha_m}\, S^p_{\alpha_j}\, S^q_{\alpha_k}\, S^r_{\alpha_\ell}\, \tilde{C}_{npqr}$$
$$= \frac{2}{D-3}\,\varepsilon\,(\delta^i{}_k\,\Omega_{j\ell} - \delta^i{}_\ell\,\Omega_{jk}) + O(\varepsilon^2)\,. \tag{5.18d}$$

The full metric of which we have computed the linearized deviation from pure de Sitter is known as the de Sitter–Schwarzchild solution. In static, spherically symmetric coordinates its line element is:

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\,d\Omega^2_{D-2}\,, \tag{5.19a}$$

$$\frac{1}{A(r)} = B(r) = 1 - H^2 r^2 - \frac{2}{D-3}\,\frac{G_D M}{r^{D-3}}\,. \tag{5.19b}$$

Were it not for this metric's coordinate singularities it would be entirely sufficient to conclude this discussion by noting that our linearized solution is related to (5.19) by the transformation:

$$Ht = \operatorname{atanh}\left[\frac{1+H^2(\|\vec{x}\|^2-u^2)}{1-H^2(\|\vec{x}\|^2-u^2)}\right] + \frac{1}{(D-3)^2}\left[\frac{\|\vec{x}\|^2}{u^2-\|\vec{x}\|^2} + \frac{1}{2}H^2\,\|\vec{x}\|^2\right]\varepsilon$$
$$+\Delta\left(\frac{Hu}{\|\vec{x}\|}\right) + O(\varepsilon^2)\,, \tag{5.20a}$$

$$Hr = \frac{\|\vec{x}\|}{u}\left[1 + \frac{1}{(D-3)^2}\,\varepsilon\right] + O(\varepsilon^2)\,, \tag{5.20b}$$

where the function Δ is defined to obey the differential equation:

$$\Delta'(w) = \frac{H^2 G_D M}{D-3}\,\frac{H^2 - 3w^2}{(H^2 - w^2)^2}\,w^{D-4}\,. \tag{5.21a}$$

It is tedious but straightforward to solve this for arbitrary dimension. For $D=4$ the solution is:

$$\lim_{D\to 4} \Delta = \left[\frac{\|\vec{x}\|^2}{u^2 - \|\vec{x}\|^2} + \frac{\|\vec{x}\|}{u}\ln\left(\frac{\|\vec{x}\| + u}{\|\vec{x}\| - u}\right)\right]\varepsilon\,. \tag{5.21b}$$

Because of the coordinate singularities it is better to compare curvatures. Since both metrics obey the same equation we need only check the Weyl tensors. Of course the de Sitter–Schwarzchild Weyl tensor can be computed to all orders:

$$C^{t}{}_{rtr} = \frac{(D-2)}{B}\,\frac{G_D\,M}{r^{D-1}}\,, \tag{5.22a}$$

$$C^{t}{}_{\alpha_i t \alpha_j} = -\frac{G_D\,M}{r^{D-3}}\,\Omega_{ij} = C^{r}{}_{\alpha_i r \alpha_j}\,, \tag{5.22b}$$

$$C^{\alpha_i}{}_{\alpha_j \alpha_k \alpha_\ell} = \frac{2}{D-3}\,\frac{G_D\,M}{r^{D-3}}\,(\delta^i{}_k\,\Omega_{j\ell} - \delta^i{}_\ell \Omega_{jk})\,. \tag{5.22c}$$

Note that all of these except $C^{t}{}_{rtr}$ are free of horizon singularities. Also note that all except $C^{t}{}_{rtr}$ agree with flat space results obtained by taking the limit $H^2 \to 0$! That the singularity in $C^{t}{}_{rtr}$ is purely a coordinate effect can be seen by computing the scalar quantity:

$$C^{\alpha}{}_{\beta\rho\sigma}\,C^{\beta}{}_{\alpha\mu\nu}\,g^{\rho\mu}\,g^{\sigma\nu} = -4\left(\frac{D-1}{D-3}\right)\left[(D-2)\,\frac{G_D\,M}{r^{D-1}}\right]^2\,. \tag{5.23}$$

The fact that this is also completely independent of H to all orders is an indication that long range fields behave in classical de Sitter space the same way as they do in flat space.

