Commun. Math. Phys. 160, 537–556 (1994)

Communications in
Mathematical
Physics

A Geometrical Presentation of the Surface Mapping Class Group and Surgery

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Received: 1 February 1993

Abstract. We construct a tangle presentation of the mapping class group similar to a natural presentation of the braid group by geometrical braids. A relation between surgery and Heegard diagrams for 3-manifolds arising in this way and different applications are studied.

1. Introduction

It is well-known that the mapping class group of the disc with n marked points has a natural presentation as the group of geometrical braids with n strings. We give a similar presentation of the mapping class group of an orientable surface of arbitrary genus (which may also easily be generalized for the case of a surface with marked points). A relation between surgery presentation of 3-manifolds and Heegard diagrams (see [6, 11]) arising in this way is investigated. This relation enables us to prove that if a 3-manifold has Heegard decomposition of genus two, it may be obtained by surgery on a framed arborescent link in S^3. We also provide a new proof (similar in spirit to [7]) of Kirby's theorem [5], which in our setting is an easy consequence of stable equivalence of Heegard splittings and Wajnryb's presentation for the mapping class group of a surface [13].

The paper is organized in the following way: in Sect. 1 we recall the notion of framed $2n$-tangles and their diagrams. In Sect. 2 Kirby calculus for framed $2n$-tangles is introduced. Section 3 is devoted to the definition of the group T_{2n} of admissible $2n$-tangles. We state our main theorem in Sect. 4; the proof is given in Sects. 5, 6. In Sects. 7, 8 we study the relation between surgery and Heegard decompositions. As a corollary of our construction in a particular case of Heegard genus two we obtain (in Sect. 7) the result mentioned above. A new proof of Kirby's theorem is established in Sect. 9.

⋆ Supported by Rotshild Fellowship

The paper was written when the first author was visiting Tel-Aviv University at the beginning of 1992. He would like to express his gratitude to Prof. M. Farber for his friendly support during the visit.

2. Framed $2n$-Tangles

For a given integer $n \geq 0$ let Y_{2n} be a set of n pairs $\{i^-, i^+\}$, $1 \leq i \leq n$ of different points in the xy-plane $\mathbb{R}^2 \subset \mathbb{R}^2 \times \mathbb{R}^1$. To make our choice explicit we will put $i^\pm = (0, i \pm 1/4)$. By a *$2n$-tangle* we mean a proper one-dimensional submanifold of $\mathbb{R}^2 \times [0,1]$ such that its boundary coincides with the set $Y_{2n} \times \{0,1\}$. A *framing* of $2n$-tangle is a trivialization of its normal bundle. We require that the restriction of the framing to $Y_{2n} \times \{0,1\}$ should be induced by the standard xy-structure of $\mathbb{R}^2$ (say, in the positive direction of y-axis).

Given two framed $2n$-tangles ξ and ζ one may define their product $\xi \cdot \zeta$ to be $2n$-tangle obtained by gluing the top of the "squeezed" copy $\zeta' \subset \mathbb{R}^2 \times [0, 1/2]$ of ζ to the bottom of the squeezed copy $\xi' \subset \mathbb{R}^2 \times [1/2, 1]$ of ξ.

Each $2n$-tangle can be presented by *tangle diagram*, i.e. by its (general position) projection to $\mathbb{R}^1 \times [0,1]$ with over- and underpasses in each crossing point indicated in the usual way. Tangle diagram determines the framing induced by the vector field orthogonal to $\mathbb{R}^1 \times [0,1] \subset \mathbb{R}^2 \times [0,1]$. Two tangle diagrams represent the same framed $2n$-tangle iff they are regularly isotopic. The addition of a kink changes the framing of corresponding component by ± 1. It will be convenient to replace positive and negative kinks by small white and black circles respectively. A pair of opposite kinks on the same string of the tangle may by cancelled, as illustrated in Fig. 1.

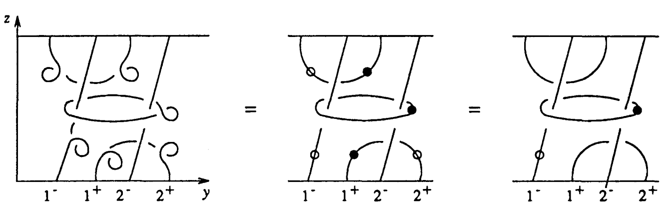


Fig. 1.

2. Kirby Moves

In [5] Kirby introduced two operations O_1, O_2 on framed links in a sphere S^3, later called *Kirby moves*. Denote by $\chi(M^3, L)$ the result of Dehn surgery of a 3-manifold M^3 along a framed link $L \subset M^3$. Then the following holds:

Kirby Theorem [5]. *Given two framed links $L_1, L_2 \in S^3$ one can pass from L_1 to L_2 by a sequence of moves O_1, O_2 iff $\chi(S^3, L_1)$ is homeomorphic to $\chi(S^3, L_2)$ (by an orientation preserving homeomorphism).*

We extend the Kirby moves to the operations on a framed $2n$-tangle $\xi \subset \mathbb{R}^2 \times [0,1]$ by introducing the following moves $K_1 - K_3$:

K_1: Add to ξ an unknotted ± 1 framed circle separated from the other strings of ξ by an embedded 2-sphere $S^2 \subset \mathbb{R}^2 \times [0,1]$. This move coincides with the Kirby move O_1.

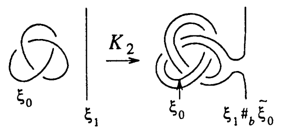


Fig. 2.

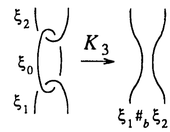


Fig. 3.

K_2: Let ξ_0 be a closed component of ξ. Add ξ_0 to ξ_1 by replacing ξ_1 with $\xi_1\#_b\tilde{\xi}_0$, where $\#_b$ is a band connected sum and $\tilde{\xi}_0$ is obtained by pushing ξ_0 off itself along the framing, as illustrated in Fig. 2. If ξ_1 is closed, this move coincides with the Kirby move O_2.

