Commun. Math Phys. 160, 1–14 (1994)

Communications in
Mathematical
Physics

Multivoice Littlewood–Paley–Meyer Wavelets and Diagonal Dominated Pseudodifferential Operators

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Received: 27 January 1993/in revised form: 31 March 1993

Abstract: Given a pseudodifferential operator $\varepsilon(p)$ satisfying certain growth and smoothness conditions in momentum space, we construct a wavelet basis of $L^2(\mathbb{R}^d)$ in which the corresponding matrix is diagonal dominated with arbitrarily small prefactor.

1. Introduction

Problems arising in several branches of mathematical physics including quantum field theory, fluid dynamics and semiclassical analysis require some sort of multiscale analysis. The techniques based on the tree expansion in quantum field theory and on pseudodifferential calculus have recently been complemented by a new tool: the wavelet bases of $L^2(\mathbb{R}^d)$ discovered by Meyer, Lemarié, Daubechies, Mallat and others. We refer to Meyer's books [M] for a review of these results and for further references.

The prototypical basis found by Meyer in 1988 is given in dimension one by a family of functions $\psi_x(\xi) \in L^2(\mathbb{R})$, $x = (s(x), \xi(x))$, $s(x) \in \mathbb{Z}$ and $\xi(x) \in 2^{-s(x)}\mathbb{Z}$, which is generated by one "mother" function $\psi(\xi) \in L^2(\mathbb{R})$ so that

$$\psi_x(\xi) = 2^{\frac{s(x)}{2}} \psi(2^{s(x)}(\xi - \xi(x))) \tag{1.1}$$

and

$$\operatorname{supp}\hat{\psi} = \left\{ p \in \mathbb{R} : \frac{\pi}{3} \leqq |p| \leqq \frac{4\pi}{3} \right\}. \tag{1.2}$$

Meyer's wavelets also have a "father" $\phi(\xi)$ in terms of which $\psi(\xi)$ is defined and which helps to generate higher dimensional wavelets by the method of tensor products. In Sect. 2, we give a selfcontained review of these constructions. This particular basis has been named by Meyer after Littlewood and Paley. We find it thus natural to call these basis functions "LPM-wavelets."

Our goal in this paper is to refine Meyer's construction of LPM wavelets in order to obtain wavelet bases with better localization properties in momentum space. The following specific question arose in an attempt to understand the Schwinger–Englert semiclassical expansions [E] within a rigorous framework [A], see also [FS] for a review of the rigorous results: given a pseudodifferential operator $\varepsilon(p)$ in a class defined below whose prototype is p^2, the problem is to construct a wavelet basis for which the matrix

$$\varepsilon(x|y) = (2\pi)^{-d} \int dp\, \varepsilon(p)\, \overline{\hat{\psi}_x(p)}\, \hat{\psi}_y(p) \tag{1.3}$$

is diagonal dominated, i.e. is such that

$$\sum_{x \neq y} |\varepsilon(x|y)| < c \cdot \varepsilon(x|x) \tag{1.4}$$

for some $c < 1$ and all x. Meyer's wavelets do not have this property for all functions $\varepsilon(p)$. In fact, even if $\varepsilon(p)$ is C^∞, there is no way to squeeze the support of $\hat{\psi}$ to a set of arbitrarily short diameter if we also insist on having only one mother ψ. In order to refine the partition of momentum space, we resort to wavelet bases forming "polygamic families" which are generated by $2^n(2^d - 1)$ father functions $\phi_\alpha(\xi)$ and as many mother functions $\psi_\alpha(\xi)$ with $n \geqq 1$. In the general d-dimensional case, this basis has the form $\{\psi_x\}_{x \in \Omega_n}$, where the index set Ω_n is

$$\Omega_n = \{x = (\alpha(x), s(x), \xi(x));\ \alpha(x) = 1, \ldots 2^n(2^d - 1), s(x) \in \mathbb{Z}, \xi(x) \in 2^{-s(x)}\mathbb{Z}^d\}\,, \tag{1.5}$$

and the functions $\psi_x(\xi)$ have the form

$$\psi_x(\xi) = 2^{\frac{d}{2}s(x)}\, \psi_{\alpha(x)}\,(2^{s(x)}(\xi - \xi(x)))\,. \tag{1.6}$$

We have

Theorem. *In all dimensions $d \geqq 1$ and for all integers $N, n \geqq 1$, there is a basis of $L^2(\mathbb{R}^d)$ of the form $\{\psi_x\}_{x \in \Omega_n}$, where Ω_n and ψ_x are given above and we have:*

(i) *For all $x \in \Omega_n$,* supp $\hat{\psi}_x$ *is invariant under reflection with respect to the coordinate planes of $\mathbb{R}^d$ and we have*

$$\text{supp}\, \hat{\psi}_x \cap \mathbb{R}^d_+ = Q_n(x) \cap \mathbb{R}^d_+\,, \tag{1.7}$$

where $\mathbb{R}^d = \{(p_1, \ldots p_d): j \geqq 0\ \forall j = 1, \ldots d\}$ and $Q_n(x)$ is a cube in $\mathbb{R}^d$ whose side has length $2^{1-n}\pi$;
(ii) *If $x_1, x_2 \in \Omega_n$ are such that*

$$Q(x_1) \cap Q(x_2) \neq \emptyset\,, \tag{1.8}$$

then $|s(x_1) - s(x_2)| \leqq 1$;
(iii) *For all multiindices $m = (m_1 \ldots m_d)$ such that $|m| \leqq N$ and all integers $\alpha \in \{1, \ldots 2^n(2^d - 1)\}$, we have*

$$\left|\frac{\partial^m}{\partial p^m} \hat{\psi}_\alpha(p)\right| \leqq c(2^{n+1}\pi)^{|m|}\,, \tag{1.9}$$

where c is a constant dependent on d and on N but not on n.