To make the comparison we must transform (5.22) to conformal coordinates. The angles suffer no transformation and one has only to convert from (t, r) to $(u, \|\vec{x}\|)$ using the zeroth order transformation read off from (5.20). The special form of (5.22) also helps, as witness the following exact result:

$$C^{u}{}_{\alpha_i u \alpha_j} = \frac{\partial u}{\partial t}\,\frac{\partial t}{\partial u}\,C^{t}{}_{\alpha_i t \alpha_j} + \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial u}\,C^{r}{}_{\alpha_i r \alpha_j} \tag{5.24a}$$

$$= \left\{\frac{\partial u}{\partial t}\,\frac{\partial t}{\partial u} + \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial u}\right\}\left[-\frac{G_D M}{r^{D-3}}\,\Omega_{ij}\right] \tag{5.24b}$$

$$= -\varepsilon\Omega_{ij}\,. \tag{5.24c}$$

The same result applies for $C^{r}{}_{\alpha_i r \alpha_j}$, and of course $C^{\alpha_i}{}_{\alpha_j \alpha_k \alpha_\ell}$ suffers no change. The only component which depends upon the detailed form of the transformation is:

$$C^{u}{}_{xux} = \left(\frac{\partial u}{\partial t}\frac{\partial r}{\partial v} + B^2 \frac{\partial u}{\partial r}\frac{\partial t}{\partial v}\right)\left(\frac{\partial t}{\partial u}\frac{\partial r}{\partial v} - \frac{\partial r}{\partial u}\frac{\partial t}{\partial v}\right) C^{t}{}_{rtr} \tag{5.25a}$$

$$= \frac{(D-2)}{\|\vec{x}\|^2}\,\varepsilon = O(\varepsilon^2)\,. \tag{5.25b}$$

In all cases there is agreement with (5.18) at order ε so we conclude that our metric is indeed a linearized version of the de Sitter–Schwarzchild solution.

6. Perturbative Quantum Gravity in De Sitter Conformal Coordinates

The purpose of this section is to specify, as completely as it can currently be done, the Feynman rules for quantum gravity in de Sitter conformal coordinates using the gauge (4.7). We first infer the propagators and then discuss the interaction vertices. The notation is in all cases that of Sect. 4. To make the final answer accessible we shall repeat the necessary definitions as they occur.

It turns out that the propagators and vertices assume their simplest forms if we use the pseudo-graviton field, $\psi_{\mu\nu}(x) \equiv \Omega^{-2}(x)\, h_{\mu\nu}(x)$. (We remind the reader that $\Omega \equiv (Hu)^{-1}$ and that indices on $\psi_{\mu\nu}$ are raised and lowered with the Lorentz metric.) The ψ propagator is defined as the expectation value in the free theory of the time-ordered product of two fields:

$$i[{}_{\rho\sigma}\Delta^{\alpha\beta}](x, x') \equiv \langle \text{in} | T[\psi_{\rho\sigma}(x)\,\psi^{\alpha\beta}(x')] | \text{out} \rangle_0\,. \tag{6.1}$$

It must give $i\delta^D(x-x')\,\delta_\mu{}^{(\alpha}\,\delta_\nu{}^{\beta)}$ when acted upon by the pseudo-graviton kinetic operator:

$$D_{\mu\nu}^{\rho\sigma} \equiv \Omega^{\frac{D}{2}-1}\,\mathscr{D}_{\mu\nu}^{\rho\sigma}\,\Omega^{\frac{D}{2}-1} \tag{6.2a}$$

$$= \delta_{(\mu}{}^{0}\,\bar{\delta}_{\nu)}{}^{(\rho}\,\delta_0{}^{\sigma)}\,D_B + \frac{1}{2}\left(\frac{D-2}{D-3}\right)\delta_\mu{}^0\,\delta_\nu{}^0\,\delta_0{}^\rho\,\delta_0{}^\sigma\,D_C$$