K_3: Let ξ_0 be a closed, 0-framed component of ξ bounding an embedded disk $D \in \mathbb{R}^2 \times [0,1]$ which intersects with $\xi \backslash \xi_0$ in exactly two points belonging to different components ξ_1, ξ_2 of ξ. Suppose also that either at least one of the components ξ_1, ξ_2 is closed or $\partial\xi_1 \subset \mathbb{R}^2 \times \{0\}$, $\partial\xi_2 \subset \mathbb{R}^2 \times \{1\}$. Then we may replace $\xi_0 \cup \xi_1 \cup \xi_2$ by $\xi_1\#_b\xi_2$, where the band b intersects D along the middle line of b, see Fig. 3.

Definition. *Two framed $2n$-tangles are said to be K-equivalent, if one can pass from one to another by a (finite) sequence of moves $K_1^{\pm 1}, K_2, K_3^{\pm 1}$; we denote it by $\xi \underset{K}{\sim} \zeta$.*

It is convenient for our further purposes to introduce some additional moves K_4, K_5 (which can be expressed via $K_1^{\pm 1}, K_2, K_3^{\pm 1}$).

The move K_4 is the deletion of unknotted ± 1-frame circle at the expense of the full left- or right-hand twist on the strings linked with it [4], as shown in Fig. 4.

Let ξ_0 be a closed, 0-framed component of ξ which bounds an embedded disk $D \subset \mathbb{R}^2 \times [0,1]$. Suppose that $D \cap (\xi \backslash \xi_0)$ consists of exactly one point lying on some component $\xi_1 \subset \xi$. Then the move K_5 is the deletion of components ξ_0, ξ_1 as illustrated in Fig. 5.

Some remarks should be made at this stage.

Remark 2.1. The move K_4 may be expressed via $K_1^{\mp 1}$ and K_2, see [4]. The same is true for the move K_5. To obtain this, note that due to presence of ξ_0 we may change overcrossings of ξ_1 with itself and with other components of ξ to undercrossings by means of K_2; this allows us to unlink (and unknot) ξ_1 so that we may consider ξ_0 and ξ_1 to form the Hopf link far from the other components of ξ. We may also put the framing of ξ_1 to be $+1$ by adding ± 1-framed unknotted circle (by K_1) and link it with ξ_1 (as above), which in view of K_4 is equivalent to changing the framing of

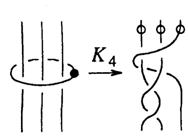


Fig. 4.

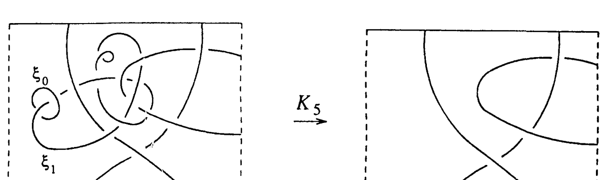


Fig. 5.

ξ_1 by ± 1. It remains to take note that now the deletion of $\xi_0 \cap \xi_1$ can be carried out by means of K_4 and K_1^{-1}.

Remark 2.2. If at least one of the components ξ_1, ξ_2 (say ξ_1) in the definition of the move K_3 is closed, then K_3 may be expressed via $K_1^{\pm 1}$ and K_2. Actually, it is a composition of the move K_2 (we add ξ_1 to ξ_2 along the band b) and the move K_5.

Remark 2.3. Any framed link $L \in S^3$ may be considered (after isotopy into $\mathbb{R}^2 \times [0,1] \subset \mathbb{R}^3 \in S^3$) as a framed $2n$-tangle for $n = 0$, hence the moves $K_1^{\pm 1}, K_2, K_3^{\pm 1}$ can also be applied to links. The equivalence relation $\underset{K}{\sim}$ for links coincides with the equivalence $\underset{\partial}{\sim}$ of Kirby [5].

3. Admissible $2n$-Tangles

Unfortunately the multiplication of $2n$-tangles does not agree with the equivalence relation $\underset{K}{\sim}$: it may occur that $\xi \underset{K}{\sim} \zeta$, but $\xi\gamma$ is not K-equivalent to $\zeta\gamma$ for some $2n$-tangle γ. The reason is that the condition on ξ_1, ξ_2 in the definition of the move K_3 may be violated after multiplication by γ. To avoid this difficulty we introduce the notion of an admissible $2n$-tangle.

Note that each geometrical $2n$-braid ξ with the ends at $Y_{2n} \times \{0,1\}$ can be considered as a $2n$-tangle. The braid ξ is called *admissible*, if the corresponding permutation preserves the decomposition of Y_{2n} into (unordered) pairs. More precisely we require that if for some i, j a string of the braid runs from $i^- \times \{0\}$ to $j^\pm \times \{1\}$, then the other string has to run from $i^+ \times \{0\}$ to $j^\mp \times \{1\}$. An arbitrary framing of the braid is allowed.

Let $\beta_i^{\pm 1}$, $1 \le i \le n$ be the framed $2n$-tangles depicted in Fig. 6.

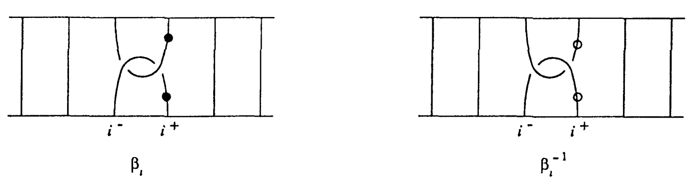


Fig. 6.

Definition. *A framed $2n$-tangle ξ is called admissible, if it can be written as $\xi = \xi_1\xi_2\cdots\xi_k$ (for some k) where each ξ_i is either $\beta_i^{\pm 1}$, $1 \le i \le n$ or a framed admissible $2n$-braid.*

Note that each admissible $2n$-tangle ξ satisfies the following

Condition $(*)$. For each unclosed component ξ_0 of ξ one of the following holds:
(i) $\partial\xi_0 = (i^- \cup i^+) \times \{0\}$ for some i, or
(ii) $\partial\xi_0 = (i^- \cup i^+) \times \{1\}$ for some i, or
(iii) $\partial\xi_0 = i^\pm \times \{0\} \cup j^\pm \times \{1\}$ for some i, j and there exists another component ξ_1 of ξ such that $\partial\xi_1 = i^\mp \times \{0\} \cup j^\mp \times \{1\}$.

One may easily see that multiplication of tangles and the moves $K_1 - K_5$ preserve the condition $(*)$. This implies that the multiplication of admissible $2n$-tangles determines correctly defined multiplication on K-equivalence classes of admissible $2n$-tangles.

