Commun Math Phys 161, 63–76 (1994)

Communications in Mathematical Physics

Low Energy Asymptotics for Schrödinger Operators with Slowly Decreasing Potentials

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Received: 12 March 1993

Abstract: Low energy behavior of Schrödinger operators with potentials which decay slowly at infinity is studied. It is shown that if the potential is positive then the zero energy is very regular and the resolvent is smooth near 0. This implies rapid local decay for the solutions of the Schrödinger equation. On the other hand, if the potential is negative then the resolvent has discontinuity at zero energy. Thus one cannot expect local decay faster than order t^{-1} as $t \to \infty$.

1. Introduction

In this paper we consider the Schrödinger operator

$$H = H_0 + V(x) = -\hbar^2 \Delta + V(x) \quad \text{on} \quad L^2(\mathbf{R}^d), \quad d \geqq 1 .$$

We will assume $V(x) \sim c|x|^{-\rho}$ as $|x| \to \infty$, and study the behavior of $(H - z)^{-1}$ near $z = 0$. If $\rho > 2$ then V is called *very short range* and the behavior of $(H - z)^{-1}$ near $z = 0$ was studied by Jensen, Kato and others (see, e.g., [JK, J, Mu]). If $d = 3$ and ρ is sufficiently large, then it is known that $(H - z)^{-1}$ has an asymptotic expansion in $z^{1/2}$:

$$(H - z)^{-1} \sim B_{-2} z^{-1} + B_{-1} z^{-1/2} + B_0 z^0 + \cdots, \quad z \to 0 .$$

The top term B_{-2} comes from the 0-energy eigenvalue, and B_{-1} comes from the 0-energy resonance. Since they are unstable under small perturbations, $(H - z)^{-1}$ is generically regular near $z = 0$.

On the other hand, if $0 < \rho < 2$, then V is called *slowly decreasing*, and it is known that $(H - z)^{-1}$ behaves quite differently near $z = 0$. For the one-dimensional case, this problem was studied by Yafaev [Y1] in detail using integral equation techniques. For the higher dimensional case, Yafaev also studied Schrödinger operators with positive slowly decreasing potentials ([Y2]). In particular, he proved that $(1 + |x|)^{-\alpha} H^{-m} (1 + |x|)^{-\beta}$ is bounded in $L^2(\mathbf{R}^d)$ if $\alpha + \beta > m\rho$. Using this estimate, the low energy asymptotics of $(H - z)^{-1}$ was studied. The aim of this paper is to prove some a priori estimates that were assumptions in Yafaev's paper, and to generalize his results.

At first we consider the positive case. Here we fix the Planck constant $\hbar = 1$, and suppose $0 < \rho < 2$.

Assumption $(A:\rho)$. (i) V is a smooth real-valued function on $\mathbf{R}^d$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(x)\right| \leqq C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathbf{R}^d ,$$

for any multi-index α.
(ii) There is $\delta > 0$ such that $V(x) \geqq \delta\langle x\rangle^{-\rho}$ for $x \in \mathbf{R}^d$.
(iii) There are $\varepsilon > 0$ and $R > 0$ such that

$$x \cdot \frac{\partial V}{\partial x}(x) \leqq -\varepsilon|x|^{-\rho} \quad \text{for} \quad |x| > R .$$

Here we have used the notation $\langle x\rangle = (1 + |x|^2)^{1/2}$ for $x \in \mathbf{R}^d$. $F(*)$ denotes the characteristic function (or the characteristic function of an operator) designated by $(*)$.

Theorem 1.1. *Suppose* $(A:\rho)$ *with* $0 < \rho < 2$. *Then there exist* $\beta, \gamma > 0$ *and* $C > 0$ *such that*

$$\|F(|x| \leqq \beta\lambda^{-1/\rho})F(H \leqq \lambda)\| \leqq C\exp(-\gamma\lambda^{-(1/\rho-1/2)}), \quad \textit{for} \quad \lambda \in (0, 1] . \tag{1.1}$$

Remark. Part (i) and (ii) of Assumption $(A:\rho)$ is not necessary for Theorem 1.1. It holds if V is bounded and satisfies $(A:\rho)$-(iii).

This result can be considered as an asymptotic estimate on the local density of states, analogous to the Lifshitz tail for random Schrödinger operators. As a direct consequence of Theorem 1.1, we obtain rapid local decay for the semigroup e^{-tH}, $t \geqq 0$:

Corollary 1.2. *There exist* $\beta, \gamma > 0$ *and* $C > 0$ *such that*

$$\|F(|x| \leqq \beta t^{2/(\rho+2)})e^{-tH}\| \leqq C\exp(-\gamma t^{(2-\rho)/(2+\rho)}), \quad t > 0 . \tag{1.2}$$

Proof. It suffices to show (1.2) for $t > 1$. Setting $\lambda = t^{-2\rho/(\rho+2)}$, we decompose the left-hand side of (1.2) as follows:

$$\begin{aligned}
&\|F(|x| \leqq \beta t^{2/(\rho+2)})e^{-tH}\| \\
&\leqq \|F(|x| \leqq \beta t^{2/(\rho+2)})F(H \leqq \lambda)\| + \|F(H > \lambda)e^{-tH}\| \\
&= \|F(|x| \leqq \beta\lambda^{-1/\rho})F(H \leqq \lambda)\| + \|F(H > \lambda)e^{-tH}\| \\
&\leqq C\exp(-\gamma\lambda^{-(1/\rho-1/2)}) + e^{-t\lambda} \\
&= C\exp(-\gamma t^{(2-\rho)/(2+\rho)}) + \exp(-t^{(2-\rho)/(2+\rho)}), \quad t > 1 .
\end{aligned}$$

Replacing γ by $\min(\gamma, 1)$ if necessary, we obtain (1.2). □

Next we consider the boundary values of the resolvent. It is well-known that under our assumption,

$$\langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma} \equiv \lim_{\varepsilon \downarrow 0} \langle x\rangle^{-\gamma}(H - \lambda \pm i\varepsilon)^{-1}\langle x\rangle^{-\gamma} \in B(L^2(\mathbf{R}^d))$$

exist for $\gamma > 1/2$ and $\lambda > 0$.