$$+ \left[\frac{1}{2}\bar{\delta}_\mu{}^{(\rho}\,\bar{\delta}_\nu{}^{\sigma)} - \frac{1}{4}\eta_{\mu\nu}\,\eta^{\rho\sigma} - \frac{1}{2(D-3)}\,\delta_\mu{}^0\,\delta_\nu{}^0\,\delta_0{}^\rho\,\delta_0{}^\sigma\right] D_A\,. \tag{6.2b}$$

Here $\mathscr{D}_{\mu\nu}^{\rho\sigma}$ is the operator defined in (4.10). The scalar kinetic operators in this expression are:

$$D_A = \Omega^{\frac{D}{2}-1}\left[\partial^2 + \left(\frac{D-2}{2}\right)\left(\frac{D}{2}\right)\frac{1}{u^2}\right]\Omega^{\frac{D}{2}-1} \tag{6.3a}$$

$$= \Omega^{\frac{D}{2}-1}\,\mathscr{D}_A\,\Omega^{\frac{D}{2}-1}\,, \tag{6.3b}$$

etc. for B and C, where $\mathscr{D}_{A-C}$ are defined in (4.11a–c). Finally, we remind the reader that a bar over a known tensor such as $\delta_\mu{}^\rho$ or $\eta^{\rho\sigma}$ indicates the suppression of its zero components.

Under the assumption that the "in" and "out" vacua are identical it follows from the canonical quantization of (4.8) that the imaginary part of expression (6.1) must be half the sum of the advanced and retarded Green's functions. We have already given in (4.25) a relation for the retarded Green's function $[{}_{\rho\sigma}G^{\alpha\beta}_{\rm ret}](x, x')$; to obtain the advanced Green's function one merely interchanges x and x'. From (3.8) we see that the de Sitter length function is symmetric, $z(x', x) = z(x, x')$, so the sum serves only to absorb the factor of $2\theta(\Delta u)$ in (4.25b):

$$\begin{aligned}\mathrm{Im}\{i[{}_{\rho\sigma}\Delta^{\alpha\beta}](x, x')\} = \pi\{&G_A(z-1)[{}_{\rho\sigma}T_A^{\alpha\beta}] \\ &+ G_B(z-1)[{}_{\rho\sigma}T_B^{\alpha\beta}] + G_C(z-1)[{}_{\rho\sigma}T_C^{\alpha\beta}]\}\ . \end{aligned} \tag{6.4}$$

The three constant matrices are:

$$[{}_{\rho\sigma}T_A^{\alpha\beta}] = \left[2\bar{\delta}_\rho{}^{(\alpha}\,\bar{\delta}_\sigma{}^{\beta)} - \frac{2}{D-3}\,\bar{\eta}_{\rho\sigma}\,\bar{\eta}^{\alpha\beta}\right], \tag{6.5a}$$

$$[{}_{\rho\sigma}T_B^{\alpha\beta}] = 4\delta_{(\rho}{}^{0}\,\bar{\delta}_{\sigma)}{}^{(\alpha}\,\delta_0{}^{\beta)}\ , \tag{6.5b}$$

$$[{}_{\rho\sigma}T_C^{\alpha\beta}] = \frac{2}{(D-3)(D-2)}\,[\eta_{\rho\sigma} + (D-2)\,\delta^0{}_\rho\,\delta^0{}_\sigma]\,[\eta^{\alpha\beta} + (D-2)\,\delta_0{}^\alpha\,\delta_0{}^\beta]\ . \tag{6.5c}$$

The three scalar functions which multiply them are defined by the integral:

$$\sqrt{\frac{uu'\Delta x}{32\pi}}\left(\frac{H^2 uu'}{2\pi\Delta x}\right)^{\mu+1}\int_0^\infty dk\, k^{\mu+1}\, J_\mu(k\Delta x)\,\{-N_\nu(ku')\,J_\nu(ku) + J_\nu(ku')\,N_\nu(ku)\}\ , \tag{6.6}$$

where the index $\mu \equiv \frac{D-3}{2}$ and ν takes the values $\frac{D-1}{2}$, $\frac{D-3}{2}$ and $\frac{D-5}{2}$ for G_A, G_B and G_C respectively.