Denote by T_{2n} the semigroup of K-equivalence classes of admissible $2n$-tangles.

Proposition 3.1. *T_{2n} is a group.*

Proof. Immediately follows from the fact that the reflection of an admissible $2n$-tangle ξ with respect to the middle plane $\mathbb{R}^2 \times \{1/2\}$ produces the inverse $2n$-tangle ξ^{-1}. $\square$

4. Main Theorem

Let $\Sigma_{n,1}$ be an orientable surface of genus n with one boundary component. The mapping class group $M_{n,1}$ of $\Sigma_{n,1}$ is the group of isotopy classes of orientation-preserving self homeomorphisms of $\Sigma_{n,1}$ fixing the boundary pointwise. We will oftenly identify $M_{n,1}$ with the mapping class group of a surface ∂H_n with an open disk $D \in \partial H_n$ removed, where H_n is the standard solid handlebody of genus n in $\mathbb{R}^3 \in S^3$.

Denote by a_i, b_i, d_i, e_i, $1 \leq i \leq n$ the right-hand Dehn twists with respect to curves labeled, by abuse of notation, with the same letters, as shown in Fig. 7 (notice that $a_1 = d_1 = e_1$). Denote by α_i, β_i, ε_i and δ_i, $1 \leq i \leq n$ the admissible $2n$-tangles depicted in Fig. 8, 6 (put $\alpha_1 = \delta_1$). Clearly, ε_i is admissible in view of K_4 and $\delta_1 \underset{K}{\sim} \varepsilon_1$.

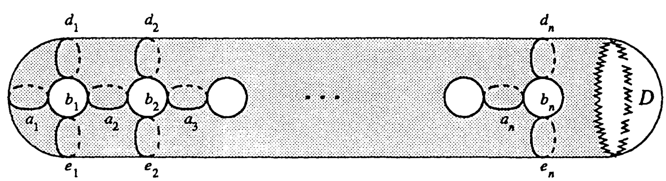


Fig. 7.

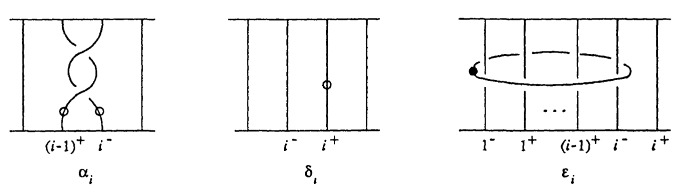


Fig. 8.

Remark 4.1. Applying the move K_3^{-1}, the regular isotopy, and the move K_3 successively, we can shift the white circle (the positive kink) of δ_i from the string $i^+ \times [0,1]$ to the string $i^- \times [0,1]$.

Theorem 4.1 (Main Theorem). *There exists an isomorphism $\phi: T_{2n} \to M_{n,1}$ which maps α_i, β_i, δ_i, ε_i to a_i, b_i, d_i, e_i, $1 \leq i \leq n$ respectively.*

Remark 4.2. One may obtain a similar result for the mapping class group $M_{n,0}$ of a closed surface, taking the factor of T_{2n} by cyclic commutator subgroup generated by full twist on $2n$ strings.

Remark 4.3. This tangle presentation of $M_{n,1}$ has a straightforward generalization to the case of mapping class group $M_{n,1,k}$ of a surface with k marked points by introducing k new strands with braiding around the other strings and between themselves (but without any new β_i generators).

5. Proof of the Main Theorem: Construction of ϕ

Consider a large disk $B \subset \mathbb{R}^2$ containing Y_{2n}. Without loss of generality we may assume that all considered $2n$-tangles are contained in $B \times [0,1]$. Let $\xi \subset B \times [0,1]$ be an admissible $2n$-tangle. For each pair $(i^-, i^+) \subset Y_{2n}$ attach to $B \times [0,1]$ index one handle $N(A_i) \subset \mathbb{R}^2 \times \mathbb{R}^1 = \mathbb{R}^3$ so that $N(A_i) \cap B \times [0,1]$ is the regular neighbourhood of $(i^- \cup i^+) \times \{1\}$. Add to ξ the cores A_i of the handles to close ξ from above. To make the construction explicit we will take $A_i = \{(0,y,z) \in \mathbb{R}^3 | (y-i)^2 + (z-1)^2 = 1/4, z \geq 1\}$. We obtain the solid handlebody $H = B \times [0,1] \cup \left(\bigcup_i N(A_i) \right) \subset \mathbb{R}^3$ with framed one-dimensional submanifold $\tilde{\xi} = \xi \cup \left(\bigcup_i A_i \right)$ inside it. Denote by ξ' the union of all unclosed components of $\tilde{\xi}$. It consists of exactly n arcs with the ends on the "bottom" $B \times \{0\}$ of H. Finally, remove from H interior of a regular neighbourhood $N(\xi')$ of ξ'. As a result we obtain 3-manifold $H_\xi = H \setminus \mathrm{Int}(N(\xi'))$ with framed link $L_\xi = \tilde{\xi} \setminus \xi'$ inside it, see Fig. 9. Note that ∂H_ξ admits natural decomposition $\partial H_\xi = \Sigma_{0\xi} \cup \partial B \times [0,1] \cup \Sigma_{1\xi}$, where $\Sigma_{j\xi}$, $j = 0,1$ is genus n surface with one boundary component and $\Sigma_{j\xi} \cap \partial B \times [0,1] = \partial B \times \{j\}$.

We would like to point out that there exist natural identifications $\kappa_j : \Sigma_{n,1} \to \Sigma_{j\xi}$. Present $\Sigma_{n,1}$ as disk B with n handles as it is depicted in Fig. 10 (compare with Fig. 7). The surface $\Sigma_{j\xi}$ is also disk $B \times \{j\}$ with n handles. Then κ_j maps B with holes identically to $B \times \{j\}$ with holes and the curves d_i, b_i are mapped, respectively, to the meridians and induced by framing "longitudes" of the corresponding handles (the images of b_i in Fig. 9 are drawn by thick lines).

Recall that $\chi(H_\xi, L_\xi)$ denotes 3-manifold obtained from H_ξ by surgery on the framed link $L_\xi \subset H_\xi$.