Theorem 1.3. *Suppose V satisfies $(A:\rho)$ with $0 < \rho < 2$ and let $\gamma > 1/2 + \rho/4$. Then*

$$\sup_{0<\lambda\leqq 1} \|\langle x\rangle^{-\gamma}(H-\lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}\| \leqq C < \infty\ . \tag{1.3}$$

Moreover, $\langle x\rangle^{-\gamma}(H-\lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}$ is Hölder continuous in λ near $\lambda = 0$.

We can prove similar estimates for powers of the resolvent:

Theorem 1.4. *Suppose V satisfies $(A:\rho)$ with $0<\rho<2$. Let $k \geqq 1$ and let $\gamma > \max(k-1/2, k(1/2+\rho/4))$. Then*

$$\sup_{0<\lambda\leqq 1} \|\langle x\rangle^{-\gamma}(H-\lambda \pm i0)^{-k}\langle x\rangle^{-\gamma}\| \leqq C_k < \infty\ . \tag{1.4}$$

Moreover, $\langle x\rangle^{-\gamma}(H-\lambda \pm i0)^{-k}\langle x\rangle^{-\gamma}$ is Hölder continuous in λ near $\lambda = 0$.

Since (at least formally),

$$\left(\frac{d}{d\lambda}\right)^k (H-\lambda \pm i0)^{-1} = k!(H-\lambda \pm i0)^{-k-1}\ ,$$

Theorem 1.4 implies differentiability of $(H-\lambda \pm i0)^{-1}$ in λ:

Corollary 1.5. *Let φ be a rapidly decreasing function on $\mathbf{R}^d$. Then $\varphi(H-\lambda \pm i0)^{-1}\varphi$ is C^∞-smooth with respect to λ (in $B(L^2(\mathbf{R}^d))$-topology).*

We denote the spectral projection of H by $E(\lambda) = F(H \leqq \lambda)$. Since

$$E'(\lambda) = (2\pi i)^{-1}((H-\lambda-i0)^{-1} - (H-\lambda+i0)^{-1})\ ,$$

we can obtain corresponding estimates for $E'(\lambda)$ from Theorems 1.4 and 1.5. In fact, we can prove slightly stronger estimates for $E'(\lambda)$ and Theorem 1.4 follow from them:

Theorem 1.6. *Suppose V satisfies $(A:\rho)$ with $0<\rho<2$. Let $k \geqq 1$ and let $\gamma > k - 1/2$. Then*

$$\left\|\langle x\rangle^{-\gamma}\left(\frac{d}{d\lambda}\right)^{k-1} E'(\lambda)\langle x\rangle^{-\gamma}\right\| \leqq C_k\lambda^\delta, \quad \lambda\in(0,1]\ , \tag{1.5}$$

where $\delta = 2\gamma/\rho - k(1/\rho + 1/2)$.

Combining these results with the method of Jensen, Mourre and Perry [JMP], we obtain the following uniform decay estimates for the Schrödinger time-evolution:

Theorem 1.7. *Suppose $(A:\rho)$ with $0<\rho<2$. Then for any $\gamma > \beta > 0$,*

$$\|\langle x\rangle^{-\gamma}e^{-itH}\langle x\rangle^{-\gamma}\| \leqq C\langle t\rangle^{-\beta}, \quad t\in\mathbf{R}\ . \tag{1.6}$$

Remark. For the free case, i.e., if $V(x) = 0$, then (1.6) holds only if $\beta \leqq d/2$. Thus a particle in a potential satisfying $(A:\rho)$ escapes from a finite region faster than a free particle.

We can also estimate $\langle x\rangle^{-\gamma}(H-z)^{-1}\langle x\rangle^{-\gamma}$ in a neighborhood of $z = 0$:

Theorem 1.8. *Suppose $(A:\rho)$ with $0<\rho<2$ and let $\gamma > 1/2+\rho/4$. Then $\langle x\rangle^{-\gamma}(H-z)^{-1}\langle x\rangle^{-\gamma}$ is uniformly bounded in $z\in\mathbf{C}\backslash[0,\infty)$.*

The idea of the proof of Theorems 1.1–1.8 is as follows: If we change the coordinate: $x = \lambda^{-1/\rho}y$, then H has the form

$$H = \lambda(-g^2\Delta_y + V_\lambda(y)) \equiv \lambda H_\lambda \,, \tag{1.7}$$

where $g = \lambda^{(1/\rho - 1/2)}$ and $V_\lambda(y) = \lambda^{-1}V(\lambda^{-1/\rho}y)$. Since $\rho < 2, g \downarrow 0$ as $\lambda \downarrow 0$, and (A : ρ) implies

$$\delta\langle y\rangle^{-\rho} \leqq V_\lambda(y) \leqq C|y|^{-\rho}, \quad y \in \mathbf{R}^d \,.$$

Thus H_λ has the form of a semiclassical Hamiltonian, and we can apply the methods of semiclassical analysis. Then, for example, Theorem 1.1 follows from the Agmon estimate for a classically forbidden region, and Theorem 1.3 follows from the semiclassical resolvent estimates (see, e.g., [S, HS, RT, GM, HN], etc.). The proof is discussed in Sect. 2.

Next we consider the negative case, i.e., $V(x) < 0$. Then the situation is quite different from the positive case. Zero is always the accumulation point of negative eigenvalues, and hence zero seems to be very singular. In order to make the problem manageable, we suppose the Planck constant $\hbar$ is sufficiently small. Note that this is equivalent to replace V by μV with a large coupling constant μ.

Assumption (B : ρ). (i) V is smooth real-valued function on $\mathbf{R}^d$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(x)\right| \leqq C_\alpha\langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathbf{R}^d$$

for any multi-index α.

(ii) There is $\delta > 0$ such that $V(x) \leqq -\delta\langle x\rangle^{-\rho}$ for $x \in \mathbf{R}^d$.

(iii) $$\sup_{x \in \mathbf{R}^d} |V(x)|^{-1}\left|x \cdot \frac{\partial V}{\partial x}(x)\right| \equiv \rho' < 2 \,.$$

Theorem 1.9. *Suppose V satisfies (B : ρ) with $0 < \rho < 2$. Then there exists $\hbar_0 > 0$ such that if $0 < \hbar < \hbar_0$ and $\gamma > 1/2 + \rho/4$ then*

$$\sup_{0<\lambda\leqq 1} \|\langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}\| \leqq C < \infty \,. \tag{1.8}$$

Moreover,

$$\langle x\rangle^{-\gamma}(H - 0 \pm i0)^{-1}\langle x\rangle^{-\gamma} \equiv \lim_{\lambda\downarrow 0} \langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma} \in B(L^2(\mathbf{R}^d)) \tag{1.9}$$

exist.