We cannot determine the real part of the pseudo-graviton propagator without knowing the "in" and "out" vacua. There seems to be no obvious prescription for these owing to the virulent time dependence of the background, which dependence extends even to asymptotically early and late times. At any given instant there is a state of lowest energy but it does not *remain* the lowest energy state; in fact its energy fails even to stay near the instantaneous minimum. The usual approach to this problem is to assume that the vacuum is de Sitter invariant and that the high momentum modes tend to behave like those of flat space. If the gauge is de Sitter invariant as well then so too will be the graviton propagator.

The approach to flat space on small scales is a consequence of the equivalence principle. The supposition that the vacuum state shares the symmetry of the background would seem similarly impervious but for the counterexample provided by the massless minimally coupled scalar. Allen and Folacci have shown that there is no normalizable state in this model which is de Sitter invariant and also has the correct short distance limit [11]. This fact has great importance for gravity because partially integrating the scalar Lagrangian:

$$\mathscr{L}_\theta = -\frac{1}{2}\,\theta_{,\mu}\,\theta_{,\nu}\,\hat{g}^{\mu\nu}\sqrt{-\hat{g}} \tag{6.7a}$$

$$= \frac{1}{2}\,\theta D_A\theta - \frac{1}{2}\,\partial^\mu(\theta\,\theta_{,\mu}\,\Omega^{D-2}) \tag{6.7b}$$

reveals the same kinetic operator as for the spatial polarizations of the pseudo-graviton field. It follows that the gravitational vacuum must break de Sitter invariance in precisely the same way that the massless minimally coupled scalar does.

Since it is not possible to assume a de Sitter invariant propagator we shall instead attempt to pass from the imaginary part (6.4) to the full propagator by analytic continuation. This process is necessarily ambiguous up to terms which are real, analytic and which are annihilated by the kinetic operator (6.2). The ambiguity reflects our lack of knowledge about the vacuum. Of course the thing being continued is the functional dependence upon x and x'. The constant matrices play a passive role:

$$i[{}_{\rho\sigma}\Delta^{\alpha\beta}](x,x')=i\Delta_A(x,x')[{}_{\rho\sigma}T_A^{\alpha\beta}]+i\Delta_B(x,x')[{}_{\rho\sigma}T_B^{\alpha\beta}]$$
$$+i\Delta_C(x,x')[{}_{\rho\sigma}T_C^{\alpha\beta}]\,, \tag{6.8}$$

but they do diagonalize $D_{\mu\nu}^{\ \ \rho\sigma}$. That is, we must have $D_I i\Delta_I(x,x')=i\delta^D(x,x')$ for $I=A,B,C$.

The integrals (6.6) are very difficult to evaluate for D odd but for even $D=2d$ and $n=d-\frac{1}{2}-\nu\leqq d$ they have the following simple and suggestive form:

$$G_n(z-1;d)\equiv-\frac{1}{4\pi}\left[\frac{H^2}{4\pi}\right]^{d-1}\sum_{k=n}^{d-1}\frac{(2d-n-k-2)!}{(d-k-1)!\,(k-n)!}\left(\frac{d}{dz}\right)^k\theta[z(x,x')-1]\,, \tag{6.9}$$

where we remind the reader of expression (3.33), $1-z(x,x')=\frac{1}{4}H^2\Omega(x)\Omega(x')(x-x')^2$. For $k>0$ we can use the Dirac identity to write:

$$\left(\frac{d}{dz}\right)^k\theta(z-1)=-\frac{1}{\pi}\,\mathrm{Im}\left\{\frac{(k-1)!}{[1-z+i\varepsilon]^k}\right\}. \tag{6.10}$$