Proposition 5.1. *For any admissible $2n$-tangle ξ the manifold $\chi(H_\xi, L_\xi)$ is homeomorphic to $\Sigma_{n,1} \times [0,1]$ and the product structure induced on $\chi(H_\xi, L_\xi)$ is an extension of the natural product structure on $\partial B \times [0,1] \subset \partial H_\xi = \partial\chi(H_\xi, L_\xi)$.*

Proof. Since the complement of a braid in $B \times [0,1]$ is homeomorphic to the complement of the trivial braid, the proposition holds for the case when ξ is an

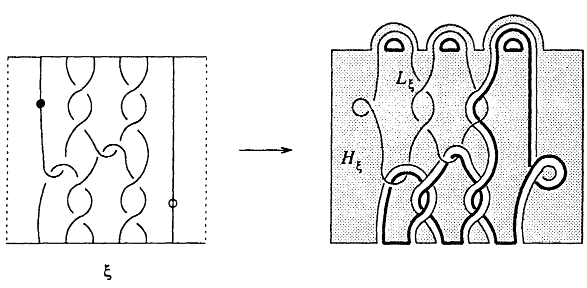


Fig. 9.

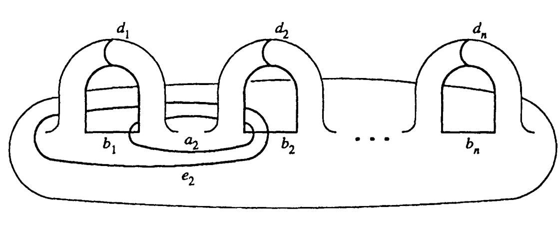


Fig. 10.

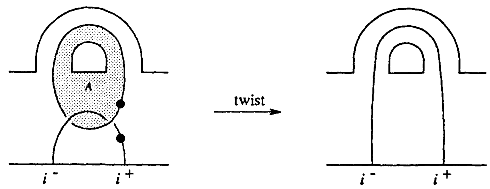


Fig. 11.

admissible $2n$-braid. Using the twist along the annulus $A \subset H_\xi$, one boundary component of which coincides with $\kappa_1(b_i) \subset \Sigma_{1\xi} \subset \partial H_\xi$ and another – with the unique closed component of $L_{\beta_i}^{\pm 1}$, one can easily prove the proposition for $\xi = \beta_i^{\pm 1}$, as shown in Fig. 11. It remains to note that by the definition of admissible $2n$-tangle product structure on $\chi(H_{\xi\zeta}, L_{\xi\zeta})$ is obtained by gluing together the product structures on $\chi(H_\xi, L_\xi)$ and $\chi(H_\zeta, L_\zeta)$, which implies the general case. □

Consider an admissible $2n$-tangle ξ. Recall that $\partial\chi(H_\xi, L_\xi) = \partial H_\xi = \Sigma_{0\xi} \cup \partial B \times [0,1] \cup \Sigma_{1\xi}$. Let $p_\xi : \Sigma_{0\xi} \to \Sigma_{1\xi}$ be the restriction on $\Sigma_{0\xi}$ of the direct product projection $p : \partial\chi(H_\xi, L_\xi) \to \Sigma_{1\xi}$. Define the homeomorphism $\phi(\xi) : \Sigma_{n,1} \to \Sigma_{n,1}$ by $\phi(\xi) = \kappa_1^{-1} p_\xi \kappa_0$. It follows from the definition that $\phi(\xi\zeta) = \phi(\xi)\phi(\zeta)$ for any two admissible $2n$-tangles ξ, ζ.

Remark 5.1. The homeomorphisms $\phi(\beta_i^{\pm 1})$ are isotopic to the twists along the curve b_i in positive and negative directions respectively, see Fig. 10 and the proof of Proposition 5.1.

Proposition 5.2. *The assignment $\xi \mapsto \phi(\xi)$ determines correctly defined homomorphism $\phi : T_{2n} \to M_{n,1}$.*

Proof. It is sufficient to show that the isotopy class of $\phi(\xi)$ depends only on the K-equivalence class of ξ. Let an admissible $2n$-tangle ζ be obtained from ξ by application of the move $K = K_1$, K_2 or K_3. Suppose that all the components of ξ involved in the move K are closed. Then the equality $\phi(\xi) = \phi(\zeta)$ is clear: the same proof as for the easy "only if" part of the Kirby theorem is valid. But all components of ξ are actually closed in $2n$-tangle $\Delta\xi\Delta$, where $\Delta = \prod_{i=1}^{n} \beta_i \beta_i^{-1}$. Therefore $\phi(\Delta\xi\Delta) = \phi(\Delta\zeta\Delta)$. The multiplicativity of ϕ and the equality $\phi(\Delta) = 1$ (see Remark 5.1) imply the desired $\phi(\xi) = \phi(\zeta)$ thus completing the proof. □

By the construction of ϕ we obtain that ϕ maps the tangles α_i, β_i, δ_i, ε_i to a_i, b_i, d_i, e_i, $1 \le i \le n$ respectively.

6. Proof of the Main Theorem: Construction of $\psi = \phi^{-1}$

Let us begin with reformulation of Wajnryb's theorem [13].

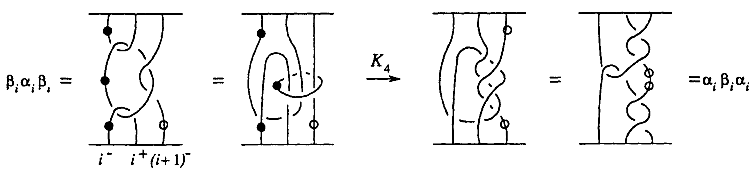
Theorem [13]. *The mapping class group $M_{n,1}$ admits a presentation with generators $a_1, b_1, \ldots, a_n, b_n, e_2$ and relations*
(A) $a_i b_i a_i = b_i a_i b_i$, $a_{i+1} b_i a_{i+1} = b_i a_{i+1} b_i$, $b_2 e_2 b_2 = e_2 b_2 e_2$, *every other pair of generators commute;*
(B) $(a_2 b_1 a_1)^4 = k e_2 k^{-1} e_2$, *where* $k = b_2 a_2 b_1 a_1^2 b_1 a_2 b_2$*;*
(C) $a_1^{-1} a_2^{-1} a_3^{-1} g_1 g_2 e_2 = w e_2 w^{-1}$, *where* $t_1 = b_1 a_2 a_1 b_1$, $t_2 = b_2 a_3 a_2 b_2$, $g_2 = t_2^{-1} e_2 t_2$, $g_1 = t_1^{-1} g_2 t_1$, $u = a_3^{-1} b_3^{-1} g_2 b_3 a_3$, $w = b_3 a_3 b_2 a_2 b_1 u a_1^{-1} b_1^{-1} a_2^{-1} b_2^{-1}$.