Remark. (i) The above result implies the existence of

$$\langle x\rangle^{-\gamma}E'(+0)\langle x\rangle^{-\gamma} \equiv \lim_{\lambda\downarrow 0} \langle x\rangle^{-\gamma}E'(\lambda)\langle x\rangle^{-\gamma}$$

for $\gamma > 1/2 + \rho/4$. In general $E'(+0) \neq 0$ (see Yafaev [Y1] for one-dimensional case). Thus we may expect at most the decay of order $O(t^{-1})$ for $\|\langle x\rangle^{-\gamma}e^{-itH}\langle x\rangle^{-\gamma}\|$ even if γ is very large.

(ii) Since zero is the accumulation point of $\sigma_p(H)$, (1.8) might look like a contradiction. But it is not, since we have additional weight $\langle x\rangle^{-\gamma}$. Instead, if $H\psi_n = \lambda_n\psi_n$, $\|\psi_n\| = 1$, $\lambda_n \uparrow 0$, then Theorem 1.9 implies $\|\langle x\rangle^\gamma\psi_n\| \geqq C|\lambda_n|^{-1/2} \to \infty$.

The idea of the proof of Theorem 1.9 is completely different from the positive case though we also use the semiclassical method. We consider a second-order elliptic operator $L_0 = (-V)^{-1/2} H_0 (-V)^{-1/2}$. As we shall see later, we can construct scattering theory for L_0, and formally we have

$$(H - 0 \pm i0)^{-1} = (H_0 - (-V) \pm i0)^{-1} = (-V)^{-1/2}(L_0 - 1 \pm i0)^{-1}(-V)^{-1/2} .$$

Thus $(H - 0 \pm i0)^{-1}$ can be represented by the boundary values of resolvent of L_0 at energy 1. This argument is justified in Sect. 3 and Theorem 1.9 follows.

2. Positive Potentials

Throughout this section we suppose V satisfies Assumption (A : ρ) with $0 < \rho < 2$, and we let $\hbar = 1$.

Let $\lambda > 0$ be an energy and we change the coordinates:

$$x \in \mathbf{R}^d \to y \in \mathbf{R}^d, \quad x = \lambda^{-1/\rho} y .$$

Then the Hamiltonian is transformed to

$$\begin{aligned} H &= -\Delta_x + V(x) \\ &= \lambda\{-\lambda^{2(1/\rho - 1/2)}\Delta_y + \lambda^{-1} V(\lambda^{-1/\rho} y)\} \\ &= \lambda\{-g^2 \Delta_y + V_\lambda(y)\} = \lambda H_\lambda , \end{aligned} \tag{2.1}$$

where $g = \lambda^{1/\rho - 1/2}$ and $V_\lambda(y) = \lambda^{-1} V(\lambda^{-1/\rho} y)$. Since $0 < \rho < 2$, $g \downarrow 0$ as $\lambda \downarrow 0$, and we can consider g as a semiclassical parameter. By (A : ρ), V_λ satisfies

$$\begin{aligned} \left| \left(\frac{\partial}{\partial y}\right)^\alpha V_\lambda(y) \right| &\leqq C_\alpha \lambda^{-1-|\alpha|/\rho} \langle \lambda^{-1/\rho} y \rangle^{-\rho-|\alpha|} \\ &\leqq C_\alpha \min(\lambda^{-1-|\alpha|/\rho}, |y|^{-\rho-|\alpha|}) , \quad y \in \mathbf{R}^d , \end{aligned} \tag{2.2}$$

$$V_\lambda(y) \geqq \delta \lambda^{-1} \langle \lambda^{-1/\rho} y \rangle^{-\rho} \geqq \delta(\lambda^{1/\rho} + |y|)^{-\rho}, \quad y \in \mathbf{R}^d . \tag{2.3}$$

Theorem 1.1 follows from the above scaling and an Agmon-type semiclassical estimate for the classically forbidden region.

Proof of Theorem 1.1. By the scaling, it suffices to show

$$\| F(|y| \leqq \beta) F(H_\lambda \leqq 1) \| \leqq C \exp(-\gamma g^{-1}), \quad \lambda > 0 , \tag{2.4}$$

for some $\beta, \gamma > 0$. We take $\lambda_0, \beta > 0$ so small that $\delta(\lambda_0^{1/\rho} + 2\beta)^{-\rho} > 2$. Thus if $0 < \lambda \leqq \lambda_0$ and $|y| \leqq 2\beta$, then $V_\lambda > 2$. In other words, $\{y \mid |y| \leqq 2\beta\}$ is in the forbidden region for a classical particle with energy less than 2. Hence by Theorem III-1 of [BCD] we obtain

$$\begin{aligned} &\| F(|y| \leqq \beta) F(H \leqq 1) \| \\ &\leqq C\left(\sup_y |V_\lambda(y)|^{1/2}\right) \exp\left(-g^{-1} \int_\beta^{2\beta} [\delta(\lambda^{1/\rho} + \mu)^{-\rho} - 3/2]^{1/2} d\mu\right) \\ &\leqq C\lambda^{-1/2} \exp(-2\gamma g^{-1}) \leqq C \exp(-\gamma g^{-1}) \end{aligned}$$

with some $\gamma, C > 0$ if g is sufficiently small. □

In order to obtain estimates for the boundary values of resolvent, we first note that we can prove an analogue of the semiclassical resolvent estimates for H_λ (cf. [RT, GM, HN, W, N2], etc.).

Lemma 2.1. *Let $k \geqq 1$ and let $\gamma > k - 1/2$. Then*

$$\| \langle y \rangle^{-\gamma} (H_\lambda - 1 \pm i0)^{-k} \langle y \rangle^{-\gamma} \| \leqq C_k g^{-k}, \quad \lambda \in (0, 1], \tag{2.5}$$

where $g = \lambda^{1/\rho - 1/2}$.