Hence we expect:

$$i\Delta_B(x,x')=\frac{1}{4\pi}\left[\frac{H^2}{4\pi}\right]^{d-1}\left\{\sum_{k=1}^{d-1}\frac{(2d-k-3)!}{(d-k-1)!}\,\frac{1}{[1-z+i\varepsilon]^k}+(\mathrm{R.A.H.})_B\right\}, \tag{6.11a}$$

and, for $D\geqq 6$:

$$i\Delta_C(x,x')=\frac{1}{4\pi}\left[\frac{H^2}{4\pi}\right]^{d-1}\left\{\sum_{k=2}^{d-1}\frac{(2d-k-4)!\,(k-1)}{(d-k-1)!}\,\frac{1}{[1-z+i\varepsilon]^k}+(\mathrm{R.A.H.})_C\right\}, \tag{6.11b}$$

where $(\mathrm{R.A.H.})_I$ stands for terms which are real, analytic and annihilated by the operator D_I. It is straightforward to check that acting D_B on (6.11a) and D_C on (6.11b) indeed gives $i\delta^D(x-x')$. For $D=4$ we have $i\Delta_B=i\Delta_C$:

$$i\Delta_{B-C}(x,x')=\frac{H^2}{16\pi^2}\left\{\frac{1}{1-z(x,x')+i\varepsilon}+(\mathrm{R.A.H.})_{B-C}\right\} \tag{6.12}$$

up to possibly different R.A.H. terms, which for $D=4$ are equal to $\Omega^{-1}(x)\,\Omega^{-1}(x')$ times harmonic functions.

One might think that G_A could be continued as well by using the identity:

$$\theta(z-1)=\frac{1}{\pi}\,\mathrm{Im}\,\{\ln[1-z+i\varepsilon]\}\ . \tag{6.13}$$

This is not so. Although (6.13) is true the operator D_A does not give $i\delta^D(x-x')$ when acted upon the implied continuation of G_A. The physical reason is the previously mentioned breaking of de Sitter invariance. The mathematical reason is that $1-z(x,x')=\frac{1}{4}H^2\Omega(x)\Omega(x')(x-x')^2$ and the always-positive conformal factors do not belong in the continuation. The correct analytic continuation necessarily breaks de Sitter invariance:

$$i\Delta_A(x,x')=\frac{1}{4\pi}\left[\frac{H^2}{4\pi}\right]^{d-1}\left\{\sum_{k=1}^{d-1}\frac{(2d-k-2)!}{(d-k-1)!\,k}\frac{1}{[1-z+i\varepsilon]^k}\right.$$
$$\left.-\frac{(2d-2)!}{(d-1)!}\ln[H^2(x-x')^2+i\varepsilon]+(\mathrm{R.A.H.})_A\right\} \tag{6.14a}$$

$$\underset{D=4}{\longrightarrow}\frac{H^2}{8\pi^2}\left\{\frac{1/2}{1-z(x,x')+i\varepsilon}-\ln[H^2(x-x')^2+i\varepsilon]+\frac{1}{2}(\mathrm{R.A.H.})_A\right\}. \tag{6.14b}$$

It is again straightforward to check that D_A acts upon this function to give $i\delta^D(x-x')$.

As stated, the R.A.H. terms are fixed by the as yet unknown vacuum. Perhaps the nicest choice for $D=4$ is given by $(\mathrm{R.A.H.})_B=(\mathrm{R.A.H.})_C=0$ and the one parameter family:

$$\frac{1}{2}(\mathrm{R.A.H.})_A=\frac{1}{2}\ln[\Omega^{-2}\Omega'^{-2}(1+E^2)(1+E'^2)]-\frac{1}{1+E^2}-\frac{1}{1+E'^2}+\frac{1}{4\alpha^2}$$
$$+8\alpha^2\left[\mathrm{acos}([1+E^2]^{-\frac{1}{2}})-\frac{E}{1+E^2}\right]$$
$$\times\left[\mathrm{acos}([1+E'^2]^{-\frac{1}{2}})-\frac{E'}{1+E'^2}\right], \tag{6.15}$$