Remark 6.1. Our formulation of Wajnryb's theorem differs slightly from the original one: we write homeomorphisms from right to left and use ε_2 instead of δ_2. The latter is possible due to existence of rotation of $\Sigma_{n,1}$ which is invariant on α_i, β_i and interchanges δ_2 and ε_2.

Proposition 6.1. *The assignment $a_i \mapsto \alpha_i$, $b_i \mapsto \beta_i$, $e_i \mapsto \varepsilon_i$ determines correctly defined homomorphism $\psi : M_{n,1} \to T_{2n}$.*

Proof. We need only check that this assignment transforms the relations (A), (B), (C) to true equalities on the tangle level.
(A) The equality $\alpha_i\beta_i\alpha_i = \beta_i\alpha_i\beta_i$ is verified in Fig. 12. Verification of the equality $\alpha_{i+1}\beta_i\alpha_{i+1} = \beta_i\alpha_{i+1}\beta_i$ is similar. Equality $\beta_2^{-1}\varepsilon_2\beta_2 = \varepsilon_2\beta_2\varepsilon_2^{-1}$ (equivalent to $\beta_2\varepsilon_2\beta_2 = \varepsilon_2\beta_2\varepsilon_2$) is verified in Fig. 13. Obviously every other pair of $2n$-tangles $\alpha_i, \beta_i, \varepsilon_2$ commutes.
(B) Let $\kappa = \beta_2\alpha_2\beta_1\alpha_1^2\beta_1\alpha_2\beta_2$ be the image of k.
From Fig. 14 we obtain that $\kappa\varepsilon_2 = \delta_2\kappa$. Therefore $\kappa\varepsilon_2\kappa^{-1} = \delta_2$. Using (A) we deduce

$$(\alpha_2\beta_1\alpha_1)^4 = (\alpha_2\beta_1\alpha_1\alpha_2\beta_1\alpha_1)^2 = (\beta_1\alpha_2\beta_1\alpha_1\beta_1\alpha_1)^2 = (\beta_1\alpha_2\alpha_1\beta_1\alpha_1^2)^2 .$$

Fig. 12.


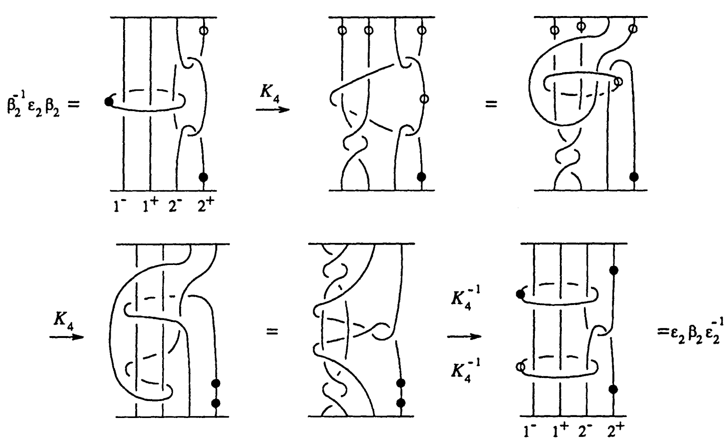


Fig. 13.

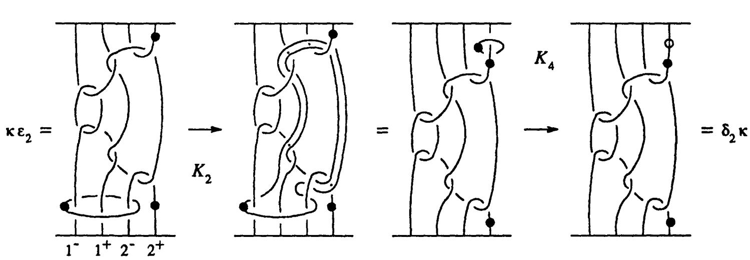


Fig. 14.

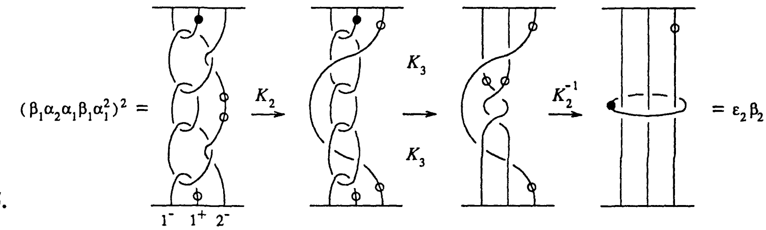


Fig. 15.

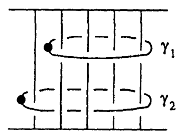


Fig. 16.

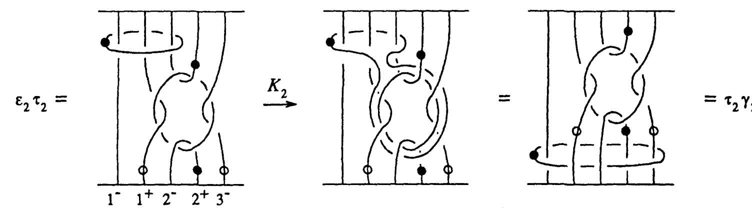


Fig. 17.

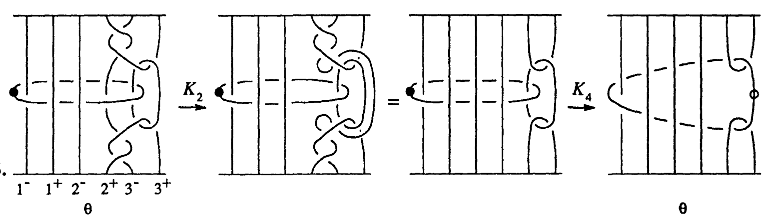


Fig. 18.