Proof. Let λ_0 and β be as in the proof of Theorem 1.1, and let $0 < \lambda \leqq \lambda_0$, $R > \lambda^{-1/\rho}\beta$, i.e., $\lambda < (\beta/R)^\rho$, where R is the constant in (A: ρ)-(iii). By the assumption, there exists $\varepsilon > 0$ such that

$$(V_\lambda(y) - 1) + \frac{1}{2} y \cdot \frac{\partial V_\lambda}{\partial y}(y) \leqq -\varepsilon \quad \text{if } V_\lambda(y) \leqq 1 + \varepsilon, \tag{2.6}$$

uniformly for small λ. Equation (2.6) implies that the classical particle in the potential V_λ is nontrapping at energy 1. Then we follow the argument in [HN]. By virtue of (2.2) and (2.3), all the estimates are uniform for small λ, and we obtain (2.5) with $k = 1$. Combining the Mourre estimate in the above proof with the method of Jensen, Mourre and Perry [PSS] (see [W] and [N2] for the semiclassical form), we obtain (2.5) for $k \geqq 2$. $\square$

Lemma 2.2. *Let β be sufficiently small and let $f \in C_0^\infty(1/2, 3/2)$. Then for any N and M, there is $C > 0$ such that*

$$\| \langle y \rangle^M f(H_\lambda) F(|y| \leqq \beta) \| \leqq C\lambda^N, \quad \lambda \in (0, 1]. \tag{2.7}$$

Proof. Instead of $\langle y \rangle$ we use the following weight function $\Lambda \in C^\infty(\mathbf{R}^d)$ such that $\Lambda(y) = \Lambda(|y|)$ is nondecreasing in $|y|$ and

$$\Lambda(y) = \begin{cases} 1 & \text{if } |y| \leqq 1, \\ |y| & \text{if } |y| \geqq 2. \end{cases}$$

We start by considering $\Lambda f(H_\lambda) F(|y| \leqq \beta)$, where β is chosen as in the Proof of Theorem 1.1,

$$\Lambda f(H_\lambda) F(|y| \leqq \beta) = f(H_\lambda) \Lambda F(|y| \leqq \beta) + [\Lambda, f(H_\lambda)] F(|y| \leqq \beta).$$

The first term in the right-hand side is of order g^N for any N as in the proof of (2.4). Thus it suffices to estimate the other term. We set

$$\mathrm{ad}_B^0(A) = A, \quad \mathrm{ad}_B^k(A) = [B, \mathrm{ad}_B^{k-1}(A)] \quad \text{for} \quad k \geqq 1.$$

Then it is known that for any $m \geqq 1$,

$$[A, f(B)] = -\sum_{k=1}^{m-1} \frac{(-1)^k}{k!} \mathrm{ad}_B^k(A) f^{(k)}(B) + \frac{(-1)^m}{2\pi i} \int_{\mathbf{C}} \partial_{\bar z} \tilde f(z) (B - z)^{-1} \mathrm{ad}_B^m(A) (B - z)^{-m} dz d\bar z,$$

where $\tilde{f}$ is an almost analytic continuation of f (see, e.g., [G], Appendix). Under our assumption,

$$\operatorname{ad}_{H_\lambda}^k(A) = \sum_{j=0}^{k} a_{kj}(\lambda, y)\left(g\frac{\partial}{\partial y}\right)^j, \quad k \geqq 1 ,$$

where a_{kj} are smooth in y, supported away from $\{y \mid |y| < 1\}$ and satisfy

$$\left|\left(\frac{\partial}{\partial y}\right)^\alpha a_{kj}(\lambda, y)\right| \leqq C_{kj\alpha} g^k \langle y\rangle^{-(k-1)-|\alpha|}, \quad y \in \mathbf{R}^d, \quad \lambda \in (0, 1] .$$

From these, we learn

$$\|\operatorname{ad}_{H_\lambda}^k(A)(H_\lambda + 1)^{-k}\| \leqq C_k g^k, \quad \lambda \in (0, 1] .$$

On the other hand, $\tilde{f}$ is compactly supported and satisfying $|\partial_{\bar{z}}\tilde{f}(z)| \leqq C_m |\operatorname{Im} z|^m$ for any m. Hence

$$\left\|\int_{\mathbf{C}} \partial_{\bar{z}}\tilde{f}(z)(H - z)^{-1}\operatorname{ad}_{H_\lambda}^m(A)(H_\lambda - z)^{-m}\,dz\,d\bar{z}\right\| \leqq C_m g^m .$$

Then we use the commutator formula to obtain

$$\begin{aligned}
&\|[A, f(H_\lambda)]F(|y| \leqq \beta)\| \\
&\leqq \sum_{k=1}^{m}\frac{1}{k!}\|\operatorname{ad}_{H_\lambda}^k(A)(H_\lambda + 1)^{-k}\| \cdot \|(H_\lambda + 1)^k f^{(k)}(H_\lambda)F(|y| \leqq \beta)\| \\
&\quad + \left\|\int_{\mathbf{C}} \partial_{\bar{z}}\tilde{f}(z)(H - z)^{-1}\operatorname{ad}_{H_\lambda}^m(A)(H_\lambda - z)^{-m}\,dz\,d\bar{z}\right\| \\
&\leqq C_m g^m
\end{aligned}$$

for any $m \geqq 1$. Here we have used an analogue of (2.4) to estimate the term: $\|(H + 1)^k f^{(k)}(H_\lambda)F(|y| \leqq \beta)\|$. Since m is arbitrary, this proves (2.7) with $M = 1$. Repeating this procedure for multiple commutators: $[A, [A, f(H_\lambda)]]$, etc., we obtain (2.7) for any M. We omit the detail. □

Lemma 2.3. *If $f \in C_0^\infty(1/2, 3/2)$, $k \geqq 1$ and $\gamma > k - 1/2$, then*

$$\|\langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-k} f(H/\lambda)\langle x\rangle^{-\gamma}\| \leqq C\lambda^{a(k,\gamma)}, \quad \lambda \in (0, 1] , \tag{2.8}$$

where $a(k, \gamma) = 2\gamma/\rho - k(1/\rho + 1/2)$.

