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On the Spectra of Schrödinger Operators

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Abstract: We give two formulas for the lowest point $\mathscr{T}$ in the spectrum of the Schrödinger operator $L = -(d/dt)p(d/dt) + q$, where the coefficients p and q are real-valued, bounded, uniformly continuous functions on the real line. We determine whether or not $\mathscr{T}$ is an eigenvalue for L in terms of a set of probability measures on the maximal ideal space of the C^*-algebra generated by the translations of p and q.

Introduction

In this paper, we will study the Schrödinger operator

$$L = -\left(\frac{d}{dt}\right)p\left(\frac{d}{dt}\right) + q$$

on $\mathscr{D}_2 \subset L^2(\mathbf{R})$. As usual, the domain $\mathscr{D}_2$ of this operator is the collection of functions $f \in L^2(\mathbf{R})$ which have the property that f and f' are absolutely continuous functions on every finite interval and $f', f'' \in L^2(\mathbf{R})$. We assume that p and q are real-valued, bounded, uniformly continuous functions on $\mathbf{R}$. In addition, we assume that p' is also a bounded, uniformly continuous function on $\mathbf{R}$ and that there is a $c > 0$ such that $p(t) \geq c$ for every $t \in \mathbf{R}$. It is well known that, under these assumptions, L is a self-adjoint operator on $\mathscr{D}_2$. The main goal of this paper is to study the lowest point $\mathscr{T} = \inf\{\lambda : \lambda \in \sigma(L)\}$ of the spectrum of L. There have been estimates of the value $\mathscr{T}$ in the literature when the coefficients p and q of the operator have recurrence properties [4]. We will give two formulas for the value $\mathscr{T}$. These formulas are related to a C^*-algebra associated with the functions p and q.

Before we state our results, some definitions are necessary. For a function f defined on $\mathbf{R}$, by a translation of f we mean a function f_s given by the formula $f_s(t) = f(t+s)$. We denote by $\mathscr{A}$ the C^*-algebra generated by all the translations of p, p', q and all

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the constant functions on $\mathbf{R}$. Let $\mathscr{A}^1 = \{f \in \mathscr{A} : f' \in \mathscr{A}\}$. For each state ϱ on $\mathscr{A}$, let H_ϱ be the Hilbert space completion of $\mathscr{A}$ with respect to the inner product $\langle f, g\rangle_\varrho = \varrho(f\bar{g})$. Let $\mathscr{M}$ be the collection of states ϱ on the C^*-algebra $\mathscr{A}$ such that $|\varrho(f')| \le C_\varrho[\varrho(|f|^2)]^{1/2}$ for every $f \in \mathscr{A}^1$, where C_ϱ is a constant depending only on ϱ. Equivalently, $\mathscr{M}$ is the collection of states ϱ for which there is a unique $h_\varrho \in H_\varrho$ such that $\varrho(f') = \langle f, h_\varrho\rangle_\varrho$ for every $f \in \mathscr{A}^1$ Let

$$\mathscr{F}_0 = \{pw^2 + (pw)' : \ w \text{ is any real-valued function in } \mathscr{A}^1\},$$

and let $\mathscr{F}$ be the closure of the convex hull of $\mathscr{F}_0$ in the norm topology. Let $d_0(\varphi) = \inf\{\|\varphi - u\|_\infty : u \in \mathscr{F}_0\}$ and $d(\varphi) = \inf\{\|\varphi - u\|_\infty : u \in \mathscr{F}\}$.

Theorem 1. (a) $d_0(f) = d(f)$ *for every* $f \in \mathscr{A}$.
(b) $\mathscr{I} = \|q\|_\infty - d(q - \|q\|_\infty)$
(c) *If* $\mathscr{I} \le 0$, *then* $I = -d(q)$
(d) $\mathscr{I} = \min\{\varrho(q) + \frac{1}{4}\langle ph_\varrho, h_\varrho\rangle_\varrho : \varrho \in \mathscr{M}\}$.
(e) *If* $d(q) > 0$, *then* $-\mathscr{I} = d(q) = \max\{-\varrho(q) - \frac{1}{4}\langle ph_\varrho, h_\varrho\rangle_\varrho : \varrho \in \mathscr{M}\}$

We particularly emphasize the fact that in (d) and (e) above, the extrema are attainable. We will explain in Sect. 3 that the fact that they are attainable makes $\mathscr{I}$ a "quasi-eigenvalue" for L. In other words, we assert that when the coefficients p and q satisfy our assumptions, the lowest point in the spectrum of $L = -(d/dt)p(d/dt) + q$ is always a quasi-eigenvalue. In fact quasi-eigenvalue is the most that one can say about $\mathscr{I}$ in general. Although $\mathscr{I}$ can be a genuine eigenvalue, in the case p and q are almost periodic functions, it is known that $\mathscr{I}$ is not an eigenvalue in probability 1.

It is obvious that for each $s \in \mathbf{R}$, the map $\varphi_s : f \mapsto f_s$ is an automorphism of the C^*-algebra $\mathscr{A}$. The fact that the functions in $\mathscr{A}$ are uniformly continuous on $\mathbf{R}$ implies that the group of automorphisms $\{\varphi_s : s \in \mathbf{R}\}$ is strongly continuous in the sense that for every $f \in \mathscr{A}$, $s \mapsto \varphi_s(f) = f_s$ is a continuous map from $\mathbf{R}$ into $\mathscr{A}$. Therefore the automorphism group $\{\varphi_s : s \in \mathbf{R}\}$ induces a strongly continuous group of homeomorphisms $\{\alpha_s : s \in \mathbf{R}\}$ of the maximal ideal space Ω of $\mathscr{A}$. In other words, the map $(\omega, s) \mapsto \alpha_s(\omega)$ from $\Omega \times \mathbf{R}$ to Ω is continuous. If we identify $\mathscr{A}$ with $C(\Omega)$, then obviously $\mathscr{A}^1$ can be regarded as the subset $C^1(\Omega)$ of $f \in C(\Omega)$ such that the limit $f' = \lim_{\varepsilon \to 0}(f \circ \alpha_\varepsilon - f)/\varepsilon$ exists in the norm topology of $C(\Omega)$. Similarly, $\mathscr{M}$ can be identified with the collection of probability measures μ on Ω such that

$$\left|\int_\Omega f'\, d\mu\right| \le C_\mu \left[\int_\Omega |f|^2 d\mu\right]^{1/2}$$

for every $f \in C^1(\Omega)$, where $C_\mu > 0$ is a constant which depends only on μ. Given a $\mu \in \mathscr{M}$, $D_\mu : f \mapsto f'$ is a linear operator from the dense subspace $C^1(\Omega)$ into $L^2(\Omega, \mu)$. It seems that the subscript of the symbol D_μ is unnecessary, for the operator itself is actually independent of the measure μ. The reason we write D_μ is that its adjoint D_μ^* does in general depend on the measure μ. If we let $\hat{p}$ and $\hat{q}$ denote the Gelfand transforms of p and q respectively, then it follows from Theorem 1 that the set

$$\mathscr{M}(p, q) = \left\{\mu \in \mathscr{M} : \mathscr{I} = \langle \hat{q}, 1\rangle_\mu + \frac{1}{4}\langle \hat{p}D_\mu^* 1, D_\mu^* 1\rangle_\mu\right\}$$

is not empty. Here, we denote the inner product in $L^2(\Omega, \mu)$ by $\langle\, .\, ,\, .\, \rangle_\mu$. Since each point $s \in \mathbf{R}$ corresponds to a maximal ideal $\hat{s} \in \Omega$, there is a continuous map $\iota_{\mathscr{A}} : s \mapsto \hat{s}$ from $\mathbf{R}$ into Ω. Obviously $\iota_{\mathscr{A}}(\mathbf{R})$ is dense in Ω. Note that unless p and q are periodic functions and have a common period, the map $\iota_{\mathscr{A}}$ is injective.

Theorem 2. *Suppose that $\iota_{\mathscr{A}}$ is injective. Then $\mathscr{T}$ is an eigenvalue of the Schrödinger operator L if and only if there is a $\mu \in \mathscr{M}(p, q)$ which is concentrated on $\iota_{\mathscr{A}}(\mathbf{R})$*

If m is a probability measure on Ω and is invariant under the group $\{\alpha_s : s \in \mathbf{R}\}$, then it is well known that

$$L_m = -D_m \hat{p} D_m + \hat{q}$$

is a self-adjoint operator on a dense domain in $L^2(\Omega, m)$.

Theorem 3. *Suppose that m is an invariant probability measure on Ω. Then $\mathscr{T}$ is an eigenvalue of L_m if and only if there is a $\mu \in \mathscr{M}(p, q)$ which is absolutely continuous with respect to m*

In the main text, we will obtain results which are slightly more general than the theorems stated above. Rather than starting with the coefficients p and q of the Schrödinger operator and build the algebra $\mathscr{A}$, we will take the following approach. We will start with a continuous flow $(X, \{\alpha_s : s \in \mathbf{R}\})$ and functions P, $Q \in C(X)$. We will consider the family of operators $\{L_x : x \in X\}$, where

$$L_x = -\left(\frac{d}{dt}\right) P_x \left(\frac{d}{dt}\right) + Q_x$$

Parts (a), (b) and (c) of Theorem 1 will be proved in Sect. 1 and 2. Parts (d) and (e) of Theorem 1 will be derived from (b) and (c) in Sect. 3 by what amounts to solving a dual extreme problem. Theorems 2 and 3 will also be proved in Sect. 3. Furthermore, we will explain in Sect. 3 that the question whether or not $\mathscr{T}$ is an eigenvalue can be converted to a question which is completely independent of the study of Schrödinger operators and which leads to what seems to be a generalization of the notion of ergodicity. In Sect. 4, we will specialize our results to the case where the flow is generated by functions on $\mathbf{R}$.

1. The Distance Formula

Let X be a compact Hausdorff space. Suppose that $\alpha = \{\alpha_t : t \in \mathbf{R}\}$ is a continuous group of homeomorphisms on X. That is, $(x, t) \mapsto \alpha_t(x)$ is a continuous map from $X \times \mathbf{R}$ to X. The dynamical system $(X, \mathbf{R}, \alpha)$ will simply be referred to as a flow. For a function φ on X, we denote $\varphi'(x) = \lim_{h \to 0} (\varphi(\alpha_h(x)) - \varphi(x))/h$ whenever such limit exists. $\varphi'(x)$ can be thought of as the derivative of φ in the direction of the flow. Let $C^1_\alpha(X)$ be the collection of *continuous* functions φ on X such that $\varphi'(x)$ exists for every $x \in X$ and $x \mapsto \varphi'(x)$ is a *continuous* function on X. For any function f on X and any $x \in X$, let f_x denote the function on $\mathbf{R}$ defined by the formula $f_x(t) = f(\alpha_t(x))$. For the rest of the paper, P will denote a function in $C^1_\alpha(X)$ such that $P(x) > 0$ for every $x \in X$.