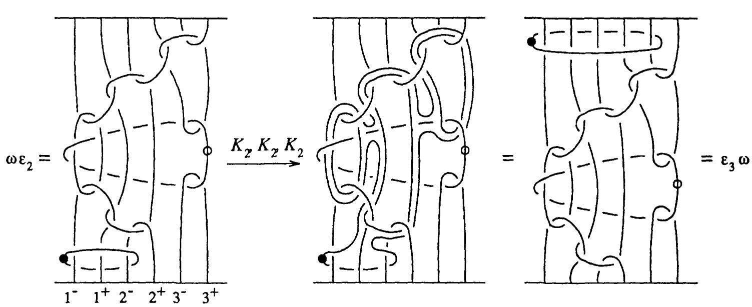


Fig. 19.

From Fig. 15 we now obtain $(\beta_1\alpha_2\alpha_1\beta_1\alpha_1^2)^2 = \delta_2\varepsilon_2$ which implis (B).
(C) Let γ_1, γ_2 be the tangles depicted in Fig. 16. It follows from Fig. 17 that $\tau_2\gamma_2 = \varepsilon_2\tau_2$, where $\tau_2 = \beta_2\alpha_3\alpha_2\beta_2$ is the image of t_2. Hence we have $\gamma_2 = \tau_2^{-1}\varepsilon_2\tau_2$. A similar trick enables us to prove that $\gamma_1 = \tau_1^{-1}\gamma_2\tau_1$ for the image $\tau_1 = \beta_1\alpha_2\alpha_1\beta_1$ of t_1. We may conclude, therefore, that the tangles γ_1, γ_2 serve as the images of g_1, g_2 respectively.

To express the tangle $\theta = \alpha_3^{-1}\beta_3^{-1}\gamma_2\beta_3\alpha_3$ (the image of u) in a more convenient form we apply the moves K_2 and K_4 as shown in Fig. 18. We deduce from Fig. 19 that

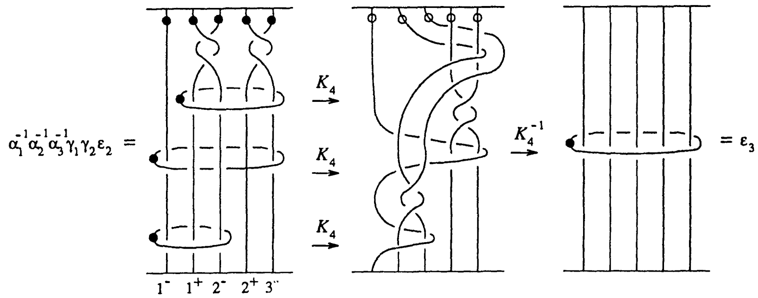


Fig. 20.

the image $\omega\varepsilon_2\omega^{-1}$ (where $\omega = \beta_3\alpha_3\beta_2\alpha_2\beta_1\theta\alpha_1^{-1}\beta_1^{-1}\alpha_2^{-1}\beta_2^{-1}$) of the RHS of relation (C) equals ε_3. The image of the LHS of relation (C) also equals ε_3, as shown in Fig. 20.

This completes the proof of the proposition. □

The Main Theorem now follows from Propositions 5.2, 6.1.

7. From Heegard Diagrams to Surgery

Let H_n denote the standard handlebody of genus n in $\mathbb{R}^2 \times \mathbb{R}^1 \subset S^3$. We shall consider a Heegard diagram of 3-manifold M as a homeomorphism $h: \partial H_n \to \partial H_n$ such that $\overline{(S^3 \backslash H_n)} \cup_h H_n = M$. Identify $\Sigma_{n,1}$ with the complement of an open disk $D \subset \partial H_n$ and the group $M_{n,1}$ with the mapping class group of ∂H_n modulo D. Particularly, one may assume that $h \subset M_{n,1}$. Let $\xi = \psi(h)$ be the admissible $2n$-tangle corresponding to h. One may express ξ as a product of tangles $\alpha_i, \beta_i, \delta_i$ according to a decomposition of h into the product of twists along the curves a_i, b_i, d_i.

Recall that the framed link $L_\xi \subset H_\xi \subset S^3$ is obtained from ξ by closing with n small semicircles from above and removing the lower strings.

Theorem 7.1. *For any admissible $2n$-tangle ξ the manifolds $\chi(S^3, L_\xi)$ and*

$$(S^3 \backslash H_n) \cup_{\phi(\xi)} H_n$$

are homeomorphic.

Proof. Let $B_1 \subset \mathbb{R}^2$ be a disk such that $\operatorname{Int} B_1 \subset B$. Denote by H_ξ^0 the handlebody $B_1 \times [-1/2, 0] \cup N(\xi') \subset \mathbb{R}^2 \times \mathbb{R}^1 \subset S^3$ and by H_ξ^1 the handlebody $B \times [-1, 1] \cup \left(\bigcup_{i=1}^{n} N(A_i)\right)$, see Fig. 21 and Sect. 5. It follows from Proposition 5.1 that $\chi(\overline{(H_\xi^1 \backslash H_\xi^0)}, L_\xi)$ is homeomorphic to $\partial H_\xi^0 \times [0, 1]$. Let $p_\xi : \partial H_\xi^0 \to \partial H_\xi^1$

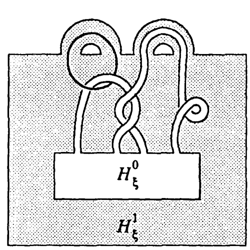


Fig. 21.

be the restriction of the direct product projection. Clearly the manifold $\chi(S^3, L_\xi)$ is homeomorphic to $\overline{(S^3\backslash H_\xi^1)} \cup_{p_\xi} H_\xi^0$. Homeomorphisms $\kappa_0 : \Sigma_{n,1} \to \Sigma_{0\xi}$ and $\kappa_1 : \Sigma_{n,1} \to \Sigma_{1\xi}$ defined in Sect. 5 can be extended to homeomorphisms $\tilde{\kappa}_0 : H_n \to H_\xi^0$ and $\tilde{\kappa}_1 : \overline{(S^3\backslash H_n)} \to \overline{(S^3\backslash H_\xi^1)}$ respectively. Since $\kappa_1^{-1} p_\xi \kappa_0 = \phi(\xi)$ by definition of ϕ, the formulas $f(x) = \tilde{\kappa}_0(x)$ if $x \in H_n$ and $f(x) = \tilde{\kappa}_1(x)$ if $x \in \overline{(S^3\backslash H_n)}$ give correctly defined homeomorphism $f : \overline{(S^3\backslash H_n)} \cup_{p_\xi} H_n \to \overline{(S^3\backslash H_\xi^1)} \cup_{p_\xi} H_\xi^0 \approx \chi(S^3, L_\xi)$. □

We now briefly recall the notion of arborescent link (see [3]). Let $\Gamma \subset \mathbb{R}^2$ be connected multigraph. Suppose that there exist a disk $D \subset \mathbb{R}^2$ such that $D \cap \Gamma = \partial D$ and ∂D consists of exactly two vertices and two edges of Γ. Collapsing D to a point we obtain a new multigraph $\Gamma' \subset \mathbb{R}^2$ as shown in Fig. 22.