where $E\equiv E^0(x)=\frac{1}{2}\Omega(x)\,[1+H^2(\|\vec{x}\|^2-u^2)]$ is the conformal embedding function (3.30a) and α is a positive real number. These are the $O(4)$ vacua discovered for the massless minimally coupled scalar by Allen and Folacci [11]. With this choice the pseudo-graviton propagator does not grow for either large timelike or spacelike separations. One of the most peculiar properties of previous solutions for the graviton propagator is the growth these solutions show at large temporal and even large spatial separations [1–3]. It is very hard to accept states which manifest powerful long range correlations between spacelike separated points. While this growth is certainly the correct result for causal, covariantly gauge fixed theories on the global manifold, we have already seen that these theories must fail to agree on even the classical level with the invariant field equations. It is comforting that the same reformulation which restores the classical connection with the invariant field equations also permits the choice of a vacuum for which large correlations between spacelike separated points are absent.

The ghost Lagrangian is obtained by varying the gauge functional F_μ:

$$F_\mu = \chi^\rho{}_{\mu,\rho} - \frac{1}{2}\chi^\rho{}_{\rho,\mu} + \left(\frac{D-2}{2}\right)\chi^\rho{}_\mu\,\phi_{,\rho} + \left(\frac{D-2}{4}\right)\chi^\rho{}_\rho\,\phi_{,\mu}\,, \tag{6.16}$$

where we remind the reader that $\chi_{\mu\nu} \equiv \Omega^{\frac{D}{2}-1}\,\psi_{\mu\nu}$ and $\phi \equiv \ln(\Omega)$. Let us consider an infinitesimal coordinate change:

$$y'^\mu = y^\mu + \kappa\varepsilon^\mu(y)\,. \tag{6.17}$$

Since the full metric transforms to:

$$g'_{\mu\nu}(x) = \frac{\partial y^\rho}{\partial y'^\mu}(x)\,\frac{\partial y^\sigma}{\partial y'^\nu}(x)\,g_{\rho\sigma}(y'^{-1}(x))\,, \tag{6.18a}$$

it follows that the variation of the graviton field is:

$$\delta h_{\mu\nu} = -\Omega^2\left[2\varepsilon_{(\mu,\nu)} + 2\eta_{\mu\nu}\varepsilon^\rho\phi_{,\rho}\right] - \kappa\left[2\varepsilon^{\rho,}{}_{(\mu}\,h_{\nu)\rho} + h_{\mu\nu,\rho}\varepsilon^\rho\right]\,. \tag{6.18b}$$

It is convenient to decompose the variation of $\chi_{\mu\nu}$ into a term of order κ^0 and one of order κ:

$$\delta_0\chi_{\mu\nu} = -\Omega^{\frac{D}{2}-1}\left[2\varepsilon_{(\mu,\nu)} + 2\eta_{\mu\nu}\varepsilon^\rho\phi_{,\rho}\right]\,, \tag{6.19a}$$

$$\delta_1\chi_{\mu\nu} = -\Omega^{\frac{D}{2}-1}\left[2\varepsilon^{\rho,}{}_{(\mu}\,\psi_{\nu)\rho} + \psi_{\mu\nu,\rho}\,\varepsilon^\rho + 2\psi_{\mu\nu}\varepsilon^\rho\phi_{,\rho}\right]\,. \tag{6.19b}$$

Note that it is the pseudo-graviton field which appears on the right of (6.19b), even though this is the variation of the rescaled field $\chi_{\mu\nu}$.