Definition 1.1. (a) Define $\mathscr{F}_0(P) = \{P\varphi^2 + (P\varphi)' : \varphi \in C^1_\alpha(X)$ and φ is real-valued$\}$.
(b) Let $\mathscr{F}(P)$ be the sup-norm closure of the convex hull of $\mathscr{F}_0(P)$.

(c) For every $f \in C(X)$, let $d_P(f) = \inf\{\|f-u\|_\infty : u \in \mathscr{F}(P)\}$, the distance from f to $\mathscr{F}(P)$.

Theorem 1.2. *If ψ is a real-valued function in $C(X)$, then $\psi + d_P(\psi) \in \mathscr{F}(P)$*

Proof. Suppose that the theorem were false. Then, by the Riezs representation theorem and the Hahn-Banch separation theorem, there would be an $a \in \mathbf{R}$ and a real-valued regular Borel measure μ on X such that

$$\int_X (\psi + d_P(\psi))\, d\mu > a \geq \int_X u\, d\mu$$

for every $u \in \mathscr{F}(P)$. For any real-valued function $f \in C^1_\alpha(X)$, $Pf^2 = (1/2)(Pf^2 + (Pf)') + (1/2)(P(-f)^2 + (P(-f))')$. Because $C^1_\alpha(X)$ is dense in $C(X)$ [2], this means that $\mathscr{F}(P)$ contains every non-negative function in $C(X)$. Hence it follows from the above inequality that $\mu(E) \leq 0$ for every Borel set $E \subset X$. It also follows from the above inequality that

$$\int_X (\psi - u)\, d\mu \geq -\mu(X) d_P(\psi) + b$$

for every $u \in \mathscr{F}(P)$, where

$$b = \int_X (\psi + d_P(\psi))\, d\mu - a > 0\,.$$

Since $-\mu(X) > 0$, $\tilde{\mu} = \mu/(-\mu(X))$ is a probability measure on X. We have

$$\|\psi - u\|_\infty \geq \int_X (\psi - u)\, d\tilde{\mu} \geq d_P(\psi) + b/(-\mu(X)) > d_P(\psi)$$

for every $u \in \mathscr{F}$. But this is inconsistent with the fact that $d_P(\psi)$ is the distance between ψ and $\mathscr{F}(P)$. □

Let Q be a real-valued function in $C(X)$. For each $x \in X$, define the Schrödinger operator

$$L_x = -\frac{d}{dt} P_x(t) \frac{d}{dt} + Q_x(t)$$

on $\mathscr{D}_2 = \{f : f$ and f' are absolutely continuous on every finite interval, f, f', $f'' \in L^2(\mathbf{R})\} \subset L^2(\mathbf{R})$. It is well-known that L_x is a self-adjoint operator on $\mathscr{D}_2$ [9]. Let $\mathscr{S}$ denote the closure of $\bigcup_{x\in X} \sigma(L_x)$. (For a linear operator A, $\sigma(A)$ denotes its spectrum.) Let $\mathscr{T}$ denote the infimum of the set $\mathscr{S}$. In the situation where it is necessary to indicate the coefficients P and Q of the Schrödinger operator to avoid confusion, we will write $L_{P,Q,x}$, $\mathscr{S}(P,Q)$ and $\mathscr{T}(P,Q)$ instead of L_x, $\mathscr{S}$ and $\mathscr{T}$.

Proposition 1.3. $-d_P(Q) \leq \mathscr{T}$.

Proof. For any $\varepsilon > 0$, there exist real-valued functions $g_1, \ldots, g_n \in C^1_\alpha(X)$ and $a_1, \ldots, a_n \in (0,1]$ with $a_1 + \ldots + a_n = 1$ such that $\left\| Q - \sum_{j=1}^n a_j (Pg_j^2 + (Pg_j)') \right\|_\infty < d_P(Q) + \varepsilon$. Let $\psi = \sum_{j=1}^n a_j (Pg_j^2 + (Pg_j)') - Q$. For each $x \in X$, let $A_{j,x}$

denote the first order differential operator $-id/dt + ig_{j,x}$. Then $A_{j,x}^* P_x A_{j,x} = -(d/dt)P_x(d/dt) + P_x g_{j,x}^2 + (Pg_j)'_x$. Hence

$$L_x + \psi_x = -\left(\frac{d}{dt}\right) P_x \left(\frac{d}{dt}\right) + \sum_{j=1}^{n} a_j (P_x g_{j,x}^2 + (Pg_j)'_x) = \sum_{j=1}^{n} a_j A_{j,x}^* P_x A_{j,x} \geq 0 .$$

Since $\|\psi\|_\infty \leq d_P(Q) + \varepsilon$, we have

$$\langle L_x f, f\rangle = \langle (L_x + \psi_x) f, f\rangle - \langle \psi_x f, f\rangle \geq -(d_P(Q) + \varepsilon)\langle f, f\rangle$$

whenever f belongs to the domain of L_x. This implies $\sigma(L_x) \subset [-d_P(Q) - \varepsilon, \infty)$ for every $x \in X$. Hence $\mathscr{S} \subset [-d_P(Q) - \varepsilon, \infty)$ for every $\varepsilon > 0$. Therefore $-d_P(Q) \leq \mathscr{I}$. □

Proposition 1.4. *Suppose that* $\lambda < \mathscr{I}$ *Then* $Q - \lambda \in \mathscr{F}_0(P)$.

Remark We will show that when $\lambda < \mathscr{I}$, the positive operator $L_x - \lambda$ admits a factorization

$$L_x - \lambda = \left(\frac{1}{i}\frac{d}{dt} - iM_x\right) P_x \left(\frac{1}{i}\frac{d}{dt} + iM_x\right)$$

with $M \in C_\alpha^1(X)$. In general, given a positive operator A, one thinks of the factorization problem $A = \sum_k B_k^* B_k$ in an algebra associated with A as a non-commutative moment problem. In the case where A is a differential operator with coefficients which are rational functions in t, such a moment problem was solved in [7]. Note that for our operators, the moment problem has a one-term solution B^*B. Also see Remark 1 after Theorem 3.2. The proof of this proposition will be given in the next section.

Theorem 1.5. (a) $\mathscr{I} = \|Q\|_\infty - d_P(Q - \|Q\|_\infty)$.
(b) *Suppose that* $\mathscr{I} \leq 0$. *Then* $\mathscr{I} = -d_P(Q)$.

Proof. Suppose that $\lambda < \mathscr{I} \leq 0$. Proposition 1.4 says that $Q - \lambda \in \mathscr{F}_0(P)$. Hence $d_P(Q) \leq |\lambda| = -\lambda$. In particular $d_P(Q) \leq -\mathscr{I} + 1/n$ for every n. Hence $d_P(Q) \leq -\mathscr{I}$ or, equivalently, $-d_P(Q) \geq \mathscr{I}$. Proposition 1.3 provides the opposite inequality. This proves (b). Because $0 \leq -(d/dt)P_x(d/dt) \leq (\|P\|_\infty)(-d^2/dt^2)$ and because $\sigma(-d^2/dt^2) = [0, \infty)$, $0 \in \sigma(-(d/dt)P_x(d/dt))$. For every $x \in X$, $Q_x - \|Q\|_\infty \leq 0$ on **R**. Hence $(-\infty, 0] \cap \sigma(L_x - \|Q\|_\infty) \neq \emptyset$. Therefore part (a) is a consequence of part (b) and the identity $\mathscr{I}(P, Q) = \|Q\|_\infty + \mathscr{I}(P, Q - \|Q\|_\infty)$. □

Proposition 1.6. $\mathscr{F}_0(P)$ *is dense in* $\mathscr{F}(P)$.

Proof. For each $u \in \mathscr{F}(P)$, $d_P(u) = 0$. If we apply Propositions 1.3 and 1.4 to $Q = u$, we see that $u + \varepsilon \in \mathscr{F}_0(P)$ for every $\varepsilon > 0$. Hence the distance between u and $\mathscr{F}_0(P)$ is 0. □

2. Weyl's m-Functions

This section is devoted to the proof of Proposition 1.4. Our proof requires only elementary techniques of differential equations and some aspects of the proof should be familiar to experts in this area. And some of the technical details perhaps need not be included. But for the convenience of the reader, we will present them anyway. If one uses the theory of spectral bundles of differential systems developed in [10], it is possible to find an alternate proof which could be somewhat shorter but not as elementary.

Recall that $\mathscr{S} = \mathscr{S}(P,Q)$ is the closure of the union of the spectra of all $L_x = L_{P,Q,x}$, $x \in X$. Let $\mathscr{D}^+$ be the collection of functions f on $[0,\infty)$ such that f and f' are absolutely continuous on every finite interval $[0,A]$ and such that f, f', $f'' \in L_2[0,\infty)$. Similarly, $\mathscr{D}^-$ denotes the collection of functions on $(-\infty,0]$ with the same properties. Define $\mathscr{D}_0^{\pm} = \{f \in \mathscr{D}^{\pm} : f(0) = f'(0) = 0\}$. For each $x \in X$, let L_x^+ (respectively L_x^-) be the symmetric operator $-(d/dt)P_x(d/dt)+Q_x$ on $\mathscr{D}_0^+$ (respectively on $\mathscr{D}_0^-$). The differential equation $-(P_xu')'(t)+Q_x(t)u(t) = zu(t)$ will be abbreviated as $\ell_xu = zu$. For $x \in X$ and $z \in \mathbf{C}$, define $\mathscr{E}_{x,z}^+ = \{u : \ell_xu = zu$ and the restriction of u to $[0,\infty)$ belongs to $\mathscr{D}^+\}$. Similarly, $\mathscr{E}_{x,z}^- = \{u : \ell_xu = zu$ and the restriction of u to $(-\infty,0]$ belongs to $\mathscr{D}^-\}$.