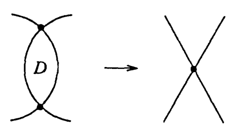


Fig. 22.

Definition. *A link $L \subset \mathbb{R}^3$ is called arborescent, if it admits a (general position) projection $\tilde{L} \subset \mathbb{R}^2$ which, if considered as multigraph, can be reduced to the one-vertex multigraph (i.e. to figure eight) by a sequence of transformations described above.*

Corollary. *Any 3-manifold of Heegard genus two may be obtained from S^3 by surgery on a framed arborescent link.*

Proof. Let ξ be an admissible 4-tangle presented as a product of α_1, α_2, β_1, β_2, δ_2 and let L_ξ be the corresponding link. One may easily deduce that L_ξ is arborescent (its natural projection becomes arborescent multigraph after untwisting small kinks). The corollary now follows from the Main Theorem and Theorem 7.1. □

8. From Surgery to Heegard Diagrams

Theorem 8.1. *For each framed link $L \subset S^3$ there exist n and an admissible $2n$-tangle ξ, such that L may be transformed into L_ξ by a sequence of Kirby moves. Tangle ξ may be chosen in the form $\xi = \eta\pi_n$, where $\pi_n = \prod_{i=1}^{n} \delta_i\beta_i\delta_i$ and η is a pure $2n$-braid.*

Remark 8.1. The first part of the theorem is an easy consequence of the Kirby theorem, the Main Theorem and Theorem 7.1, but we will provide a constructive proof without using the Kirby theorem.

Remark 8.2. Theorem 8.1 allows one to construct a Heegard diagram of 3-manifold M starting from any surgery presentation: from $M = \chi(S^3, L)$ we pass to $M = \chi(S^3, L_\xi)$ and then to the presentation $M = \overline{(S^3\backslash H_n)} \cup_{p_\xi} H_n$ by Theorem 7.1.

Definition [1, 10]. *An n-component link $L \subset \mathbb{R}^3$ is said to be represented by pure $2n$-plat if it admits a diagram with n local maxima (with respect to the projection on the z-axis).*

Remark 8.3. Each pure $2n$-plat is regularly isotopic to the plat closure of pure $2n$-braid.

Let us prove the following

Lemma 8.1. *Each framed link can be transformed to a pure $2n$-plat by a sequence of Kirby moves.*

Proof. Denote by $m(L)$ the number of local maxima and by $n(L)$ the number of components of L. Suppose that $d(L) = m(L) - n(L) > 0$. Then there exists a component L_i of L with at least two points of local maxima. Using the move K_3^{-1} we may pass to a link L' with $d(L') = d(L) - 1$ as shown in Fig. 23 (the diagrams coincide inside the square). After $d(L)$ steps we will obtain a pure $2n$-plat. □

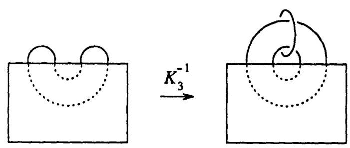


Fig. 23.

We are in a position to prove Theorem 8.1.

Proof of Theorem 8.1. By Lemma 8.1 transform L to a pure $2n$-plat L' which is a plat closure of some pure $2n$-braid η. Then $\xi = \eta\pi_n$ satisfies the theorem since $L_\xi = L'$ by the definition of L_ξ and the proof is complete. □

The following corollary can be considered as a version of Birman's theorem on the existence of special Heegard diagrams [2, 10]:

Corollary. *Each closed orientable 3-manifold can be obtained by pasting together two copies of the standard handlebody $H_n \subset S^3$ via a homeomorphism $u: \partial H_n \to \partial H_n$ which is fixed on all longitudes b_i, $1 \le i \le n$ (and therefore is extendable to $\overline{(S^3 \backslash H_n)}$).*

Proof. From Remark 8.2 we obtain that $M = \overline{(S^3 \backslash H_n)} \cup_h H_n$, where homeomorphism $h: \partial H_n \to \partial H_n$ has the form $h = \phi(\eta\pi_n)$ for some pure $2n$-braid η. One can easily verify that the homeomorphism $p_n = \phi(\pi_n)$ maps each meridian d_i; $1 \le i \le n$ to the corresponding longitude b_i and vice versa and, hence, may be extended to homeomorphism of $\overline{(S^3 \backslash H_n)}$ onto H_n. This implies that M is homeomorphic to $H_n \cup_u H_n$ for $u = p_n^{-1}h$. Since η is a pure braid, homeomorphism $\phi(\eta)$ preserves meridians. Therefore homeomorphism $u = p_n^{-1}h = p_n^{-1}\phi(\eta)p_n$ preserves longitudes and the corollary follows. □

9. Proof of the Kirby Theorem

We start with some preliminary lemmas.

Lemma 9.1. *Let ξ, ζ be K-equivalent admissible $2n$-tangles. Then the corresponding links L_ξ, L_ζ are K-equivalent.*

Proof. Let ζ be obtained from ξ by the move $K = K_1$, K_2 or K_3. Assume first that components of ξ with the ends on the bottom $B \times \{0\}$ of $B \times [0, 1]$ do not take part in K. Then L_ζ is obtained from L_ξ by the same move K. If $K = K_2$ or K_3 and components of ξ with the ends on $B \times \{0\}$ do take part in K, then L_ζ either coincides with L_ξ or is obtained from L_ξ by the move K_5. □

The following two lemmas assure us that multiplication of ξ by a tangle corresponding to a homeomorphism extendable to the inner or outer handlebody does not change K-equivalence class of L_ξ.