Proof. It suffices to show (2.8) for small λ. We show

$$\|(\lambda^{1/\rho} + |y|)^{-\gamma}(H_\lambda - 1 \pm i0)^{-k} f(H_\lambda)(\lambda^{1/\rho} + |y|)^{-\gamma}\| \leqq C g^{-k} . \tag{2.9}$$

By Lemma 2.1 we have

$$\begin{aligned}
&\|(\lambda^{1/\rho} + |y|)^{-\gamma}F(|y| \geqq \beta)(H_\lambda - 1 \pm i0)^{-k} f(H_\lambda) \\
&\qquad \times F(|y| \geqq \beta)(\lambda^{1/\rho} + |y|)^{-\gamma}\| \leqq C g^{-k} ,
\end{aligned}$$

where β is chosen so small as in Lemma 2.2. On the other hand we also have

$$\begin{aligned}
&\|(\lambda^{1/\rho}+|y|)^{-\gamma}F(|y|<\beta)(H_\lambda-1\pm i0)^{-k}f(H_\lambda)\\
&\quad\times F(|y|\geqq\beta)(\lambda^{1/\rho}+|y|)^{-\gamma}\|\\
&\leqq C\lambda^{-\gamma/\rho}\|F(|y|<\beta)f(H_\lambda)\langle y\rangle^{\gamma}\|\;\|\langle y\rangle^{-\gamma}(H_\lambda-1\pm i0)^{-k}\langle y\rangle^{-\gamma}\|\\
&\leqq C_m g^m
\end{aligned}$$

for any $m\geqq 0$. Similarly we can show

$$\begin{aligned}
&\|(\lambda^{1/\rho}+|y|)^{-\gamma}F(|y|<\beta)(H_\lambda-1\pm i0)^{-k}f(H_\lambda)\\
&\quad\times F(|y|<\beta)(\lambda^{1/\rho}+|y|)^{-\gamma}\|\leqq C_m g^m\,,
\end{aligned}$$

for any $m\geqq 0$. Combining them we obtain (2.9).

Now we change the coordinates to obtain estimates for operators in x-space. Then we have

$$\begin{aligned}
&\|(1+|x|)^{-\gamma}(H-\lambda\pm i0)^{-k}f(H/\lambda)(1+|x|)^{-\gamma}\|\\
&=\|\{\lambda^{-1/\rho}(\lambda^{1/\rho}+|y|)\}^{-\gamma}\lambda^{-k}(H_\lambda-1\pm i0)^{-k}f(H_\lambda)\\
&\quad\times\{\lambda^{-1/\rho}(\lambda^{1/\rho}+|y|)\}^{-\gamma}\|\\
&\leqq C\lambda^{-k}\lambda^{2\gamma/\rho}g^{-k}=C\lambda^{a(k,\gamma)}.\quad\square
\end{aligned}$$

As a direct consequence of Lemma 2.3 we obtain Theorem 1.6 since $f(H/\lambda)E'(\lambda)=E'(\lambda)$ if $f(1)=1$. Thus if $\gamma>k-1/2$ and $\gamma\geqq k(1/2+\rho/4)$ then we observe that $\langle x\rangle^{-\gamma}(d/d\lambda)^{k-1}E'(\lambda)\langle x\rangle^{-\gamma}$ is bounded for $\lambda\in(0,1]$. Moreover, using complex interpolation with respect to γ, we learn that if $\gamma>k(1/2+\rho/4)$ then $\langle x\rangle^{-\gamma}(d/d\lambda)^{k-1}E'(\lambda)\langle x\rangle^{-\gamma}$ is Hölder continuous. Now we note that

$$\sup_{0<\varepsilon\leqq 1}\left|\int_{-1}^{1}\frac{\varphi(x)}{x\pm i\varepsilon}dx\right|\leqq C_\alpha\|\varphi\|_{C^\alpha},\quad \varphi\in C^\alpha(\mathbf{R}),\quad \alpha>0\,,$$

where $C^\alpha(\mathbf{R})$ denotes the Hölder space of order α. Then, since

$$(H-z)^{-1}F(H\leqq 1)=\int_0^1 E'(\lambda)(\lambda-z)^{-1}d\lambda\,,$$

Theorem 1.3 and Theorem 1.8 follow from the Hölder continuity of $\langle x\rangle^{-\gamma}E'(\lambda)\langle x\rangle^{-\gamma}$. Theorem 1.4 also follows from the Hölder continuity of $\langle x\rangle^{-\gamma}(d/d\lambda)^{k-1}E'(\lambda)\langle x\rangle^{-\gamma}$ and by using integration by parts.

At last we prove Theorem 1.7. By mimicking the proof of Theorem 5.1 of [JMP], we conclude from Theorem 1.6 that for any $\gamma>\beta>0$,

$$\|\langle x\rangle^{-\gamma}e^{-itH}\chi(H)\langle x\rangle^{-\gamma}\|\leqq C\langle t\rangle^{-\beta},\quad t\in\mathbf{R}\,,$$

where $\chi\in C_0^\infty(-1,1)$ is 1 in a neighborhood of 0. On the other hand, it is well-known that for any $\gamma>0$,

$$\|\langle x\rangle^{-\gamma}e^{-itH}\tilde{\chi}(H)\langle x\rangle^{-\gamma}\|\leqq C\langle t\rangle^{-\gamma},\quad t\in\mathbf{R}\,,$$

if $\tilde{\chi}$ is smooth, bounded and supported away from 0 (see, e.g., [I]). Theorem 1.7 follows from these.

3. Negative Potentials

In this section we consider Schrödinger operators with Planck constant $\hbar > 0$ and we always suppose V satisfies Assumption $(\mathrm{B}:\rho)$ with $0 < \rho < 2$. Let $\lambda \in [0, 1]$ be an energy. We set $W_\lambda(x) = (-V(x) + \lambda)^{-1/2}$, $x \in \mathbf{R}^d$. Then $W_\lambda(x)$ satisfies

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha W_\lambda(x)\right| \leqq C_\alpha \min(\langle x\rangle^{\rho/2-|\alpha|}, \lambda^{-1/2-|\alpha|}), \quad x \in \mathbf{R}^d, \quad \lambda \in [0, 1)$$

for any α. We consider a second order elliptic operator

$$L_\lambda = W_\lambda H_0 W_\lambda = (-V(x) + \lambda)^{-1/2}(-\hbar^2\Delta)(-V(x) + \lambda)^{-1/2} .$$

It is easy to see that L_λ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. The dilation generator A is defined by

$$A = \frac{1}{2i}\left(x \cdot \frac{\partial}{\partial x} + \frac{\partial}{\partial x} \cdot x\right) .$$