Lemma 2.1. *Let $z \in \mathbf{C}$ and $x \in X$ If u is a solution of $\ell_xu = zu$ and $|u|^2$ is integrable on $[0,\infty)$ (respectively on $(-\infty,0]$), then $u \in \mathscr{E}_{x,z}^+$ (respectively $u \in \mathscr{E}_{x,z}^-$)*

Proof. Suppose that $\int_0^\infty |u(t)|^2dt < \infty$. Since u and u' are continuously differentiable, it suffices to show that $\int_0^\infty |u'(t)|^2dt < \infty$. Straightforward differentiation shows that $(P_x(|u|^2)')' = 2(Q_x - \mathrm{Re}(z))|u|^2 + 2P_x|u'|^2$. Hence $P_x|u'|^2 = (P_x(|u^2|)')'/2 - (Q_x - \mathrm{Re}(z))|u|^2$. Consequently we have

$$\int_0^T P_x(t)|u'(t)|^2dt = \frac{1}{2}[(|u^2|)'(T)P_x(T) - (|u|^2)'(0)P_x(0)] - \int_0^T (Q_x(t) - \mathrm{Re}(z))|u(t)|^2dt\,.$$

Since $\int_0^T |u^2(t)|\,dt < \infty$, $(|u^2|)'(t)$ cannot be strictly positive any interval $[A,\infty)$. Hence $P_x|u'|^2$ is integrable on $[0,\infty)$. By the assumption on P, there is a $\delta > 0$ such that $P \geq \delta$ on X. Hence $u' \in L^2[0,\infty)$. The case $\int_{-\infty}^0 |u(t)|^2dt < \infty$ is similarly treated. □

Proposition 2.2. *Suppose that $z \in \mathbf{C}\backslash\mathscr{S}$. Then for any $x \in X$, $\dim(\mathscr{E}_{x,z}^+) = \dim(\mathscr{E}_{x,z}^-) = 1$.*

Proof Suppose that $z \in \mathbf{C}\backslash\mathbf{R}$. By Weyl's limit point-limit circle alternatives, for both symmetric operators L_x^+ and L_x^-, the deficiency indices are either $(1,1)$ or $(2,2)$ (see, for example, [1 or 9]). By Lemma 2.1, this means that $\dim(\mathscr{E}_{x,z}^+)$ and $\dim(\mathscr{E}_{x,z}^-)$ are either 1 or 2. If $\dim(\mathscr{E}_{x,z}^+) = 2$, then $\mathscr{E}_{x,z}^+ \supset \mathscr{E}_{x,z}^-$. Hence z is an eigenvalue for L_x. But this is not possible due to the fact that L_x is self-adjoint and $z \in \mathbf{C}\backslash\mathbf{R}$. Hence $\dim(\mathscr{E}_{x,z}^+) = 1$ and, similarly, $\dim(\mathscr{E}_{x,z}^-) = 1$ if $z \in \mathbf{C}\backslash\mathbf{R}$. If $z \notin \mathscr{S}$, then $\|(L_x^+ - z)f\|_2 = \|(L_x - z)f\|_2 \geq d(z,\mathscr{S})\|f\|_2$ for every $f \in \mathscr{D}_0^+$ ($\mathscr{D}_0^+$ is considered as a subset of $L^2(\mathbf{R})$ in the natural way). Therefore $(L_x^+ - z)\mathscr{D}_0^+$ is closed. It follows from Corollary 2 on p. 42 of [9] that $\mathscr{E}^+_{x,z} = L^2[0,\infty) \ominus (L^+_x - z)\mathscr{D}_0^+$ and that $\dim(\mathscr{E}^+_{x,z})$ remains constant on $\mathbf{C}\setminus\mathscr{S}$. □

Definition 2.3. For each pair $(x,z) \in X \times (\mathbf{C}\setminus\mathscr{S})$, $E^+_{x,z}$ denotes the orthogonal projection from $L^2[0,\infty)$ onto the one-dimensional subspace $\{f \mid [0,\infty) \cdot f \in \mathscr{E}^+_{x,z}\}$. Similarly, $E^-_{x,z}$ denotes the orthogonal projection from $L^2(-\infty,0]$ onto $\{g \mid (-\infty,0]$ $g \in \mathscr{E}^-_{x,z}\}$.

Proposition 2.4. *The map $(x,z) \mapsto E^+_{x,z}$* (resp. *$(x,z) \mapsto E^-_{x,z}$) is continuous from $X \times (\mathbf{C}\setminus\mathscr{S})$ into $\mathscr{L}(L^2[0,\infty))$* (resp *into $\mathscr{L}(L^2(-\infty,0]))$ equipped with the operator norm topology*

Proof Let $(x_0,z_0) \in X \times (\mathbf{C}\setminus\mathscr{S})$ be given and let $g_0 \in E^+_{x_0,z_0}L^2[0,\infty)$ be a unit vector Note that for any $z \in \mathbf{C}\setminus\mathscr{S}$, $\|(L^+_x - z)f\|_2 = \|(L_x - z)f\|_2 \geq d(z,\mathscr{S})\|f\|_2$ if $f \in \mathscr{D}_0^+$. Since $E^+_{x,z}L^2[0,\infty) = L^2[0,\infty) \ominus (L^+_x - z)\mathscr{D}_0^+$, we have

$$
\begin{aligned}
\|g_0 - E^+_{x,z}g_0\| &= \sup\left\{\frac{|\langle g_0,(L^+_x - z)f\rangle|}{\|L^+_x - z)f\|_2} : f \in \mathscr{D}_0^+,\ f \neq 0\right\} \\
&\leq \frac{1}{d(z,\mathscr{S})}\sup\left\{\frac{|\langle g_0,(L^+_x - z)f\rangle|}{\|f\|_2}\quad f \in \mathscr{D}_0^+,\ f \neq 0\right\} \\
&= \frac{1}{d(z,\mathscr{S})}\sup\{|\langle g_0,[(L^+_x - z) - (L^+_{x_0} - z_0)]f\rangle| : f \in \mathscr{D}_0^+,\ \|f\|_2 = 1\} \\
&= \frac{1}{d(z,\mathscr{S})}\sup\{|\langle g_0,[-(d/dt)(P_x - P_{x_0})(d/dt) \\
&\quad - (Q_x - Q_{x_0}) - (z - z_0)]f\rangle| \cdot f \in \mathscr{D}_0^+,\ \|f\|_2 = 1\} \\
&\leq \frac{\|(P'_x - P'_{x_0})g'_0\|_2 + \|(P_x - P_{x_0})g''_0\|_2 + \|(Q_x - Q_{x_0})g_0\|_2 + |z - z_0|}{d(z_0,\mathscr{S}) - |z - z_0|}
\end{aligned}
$$

Because $g_0 \in \mathscr{D}^+$ and $P' \in C(X)$, the maps $x \mapsto P'_x g'_0$, $x \mapsto P_x g''_0$ and $x \mapsto Q_x g_0$ are continuous from X into $L^2[0,\infty)$ Hence for any $\varepsilon > 0$, there is a neighborhood U of x_0 and a $\delta > 0$ such that $\|g_0 - E^+_{x,z}g_0\| < \varepsilon$ if $x \in U$ and $|z - z_0| < \delta$ The desired continuity follows from the formulas $E^+_{x_0,z_0}h = \langle h,g_0\rangle g_0$ and $E^+_{x,z}h = \langle h,E^+_{x,z}g_0\rangle E^+_{x,z}g_0/\|E^+_{x,z}g_0\|_2^2$, The continuity of $(x,z) \mapsto E^-_{x,z}$ can be similarly proved. □

Proposition 2.5. (a) *Let $\varphi \in L^2[0,\infty)$* (resp. *$\psi \in L^2(-\infty,0]$) and $t \in (0,\infty)$* (resp. *$t \in (-\infty,0)$) be given Then $(x,z) \mapsto (E^+_{x,z}\varphi)(t)$*[1] (resp. *$(x,z) \mapsto (E^-_{x,z}\psi)(t)$) is a continuous function from $X \times (\mathbf{C}\setminus\mathscr{S})$ into $\mathbf{C}$*
(b) *Let $\varphi \in L^2[0,\infty)$* (resp. *$\psi \in L^2(-\infty,0]$) and $t \in (0,\infty)$* (resp *$t \in (-\infty,0)$) be given Then $(x,z) \mapsto (E^+_{x,z}\varphi)'(t)$* (resp. *$(x,z) \mapsto (E^-_{x,z}\psi)'(t)$) is a continuous function from $X \times (\mathbf{C}\setminus\mathscr{S})$ into $\mathbf{C}$*

[1] Strictly speaking, $E^+_{x,z}\varphi$ is an element in $L^2[0,\infty)$ and represents a class of functions which differ from each other on sets of measure zero So it is necessary to give a clear definition for $(E^+_{x,z}\varphi)(t)$, the value of $E^+_{x,z}\varphi$ at t There is a solution of the differential equation $\ell_x u = zu$ on $\mathbf{R}$ such that $u|[0,\infty)$ represents the same element $E^+_{x,z}\varphi \in L^2[0,\infty)$ The value $(E^+_{x,z}\varphi)(t)$ is defined to be $u(t)$ Also when we write $(E^+_{x,z}\varphi)'(t)$ we mean $u'(t)$ $(E^-_{x,z}\psi)(t)$ and $(E^-_{x,z}\psi)'(t)$ are defined similarly

The proof of this proposition will be divided into several steps. The main ingredients in the proof are Proposition 2.4 and a sequence of lemmas in elementary analysis. We will now establish these lemmas.