Denote by $TI_{2n} \subset T_{2n}$ a subgroup of admissible $2n$-tangles corresponding (via ϕ) to homeomorphisms of ∂H_n which are extendable to H_n.

Lemma 9.2. *Let γ_I be an arbitrary tangle in the subgroup TI_{2n}. Then for any admissible $2n$-tangle ξ links L_ξ and L_ζ for $\zeta = \xi\gamma_I$ are K-equivalent.*

Proof. Obviously, all admissible $2n$-braids belong to TI_{2n} since the corresponding homeomorphisms of ∂H_n are invariant on the union $\bigcup_{i=1}^{n} d_i$ of meridians of H_n (and vice versa, any homeomorphism which is invariant on the union of meridians may be obtained as $\phi(\eta)$ for some admissible $2n$-braid η). Moreover, one can easily check that $2n$-tangle $\tau_1 = \beta_1\alpha_2\alpha_1\beta_1$ also belongs to TI_{2n} since the corresponding homeomorphism $t_1 = \phi(\tau_1): \partial H_n \to \partial H_n$ maps d_1 to a_2 and is fixed on all other meridians $d_2, d_3, \dots, d_n$. It is known that each homeomorphism of ∂H_n which is extendable to H_n can be expressed via t_1 and homeomorphisms of ∂H_n which are invariant on the union $\bigcup_{i=1}^{n} d_i$ of meridians of H_n (see [12, 8]). Therefore the subgroup TI_{2n} is generated by admissible $2n$-braids and τ_1 and it suffices to check the statement of the lemma for γ_I being admissible $2n$-braid and for $\gamma_I = \tau_1$. If γ_I is an admissible $2n$-braid then $L_\zeta = L_\xi$ by the construction of L_ζ (removing of the lower strings of ζ removes all strings of γ_I). For $\gamma_I = \tau_1$ we may obtain L_ξ from L_ζ by the move K_5 and the lemma follows. □

Denote by $TO_{2n} \subset T_{2n}$ a subgroup of admissible $2n$-tangles corresponding (via ϕ) to homeomorphisms of ∂H_n which are extendable to $\overline{(S^3 \backslash H_n)}$.

Lemma 9.3. *Let γ_O be an arbitrary tangle in the subgroup TO_{2n}. Then for any admissible $2n$-tangle ξ links L_ξ and L_ζ for $\zeta = \gamma_O\xi$ are K-equivalent.*

Proof. Recall that homeomorphism $p_n = \phi(\pi_n)$ where $\pi_n = \prod_{i=1}^{n} \delta_i\beta_i\delta_i$ permutes each meridian d_i with the corresponding longitude b_i and, hence, may be extended to homeomorphisms of H_n to $\overline{(S^3 \backslash H_n)}$ and of $\overline{(S^3 \backslash H_n)}$ to H_n. This implies that $\pi_n\gamma_O\pi_n^{-1}$ belongs to TI_{2n} (which is generated by admissible $2n$-braids and τ_1). So it is sufficient to prove the lemma for $\gamma_O = \pi_n^{-1}\gamma_I\pi_n$, where γ_I is either admissible $2n$-braid or τ_1. In the first case the link L_ξ is obtained from L_ζ by n moves K_5, in the second – by $n + 1$ moves K_5 as shown in Fig. 24. □

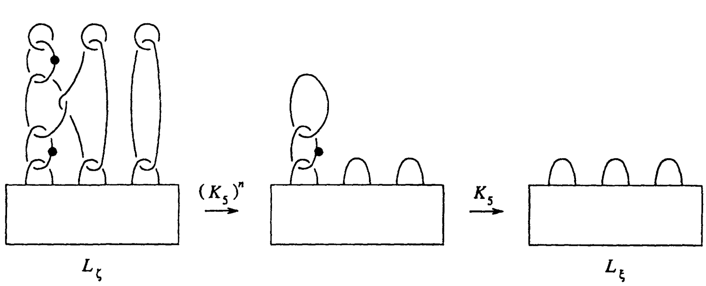


Fig. 24.

Note that for any m, n there exists a natural embedding $i_{m,n}: T_{2m} \to T_{2n}$ generated by the addition of $2(n-m)$ new vertical strings (of the form $\{i^{\pm}\} \times [0,1] \subset \mathbb{R}^2 \times [0,1]$, $m+1 \leq i \leq n$) to each admissible $2n$-tangle ξ. The addition of new strings does not change L_ξ. This observation motivates the following

Definition. *Two admissible tangles $\xi \in T_{2m}$, $\zeta \in T_{2n}$ are called stably equivalent (we write $\xi \underset{st}{\sim} \zeta$), if $i_{m,N}(\xi) = \gamma_O i_{n,N}(\zeta)\gamma_I$ for some $N \geq m, n$ and $\gamma_O \in TO_{2N}$, $\gamma_I \in TI_{2N}$.*

Lemma 9.4. *If $\xi \underset{st}{\sim} \zeta$, then $L_\xi \underset{K}{\sim} L_\zeta$.*

Proof. Immediately follows from Lemmas 9.1–9.3. □

Proof of the Kirby theorem. Let $L_1, L_2 \subset S^3$ be two links such that $\chi(S^3, L_1) = \chi(S^3, L_2)$. Using Theorem 8.1 we may construct (for some m, n) admissible tangles $\xi \in T_{2m}$, $\zeta \in T_{2n}$ so that $L_1 \underset{K}{\sim} L_\xi$, $L_2 \underset{K}{\sim} L_\zeta$. Obviously we have $\chi(S^3, L_\xi) = \chi(S^3, L_\zeta)$ which (by Theorem 7.1) is equivalent to $\overline{(S^3 \backslash H_m)} \cup_{\phi(\xi)} H_m = \overline{(S^3 \backslash H_n)} \cup_{\phi(\zeta)} H_n$. Now use the Reidemeister-Singer theorem [9] which states that any two Heegard diagrams of the same 3-manifold are stably equivalent. Translating this theorem to admissible tangle setting by means of Theorem 4.1 we obtain that tangles ξ and ζ are stably equivalent. It follows from Lemma 9.4 that $L_\xi \underset{K}{\sim} L_\zeta$ which implies the theorem. □

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Communicated by N. Yu. Reshetikhin