We also have

Corollary. *Let $d \geqq 1$ and let $\varepsilon \in \mathscr{C}^{d+1}(\mathbb{R}^d)$ be a function which is symmetric under all reflections with respect to the coordinate planes and such that for all multiindices $m = (m_1 \ldots m_d)$ with $|m| \leqq d+1$ we have*

$$\left| p^m \frac{\partial^{|m|}}{\partial p^m} \varepsilon(p) \right| \leqq c \cdot \varepsilon(p) \tag{1.10}$$

for some constant c. Then, if $n \geqq 1$ and $\{\psi_x\}_{x \in \Omega_n}$ is the wavelet basis in the theorem above corresponding to $N = d+1$, we have

$$\sum_{x \neq y} |\varepsilon(x|y) \leqq c \cdot 2^{-(d+1)(n-1)} \varepsilon(x|x) \tag{1.11}$$

for all $x \in \Omega_n$, where c is a constant dependent on d but not on n. In particular, if n is large enough then the matrix $\varepsilon(x|y)$ is diagonal dominated.

2. Littlewood–Paley–Meyer (LPM) Wavelets

Here we review Meyer's construction of LPM wavelets.

Let $\phi \in L^2(\mathbb{R})$ be a real valued function whose Fourier transform has the following properties:

$$\hat{\phi}(p) = 1 \quad \forall |p| \leqq \pi - \delta_0 , \tag{2.1}$$

$$0 < \hat{\phi}(p) < 1 \quad \forall \pi - \delta_0 < |p| \leqq \pi + \delta_0 , \tag{2.2}$$

$$\hat{\phi}(p) = 0 \quad \forall |p| \geqq \pi + \delta_0 , \tag{2.3}$$

$$\hat{\phi}(p)^2 + \hat{\phi}(2\pi - p)^2 = 1 \quad \forall 0 \leqq p \leqq 2\pi , \tag{2.4}$$

where $\delta_0 \in (0, \pi)$ is an adjustable parameter that we fix below. Let us remark that, as a consequence of (2.1)–(2.4), we have

$$\sum_{k \in \mathbb{Z}} \hat{\phi}(p + 2\pi k)^2 = 1 . \tag{2.5}$$

Let $L^2_{\mathrm{per}}(2\pi)$ be the Hilbert space of the functions $m \in L^2_{\mathrm{loc}}(\mathbb{R})$ of period 2π with the following scalar product:

$$(m_1, m_2) = \int_0^{2\pi} dp\, m_1(p) m_2(p) \tag{2.6}$$

and let V_s, $s \in \mathbb{Z}$, be the subspace of $L^2(\mathbb{R})$ of the functions whose Fourier transform is of the form

$$m(2^{-s}p)\hat{\phi}(2^{-s}p) \tag{2.7}$$

with $m \in L^2_{\mathrm{per}}(2\pi)$. If $m_1, m_2 \in L^2_{\mathrm{per}}(2\pi)$, then thanks to (2.5) we have

$$\int_{-\infty}^{\infty} dp\, \overline{m_1(2^{-s}p)}\, m_2(2^{-s}p)\, \hat{\phi}(2^{-s}p)^2 = 2^s \int_0^{2\pi} dp\, \overline{m_1(p)}\, m_2(p) . \tag{2.8}$$

In particular, for every fixed scale $s \in \mathbb{Z}$, the functions $\phi_x(\xi)$, $x \in \Omega_1$, $s(x) = s$, such that

$$\hat{\phi}_x(p) = 2^{-\frac{1}{2}s(x)} \exp(i \cdot \xi(x)\, p)\, \hat{\phi}(2^{-s(x)}p) \tag{2.9}$$

form an orthonormal basis of V_s. Let us notice that we also have

$$\phi_x(\xi) = 2^{\frac{1}{2}s(x)}\, \phi(2^{s(x)}(\xi - \xi(x)))\,. \tag{2.10}$$

Lemma 2.1. (Meyer). *For all* $s \in \mathbb{Z}$, *we have*

(i) $\phi_x(\xi)$, $x \in \Omega_1$, $s(x) = s$, *is an orthonormal basis of* V_s;
(ii) $V_s \subset V_{s+1}$;
(iii) $\bigcap_{s\in\mathbb{Z}} V_s = \{0\}$ *and* $\overline{(\bigcup_{s\in\mathbb{Z}} V_s)} = L^2(\mathbb{R})$;
(iv) $f(\xi) \in V_s \Leftrightarrow f(2\xi) \in V_{s+1}$;
(v) *For all* $\xi_0 \in 2^{-s}\mathbb{Z}$, *if* $f(\xi) \in V_s$, *then* $f(\xi - \xi_0) \in V_s$.

Proof. Only (ii) requires a proof. We have to show that for all $m_1 \in L^2_{\text{per}}(2\pi)$ there exists an $m_2 \in L^2_{\text{per}}(2\pi)$ such that

$$m_1(2^{-s}p)\, \hat{\phi}(2^{-s}p) = m_2(2^{-s-1}p)\, \hat{\phi}(2^{-s-1}p)\,. \tag{2.11}$$

It suffices to consider the case $s = -1$. One can set $m_2(p) = 0$ for $|p| \geqq \frac{1}{2}(\pi + \delta_0)$ because in this case $\hat{\phi}(2p) = 0$. On the other hand, if $|p| \leqq \frac{1}{2}(\pi + \delta_0)$, we have $\hat{\phi}(p) = 1$ so that

$$m_2(p) = m_1(2p)\, \hat{\phi}(2p) \tag{2.12}$$

Q.E.D.

A sequence of subspaces V_s, $s \in \mathbb{Z}$, of $L^2(\mathbb{R})$ with the properties above is said to provide a *multiscale decomposition* of $L^2(\mathbb{R})$.

Let W_s be the orthogonal complement of V_s in V_{s+1}, so that we have $V_{s+1} = V_s \oplus W_s$. Thanks to Lemma 2.1, we ahve

$$L^2(\mathbb{R}) = \bigoplus_{s\in\mathbb{Z}} W_s\,. \tag{2.13}$$

We also have

Lemma 2.2. (Meyer). *For all* $s \in \mathbb{Z}$, *the space* $\mathscr{F}\, W_s$ *is spanned by the functions of the form*

$$m(2^{-s}p)\, \hat{\psi}(2^{-s}p) \tag{2.14}$$

with $m \in L^2_{\text{per}}(2\pi)$ *and*

$$\hat{\psi}(p) = e^{-\frac{i}{2}p}\left(\hat{\phi}\left(\frac{1}{2}p\right)^2 - \hat{\phi}(p)^2\right)^{1/2}. \tag{2.15}$$

Moreover the application $\mathscr{R}$: $L^2_{\text{per}}(2\pi) \to \mathscr{F}\, W_s$ *such that*

$$\mathscr{R}(m)(p) = 2^{-\frac{s}{2}} \cdot m(2^{-s}p) \cdot \hat{\psi}(2^{-s}p) \tag{2.26}$$

is an isometry.