We use the $\hbar$-pseudodifferential operator calculus (see, e.g., [R, N1]). We denote $a \in S(m, g)$, $g = dx^2/\langle x\rangle^2 + d\xi^2/\langle \xi\rangle^2$, if

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta a(\hbar; x, \xi)\right| \leqq C_{\alpha\beta} m(\hbar; x, \xi)\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|} ,$$

$$x, y \in \mathbf{R}^d, \quad 0 < \hbar \leqq 1 ,$$

for any α and β. The Weyl operator $a(\hbar; x, \hbar D)$ is defined by

$$a(\hbar; x, \hbar D)\varphi(x) = (2\pi\hbar)^{-d} \int e^{i(x-y)\xi/\hbar} a(\hbar; (x + y)/2, \xi)\varphi(y)\, dy d\xi$$

for $\varphi \in \mathscr{S}(\mathbf{R}^d)$.

We first prove the limiting absorption principle for L_λ.

Lemma 3.1. *There exists $\hbar_0 > 0$ such that if $0 < \hbar \leqq \hbar_0$ then for any $\gamma > 1/2$,*

$$\sup_{0<\varepsilon\leqq 1} \|\langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i\varepsilon)^{-1}\langle A\rangle^{-\gamma}\| \leqq C < \infty , \quad \lambda \in [0, 1] , \tag{3.1}$$

where C depends only on $\hbar$, γ and V. Moreover,

$$\lim_{\varepsilon \downarrow 0} \langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i\varepsilon)^{-1}\langle A\rangle^{-\gamma} = \langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i0)^{-1}\langle A\rangle^{-\gamma} \tag{3.2}$$

exist.

Proof. We use the Mourre theory of limiting absorption principle (see, e.g., [Mo, PSS, CFKS]). By direct computations we can show

$$[L_\lambda, iA](L_\lambda + 1)^{-1}, [[L_\lambda, iA], iA](L_\lambda + 1)^{-1} \in B(L^2(\mathbf{R}^d)) .$$

Hence it suffices to show the Mourre estimate:

$$E_I[L_\lambda, iA]E_I \geqq \delta E_I, \quad \delta > 0, \quad I = [1/2, 2] . \tag{3.3}$$

Since the principal symbols of L_λ are A and given by $(-V+\lambda)^{-1}\xi^2$ and $\hbar^{-1}x\cdot\xi$, respectively, the principal symbol of $[L_\lambda, iA]$ is given by

$$\{(-V+\lambda)^{-1}\xi^2, x\cdot\xi\} = \left(2 - \frac{x\cdot\partial V}{-V+\lambda}\right)(-V+\lambda)^{-1}\xi^2$$
$$\geqq (2-\rho')(-V+\lambda)^{-1}\xi^2\,,$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Moreover, the lower order symbol is in $S(\hbar^2\langle x\rangle^{\rho-2}, g)$. Hence by the Fefferman-Phong inequality we obtain

$$[L_\lambda, iA] \geqq (2-\rho')L_\lambda - C\hbar^2, \quad \hbar\in(0,1]\,, \tag{3.4}$$

uniformly in $\lambda\in[0,1]$. If we take $\hbar_0$ sufficiently small, (3.4) implies (3.3) with $0<\delta<2-\rho'$. □

In what follows we fix $\hbar>0$ such that $0<\hbar\leqq\hbar_0$.

Lemma 3.2. *For each* $0<\lambda\leqq 1, \gamma>1/2$,

$$\sup_{0<\varepsilon\leqq 1} \|\langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i\varepsilon W_\lambda^2)^{-1}\langle A\rangle^{-\gamma}\| \leqq C < \infty\,. \tag{3.5}$$

Moreover,

$$\lim_{\varepsilon\downarrow 0}\langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i\varepsilon W_\lambda^2)^{-1}\langle A\rangle^{-\gamma} = \langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i0)^{-1}\langle A\rangle^{-\gamma}\,. \tag{3.6}$$

Remark. The constant C in (3.5) depends on λ. The right-hand side of (3.6) is defined by (3.2).

Proof. Given the Mourre estimate (3.3), the proof of (3.5) is almost identical with the proof of Lemma 3.1. We only note that we use $\sup|W_\lambda| = \lambda^{-1} < \infty$ and $\sup|[W_\lambda^2, iA]| \leqq C\lambda^{-2} < \infty$, and hence the estimates depends on λ. It remains only to show (3.6). We mimic the proof of the Hölder continuity of $F(z) = \langle A\rangle^{-\gamma}(L_\lambda - z)^{-1}\langle A\rangle^{-\gamma}$ in z (cf. [PSS]). Instead of $\|F(z)-F(z')\|$ we estimate $\|F(1+i0) - F(1+i0\cdot W_\lambda^2)\|$. We omit the detail because the modification is obvious and the notations are rather involved. □

Lemma 3.3. *For any* $\beta,\gamma\in\mathbf{R}$, $\lambda\in[0,1]$,

$$\langle P\rangle^{\beta+2}\langle x\rangle^{\gamma}(L_\lambda+1)^{-1}\langle x\rangle^{-\gamma}\langle P\rangle^{-\beta}\in B(L^2(\mathbf{R}^d))\,, \tag{3.7}$$

where $P = -i\hbar\partial/\partial x$. *Moreover the operator norm is uniformly bounded for* $\lambda\in[0,1]$.