Lemma 2.6. *Let $f \geq 0$ be a C^1 function on $[a,b]$.*

(a) *There is an $s \in (a,b)$ such that $f'(s) \geq -8 \int_a^b f(t)\,dt/(b-a)^2$*

(b) *There is an $r \in (a,b)$ such that $f'(r) \leq 8 \int_a^b f(t)\,dt/(b-a)^2$.*

Proof. Let $\eta = 8 \int_a^b f(t)\,dt/(b-a)^2$. (a) Suppose that such an s did not exist. Then $f'(t) < -\eta$ for every $t \in (a,b)$. Hence $f((b+a)/2) - f(\lambda) = \int_\lambda^{(b+a)/2} f'(t)\,dt < -\eta\Big(\dfrac{b+a}{2} - \lambda\Big)$ for every $\lambda \in (a,(b+a)/2]$. That is, $f(\lambda) > f((b+a)/2) + \eta\Big(\dfrac{b+a}{2} - \lambda\Big)$ whenever $\lambda \in (a,(b+a)/2]$. Since $f \geq 0$ on $[a,b]$, we have $\int_a^b f(\lambda)\,d\lambda \geq \int_a^{(b+a)/2} f(t)\,dt > \eta \int_a^{(b+a)/2} \Big(\dfrac{b+a}{2} - \lambda\Big)\,d\lambda = \eta(b-a)^2/8$, which is a contradiction. This proves (a). (b) is proved in a similar way. □

Lemma 2.7. *Let B be a bounded subset of $\mathbf{C}\backslash\mathscr{S}$ and let $\varepsilon > 0$. Then there is an $N = N(P,Q,B,\varepsilon) > 0$ such that for every $(x,z) \in X \times B$,*

$$\int_\varepsilon^\infty |(E_{x,z}^+\varphi)'(t)|^2 dt \leq N$$

if φ is a unit vector in $L^2[0,\infty)$ and

$$\int_{-\infty}^\varepsilon |(E_{x,z}^-, \psi)'(t)|^2 dt \leq N$$

if ψ is a unit vector in $L^2(-\infty,0]$

Proof. Suppose that $\varphi \in L^2[0,\infty]$ is a unit vector. Since $\|\chi_{[0,\varepsilon]}|E_{x,z}^+\varphi|^2\|_1 \leq 1$, it follows from Lemma 2.6 (a) that there is an $s = s(\varphi,x,z) \in (0,\varepsilon)$ such that

$$(|E_{x,z}^+\varphi|^2)'(s) \geq -8/\varepsilon^2 .$$

Write u for $E_{x,z}^+\varphi$ for the moment. Since $-(P_x u')' + Q_x u = zu$ on $(0,\infty)$, we have $(P_x(|u|^2)')' = 2P_x|u'|^2 + 2(Q_x - \mathrm{Re}(z))|u|^2$. Suppose that $P(x) \geq \delta > 0$ for every

$x \in X$. Then

$$\delta \int_\varepsilon^T |(E^+_{x,z}\varphi)'(t)|^2 dt \le \int_s^T P_x(t)|u'|^2(t)\,dt$$

$$= \frac{1}{2}[P_x(T)(|u|^2)'(T) - P_x(s)(|u|^2)'(s)] - \int_s^T (Q_x(t) - \mathrm{Re}(z))|u(t)|^2 dt$$

$$\le P_x(T)\frac{1}{2}(|u|^2)'(T) + \|P\|_\infty(4/\varepsilon^2) + \|Q\|_\infty + \sup\{|w| : w \in B\}.$$

Since the function $|u|^2$ is integrable on $[0,\infty)$, for any $A > 0$, $\inf\{(|u|^2)'(t) : t > A\} \le 0$. Hence

$$\int_\varepsilon^\infty |(E^+_{x,z}\varphi)'(t)|^2 dt \le \frac{1}{\delta}[\|P\|_\infty(4/\varepsilon^2) + \|Q\|_\infty + \sup\{|w| : w \in B\}].$$

The proof for the other inequality is similar and will be omitted. □

Lemma 2.8. *Let B be a bounded subset of $\mathbf{C}\backslash\mathscr{S}$ and let $\varepsilon > 0$. There there is a $C = C(P, Q, B, \varepsilon) > 0$ such that for any $(x, z) \in X \times B$ and $t > \varepsilon$,*

$$|(E^+_{x,z}\varphi)'(t)|^2 \le C$$

if φ is a unit vector in $L^2[0,\infty)$ and

$$|(E^-_{x,z}\psi)'(-t)|^2 \le C$$

if ψ is a unit vector in $L^2(-\infty, 0]$.

Proof For any $u \in \mathscr{D}^+$,

$$\int_t^\infty P_x(s)u'(s)\bar{u}''(s)\,ds = -P_x(t)|u'(t)|^2 - \int_t^\infty (P_x u')'(s)\bar{u}'(s)\,ds.$$

Therefore

$$P_x(t)|u'(t)|^2 = -\int_t^\infty P_x(s)u'(s)\bar{u}''(s)\,ds - \int_t^\infty (P_x u')'(s)\bar{u}'(s)\,ds$$

$$= -\int_t^\infty u'(s)(P_x\bar{u}')'(s)\,ds - \int_t^\infty (P_x u')'(s)\bar{u}'(s)\,ds$$

$$+ \int_t^\infty P'_x(s)|u'(s)|^2 ds$$

$$\le 2\left[[\int_t^\infty |u'(s)|^2 ds\right]^{1/2} \|(P_x u')'\|_2 + \|P'\|_\infty \int_t^\infty |u'(s)|^2 ds.$$

Let $N_1 = \|Q\|_\infty + \sup\{|w| : w \in B\}$. Then $\|(P_x(E^+_{x,z}\varphi)')'\|_2 \le N_1\|E^+_{x,z}\varphi\|_2$ whenever $(x,z) \in X \times B$ and $\varphi \in L^2[0,\infty)$. Let N be the constant provided by the previous lemma. Then obviously $C = (2N_1\sqrt{N} + \|P'\|_\infty N)/\delta$ will do. □

An immediate consequence of this lemma is the following:

Corollary 2.9. *Let B be a bounded subset of $\mathbf{C}\backslash\mathscr{S}$ and let ε be a positive number Then there is a $K = K(P,Q,B,\varepsilon) > 0$ such that for any $(x,z) \in X \times B$, and $s,t \in [\varepsilon,\infty)$,*

$$|(E^+_{x,z}\varphi)(t) - (E^+_{x,z}\varphi)(s)| \le K|t-s|$$

if φ is a unit vector in $L^2[0,\infty)$; and

$$|(E^-_{x,z}\psi)(-t) - (E^-_{x,z}\psi)(-s)| \le K|t-s|$$

if ψ is a unit vector in $L^2(-\infty,0]$.

Proof of Proposition 2.5. (a) *Let $t \in (0,\infty)$ and $\varphi \in L^2[0,\infty)$ be given. Let $(x_0,z_0) \in X \times (\mathbf{C}\backslash\mathscr{S})$ and let B be a bounded neighborhood of z_0 in $\mathbf{C}\backslash\mathscr{S}$ By the previous lemma, there is a $K > 0$ such that*

$$|(E^+_{x,z}\varphi)(s_1) - (E^+_{x,z}\varphi)(s_2)| \le K|s_1 - s_2|$$

whenever $(x,z) \in X \times B$ and s_1, $s_2 \in [t/2,\infty)$. Let $n_0 > 2/t$ and define

$$g_n(x,z) = \frac{n}{2}\int_{t-(1/n)}^{t+(1/n)} (E^+_{x,z}\varphi)(s)\,ds = \frac{n}{2}\langle E^+_{x,z}\varphi, \chi_{[t-(1/n),t+(1/n)]}\rangle$$

for $n \ge n_0$. It follows from Proposition 2.4 that each g_n is a continuous function on $X \times (\mathbf{C}\backslash\mathscr{S})$. On the other hand, for $(x,z) \in X \times B$,

$$\begin{aligned} |g_n(x,z) - (E^+_{x,z}\varphi)(t)| &= \frac{n}{2}\left|\int_{t-(1/n)}^{t+(1/n)} [(E^+_{x,z}\varphi)(s) - (E^+_{x,z}\varphi)(t)]\,ds\right| \\ &\le \frac{n}{2}\int_{t-(1/n)}^{t+(1/n)} K|t-s|\,ds \le \frac{1}{2}\int_{t-(1/n)}^{t+(1/n)} K\,ds = K/n\,. \end{aligned}$$

This shows that on $X \times B$, the function $(x,z) \mapsto (E^+_{x,z}\varphi)(t)$ is the uniform limit the continuous functions $\{g_n\}$. Therefore it is continuous. The continuity of $(E^-_{x,z}\psi)(t)$ can be established similarly.

(b) Let $s < t$. Then

$$P_x(t)(E^+_{x,z}\varphi)'(t) - P_x(s)(E^+_{x,z}\varphi)'(s) = \int_s^t (P_x(E^+_{x,z}\varphi)')'(\lambda)\,d\lambda$$

$$= \int_s^t (Q_x(\lambda) - z)(E^+_{x,z}\varphi)(\lambda)\,d\lambda = \langle (Q_x - z)E^+_{x,z}\varphi, \chi_{[s,t]}\rangle$$

For any $g \in L^2[0,\infty)$, $x \mapsto Q_x g$ is a continuous map from X into $L^2[0,\infty)$. Hence the map $(x,z,s) \mapsto \langle (Q_x - z)E^+_{x,z}\varphi, \chi_{[s,t]}\rangle$ is continuous on $X \times (\mathbf{C}\backslash\mathscr{S}) \times [t/3, t/2]$. This implies that $(x,z) \mapsto P_x(t)(E^+_{x,z}\varphi)'(t) - P_x(.)(E^+_{x,z}\varphi)'(.)$ is a continuous map from $X \times (\mathbf{C}\backslash\mathscr{S})$ into $C[t/3,t/2]$. Therefore

$$(x,z) \mapsto \int_{t/3}^{t/2} [P_x(s)]^{-1}[P_x(t)(E^+_{x,z}\varphi)'(t) - P_x(s)(E^+_{x,z}\varphi)'(s)]\,ds$$

$$= P_x(t)(E^+_{x,z}\varphi)'(t)\left(\int_{t/3}^{t/2}[P_x(s)]^{-1}\,ds\right) - (E^+_{x,z}\varphi)(t/2) + (E^+_{x,z}\varphi)(t/3)$$

is a continuous function on $X \times (\mathbf{C}\backslash\mathscr{S})$. Since the last two terms are continuous by part (a), so is the first term. The continuity of $(E^-_{x,z}\psi)'(t)$ is established similarly. □

Suppose that $z \in \mathbf{C}\backslash\mathscr{S}$. Given a $\varphi \in L^2[0,\infty)$, there is a unique solution u of the differential equation $\ell_x u = zu$ on $\mathbf{R}$ such that $E^+_{x,z}\varphi$ is represented by the restriction of u to $[0,\infty)$ in the space $L^2[0,\infty)$. We can also think of u as the natural extension of $E^+_{x,z}\varphi$ to $\mathbf{R}$. For the sake of convenience, we will use the same symbol $E^+_{x,z}\varphi$ to denote this function on $\mathbf{R}$. Similarly, for $\psi \in L^2(-\infty,0]$, we also use $E^-_{x,z}\psi$ to denote its natural extension to $\mathbf{R}$.