Proof. It suffices to consider the case $s = -1$. Let us introduce the following 2π-periodic function:

$$m_0(p) = \sum_{\xi_0 \in \mathbb{Z}} \frac{1}{2} e^{ip\xi_0} \int_{-\infty}^{\infty} d\xi\, \overline{\phi(\xi + \xi_0)}\, \phi\left(\frac{1}{2}\xi\right). \tag{2.17}$$

Since $V_{-1} \subset V_0$, we can write

$$\frac{1}{2}\phi\left(\frac{1}{2}\xi\right) = \sum_{\xi_0 \in \mathbb{Z}} \left[\frac{1}{2}\int_{-\infty}^{\infty} d\xi\, \overline{\phi(\xi + \xi_0)}\, \phi\left(\frac{1}{2}\xi\right)\right] \phi(\xi + \xi_0). \tag{2.18}$$

By taking the Fourier transform of both members, we find

$$\hat{\phi}(2p) = m_0(p)\hat{\phi}(p). \tag{2.19}$$

Moreover, thanks to the basic equality (2.5), we have

$$\begin{aligned} 1 &= \sum_{k \in \mathbb{Z}} \hat{\phi}(2p + 2\pi k)^2 = \sum_{k \in \mathbb{Z}} m_0(p + \pi k)^2\, \hat{\phi}(p + \pi k)^2 \\ &= \sum_{k \in \mathbb{Z}} [m_0(p + 2\pi k)^2\, \hat{\phi}(p + 2\pi k)^2 + m_0(p + \pi(2k+1))^2\, \hat{\phi}(p + \pi(2k+1))^2] \\ &= m_0(p)^2 + m_0(p + \pi)^2. \end{aligned} \tag{2.20}$$

We have

$$V_{-1} = \{m(2p)m_0(p)\,\hat{\phi}(p),\, m \in L^2_{\text{per}}(2\pi)\} \tag{2.21}$$

and W_{-1} is the space of the functions of the form $l(p)\hat{\phi}(p)$ with $l \in L^2_{\text{per}}(2\pi)$ such that

$$0 = \int_{-\infty}^{\infty} dp\, \overline{l(p)}\, m(2p)\, m_0(p)\, \hat{\phi}(p)^2 = \int_0^{2\pi} dp\, \overline{l(p)}\, m(2p)\, m_0(p) \tag{2.22}$$

for all $m(p) \in L^2_{\text{per}}(2\pi)$. Hence, $\overline{l(p)}\, m_0(p)$ has a Fourier series of the form

$$\overline{l(p)}\, m_0(p) = \sum_{\xi_0 \in \mathbb{Z}} c(\xi_0) \exp(i(2\xi_0 + 1)p), \tag{2.23}$$

i.e. we have

$$\overline{l(p)}\, m_0(p) + \overline{l(p + \pi)}\, m_0(p + \pi) = 0. \tag{2.24}$$

Due to (2.17), (2.20) and to the symmetry property $\phi(\xi) = \phi(-\xi)$, $(m_0(p), m_0(p + \pi))$ is a real vector of unit norm of $\mathbb{C}^2$. Hence, the vector $(l(p), l(p + \pi))$ must be proportional to the orthogonal vector $(m_0(p + \pi), -m_0(p))$ for all p. We thus find

$$l(p) = e^{-ip}\, m_0(p + \pi)m(2p) \tag{2.25}$$

for some $m \in L^2_{\text{per}}(2\pi)$. Finally, thanks to (2.18) and (2.19), we have

$$\begin{aligned} l(p)\,\hat{\phi}(p) &= e^{-ip}(1 - m_0(p)^2)^{1/2}\, \hat{\phi}(p)m(p) \\ &= e^{-ip}\left(1 - \frac{\hat{\phi}(2p)^2}{\hat{\phi}(p)^2}\right)^{1/2} \hat{\phi}(p)m(2p) \\ &= e^{-ip}(\hat{\phi}(p)^2 - \hat{\phi}(2p)^2)^{1/2}\, m(2p). \end{aligned} \tag{2.26}$$

Finally, if $m \in L^2_{\text{per}}(2\pi)$, thanks to (2.5) we have

$$\|\mathscr{R}(m)(p)\|_2^2 = 2^{-s} \int_{-\infty}^{\infty} dp\, m(2^{-s}p)^2\, |\hat{\psi}(2^{-s}p)|^2$$
$$= \int_{-\infty}^{\infty} dp\, m(p)^2 \left(\hat{\phi}\left(\frac{p}{2}\right)^2 - \hat{\phi}(p)^2 \right) = \int_0^{2\pi} dp\, m(p)^2 \,. \tag{2.27}$$

Q.E.D.

This completes the construction of LPM wavelets in the one dimensional case. In dimension $d \geqq 1$ we can also obtain LPM wavelets by using the method of tensor products described below in Sect. 4.