Proof. It suffices to show

$$\langle x\rangle^{\gamma}(L_\lambda+1)^{-1}\langle x\rangle^{-\gamma}\in B(L^2(\mathbf{R}^d)) \tag{3.8}$$

and

$$\langle P\rangle^{\beta+2}(L_\lambda+1)^{-1}\langle P\rangle^{-\beta}\in B(L^2(\mathbf{R}^d))\,. \tag{3.9}$$

Let $\beta_0 = 1 - \rho/2$ and let $M_\lambda = PW_\lambda$. Then $L_\lambda = M_\lambda^* M_\lambda$ and hence M_λ is $L_\lambda^{1/2}$-bounded. By easy computations, M_λ^* is also $L_\lambda^{1/2}$-bounded. We compute

$$\begin{aligned}[\langle x\rangle^{\beta_0}, (L_\lambda + 1)^{-1}] &= (L_\lambda + 1)^{-1}[L_\lambda, \langle x\rangle^{\beta_0}](L_\lambda + 1)^{-1} \\ &= (L_\lambda + 1)^{-1}[M_\lambda^*\{M_\lambda, \langle x\rangle^{\beta_0}] \\ &\quad + [M_\lambda^*, \langle x\rangle^{\beta_0}]M_\lambda\}(L_\lambda + 1)^{-1}.\end{aligned}$$

It is easy to see $[M_\lambda, \langle x\rangle^{\beta_0}] \in B(L^2)$, etc., and hence $[\langle x\rangle^{\beta_0}, (L_\lambda + 1)^{-1}] \in B(L^2)$. This implies (3.8) with $\gamma = \beta_0$. Iterating this procedure we can obtain (3.8) for $\gamma = m\beta_0, m \geqq 1$. Then the claim follows by complex interpolation.

On the other hand, using $L_\lambda^{1/2}$-boundedness of M_λ again, we have

$$\begin{aligned}&W_\lambda^2 P^2 (L_\lambda + 1)^{-1} \\ &= W_\lambda P^2 W_\lambda (L_\lambda + 1)^{-1} + (W_\lambda P[W_\lambda, P] + W_\lambda[W_\lambda, P]P)(L_\lambda + 1)^{-1} \\ &= L_\lambda (L_\lambda + 1)^{-1} + 2[W_\lambda, P]M_\lambda^*(L_\lambda + 1)^{-1} + W_\lambda[P, [W_\lambda, P]](L_\lambda + 1)^{-1} \\ &\quad \in B(L^2(\mathbf{R}^d)),\end{aligned}$$

since M_λ^* is L_λ-bounded and $|W_\lambda[P, [W_\lambda, P]]| \leqq C\langle x\rangle^{\rho-2}$. This implies (3.9) with $\beta = 0$. Noting that $[P, M_\lambda] \in B(L^2)$, (3.9) for general β is proved by commutator calculations as in the proof of (3.8). □

Lemma 3.4. *For any* $0 \leqq \gamma \leqq 1$,

$$\|\langle A\rangle^\gamma (L_\lambda + 1)^{-1}\langle x\rangle^{-\gamma} W_\lambda^\gamma\| \leqq C < \infty, \quad \lambda \in [0, 1]. \tag{3.10}$$

Proof. We show $\|A(L_\lambda + 1)^{-1}\langle x\rangle^{-1} W_\lambda\| \leqq C < \infty$. Then (3.10) holds for $\gamma = 1$ and (3.10) with $\gamma = 0$ is obvious. Hence (3.10) for $0 < \gamma < 1$ follows by complex interpolation. The above estimate follows from commutator calculations as in the proof of Lemma 3.3, and the fact $M_\lambda = PW_\lambda$ is L_λ-bounded. □

Lemma 3.5. *For* $1/2 < \gamma \leqq 1$,

$$\sup_{0 \leqq \lambda \leqq 1} \|\langle x\rangle^{-\gamma} W_\lambda^\gamma (L_\lambda - 1 \pm i0)^{-1} W_\lambda^\gamma \langle x\rangle^{-\gamma}\| \leqq C < \infty. \tag{3.11}$$

Proof. We use Lemmas 3.1 and 3.4. Then

$$\begin{aligned}&\|\langle x\rangle^{-\gamma} W_\lambda^\gamma (L_\lambda - 1 \pm i0)^{-1} W_\lambda^\gamma \langle x\rangle^{-\gamma}\| \\ &= \|\langle x\rangle^{-\gamma} W_\lambda^\gamma \{(L_\lambda + 1)^{-1} + 2(L_\lambda + 1)^{-2} \\ &\quad + 4(L_\lambda + 1)^{-1}(L_\lambda - 1 \pm i0)^{-1}(L_\lambda + 1)^{-1}\} W_\lambda^\gamma \langle x\rangle^{-\gamma}\| \\ &\leqq C + \|\langle x\rangle^{-\gamma} W_\lambda^\gamma (L_\lambda + 1)^{-1}\langle A\rangle^\gamma\|^2 \|\langle A\rangle^{-\gamma}(L_\lambda - 1 \pm i0)^{-1}\langle A\rangle^{-\gamma}\| \\ &\leqq 2C < \infty. \quad \square\end{aligned}$$

Now we can prove the first part of Theorem 1.9.

Lemma 3.6. *If* $\gamma > 1/2 + \rho/4$, *then*

$$\sup_{0 < \lambda \leqq 1} \|\langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}\| \leqq C < \infty. \tag{3.12}$$

Proof. We first note that by Lemmas 3.2 and 3.4,

$$\lim_{\varepsilon\downarrow 0} \langle x\rangle^{-\gamma} W_\lambda^\gamma (L_\lambda - 1 \pm i\varepsilon W_\lambda^2)^{-1} W_\lambda^\gamma \langle x\rangle^{-\gamma}$$
$$= \langle x\rangle^{-\gamma} W_\lambda^\gamma (L_\lambda - 1 \pm i0)^{-1} W_\lambda^\gamma \langle x\rangle^{-\gamma} .$$

On the other hand, it is easy to see

$$(H - \lambda \pm i\varepsilon)^{-1} = W_\lambda (L_\lambda - 1 \pm i\varepsilon W_\lambda^2)^{-1} W_\lambda .$$

Hence we obtain

$$\lim_{\varepsilon\downarrow 0} \langle x\rangle^{-\gamma} W_\lambda^{-(1-\gamma)} (H - \lambda \pm i\varepsilon)^{-1} W_\lambda^{-(1-\gamma)} \langle x\rangle^{-\gamma}$$
$$= \langle x\rangle^{-\gamma} W_\lambda^\gamma (L_\lambda - \lambda \pm i0)^{-1} W_\lambda^\gamma \langle x\rangle^{-\gamma} .$$

Thus by Lemma 3.5,

$$\sup_{0<\lambda\leqq 1} \| \langle x\rangle^{-\gamma} W_\lambda^{-(1-\gamma)} (H - \lambda \pm i0)^{-1} W_\lambda^{-(1-\gamma)} \langle x\rangle^{-\gamma} \| \leqq C < \infty .$$