Proposition 2.5′. (a) *Let $\varphi \in L^2,[0,\infty)$ (*resp. *$\psi \in L^2(-\infty,0]$) and $t \in \mathbf{R}$ be given Then $(x,z) \mapsto (E^+_{x,z}\varphi)(t)$ (*resp. *$(x,z) \mapsto (E^-_{x,z}\psi(t))$ is a continuous function from $X \times (\mathbf{C}\backslash\mathscr{S})$ into $\mathbf{C}$.*
(b) *Let $\varphi \in L^2[0,\infty)$ (*resp. *$\psi \in L^2(-\infty,0]$) and $t \in \mathbf{R}$ be given. Then $(x,z) \mapsto (E^+_{x,z}\varphi)'(t)$ (*resp. *$(x,z) \mapsto (E^-_{x,z}\psi)'(t)$) is a continuous function from $X \times (\mathbf{G}\backslash\mathscr{S})$ into $\mathbf{C}$*

Proof Let $\xi'(t) = A(x,z,t)\xi(t)$ be the standard 2×2 first order linear differential equation system which is the equivalent of the second order equation $\ell_x u = zu$. For any $a \in \mathbf{R}$, let $\Phi_a(x,z,t)$ be the fundamental solution matrix of $\xi'(t) = A(x,z,t)\xi(t)$ such that $\Phi_a(x,z,a)$ is the 2×2 identity matrix. Proposition 2.5′ is an immediate consequence of Proposition 2.5 and the well-known fact that for any fixed $t \in \mathbf{R}$, $(x,z) \mapsto \Phi_a(x,z,t)$ is a continuous map from $X \times \mathbf{C}$ into M_2. □

Suppose that $z \in \mathbf{C}\backslash\mathbf{R}$. Then it is well-known that a function in $\mathscr{E}^\pm_{x,z}$ which is not identically zero cannot vanish on $\mathbf{R}$. This is due to the simple fact that the natural extensions of the operators $L^\pm_x$ to $\{f \in \mathscr{D}^\pm : f(0) = 0\}$ are self-adjoint and, therefore, cannot have any non-real eigenvalues. Our next objective is to establish the same result for real values $z \in (-\infty, \mathscr{T})$. But since the possibility of the self-adjoint extensions of $L^\pm_x$ having eigenvalues in $(-\infty,\mathscr{T})$ cannot be excluded, the proof becomes quite involved.

Proposition 2.10. *Suppose $\lambda < \mathscr{T}$. If $\varphi \in \mathscr{E}^\pm_{x,\lambda}$ and φ is not the constant function 0, then $\varphi(t) \neq 0$ for every $t \in \mathbf{R}$*

Proof. We first assume that $\lambda < \|Q\|_\infty$. Suppose $\varphi \in \mathscr{E}^+_{x,\lambda}$ and $\varphi(t_0) = 0$. Define a function $\hat\varphi$ on $\mathbf{R}$ such that $\hat\varphi(t) = \varphi(t)$ for $t \geq t_0$ and $\hat\varphi(t) = -\varphi(2t_0 - t)$ for $t < t_0$. It is easy to see that since $\varphi(t_0) = 0$, $\hat\varphi$ is absolutely continuous on every bounded interval.

The definition of $\hat{\varphi}$ also ensures that the left derivative and the right derivative of $\hat{\varphi}$ at t_0 agree. Therefore $\hat{\varphi}'$ is also absolutely continuous on every bounded interval. It is obvious that $\hat{\varphi}$, $\hat{\varphi}'$ and $\hat{\varphi}''$ all belong to $L^2(\mathbf{R})$. Hence $\hat{\varphi} \in \mathscr{D}_2$. Define $U(t) = P_x(t)$ and $V(t) = Q_x(t)$ for $t \geq t_0$ and define $U(t) = P_x(2t_0 - t)$ and $V(t) = Q_x(2t_0 - t)$ for $t < t_0$. Since $-(P_x\varphi')' + Q_x\varphi = \lambda\varphi$ on $\mathbf{R}$, it follows that $-(U\hat{\varphi}')' + V\hat{\varphi} = \lambda\hat{\varphi}$ on $\mathbf{R}$. If φ were not identically zero, then λ would be an eigenvalue for the self-adjoint operator $\hat{L} = -(d/dt)U(d/dt) + V$. But this is impossible because $\lambda < -\|Q\|_\infty$ and $\hat{L} \geq -\min\{P(y) : y \in X\}d^2/dt^2 - \|Q\|_\infty \geq -\|Q\|_\infty$.

For the general case $\lambda < \mathscr{T}$, we need the following function: For $(y, z) \in X \times (\mathbf{C}\backslash\mathscr{S})$, define

$$\Gamma(y, z, t) = \frac{\psi_+(t)\psi_-(t)}{\psi_+(t)\psi_-'(t) - \psi_+'(t)\psi_-(t)},$$

where ψ_+ is any nonzero function in $\mathscr{E}^+_{y,z}$ and ψ_- is any nonzero function in $\mathscr{E}^-_{y,z}$. That the spaces $\mathscr{E}^\pm_{y,z}$ are one-dimensional guarantees that the definition of Γ is independent of the choice of $\psi_\pm$. Since $P_y(\psi_+\psi_-' - \psi_+'\psi_-)$ is the Wronskian for the differential equation $\ell_y u = zu$, there is a nonzero constant c such that $\psi_+\psi_-' - \psi_+'\psi_- = c/P_y$. Hence Γ is well defined. (Actually, $\Gamma(y, z, t) = G(y, z, t, t)/P_y(t)$, where G is the Green's function for $\ell_y u = zu$.) For any $s \in \mathbf{R}$, the functions $t \mapsto \psi_\pm(t + s)$ belong to $\mathscr{E}^\pm_{\alpha_s(y),z}$. Hence $\Gamma(y, z, t + s) = \Gamma(\alpha_s(y), z, t)$. Let $\eta_+ = \psi_+|[0, \infty)$ and $\eta_- = \psi_-|(-\infty, 0]$. Then there is a neighborhood U of (y, z) in $X \times (\mathbf{C}\backslash\mathscr{S})$ such that $E^+_{\xi,w}\eta_+$ and $E^-_{\xi,w}\eta_-$ are not identically zero for $(\xi, w) \in U$. Hence it follows from Proposition 2.5$'$ that for each $t \in \mathbf{R}$, $(y, z) \mapsto \Gamma(y, z, t)$ is a continuous function on $X \times (\mathbf{C}\backslash\mathscr{S})$.

From the relation $\Gamma(\alpha_t(y), z, 0) = \Gamma(y, z, t)$ it follows that for a fixed z, the function $G_z(y) = \Gamma(y, z, 0)$ has a zero on X if and only if there is a $\xi \in X$ such that one of the spaces $\mathscr{E}^\pm_{\xi,z}$ contains a nonzero function which vanishes somewhere on $\mathbf{R}$. That is, G_z is not an invertible element in $C(X)$ if and only if one of the spaces $\mathscr{E}^\pm_{\xi,z}$ contains a nonzero function which vanishes somewhere on $\mathbf{R}$. Suppose that the proposition were false. That is, there were an $x \in X$ and a $\lambda < \mathscr{T}$ such that $\mathscr{E}^+_{x,\lambda} \cup \mathscr{E}^-_{x,\lambda}$ contains a nonzero function which vanishes on $\mathbf{R}$. Then the set $Z = \inf\{z \in (-\infty, \mathscr{T}) : G_z$ is not invertible in $C(X)\}$ would not be empty. Since $(y, z) \mapsto \Gamma(y, z, 0)$ is continuous and X is compact, the map $z \mapsto G_z$ from $\mathbf{C}\backslash\mathscr{S}$ into $C(X)$ is also continuous. Because the subset of non-invertible elements in the C^*-algebra $C(X)$ is closed, Z is closed in $(-\infty, \mathscr{T})$. We will next show that Z is also an open subset of $(-\infty, \mathscr{T})$.

Suppose that $\lambda_0 \in Z$ and suppose that $\mathscr{E}^+_{\xi,\lambda_0}$ contains a non-trivial function φ such that $\varphi(a) = 0$. Since λ_0 is real, we may assume that φ is a real-valued function. Furthermore, we may assume that $\varphi = E^+_{\xi,\lambda_0}\eta$ with some appropriate real-valued function η. That φ is not identically zero ensures that $\varphi'(a) \neq 0$. Hence φ has positive as well as negative values on $\mathbf{R}$. It follows from Proposition 2.5$'$ that there is an $\varepsilon > 0$ such that for $\omega \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, $E^+_{\xi,\omega}\eta$ also has both positive and negative values on $\mathbf{R}$. This means that $E^+_{\xi,\omega}\eta$ must vanish on $\mathbf{R}$. Hence G_ω is not invertible for $\omega \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. If $\mathscr{E}^-_{\xi,\lambda_0}$ contains a non-trivial function which vanishes on $\mathbf{R}$, then we can similarly show that G_ω vanishes for ω in a neighborhood of λ_0. Therefore the set Z is open. The first paragraph of the proof tells us that $Z \neq (-\infty, \mathscr{T})$. Thus we have created a nonempty, proper subset of $(-\infty, \mathscr{T})$ which is both open and closed. This contradiction completes the proof. $\square$

Proof of Proposition 1 4. For $(x, z, t) \in X \times [(\mathbf{C}\backslash\mathbf{R}) \cup (-\infty, \mathscr{T})] \times \mathbf{R}$, define

$$M_+(x, z, t) = \varphi_+'(t)/\varphi_+(t)$$

and

$$M_-(x, z, t) = \varphi_-'(t)/\varphi_-(t)\,,$$

where $\varphi_\pm \in \mathscr{E}^\pm_{x,z}$ are functions which are not identically zero. The fact that $\dim(\mathscr{E}^\pm_{x,z}) = 1$ guarantees that $M_\pm$ are independent of the choice of $\varphi_\pm$. It follows from Proposition 2.10 and the discussion preceding it that the values of $M_\pm$ are finite numbers. It follows from Proposition 2.5$'$ that for each $t \in \mathbf{R}$, the functions $(x, z) \mapsto M_\pm(x, z, t)$ are continuous on $X \times [(\mathbf{C}\backslash\mathbf{R}) \cup (-\infty, \mathscr{T})]$. Finally, if $\varphi \in \mathscr{E}^\pm_{x,z}$, then $\varphi(s + .) \in \mathscr{E}^\pm_{\alpha_s(x),z}$. Therefore $M_\pm(x, z, t) = M_\pm(\alpha_t(x), z, 0)$. For $\lambda \in (-\infty, \mathscr{T})$, it is straightforward to verify that

$$P(x)M_\pm^2(x, \lambda, 0) + \frac{d}{dt}P(\alpha_t(x))M_\pm(\alpha_t(x), \lambda, 0)\Big|_{t=0} = Q(x) - \lambda\,. \qquad \square$$

3. The Dual Extreme Problem and Quasi-Eigenvalues

We will next investigate the extreme problem dual to $\mathscr{T} = -d_P(Q) = -\inf\{\|u - Q\|_\infty : u \in \mathscr{F}(P)\}$ in the case $\mathscr{T} < 0$. In other words we will express $d_P(Q)$ as a supremum. This dual extreme problem involves probability measures on X. By a probability measure on X we mean a positive, regular Borel measure on X whose total mass is 1. Suppose that μ is such a measure and consider the space $L^2(X, \mu)$. Let $\|.\|_{\mu,2}$ and $\langle\, .\,, .\,\rangle_\mu$ denote the norm and the inner product on $L^2(X, \mu)$ respectively. On the subset $C^1_\alpha(X) \subset L^2(X, \mu)$, we denote by D_μ the differential operator $(D_\mu f)(x) = \lim_{\varepsilon\to 0}(f(\alpha_\varepsilon(x)) - f(x))/\varepsilon = f'(x)$. (The operator D_μ is actually independent of μ; it is its adjoint D^*_μ that depends on μ.) Let $\mathscr{M}$ be the collection of probability measures μ on X such that the constant function 1 belongs to the domain of D^*_μ. It is obvious that a probability measure μ on X belongs to $\mathscr{M}$ if and only if there is a $C \geq 0$ such that

$$\left|\int_X (D_\mu f)(x)\, d\mu(x)\right| \leq C\|f\|_{\mu,2}$$

for every $f \in C^1_\alpha(X)$. Because $C^1_\alpha(X)$ is dense in $L^2(X, \mu)$, in the case $\mu \in \mathscr{M}$, the smallest such value C is $\|D^*_\mu 1\|_{\mu,2}$.