3. Decomposition of the Space $L^2_{\text{per}}(2\pi)$

Let us introduce a family of real valued functions $\theta_0(\delta; p) \in L^2_{\text{per}}(2\pi)$ parametrized by $\delta \in \left(0, \frac{\pi}{2}\right)$ and such that

$$\theta_0(\delta; p) = \theta_0(\delta; -p) \quad \forall p \in \mathbb{R} \,, \tag{3.1}$$

$$\theta_0(\delta; p) = 1 \quad \text{if } \frac{\pi}{2} + \delta \leqq p \leqq \frac{3\pi}{2} - \delta \,, \tag{3.2}$$

$$0 < \theta_0(\delta; p) < 1 \quad \text{if } \frac{3\pi}{2} - \delta < p < \frac{3\pi}{2} + \delta \,, \tag{3.3}$$

$$\theta_0(\delta; p)^2 + \theta_0(\delta; p + \pi)^2 = 1 \quad \forall p \in \mathbb{R} \,. \tag{3.4}$$

Let $\theta_1(\delta; p)$ be the function such that

$$\theta_1(\delta; p) = e^{ip}\, \theta_0(\delta; p + \pi) \tag{3.5}$$

and let

$$\mathscr{T}_\delta \colon L^2_{\text{per}}(\pi) \oplus L^2_{\text{per}}(\pi) \to L^2_{\text{per}}(2\pi) \tag{3.6}$$

be the operator defined in such a way that if $m_0, m_1 \in L^2_{\text{per}}(\pi)$, then we have

$$\mathscr{T}_\delta(m_0, m_1)(p) = m_0(p)\theta_0(\delta; p) + m_1(p)\theta_1(\delta; p) \,. \tag{3.7}$$

Lemma 3.1. *The operator $\mathscr{T}_\delta$ is an isometry,* i.e.

$$\|\mathscr{T}_\delta(m_0, m_1)\|^2_{L^2_{\text{per}}(2\pi)} = \|m_0\|^2_{L^2_{\text{per}}(\pi)} + \|m_1\|^2_{L^2_{\text{per}}(\pi)} \tag{3.8}$$

for all $m_0, m_1 \in L^2_{\text{per}}(\pi)$ and

$$\mathscr{R}an\, \mathscr{T}_\delta = L^2_{\text{per}}(2\pi) \,. \tag{3.9}$$

Proof. Let us start by proving that for all $m_0, m_1 \in L^2_{\text{per}}(\pi)$, we have

$$\int_0^{2\pi} dp\, (\overline{m_0(p)}\, \theta_0(\delta; p) + \overline{m_1(p)}\, \overline{\theta_1(\delta; p)})\, (m_0(p)\theta_0(\delta; p) + m_1(p)\theta_1(\delta; p))$$
$$= \int_0^{\pi} dp (|m_0(p)|^2 + |m_1(p)|^2) \,. \tag{3.10}$$

By using (3.4), we find

$$\int_0^{2\pi} dp|m_0(p)|^2\,\theta_0(\delta;p)^2 = \int_0^{\pi} dp|\,m_0(p)|^2\,, \tag{3.11}$$

$$\int_0^{2\pi} dp|m_1(p)|^2\,|\theta_1(\delta;p)|^2 = \int_0^{\pi} dp|m_1(p)|^2\,. \tag{3.12}$$

Moreover, we have

$$\begin{aligned}&\int_0^{2\pi} dp\,\overline{m_1(p)}\,m_0(p)\,\overline{\theta_1(\delta;p)}\,\theta_0(\delta;p)\\ &\quad = \int_0^{2\pi} dp\,\overline{m_1(p)}\,m_0(p)\,e^{-ip}\theta_0(\delta;p)\theta_0(\delta;p+\pi) = 0\end{aligned} \tag{3.13}$$

because

$$e^{-i(p+\pi)}\theta_0(\delta;p+\pi)\theta_0(\delta;p+2\pi) = -\,e^{-ip}\,\theta_0(\delta;p)\theta_0(\delta;p+\pi)\,. \tag{3.14}$$

Similarly, one can show that

$$\int_0^{2\pi} dp\,\overline{m_0(p)}\,m_1(p)\theta_0(\delta;p)\theta_1(\delta;p) = 0\,. \tag{3.15}$$

It remains to prove that $\mathscr{R}an\,\mathscr{T}_\delta = L^2_{\text{per}}(2\pi)$. The functions of the form $m(p)\theta_0(\delta;p)$ (resp. $m(p)\theta_1(\delta;p)$) with $m\in L^2_{\text{per}}(\pi)$, span the subspace of $L^2_{\text{per}}(2\pi)$ of the functions with support in $\left[\frac{\pi}{2}+\delta,\frac{3\pi}{2}-\delta\right]+2\pi\mathbb{Z}$ (resp. $\left[-\frac{\pi}{2}+\delta,\,+\frac{\pi}{2}-\delta\right]+2\pi\mathbb{Z}$). Moreover, if $f\in L^2_{\text{per}}(2\pi)$ is a function with support in $\left(\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta\right)+2\pi\mathbb{Z}$, then there are two functions $m_0, m_1\in L^2_{\text{per}}(\pi)$ with support in $\left(\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta\right)+\pi\mathbb{Z}$ such that

$$m_0(p)\theta_0(\delta;p) + m_1(p)\theta_1(\delta;p) = \varepsilon(p) \tag{3.16}$$

for all $p\in\mathbb{R}$. To find these functions, let us observe that for all $p\in\left(\frac{\pi}{2}-\delta,\frac{\pi}{2}+\delta\right)+2\pi\mathbb{Z}$, we must have

$$\begin{aligned}0 &= m_0(p+\pi)\theta_0(\delta;p+\pi) + m_1(p+\pi)\theta_1(\delta;p+\pi)\\ &= m_0(p)e^{-ip}\theta_1(\delta;p) - m_1(p)e^{ip}\theta_0(\delta;p)\,,\end{aligned} \tag{3.17}$$

and hence

$$m_0(p) = e^{2ip}\,m_1(p)\theta_0(\delta;p)\theta_1(\delta;p)^{-1}\,. \tag{3.18}$$

For such values of p, we also have

$$\begin{aligned}f(p) &= m_1(p)[e^{2ip}\,\theta_0(\delta;p)^2\,\theta_1(\delta;p)^{-1} + \theta_1(\delta;p)]\\ &= m_1(p)\theta_1(\delta;p)^{-1}\,e^{2ip}[\theta_0(\delta;p)^2 + \theta_0(\delta;p+\pi)^2]\\ &= m_1(p)\theta_1(\delta;p)^{-1}\,e^{2ip}\,.\end{aligned} \tag{3.19}$$

Hence, we find the following explicit expressions for m_0 and m_1:

$$m_1(p) = f(p)\theta_1(\delta; p)e^{-2ip} + f(p+\pi)\,\theta_1(\delta; p+\pi)e^{-2ip}\,, \tag{3.20}$$

$$m_0(p) = f(p)\theta_0(\delta; p) + f(p+\pi)\theta_0(\delta; p+\pi)\,. \tag{3.21}$$

Q.E.D. . . . .