Since $|W_\lambda(x)| \leqq C\langle x\rangle^{\rho/2}$ for $x \in \mathbf{R}^d$, $\lambda \in [0, 1]$, it implies

$$\sup_{0<\lambda\leqq 1} \| \langle x\rangle^{-\gamma'} (H - \lambda \pm i0)^{-1} \langle x\rangle^{-\gamma'} \| \leqq C < \infty ,$$

with $\gamma' = \gamma + (1-\gamma)\rho/2$. Since $\gamma > 1/2$ if and only if $\gamma' > 1/2 + \rho/4$, this implies the assertion. □

In order to prove the existence of $(H - 0 \pm i0)^{-1}$, we show

$$\lim_{\lambda\downarrow 0} \langle x\rangle^{-\gamma} (H - \lambda \pm i0)^{-1} \langle x\rangle^{-\gamma} = \langle x\rangle^{-\gamma} W_0 (L_0 - 1 \pm i0)^{-1} W_0 \langle x\rangle^{-\gamma} . \tag{3.13}$$

Lemma 3.7. *As* $\lambda \downarrow 0$, $L_\lambda \to L_0$ *in the strong resolvent sense.* i.e., *for any* $z \in \mathbf{C}\backslash[0, \infty)$,

$$\operatorname*{s-lim}_{\lambda\downarrow 0} (L_\lambda - z)^{-1} = (L_0 - z)^{-1} . \tag{3.14}$$

Proof. It suffices to show

$$\lim_{\lambda\downarrow 0} (L_\lambda + 1)^{-1}\varphi = (L_0 + 1)^{-1}\varphi$$

for $\varphi \in \mathscr{S}(\mathbf{R}^d)$ (see [RS, Sect. VIII.7]). We have

$$(L_\lambda + 1)^{-1}\varphi - (L_0 + 1)^{-1}\varphi = (L_\lambda + 1)^{-1}(L_0 - L_\lambda)(L_0 + 1)^{-1}\varphi$$
$$= (L_\lambda + 1)^{-1}(W_0 P^2 W_0 - W_\lambda P^2 W_\lambda)\langle P\rangle^{-2}\langle x\rangle^{-2}$$
$$\times (\langle x\rangle^2 \langle P\rangle^2 (L_0 + 1)^{-1} \langle P\rangle^{-2}\langle x\rangle^{-2})(\langle x\rangle^2\langle P\rangle^2\varphi) ,$$

and hence by Lemma 3.3,

$$\| (L_\lambda + 1)^{-1}\varphi - (L_0 + 1)^{-1}\varphi \| \leqq C\lambda \| \langle x\rangle^2\langle P\rangle^2\varphi \| \to 0$$

as $\lambda \downarrow 0$. □

Lemma 3.8. *For* $\gamma > 1/2$,

$$\operatorname*{s\text{-}lim}_{\lambda \downarrow 0} \langle x\rangle^{-\gamma}(L_\lambda - 1 \pm i0)^{-1}\langle x\rangle^{-\gamma} = \langle x\rangle^{-\gamma}(L_0 - 1 \pm i0)^{-1}\langle x\rangle^{-\gamma}\,. \tag{3.15}$$

Proof. We note that the convergence:

$$\langle x\rangle^{-\gamma}(L_\lambda - 1 \pm i\varepsilon)^{-1}\langle x\rangle^{-\gamma} \to \langle x\rangle^{-\gamma}(L_\lambda - 1 \pm i0)^{-1}\langle x\rangle^{-\gamma} \quad \text{as } \varepsilon \downarrow 0$$

is uniform in $\lambda \in [0, 1]$ since all the estimates in the Mourre theory is uniform in λ. Hence Lemma 3.7 implies (3.15). □

Proof of Theorem 1.9. Let $\varphi, \psi \in \mathscr{S}(\mathbf{R}^d)$. Then it follows from Lemma 3.8 that if $\gamma > 1/2 + \rho/4$ and $\beta > 1/2$,

$$\begin{aligned}
&(\varphi, \langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}\psi)\\
&\quad = (\varphi, \langle x\rangle^{-\gamma} W_\lambda(L_\lambda - 1 \pm i0)^{-1} W_\lambda\langle x\rangle^{-\gamma}\psi)\\
&\quad = ((W_\lambda\langle x\rangle^{\beta-\gamma}\varphi), [\langle x\rangle^{-\beta}(L_\lambda - 1 \pm i0)^{-1}\langle x\rangle^{-\beta}](W_\lambda\langle x\rangle^{-\beta-\gamma}\psi))\\
&\qquad \to ((W_0\langle x\rangle^{\beta-\gamma}\varphi), [\langle x\rangle^{-\beta}(L_0 - 1 \pm i0)^{-1}\langle x\rangle^{-\beta}](W_0\langle x\rangle^{\beta-\gamma}\psi))\\
&\quad = (\varphi, \langle x\rangle^{-\gamma} W_0(L_0 - 1 \pm i0)^{-1} W_0\langle x\rangle^{-\gamma}\psi)\,.
\end{aligned}$$

This proves (3.13) in the weak sense. Let $\gamma > \gamma' > 1/2 + \rho/4$. Then

$$\begin{aligned}
&\langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}\\
&\quad = \langle x\rangle^{-\gamma}(H + 1)^{-1}\langle x\rangle^{-\gamma} + (1 + \lambda)\langle x\rangle^{-\gamma}(H + 1)^{-2}\langle x\rangle^{-\gamma}\\
&\qquad + (1 + \lambda)^2(\langle x\rangle^{-\gamma}(H + 1)^{-1}\langle x\rangle^{\gamma'})\\
&\qquad \times(\langle x\rangle^{-\gamma'}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma'})(\langle x\rangle^{\gamma'}(H + 1)^{-1}\langle x\rangle^{-\gamma})\,.
\end{aligned}$$

Noting $\langle x\rangle^{\gamma'}(H + 1)^{-1}\langle x\rangle^{-\gamma}$ is compact, we learn that the weak convergence of $\langle x\rangle^{-\gamma'}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma'}$ implies the norm convergence of $\langle x\rangle^{-\gamma}(H - \lambda \pm i0)^{-1}\langle x\rangle^{-\gamma}$. This completes the proof of Theorem 1.9. □

Acknowledgement The author wishes to thank Professor A Jensen and Professor A.V. Sobolev for valuable discussions.

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Communicated by H. Araki