Theorem 3.1. (a) *Suppose that $d_P(Q) > 0$ Then*

$$d_P(Q) = \sup\left\{ -\langle Q, 1\rangle_\mu - \frac{1}{4}\langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu : \mu \in \mathscr{M}\right\}.$$

(b) *In any case,*

$$\mathscr{T} = \inf\left\{\langle Q, 1\rangle_\mu + \frac{1}{4}\langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu : \mu \in \mathscr{M}\right\}$$

Proof. (a) Suppose that $0 < \lambda < d_P(Q)$. Then, since $Q + \lambda \notin \mathscr{F}(P)$, there is a real-valued regular Borel measure ν and a number $a \in \mathbf{R}$ such that

$$\int_X (Q+\lambda)\,d\nu < -a \le \int_X u\,d\nu \tag{3.1}$$

for every $u \in \mathscr{F}(P)$. Since $\mathscr{F}(P)$ contains every non-negative function in $C(X)$, $a \ge 0$ and ν is a positive measure. Let $\mu = \nu/\nu(X)$ and

$$-b = \inf\left\{ \int_X u\,d\mu : u \in \mathscr{F}(P) \right\}.$$

Hence $\int_X Pf^2 d\mu + b \ge \int_X (Pf)'\,d\mu$ for every real function $f \in C^1_\alpha(X)$. This implies that

$$\int_X Pf^2 d\mu + b \ge \left| \int_X D_\mu Pf\;d\mu \right|$$

for every real function $f \in C^1_\alpha(X)$. Replacing f by tf with $t \in (0,\infty)$, we see that

$$t\int_X Pf^2 d\mu + b/t \ge \left| \int_X D_\mu Pf\,d\mu \right|.$$

If we set $t = \left[b/\int_X Pf^2 d\mu \right]^{1/2}$, then

$$2\sqrt{b}[\langle Pf, f\rangle_\mu]^{1/2} \ge |\langle D_\mu Pf, 1\rangle_\mu|\,.$$

Hence the constant function 1 belongs to the domain of D^*_μ. The above inequality can be rewritten as

$$2\sqrt{b}\|f\|_{\mu_P,2} \ge |\langle f, D^*_\mu 1\rangle_{\mu_P}|\,,$$

where $\mu_P(E) = \int_E P\,d\mu$. It follows from the equality $\langle g, D^*_\mu 1\rangle_\mu = \langle D_\mu g, 1\rangle_\mu$ that $D^*_\mu 1$ is a real-valued function in $L^2(X,\mu)$. Therefore the above inequality implies that

$$2\sqrt{b} \ge \|D^*_\mu 1\|_{\mu_P,2}\,. \tag{3.2}$$

If $b = 0$, $\|D^*_\mu 1\|_{\mu_P,2} = 0$. Suppose that $b > 0$. Because $\mathscr{F}_0(P)$ is dense in $\mathscr{F}(P)$ (Proposition 1.6), for any $\varepsilon \in (0,b)$ there is a real-valued $g \in C^1_\alpha(X)$ such that

$$\int_X (Pg^2 + (Pg)')\,d\mu \le -b + \varepsilon\,,$$

i.e.

$$\|g\|^2_{\mu_P,2} + b - \varepsilon \le |\langle g, D^*_\mu 1\rangle_{\mu_P}|\,.$$

Therefore

$$2\sqrt{b-\varepsilon}\|g\|_{\mu_P,2} \le \|g\|^2_{\mu_P,2} + b - \varepsilon \le |\langle g, D^*_\mu 1\rangle_{\mu_P}|\,.$$

This implies that $\|D\mu^*1\|_{\mu_P,2} \geq 2\sqrt{b}$. Combining this with (3.2), we have

$$b = \frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 .$$

It follows from (3.1) that

$$\int_X (Q+\lambda)\,d\mu < -b = \frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 .$$

That is,

$$\lambda < -\int_X Q\,d\mu - \frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 .$$

This proves that

$$d_P(Q) \leq \sup\left\{ -\langle Q,1\rangle_\mu - \frac{1}{4}\langle PD_\mu^*1, D_\mu^*1\rangle_\mu : \mu \in \mathscr{M}\right\}. \tag{3.3}$$

Suppose now that μ is an arbitrary measure in $\mathscr{M}$. For any real-valued function $f \in C_\alpha^1(X)$, we have

$$\begin{aligned} -\int_X (Pf)'\,d\mu &\leq |\langle D_\mu Pf, 1\rangle_\mu| = |\langle f, D_\mu^*1\rangle_{\mu_P}| \\ &\leq 2\|f\|_{\mu_P,2}\left(\frac{1}{2}\|D_\mu^*1\|_{\mu_P,2}\right) \leq \|f\|_{\mu_P,2}^2 + \frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 . \end{aligned}$$

In other words,

$$-\frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 \leq \int_X (Pf^2 + (Pf)')\,d\mu .$$

Adding $-\int_X Q\,d\mu$ to both sides, we have

$$-\int_X Q\,d\mu - \frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 \leq \int_X (Pf^2 + (Pf)' - Q)\,d\mu \leq \|Pf^2 + (Pf)' - Q\|_\infty .$$

Since $\mathscr{F}_0(P)$ is dense in $\mathscr{F}(P)$, we have $-\int_X Q\,d\mu - \frac{1}{4}\|D_\mu^*1\|_{\mu_P,2}^2 \leq d_P(Q)$. Combining this with (3.3), we have

$$d_P(Q) = \sup\left\{ -\langle Q,1\rangle_\mu - \frac{1}{4}\langle PD_\mu^*1, D_\mu^*1\rangle_\mu : \mu \in \mathscr{M}\right\}.$$

(b) Take any $\lambda \in \mathbf{R}$ such that $\mathscr{T}(P, Q-\lambda) < 0$. Then it follows from part (a) and Theorem 1.5 that

$$\begin{aligned} \mathscr{T}(P,Q) &= \mathscr{T}(P,Q-\lambda) + \lambda = -d_P(Q-\lambda) + \lambda \\ &= \inf\left\{ \langle Q-\lambda, 1\rangle_\mu + \frac{1}{4}\langle PD_\mu^*1, D_\mu^*1\rangle_\mu : \mu \in \mathscr{M}\right\} + \lambda \\ &= \inf\left\{ \langle Q, 1\rangle_\mu + \frac{1}{4}\langle PD_\mu^*1, D_\mu^*1\rangle_\mu : \mu \in \mathscr{M}\right\}. \quad \square \end{aligned}$$

Define

$$\mathscr{M}(P,Q)=\left\{\mu\in\mathscr{M}:\mathscr{T}=\langle Q,1\rangle_{\mu}+\frac{1}{4}\langle PD_{\mu}^{*}1,D_{\mu}^{*}1\rangle_{\mu}\right\}.$$

Theorem 3.2. *Suppose that x is a point in X such that $\alpha_t(x)\neq x$ for every $t\neq 0$ Then the following are equivalent.*
(a) *There exists a $\mu\in\mathscr{M}(P,Q)$ which is concentrated on the orbit $\{\alpha_t(x):x\in\mathbf{R}\}$.*
(b) *$\mathscr{T}$ is an eigenvalue for the self-adjoint operator L_x*

Proof. (b) $\Rightarrow$ (a): Suppose that $u\in L^2(\mathbf{R})$ is a real-valued unit vector such that $L_x u=\mathscr{T}u$. Then $\mu(\{\alpha_t(x):t\in E\})=\int_E u^2(s)\,ds$ define a probability measure concentrated on the orbit $\{\alpha_t(x):t\in\mathbf{R}\}$. It is easy to see that $\int_X f\,d\mu=\int_{\mathbf{R}} f(\alpha_t(x))u^2(t)\,dt$ for $f\in C(X)$. Furthermore, for $f\in C_\alpha^1(X)$, $\int_X f'\,d\mu=\int_{\mathbf{R}} u^2(t)\frac{d}{dt}f(\alpha_t(x))\,dt=-2\int_{\mathbf{R}} f(\alpha_t(x))u(t)u'(t)\,dt$. Hence $(D_\mu^*1)(\alpha_t(x))$ $=-2u'(t)/u(t)$. (Because u is a non-trivial solution of the differential equation $\ell_x u=\mathscr{T}u$, the set $\{t\in\mathbf{R}:u(t)=0\}$ has no limit points. Therefore $u'(t)/u(t)$ is well defined for a.e. $t\in\mathbf{R}$.) Thus $\frac{1}{4}\langle PD_\mu^*1,D_\mu^*1\rangle_\mu=\int_{\mathbf{R}} P_x(t)(u'(t))^2dt=-\int_{\mathbf{R}} u(t)[P_x(t)u'(t)]'\,dt$. Therefore $\langle Q,1\rangle_\mu+\frac{1}{4}\langle PD_\mu^*1,D_\mu^*1\rangle_\mu=\mathscr{T}$.