4. 2^n-voice LPM Wavelets

In this section, by combining Meyer's construction of LPM wavelets and the decomposition of the space $L^2_{\mathrm{per}}(2\pi)$ given above, we construct multivoice LPM wavelets which have all the properties required by the theorem in Sect. 1 and by its corollary.

The decomposition of the space $L^2_{\mathrm{per}}(2\pi)$ provided by the operators $\mathscr{T}_\delta$ in (3.6) can be iterated. Namely, for all integers $n \geqq 1$ and $\delta \in (0, 2^{-n}\pi)$ one can define an isometry

$$\mathscr{T}_\delta^{(n)}: \bigoplus_{j=1 \quad 2^n} L^2_{\mathrm{per}}(2^{-n}\pi) \to L^2_{\mathrm{per}}(2\pi) \tag{4.1}$$

such that if $m_\sigma \in L^2_{\mathrm{per}}(2^{1-n}\pi)$ is a family of functions labelled by elements σ in the set

$$s_n = \{\sigma = (\sigma_1 \ldots \sigma_n) | \sigma_m \in \{0, 1\}\ \forall m = 1 \ldots n\} \tag{4.2}$$

we have

$$\mathscr{T}_\delta^n (m_\sigma)(p) = \sum_\sigma m_\sigma(p)\theta_\sigma(\delta; p)\,. \tag{4.3}$$

Here, we set

$$\theta_\sigma(\delta; p) = \theta_{\sigma_1}(\delta; p)\cdot \prod_{m=2, \quad n} \theta_{1+\sigma_m+\sigma_{m-1}}(\delta_m; 2^{m-1}p) \tag{4.4}$$

where

$$\delta_m = 2^{m-1}\delta\,, \tag{4.5}$$

and the sum $(1 + \sigma_m + \sigma_{m-1})$ is computed mod 2 and has values in $\{0, 1\}$. With this definition of θ_σ, we have

Lemma 4.1. *For all fixed $n \geqq 1$ and all $\delta \in (0, 2^{1-n}\pi)$, we have*

$$\theta_\sigma(\delta; -p) = \overline{\theta_\sigma(\delta; p)} \tag{4.6}$$

and

$$\operatorname{supp}\theta_\sigma(\delta; \cdot) \cap [\pi - \delta, 2\pi + \delta] = \pi + \sum_{m=1}^{n} \sigma_m 2^{-m}\pi + [-\delta, 2^{-n}\pi + \delta]\,. \tag{4.7}$$

Proof. Equation (4.6) is obvious. If $n = 1$, then (4.7) follows from (3.1) and (3.5). Otherwise, if $n \geqq 2$, (4.7) can be proved by induction in n. In fact, let us suppose (4.7) holds for all $\sigma \in s_{n-1}$ and let us fix a $\delta \in (0, 2^{-n}\pi)$ and a $\sigma \in s_n$. We have

$$\theta_\sigma(\delta; p) = \theta_{\sigma'}(\delta; p)\,\theta_{1+\sigma_n+\sigma_{n-1}}(\delta_n; 2^{n-1}p)\,, \tag{4.8}$$

where $\sigma' = (\sigma_1 \ldots \sigma_{n-1}) \in s_{n-1}$. We also have

$$\begin{aligned}
&\operatorname{supp} \theta_{1+\sigma_n+\sigma_{n-1}}(\delta_n; 2^{n-1} p) \\
&\quad = [2^{-n}\pi - \delta,\ 3 \cdot 2^{-n}\pi + \delta] + (1 + \sigma_n + \sigma_{n-1}) \cdot 2^{1-n}\pi + 2^{2-n}\pi\mathbb{Z} \\
&\quad = [2^{-n}(2\sigma_n + 3)\pi - \delta, 2^{-n}(2\sigma_n + 5)\pi + \delta] + \sigma_{n-1} \cdot 2^{1-n}\pi + 2^{2-n}\pi\mathbb{Z} \\
&\quad = [2^{-n}(2\sigma_n - 1)\pi - \delta, 2^{-n}(2\sigma_n + 1)\pi + \delta] + \pi \\
&\qquad + \sum_{m=1}^{n-1} \sigma_m \cdot 2^{-m}\pi + 2^{2-n}\pi\mathbb{Z} \,. \qquad (4.9)
\end{aligned}$$

Hence

$$\begin{aligned}
&\operatorname{supp} \theta_\sigma(\delta; \cdot) \cap [\pi - \delta, 2\pi + \delta] \\
&\quad = \left(\pi + \sum_{m=1}^{n-1} \sigma_m 2^{-m}\pi + [-\delta, 2^{-n+1}\pi + \delta]\right) \\
&\qquad \cap \left(\pi + \sum_{m=1}^{n-1} \sigma_m 2^{-m}\pi + [2^{-n}(2\sigma_n - 1)\pi - \delta,\right. \\
&\qquad\qquad \left. 2^{-n}(2\sigma_n + 1)\pi + \delta] + 2^{2-n}\pi\mathbb{Z}\right) \\
&\quad = \pi + \sum_{m=1}^{n-1} \sigma_m 2^{-m}\pi + [2^{-n}\sigma_n\pi - \delta, 2^{-n}(\sigma_n + 1)\pi + \delta] \\
&\quad = \pi + \sum_{m=1}^{n} \sigma_m 2^{-m}\pi + [-\delta, 2^{-n}\pi + \delta] \,. \qquad (4.10)
\end{aligned}$$

Q.E.D.