We now let ε_μ become an anticommuting ghost field and take the ghost Lagrangian to be:

$$\mathscr{L}_{\text{gh}} = -\Omega^{\frac{D}{2}-1}\gamma^\mu\delta F_\mu\,, \tag{6.20}$$

where γ^μ is the antighost field. The curious factor of $\Omega^{\frac{D}{2}-1}$ is inserted for convenience; it could always be absorbed into the antighost by a multiplicative field redefinition. The quadratic part of the ghost action is:

$$\mathscr{L}^2_{\text{gh}} = -\Omega^{\frac{D}{2}-1}\gamma^\mu\delta_0 F_\mu \tag{6.21a}$$

$$= \gamma^\mu\left[\bar{\delta}_\mu{}^\rho\,D_A + \delta_\mu{}^0\,\delta_0{}^\rho\,D_B\right]\varepsilon_\rho\,. \tag{6.21b}$$

It follows that the ghost propagator is:

$$i\left[{}_\rho\Delta^\alpha\right](x,x') = i\Delta_A(x,x')\,\bar{\delta}_\rho{}^\alpha + i\Delta_B(x,x')\,\delta_\rho{}^0\,\delta_0{}^\alpha\,. \tag{6.22}$$

Note that the same choices must be made for the R.A.H. terms in the functions $i\Delta_A(x,x')$ and $i\Delta_B(x,x')$ as for the pseudo-graviton.

Up to a surface term the ghost interaction Lagrangian is:

$$\begin{aligned}\mathscr{L}^3_{\text{gh}} - \mathscr{S}^\nu{}_{,\nu} = &-2\kappa\Omega^{D-2}\gamma^{\mu,\nu}\left[\psi^\rho{}_{(\mu}\partial_{\nu)} + \tfrac{1}{2}\psi_{\mu\nu}{}^{,\rho} + \psi_{\mu\nu}\phi^{,\rho}\right]\varepsilon_\rho\\ &+\kappa(\Omega^{D-2}\gamma^\mu)_{,\mu}\left[\psi^{\rho\sigma}\partial_\sigma + \tfrac{1}{2}\psi_\sigma{}^{\sigma,\rho} + \psi_\sigma{}^\sigma\phi^{,\rho}\right]\varepsilon_\rho\end{aligned} \tag{6.23}$$

Self interactions of the pseudo-graviton field are obtained by expanding expression (4.3b):

$$\mathscr{L}_{\text{inv}} - \mathscr{S}^{\nu}{}_{,\nu} = -\frac{1}{2}\sqrt{-\tilde{g}}\,\tilde{g}^{\rho\sigma}\tilde{g}^{\mu\nu}\psi_{\rho\sigma,\mu}\,\psi_{\nu}{}^{\alpha}(\,\Omega^{D-2})_{,\alpha}$$

$$+\sqrt{-\tilde{g}}\,\tilde{g}^{\alpha\beta}\,\tilde{g}^{\rho\sigma}\,\tilde{g}^{\mu\nu}\left[\frac{1}{2}\psi_{\alpha\rho,\mu}\,\psi_{\nu\sigma,\beta} - \frac{1}{2}\psi_{\alpha\beta,\rho}\,\psi_{\sigma\mu,\nu}\right.$$

$$\left. + \frac{1}{4}\psi_{\alpha\beta,\rho}\,\psi_{\mu\nu,\sigma} - \frac{1}{4}\psi_{\alpha\rho,\mu}\,\psi_{\beta\sigma,\nu}\right]\Omega^{D-2}\,, \tag{6.24}$$

where we remind the reader that $\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa\psi_{\mu\nu}$. Note that almost all of the interaction vertices are just those of flat space times a factor of Ω^{D-2}.

It ought to be possible to at least estimate infrared graviton effects by using the resulting perturbative apparatus. Of course the mathematical aesthete will insist upon a formulation which extends to the full manifold. We welcome such critics to indulge their prejudice in the resulting canonical quagmire. In the meantime we shall go on to learn what the open formulation can tell us. If the resulting infrared effects are sufficiently interesting then more resources will no doubt be directed towards developing an efficient global formalism. Finally, it should be noted that there are interesting and relevant situations for which restricting quantization to an open submanifold is the rigorously correct thing to do. An example would be the case of an open Friedmann universe which enters an inflationary phase. We are therefore cognizant of the partial nature of our solution but thoroughly impenitent.

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