(a) $\Rightarrow$ (b): Suppose that $\mu\in\mathscr{M}(P,Q)$ is concentrated on $\{\alpha_t(x):t\in\mathbf{R}\}$. Then $\tilde{\mu}(E)=\mu(\{\alpha_t(x):t\in E\})$ is a probability measure on $\mathbf{R}$. It is easy to see that $\int_X\varphi\,d\mu=\int_{\mathbf{R}}\varphi_x\,d\tilde{\mu}$ for every $\varphi\in C(X)$. Therefore,

$$\langle\varphi_x,\psi_x\rangle_{\tilde{\mu}}=\langle\varphi,\psi\rangle_\mu$$

for all φ, $\psi\in C(X)$ and, in particular,

$$\langle Q_x,1\rangle_{\tilde{\mu}}=\langle Q,1\rangle_\mu\,.$$

Let $\mathscr{C}=\{\varphi_x:\varphi\in C_\alpha^1(X)\}$. We claim that $\mathscr{C}$ is dense in $L^2(\mathbf{R},\tilde{\mu})$. Let $f\in C_c(\mathbf{R})$, let g be a C^∞ function on $\mathbf{R}$ with a support contained in $[-A,A]$, and let $\varepsilon>0$ be given. There is an $N>0$ such $\tilde{\mu}([-N,N])=\mu(\{\alpha_t(x):-N\le t\le N\})\ge 1-\varepsilon$. The function $F(\alpha_t(x))=f(t)$ is continuous on the compact subset $K=\{\alpha_t(x):-A-N-1\le t\le A+N+1\}$. By Tietze's extension theorem, F can be extended to a continuous function on X such that $\|F\|_\infty=\|f\|_\infty$. Obviously the function $g*F(z)=\int_{\mathbf{R}} g(s)F(\alpha_s(z))\,ds$ belongs to $C_\alpha^1(X)$. Since g is supported on $[-A,A]$, we have $(g*F)_x(t)=\int_{\mathbf{R}} g(s)F(\alpha_{s+t}(x))\,ds=\int_{\mathbf{R}} g(s)f(t+s)\,ds$ for $-N\le t\le N$. Hence

$$\int_{\mathbf{R}}|g*f-(g*F)_x|^2d\tilde{\mu}=\int_{|t|\ge N}|g*f-(g*F)_x|^2d\tilde{\mu}$$
$$\le\varepsilon\|g*f-(g*F)_x\|_\infty^2\le 4\varepsilon\|f\|_\infty^2\|g\|_1^2\,.$$

This shows that $g*f$ is contained in the closure of $\mathscr{C}$ in $L^2(\mathbf{R},\tilde{\mu})$. Hence $\mathscr{C}$ is dense in $L^2(\mathbf{R},\tilde{\mu})$.

Define $D_{\tilde{\mu}}(\varphi_x) = (D_\mu \varphi)_x = (\varphi')_x = (\varphi_x)'$ for $\varphi \in C^1_\alpha(X)$. Because

$$\sup\left\{\left|\int_{\mathbf{R}} D_{\tilde{\mu}}(\varphi_x)\,d\mu\right| : \varphi \in C^1_\alpha(X),\ \|\varphi_x\|_{\tilde{\mu},2} = 1\right\}$$
$$= \sup\left\{\left|\int_{\mathbf{R}} D_\mu \varphi\,d\mu\right| : \varphi \in C^1_\alpha(X)\,,\ \|\varphi\|_{\mu,2} = 1\right\} = \|D^*_\mu 1\|_{\mu,2}\,,$$

and because $\mathscr{C}$ is dense in $L^2(\mathbf{R}, \tilde{\mu})$, there is a unique $h = D^*_{\tilde{\mu}} 1 \in L^2(\mathbf{R}, \tilde{\mu})$ such that

$$\int_{\mathbf{R}} f'\,d\mu = \langle f, h\rangle_{\tilde{\mu}}$$

for every $f \in \mathscr{C}$. Suppose that g is a compactly supported C^∞ function on $\mathbf{R}$. Then

$$\int_{\mathbf{R}} g' * (\varphi_x)\,d\tilde{\mu} = \int_{\mathbf{R}} [g' * \varphi]_x d\tilde{\mu} = -\int_{\mathbf{R}} [(g * \varphi)']_x\,d\tilde{\mu}$$
$$= -\int_{\mathbf{R}} [(g * \varphi)_x]'\,d\tilde{\mu} = -\langle (g * \varphi)_x, h\rangle_{\tilde{\mu}} = -\langle g * (\varphi_x), h\rangle_{\tilde{\mu}}$$

for every $\varphi \in C(X)$. Let $\xi \in C_c(\mathbf{R})$. Using Tietze's extension theorem once more, we see that there is a sequence $\{\varphi_n\} \subset C(X)$ such that $(\varphi_n)_x = \xi$ on $[-n, n]$ and $\|\varphi_n\|_\infty = \|\xi\|_\infty$. Therefore $\|g' * \varphi_n\|_\infty \le \|g'\|_1 \|\xi\|_\infty$ and $\lim_{n\to\infty} [g' * ((\varphi_n)_x)](t) = g' * \xi(t)$ for every $t \in \mathbf{R}$. By the dominated convergence theorem, we have

$$\int_{\mathbf{R}} (g') * \xi\,d\tilde{\mu} = -\langle g * \xi, h\rangle_{\tilde{\mu}}\,.$$

In particular, if $\xi \in C^1_c(\mathbf{R})$, then

$$\int_{\mathbf{R}} g * (\xi')\,d\tilde{\mu} = -\int_{\mathbf{R}} (g') * \xi\,d\tilde{\mu}\,. = \langle g * \xi, h\rangle_{\tilde{\mu}}$$

Hence

$$\int_{\mathbf{R}} \xi'\,d\tilde{\mu} = \langle \xi, h\rangle_{\tilde{\mu}}$$

for every $\xi \in C^1_c(\mathbf{R})$. This in particular implies that h is a real-valued function. The map $\varphi \mapsto \varphi_x$, $\varphi \in C^1_\alpha(X)$, extends to a unitary operator U from $L^2(X, \mu)$ onto $L^2(\mathbf{R}, \tilde{\mu})$. Since $\langle \varphi, D^*_\mu 1\rangle_\mu = \langle \varphi', 1\rangle_\mu = \langle (\varphi_x)', 1\rangle_{\tilde{\mu}} = \langle U\varphi, h\rangle_{\tilde{\mu}} = \langle \varphi, U^* h\rangle_\mu$ for every $\varphi \in C^1_\alpha(X)$, we have $h = UD^*_\mu 1$. Similarly, it follows from $\langle UPD^*_\mu 1, \varphi_x\rangle_{\tilde{\mu}} = \langle PD^*_\mu 1, \varphi\rangle_\mu = \langle D^*_\mu 1, P\varphi\rangle_\mu = \langle h, P_x \varphi_x\rangle_{\tilde{\mu}} = \langle P_x h, \varphi_x\rangle_{\tilde{\mu}}$ that $P_x h = UPD^*_\mu 1$. Hence

$$\langle Q_x, 1\rangle_{\tilde{\mu}} + \frac{1}{4}\langle P_x h, h\rangle_{\tilde{\mu}} = \langle Q, 1\rangle_\mu + \frac{1}{4}\langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu = \mathscr{I}\,.$$

For any $f \in C^1_\alpha(X)$, we have

$$f \circ \alpha_t - f \circ \alpha_s = \int_s^t (f \circ \alpha_\lambda)' \, d\lambda \, .$$

Therefore

$$\left| \int_X (f \circ \alpha_t - f \circ \alpha_s) \, d\mu \right| \leq \int_s^t \left| \int_X (f \circ \alpha_\lambda)' \, d\mu \right| d\lambda$$
$$\leq |t-s| \max_{s \leq \lambda \leq t} \|f \circ \alpha_\lambda\|_{\mu,2} \|D^*_\mu 1\|_{\mu,2} \leq \|f\|_\infty \|D^*_\mu 1\|_{\mu,2} |t-s|$$

for every $f \in C^1_\alpha(X)$. Since $C^1_\alpha(X)$ is dense in $C(X)$, we have

$$\left| \int_X (f \circ \alpha_t - f \circ \alpha_s) \, d\mu \right| \leq \|f\|_\infty \|D^*_\mu 1\|_{\mu,2} |t-s| \qquad (3\ 3)$$

for every $f \in C(X)$. By the regularity of μ, this implies that if Δ is a Borel set in X, then

$$|\mu(\alpha_t(\Delta)) - \mu(\alpha_s(\Delta))| \leq \|D^*_\mu 1\|_{\mu,2} |t-s| \, .$$

Hence for every Borel subset $E \subset \mathbf{R}$,

$$|\tilde{\mu}(E+t) - \tilde{\mu}(E+s)| \leq \|D^*_\mu 1\|_{\mu,2} |t-s|$$

In particular, the function $t \mapsto \tilde{\mu}(E+t)$ is continuous. It follows from this continuity that $\tilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$. Therefore $d\tilde{\mu}(t) = g_0(t)\, dt$ with some $g_0 \in L^1(\mathbf{R}, dt)$. Since $h \in L^2(\mathbf{R}, \tilde{\mu}) \subset L^1(\mathbf{R}, \tilde{\mu}) = L^1(\mathbf{R}, g_0\, dt)$, hg_0 is Lebesgue integrable on $\mathbf{R}$. The equality

$$\int_{\mathbf{R}} f'(t) g_0(t) \, dt = \int_{\mathbf{R}} f(t) h(t) g_0(t) \, dt$$

holds for every $f \in C^1_c(\mathbf{R})$. From this it is easy to see that if s and t are Lebesgue points for g_0, then

$$g_0(t) - g_0(s) = \int_s^t h(\lambda) g_0(\lambda) \, d\lambda \, .$$

Since $g_0 \in L^1(\mathbf{R}, dt)$, there exists a sequence $\{s_n\}$ of Lebesgue points of g_0 such that $s_n \to -\infty$ and $g_0(s_n) \to 0$. Hence

$$g_0(t) = \int_{-\infty}^t h(\lambda) g_0(\lambda) \, d\lambda$$

if t is a Lebesgue point of g_0. Now define

$$g_1(t) = \int_{-\infty}^t h(\lambda) g_0(\lambda) \, d\lambda \, .$$

for *every* $t \in \mathbf{R}$. Then g_1 is absolutely continuous on $\mathbf{R}$ and we have $d\tilde{\mu}(t) = g_1(t)\,dt$. Let $u = \sqrt{g_1}$ on $\mathbf{R}$. Then $hu \in L^2(\mathbf{R})$ and

$$\langle Q_x u, u\rangle + \frac{1}{4}\langle P_x hu, hu\rangle = \langle Q_x, 1\rangle_{\tilde{\mu}} + \frac{1}{4}\langle Ph, h\rangle_{\tilde{\mu}} = \mathscr{T}\,.$$

We claim that u is absolutely continuous on every finite interval and that $(u')^2 = h^2 g_1/4 = (hu)^2/4$.