One can now construct 2^n-voices LPM wavelets. The fathers ϕ_σ and the mothers ψ_σ, $\sigma \in s_n$, are defined as the functions such that

$$\hat{\phi}_\sigma(p) = \theta_\sigma(\delta; p)\, \hat{\phi}(p) \,, \qquad (4.11)$$

$$\hat{\psi}_\sigma(p) = \theta_\sigma(\delta; p)\, \hat{\psi}(p) \,, \qquad (4.12)$$

where $\delta \in (0, 2^{-n}\pi)$, $\hat{\phi}(p)$ is a function satisfying (2.1)–(2.4), and, according to Lemma 2.1, $\hat{\psi}(p)$ is given by

$$\hat{\psi}(p) = e^{-\frac{i}{2}p}\left(\hat{\phi}\left(\frac{1}{2}p\right)^2 - \hat{\phi}(p)^2\right)^{1/2} . \qquad (4.13)$$

If we set

$$\phi_{\sigma s \xi_0}(\xi) = 2^{\frac{1}{2}s}\, \phi_\sigma\,(2^s(\xi - \xi_0)) \,, \qquad (4.14)$$

$$\psi_{\sigma s \xi_0}(\xi) = 2^{\frac{1}{2}s}\, \psi_\sigma\,(2^s(\xi - \xi_0)) \,, \qquad (4.15)$$

then thanks to the preceding lemma, we have

Lemma 4.2. *For all $n \geqq 1$, $\delta \in (0, 2^{-n}\pi)$ and $\sigma \in s_n$, there exist functions $\psi_\sigma(\xi)$, $\phi_\sigma(\xi)$ such that*

(i)

$$\hat{\psi}_\sigma(p) = \overline{\hat{\psi}_\sigma(-p)}, \quad \hat{\phi}_\sigma(p) = \overline{\hat{\phi}_\sigma(-p)}\,; \tag{4.16}$$

(ii)

$$\operatorname{supp} \hat{\psi}_\sigma \cap \mathbb{R}_+ = \pi + \sum_{m=1}^{n} \sigma_m 2^{-m}\pi + [-\delta, 2^{-n}\pi + \delta] \tag{4.17}$$

and

$$\operatorname{supp} \hat{\phi}_\sigma \cap \mathbb{R}_+ = \left(\sum_{m=1}^{n} (1 - \sigma_m) 2^{-m}\pi + [-\delta, 2^{-n}\pi + \delta] \right) \cap \mathbb{R}_+\,; \tag{4.18}$$

(iii)

$$\langle \phi_{\sigma s \xi_0} | \phi_{\sigma' s \xi_0'} \rangle = \delta_{\sigma\sigma'}\, \delta_{\xi_0 \xi_0'} \tag{4.19}$$

and

$$\langle \psi_{\sigma s \xi_0} | \psi_{\sigma' s' \xi_0'} \rangle = \delta_{\sigma\sigma'}\, \delta_{ss'}\, \delta_{\xi_0 \xi_0'} \tag{4.20}$$

for all $\sigma, \sigma' \in s_n$, $s, s' \in \mathbb{Z}$, $\xi_0 \in 2^{n-s}\mathbb{Z}$, $\xi'_0 \in 2^{n-s'}\mathbb{Z}$;

(iv) *The spaces*

$$V_s = \overline{\operatorname{span}\{\phi_{\sigma s \xi_0}, \sigma \in s_n, \xi_0 \in 2^{n-s}\mathbb{Z}\}}\,, \tag{4.21}$$

$s \in \mathbb{Z}$, give a multiscale decomposition of $L^2(\mathbb{R})$;
(v) *If $s \in \mathbb{Z}$ and*

$$W_s = \overline{\operatorname{span}\{\psi_{\sigma s \xi_0}, \sigma \in s_n, \xi_0 \in 2^{n-s}\mathbb{Z}\}}\,, \tag{4.22}$$

then

$$V_{s+1} = W_s \oplus V_s\,. \tag{4.23}$$

(*Remark. These spaces are the same ones defined in Sect.* 2.) *In particular, we have*

$$L^2(\mathbb{R}) = \bigoplus_{s \in \mathbb{Z}} W_s \tag{4.24}$$

and $\psi_{\sigma s \xi_0}\{\sigma \in s_n, s \in \mathbb{Z}, \xi_0 \in 2^{n-s}\mathbb{Z}\}$, is an orthonormal basis of $L^2(\mathbb{R})$;

(vi) *For all integers $m \in [1, N]$ and all $\delta \in (0, 2^{-n}\pi)$ we have*

$$\sup_p \left| \frac{d^m}{dp^m} \hat{\phi}_\sigma(p) \right| \leqq c\delta^{-m}\,, \tag{4.25}$$

$$\sup_p \left| \frac{d^m}{dp^m} \hat{\psi}_\sigma(p) \right| \leqq c\delta^{-m}\,, \tag{4.26}$$

where c is a constant independent of both δ and n.

Proof. Equation (4.19) follows from (2.8) and (3.13), while (4.20) can be derived from Lemma 2.1 (ii) and from (2.2), (2.27) and (3.13). Only the last statement requires a proof. Let us consider the following function:

$$\mu(p) = \begin{cases} 0 & \text{if } p \leqq 0 \\ \mu_0 \int_0^p dp'\, p'^{N-1}(p'-\delta)^{N-1} & \text{if } 0 \leqq p \leqq \frac{1}{2}\delta\,, \\ 1/2 & \text{if } p > \frac{1}{2}\delta \end{cases} \tag{4.27}$$

where μ_0 is the following normalization constant:

$$\mu_0 \equiv \frac{1}{2}\left(\int_0^{\frac{\delta}{2}} dp'\, p'^{N-1}(p'-\delta)^{N-1}\right)^{-1} \leqq \text{const.}\, \delta^{-2N+1}\,. \tag{4.28}$$

If m is an integer in $[1, N]$ and $p \in [0, \frac{1}{2}\delta]$, we have

$$\frac{d^m}{dp^m}\mu(p) = \mu_0 \left|\frac{d^{m-1}}{dp^{m-1}} p^{N-1}(p-\delta)^{N-1}\right| \leqq \text{const.}\, \delta^{-m} \tag{4.29}$$

and

$$\frac{d^m}{dp^m}\sqrt{1-\mu(p)^2} \leqq \text{const.}\, \delta^{-m}\,. \tag{4.30}$$

Let us define $\hat{\phi}(p) \in L^2(\mathbb{R})$ to be the function satisfying (2.1–4) with $\delta_0 = \delta$ and such that

$$\hat{\phi}(p) = \mu(p+\pi+\delta) \quad \text{if } -\pi-\delta \leqq p \leqq -\pi\,,$$

$$\hat{\phi}(p) = \sqrt{1-\mu(p+\pi)^2} \quad \text{if } -\pi \leqq p \leqq -\pi+\delta\,. \tag{4.31}$$

We clearly have

$$\left|\frac{d^m}{dp^m}\hat{\phi}(p)\right| \leqq \text{const.}\, \delta^{-m} \tag{4.32}$$

for all $1 \leqq m \leqq N$. Similarly, if $\theta(\delta; p)$ is a function satisfying (3.1)–(3.4) and such that

$$\theta_0(\delta; p) = \mu\left(p - \frac{\pi}{2} + \delta\right) \quad \text{if } \frac{\pi}{2} - \delta \leqq p \leqq \frac{\pi}{2}\,,$$

$$\theta_0(\delta; p) = \sqrt{1 - \mu\left(p - \frac{\pi}{2}\right)^2} \quad \text{if } \frac{\pi}{2} \leqq p \leqq \frac{\pi}{2} + \delta\,, \tag{4.33}$$

we have

$$\frac{d^m}{dp^m}\theta_0(\delta; p) \leqq \text{const.}\, \delta^{-m} \tag{4.34}$$

for all $1 \leqq m \leqq N$. Thanks to (4.4), we also have

$$\frac{d^m}{dp^m}\theta_\sigma(\delta; p) \leqq \text{const.}\, \delta^{-m} \tag{4.35}$$

for all $n \geqq 1$ and all $\sigma \in s_n$. Hence, (4.25) and (4.26) are also valid. Q.E.D.

Once one knows a family of fathers $\phi_\sigma(\xi)$ and mothers $\psi_\sigma(\xi)$, $\sigma \in s_n$, of one dimensional, 2^n-voices LPM wavelets, one can construct similar wavelet bases in any dimension $d \geqq 2$ by the method of tensor products. In fact, let us consider the subspaces $\mathscr{V}_s$ and $\mathscr{W}_s$, $s \in \mathbb{Z}$, of $L^2(\mathbb{R}^d)$ such that

$$\mathscr{V}_s = \bigotimes_{j=1, \ldots d} V_s, \quad \mathscr{V}_{s+1} = \mathscr{V}_s \oplus \mathscr{W}_s . \tag{4.36}$$

We have

$$\mathscr{W}_s = \bigoplus_{\sigma \in s_n, \tau \in t_d} \bigoplus_{j=1, \ldots d} \mathscr{F}^{\tau_j}_{\sigma s} , \tag{4.37}$$

where

$$t_d = \{\tau = (\tau_1 \ldots \tau_d) \colon \tau_j \in \{0, 1\}, j = 1 \ldots d, \text{ and } (\tau_1 \ldots \tau_d) \neq (0, \ldots 0)\} \tag{4.38}$$

and

$$F^0_{\sigma s} = \overline{\operatorname{span}\{\phi_{\sigma s \xi_0}, \xi_0 \in 2^{n-s}\mathbb{Z}\}} ,$$
$$F^1_{\sigma s} = \overline{\operatorname{span}\{\psi_{\sigma s \xi_0}, \xi_0 \in 2^{n-s}\mathbb{Z}\}} . \tag{4.39}$$

If $\alpha = (\sigma, \tau) \in s_n \times t_d$, let us introduce the functions

$$\psi_\alpha(\xi) = \prod_{j=1 \ldots d} f^{\tau_j}_\sigma(\xi_j) , \tag{4.40}$$

where $\xi = (\xi \ldots \xi_d) \in \mathbb{R}^d$ and

$$f^0_\sigma(\xi_j) = \phi_\sigma(\xi_j), \quad f^1_\sigma(\xi_j) = \psi_\sigma(\xi_j) . \tag{4.41}$$

Let Ω_n be the set in (1.5), i.e.

$$\Omega_n = \{(\alpha(x), s(x), \xi(x)), \alpha(x) \in s_n \times t_d, s(x) \in \mathbb{Z}, \xi(x) \in 2^{n-s}\mathbb{Z}^d\} . \tag{4.42}$$

If $x \in \Omega_n$, let ψ_x be the function such that

$$\psi_x(\xi) = 2^{\frac{d}{2}s(x)} \psi_{\alpha(x)}(2^{s(x)}(\xi - \xi(x))) . \tag{4.43}$$

Then, for all $x_1, x_2 \in \Omega_n$ we have

$$\langle \psi_{x_1} | \psi_{x_2} \rangle = \delta_{x_1 x_2} \tag{4.44}$$

and $\{\psi_x\}_{x \in \Omega_n}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

It is easy to see that if the functions ϕ_σ and ψ_σ are chosen so that they have all the properties in Lemma 4.2 for $\delta = 2^{-n-1}\pi$, then the basis $\{\psi_x\}_{x \in \Omega_n}$ has all the properties required in the theorem of Sect. 1. In fact, supp $\hat{\psi}_x$ is invariant under all reflections with respect to one of the coordinate planes of $\mathbb{R}^d$. Moreover, we have

$$\operatorname{supp} \hat{\psi}_x \cap \mathbb{R}^d_+ = Q(x) \cap \mathbb{R}^d_+ , \tag{4.45}$$

where $Q(x)$ is a cube in $\mathbb{R}^d$ whose side has length $2^{1-n-s(x)}\pi$. The center $p(x) = (p_1(x), \ldots p_d(x)) \in \mathbb{R}^d$ of $Q(x)$ is such that, for all $j = 1, \ldots d$, we have

$$p_j(x) = 2^s\pi \left[2^{-n-1} + \sum_{m=1}^{n} (1 - \sigma_m) 2^{-m} \right] \tag{4.46}$$

in case $\tau_j(x) = 0$ and

$$p_j(x) = 2^s \pi \left[1 + 2^{-n-1} + \sum_{m=1}^{n} \sigma_m 2^{-m} \right] \tag{4.47}$$

if $\tau_j(x) = 1$. In particular, for all $x \in \Omega_n$ there are no more than $(2^d + 2d - 2)$ wavelets $y \in \Omega_n$ with $\xi(x) = \xi(y)$ such that

$$Q(x) \cap Q(y) \neq \emptyset , \tag{4.48}$$

up to irrelevant boundaries of zero Lebesgue measure. Finally, let us pass to the proof of the corollary in the introduction. Let $\varepsilon \in \mathscr{C}^{d+1}(\mathbb{R}^d)$ be a function which is symmetric under all reflections with respect to the coordinate planes and such that for all multiindices $m = (m_1 \ldots m_d)$ with $|m| \leqq d + 1$ we have

$$\left| p^m \frac{\partial^{|m|}}{\partial p^m} \varepsilon(p) \right| \leqq c\varepsilon(p) \tag{4.49}$$

for some constant $c > 0$. We have to prove that if $n \geqq 0$ and $\{\psi_x\}_{x \in \Omega_n}$ is the wavelet basis constructed above and satisfying all the conditions in the theorem in Sect. 1 with $N = d + 1$, then we have

$$\sum_{x_2 \neq x_1} |\varepsilon(x_1|x_2)| \leqq c 2^{-(d+1)n} \varepsilon(x_1|x_1) \tag{4.50}$$

for all $x_1 \in \Omega_n$.

Let $x_i \in \Omega_n$ and let us introduce the shorthand notations

$$\alpha_i = \alpha(x_i), \quad s_i = s(x_i), \quad \xi_{0i} = \xi(x_i), \quad p_i = p(x_i), \quad Q_i = Q(x_i) \tag{4.51}$$

and

$$\varepsilon_i = \max_{p \in Q_i} |\varepsilon(p)| . \tag{4.52}$$

We have

$$\begin{aligned} &\varepsilon(x_1|x_2)\,(2\pi)^{-d} \int dp\, \varepsilon(p)\, \overline{\hat{\psi}_{x_1}(p)}\, \hat{\psi}_{x_2}(p) \\ &= (2\pi)^{-d}\, 2^{-\frac{d}{2}(s_1+s_2)} \int dp\, \varepsilon(p)\, \hat{\psi}_{a_1}(2^{-s_1} p_1)\, \hat{\psi}_{a_2}(2^{-s_2} p) \\ &\qquad \cdot \exp(i(\xi_{02} - \xi_{01}) \cdot p) \\ &= (2\pi)^{-d}\, \varepsilon(p_1)\, \delta_{x_1 x_2} \\ &\quad + (2\pi)^{-d}\, 2^{-\frac{d}{2}(s_1+s_2)} \int dp\, [\varepsilon(p) - \varepsilon(p_1)]\, \overline{\hat{\psi}_{a_1}(2^{-s_1} p)}\, \hat{\psi}_{a_2}(2^{-s_2} p) \\ &\qquad \cdot \exp(i(\xi_{02} - \xi_{01}) \cdot p) . \end{aligned} \tag{4.53}$$

Let

$$\tilde{\varepsilon}(x_1|x_2) \equiv \varepsilon(x_1|x_2) - (2\pi)^{-d}\, \varepsilon(p_1)\, \delta_{x_1 x_2} . \tag{4.54}$$

If $Q_1 \cap Q_2 = \emptyset$, then we have $\varepsilon(x_1|x_2) = 0$. Moreover, for all multiindices $m = (m_1 \ldots m_d)$ such that $|m| = d + 1$, we have

$$\prod_{j=1}^{d} |\xi_{02,j} - \xi_{01,j}|^{m_j} |\tilde{\varepsilon}(x_1|x_2)|$$
$$\leqq c2^{-ds_1} \int_{Q_1} dp \left| \frac{\partial^m}{\partial p^m} [\varepsilon(p) - \varepsilon(p_1)] \overline{\hat{\psi}_{a_1}(2^{2-s_1}p)} \hat{\psi}_{a_2}(2^{-s_2}p) \right|$$
$$\leqq c2^{-ds_1} \int_{Q_1} dp \left\{ \sup_{p' \in Q_1} |\nabla\varepsilon(p')| 2^{s_1 - n} \cdot 2^{|m|(n-s_1)} + \sum_{\substack{m_1 + m_2 = m \\ m_1 \neq 0}} \sup_{p' \in Q_1} \left| \frac{\partial^{m_1}}{\partial p^{m_1}} \varepsilon(p)' \right| 2^{|m_2|(n-s_1)} \right\}$$
$$\leqq c2^{-ds_1} \cdot |\varepsilon_1| \cdot 2^{-n+(d+1)(n-s_1)} \cdot 2^{-d(n-s_1)}$$
$$\leqq c2^{-(d+1)s_1} |\varepsilon_1| , \tag{4.55}$$

where the constants c do not depend on n. Hence, we have

$$|\tilde{\varepsilon}(x_1|x_2)| \leqq c2^{-(d+1)s_1} (1 + |\xi_{02} - \xi_{01}|)^{-(d+1)} . \tag{4.56}$$

Since there are at most $(2^d + 2d - 2)$ possible values for $\alpha(x_2) \in s_n \times t_d$ such that $\tilde{\varepsilon}(x_1|x_2) \neq 0$, we have

$$\sum_{x_2 \cdot \xi_2 \neq \xi_1} |\tilde{\varepsilon}(x_1|x_2)| \leqq c2^{-(d+1)(n-1)} |\varepsilon_1| . \tag{4.57}$$

Finally, if $\xi_1 = \xi_2$, we have

$$|\tilde{\varepsilon}(x_1|x_2)| \leqq c2^{-ds_1} \int_{Q_1} dp |\varepsilon(p) - \varepsilon(p_1)| \overline{\hat{\psi}_{a_1}(2^{-s_1}p)} \hat{\psi}_{a_2}(2^{-s_2}p)$$
$$\leqq c2^{-ds_1} \left(\sup_{p \in Q_1} |\nabla\varepsilon(p)| \right) 2^{(d+1)(s_1 - n)}$$
$$\leqq c|\varepsilon_1| 2^{-(d+1)n} . \tag{4.58}$$

The proof of the corollary in Sect. 1 is thus completed. Q.E.D.

Acknowledgement. This paper was written in 1991 during a visit to the Physics Department of Princeton University. I would like to thank Elliott Lieb for the hospitality and the U S. National Science Foundation for support under grant # PHY-8515288-A04.

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Communicated by B. Simon