To prove this claim, we start with the identity

$$\int_{\mathbf{R}} f(t)h(t)g_1(t)\,dt = \int_{\mathbf{R}} f'(t)g_1(t)\,dt = -\int_{\mathbf{R}} f(t)g_1'(t)\,dt$$

for $f \in C_c^1(\mathbf{R})$. Since the closed set $F = \{s \in \mathbf{R} : g_1(s) = 0\}$ has at most a countable number of isolated points, we have $g_1'(t) = 0$ for almost every $t \in F$ (with respect to the Lebesgue measure). Therefore we may choose $h(t) = -g_1'(t)/g_1(t)$ if $g_1(t) \neq 0$ and $h(t) = 0$ if $g_1(t) = 0$. That $h \in L^2(\mathbf{R}, g_1 dt)$ means that $(g_1')^2/g_1 = h^2 g_1$ is Lebesgue integrable on $\mathbf{R}\backslash F$. Therefore for any finite interval $[a, b]$, the function $g_1'/\sqrt{g_1} = [(g_1')^2/g_1]^{1/2} = [h^2/g_1]^{1/2}$ is Lebesgue integrable on $[a, b]\backslash F$. We will now use this fact to prove that $u = \sqrt{g_1}$ is absolutely continuous on any finite interval.

Since $g_1 \geq 0$, for any positive integer n, $u_n = \sqrt{g_1 + (1/n)}$ is absolutely continuous on finite intervals. Hence for any $a < b$,

$$\begin{aligned} u_n(b) - u_n(a) &= \int_a^b u_n'(t)\,dt = \int_a^b \frac{g_1'(t)}{2\sqrt{g_1(t) + (1/n)}}\,dt \\ &= \int_{[a,b]\backslash F} \frac{g_1'(t)}{2\sqrt{g_1(t) + (1/n)}}\,dt\,. \end{aligned}$$

Since $g_1'/\sqrt{g_1}$ is Lebesgue integrable on $[a, b]\backslash F$, it follows from the dominated convergence theorem that

$$u(b) - u(a) = \int_{[a,b]\backslash F} [g_1'(t)/2\sqrt{g_1(t)}]\,dt\,.$$

This proves that u is absolutely continuous on any given interval.

On the other hand, since $\mathbf{R}\backslash F$ is open, $u = \sqrt{g_1}$ is almost everywhere differentiable on $\mathbf{R}\backslash F$ and $2uu' = g_1'$. Hence the equality

$$(u')^2 = (g_1')^2/4g_1 = h^2 g_1/4$$

holds on on $\mathbf{R}\backslash F$ a.e. This shows that $u' \in L^2(\mathbf{R}) = L^2(\mathbf{R}, dt)$. To complete the proof, we will show that u belongs to the domain of L_x and $L_x u = \mathscr{T} u$. Let $\mathscr{D}_1 = \{f \in L^2(\mathbf{R}) : f$ is absolutely continuous on every finite interval and $f' \in L^2(\mathbf{R})\}$. Define

$$H(f, g) = \langle Q_x f, g\rangle + \langle P_x f', g'\rangle - \mathscr{T}\langle f, g\rangle$$

for $f, g \in \mathscr{D}_1$. Since $\langle u, u\rangle = \int_{\mathbf{R}} g_1(t)\,dt = 1$, we have

$$H(u,u) = \langle Q_x u, u\rangle + \langle P_x u', u'\rangle - \mathscr{T}$$
$$= \langle Q_x u, u\rangle + \frac{1}{4}\langle P_x hu, hu\rangle - \mathscr{T} = 0\,.$$

We claim that $H(g,g) \geq 0$ for every $g \in \mathscr{D}_1$. Suppose that g belongs to the domain $\mathscr{D}_2 = \{f \in \mathscr{D}_1 : f' \in \mathscr{D}_1\}$. Then $H(g,g) = \langle (L_x - \mathscr{T})g, g\rangle \geq 0$. For an arbitrary $g \in \mathscr{D}_1$, one find a sequence $\{g_n\} \subset \mathscr{D}_2$ such that $\|g_n - g\|_2 \to 0$ and $\|g_n' - g'\|_2 \to 0$ Hence we always have $H(g,g) \geq 0$ for $g \in \mathscr{D}_1$. The positivity of $H(g,g)$ implies that the Cauchy-Schwarz inequality holds for the Hermitian form H. Hence

$$|H(f,u)| \leq [H(f,f)]^{1/2}[H(u,u)]^{1/2} = 0$$

for every $f \in \mathscr{D}_1$. In particular, if $f \in \mathscr{D}_2$, we have

$$\langle (L_x - \mathscr{T})f, u\rangle = H(f,u) = 0\,.$$

This implies $u \in \mathscr{D}_2$ and $L_x u = \mathscr{T} u$. □

Remark 1 The proof clearly shows that in the case $\mathscr{T}$ is an eigenvalue for L_x, there is an eigenvector u which is non-negative on **R**. Since u and u' cannot have common zeros, we have $u(t) > 0$ for every $t \in \mathbf{R}$. That is, non-trivial eigenvectors of L_x corresponding to the eigenvalue $\mathscr{T}$ do not vanish on **R**.

Remark 2 The proof also shows that if μ is a finite positive Borel measure on **R** such that

$$\left|\int_{\mathbf{R}} f' d\mu\right| \leq C\left[\int_{\mathbf{R}} |f|^2 d\mu\right]^{1/2}$$

for every $f \in C_c^1(\mathbf{R})$, then there is a $u \geq 0$ in $\mathscr{D}_1$ such that

$$d\mu(t) = u^2(t)\,dt\,.$$

The analogue of this result on the unit circle T can be established using the same argument. (Actually the case of unit circle is covered by Theorem 3.3 below.) It was proved in [6] that if a finite positive Borel measure ν on T has the property that

$$\left|\int_T p' d\nu\right| \leq C\left[\int_T |p|^2 d\nu\right]^{1/2}$$

for every trigonometric polynomial p, then

$$d\nu(t) = \sum_k |w_k(t)|^2 dt\,,$$

where $w_k' s$ are absolutely continuous functions on T such that $\sum_k |w_k'|^2 \in L^1$. In [6], this result was obtained through an operator-theoretical approach. And the terms w_k, while not unique, have meanings in the related operator theory. Our approach, which uses real analysis only, yields an absolutely continuous function $u \geq 0$ on T with $|u'|^2 \in L^1$ such that $d\nu(t) = u^2(t)\,dt$. In other words, $\sum_k |w_k|^2$ always has a square root u in the $\mathscr{D}_1$ of the unit circle.

Remark 3 There are examples of almost periodic potential Q (with $P = 1$), where $\mathscr{T}$ is an eigenvalue for L_x for some x (see [3, 5]). On the other hand, if the flow $(X, \{\alpha_s : s \in \mathbf{R}\})$ is minimal, it is easy to show using [8, Proposition 2.11] that for almost every $x \in X$ (with respect to any given ergodic measure), $\mathscr{T}$ is not an eigenvalue for L_x.

Remark 4 The theorem is false when the assumption $\alpha_t(x) \neq x$ for every $t \neq 0$ is dropped. In fact if $\alpha_a(x) = x$ with some $a \neq 0$, then Q_x and P_x are periodic functions. In this case the spectrum of the operator L_x is known to be absolutely continuous. But in the case $\alpha_a(x) = x$, the orbit $\{\alpha_t(x) : t \in \mathbf{R}\}$ is either a circle or a single point. Therefore there is an invariant probability measure of the flow concentrated on the orbit. In general $\mathscr{T}$ may still be an eigenvalue for a Schrödinger operator with the same coefficients P and Q but on a different L^2-space.

Suppose that m is an invariant probability measure of the dynamical system $(X, \mathbf{R}, \alpha)$. Then $D_m\varphi = \varphi'$ is a skew-symmetric operator on $L^2(X, m)$. In fact it is the infinitesimal generator of the unitary group $u_t\varphi = \varphi \circ \alpha_t$, $\varphi \in L^2(X, m)$, $t \in \mathbf{R}$. Let $\mathscr{D}_1^m$ denote the domain of D_m and let $\mathscr{D}_2^m$ denote the domain of $D_m^* D_m = -D_m^2$. Then $\mathscr{D}_2^m = \{\varphi \in \mathscr{D}_1^m : D_m\varphi \in \mathscr{D}_1^m\}$ and it is also the domain of the self-adjoint operator

$$L_m = -D_m Q D_m + P$$

Suppose that $\eta \in \mathscr{D}_2^m$ with $\|\eta\|_{dm,2} = 1$. Then $d\mu_\eta = |\eta|^2 dm$ is a probability measure on X. For $f \in C_\alpha^1(X)$, we have

$$\int_X f' d\mu_\eta = \langle \eta D_m f, \eta \rangle_m = \langle D_m f\eta, \eta \rangle_m - \langle f D_m \eta, \eta \rangle_m$$

$$= -[\langle f\eta, D_m\eta \rangle_m + \langle f D_m\eta, \eta \rangle_m] = \int_X f[-\eta(D_m\bar{\eta}) - \bar{\eta}(D_m\eta)]\, dm$$

$$= \int_X f h_\eta |\eta|^2 dm = \langle f, h_\eta \rangle_{\mu_\eta},$$

where $h_\eta(x) = -2\,\mathrm{Re}[(D_m\eta)(x)/\eta(x)]$ when $\eta(x) \neq 0$ and $h_\eta(x) = 0$ when $\eta(x) = 0$. This means that each $\eta \in \mathscr{D}_2^m$ gives rise to a $\mu_\eta \in \mathscr{M}$ with $D_{\mu_\eta}^* 1 = h_\eta$. Furthermore,

$$\langle PD_m\eta, D_m\eta \rangle_m = \int_X P|D_m\eta|^2 dm \geq \frac{1}{4}\int_X P h_\eta^2 |\eta|^2 dm = \frac{1}{4}\langle PD_{\mu_\eta}^* 1, D_{\mu_\eta} 1 \rangle_{\mu_\eta}$$

Hence, by Theorem 3.1, we have

$$\begin{aligned} \langle L_m\eta, \eta \rangle_m &= \langle Q\eta, \eta \rangle_m + \langle PD_m\eta, D_m\eta \rangle_m \\ &\geq \langle Q, 1 \rangle_{\mu_\eta} + \frac{1}{4}\langle PD_{\mu_\eta}^* 1, D_{\mu_\eta} 1 \rangle_{\mu_\eta} \geq \mathscr{T}. \end{aligned}$$

Theorem 3.3. *Let m be an invariant probability measure of the flow $(X, \mathbf{R}, \alpha)$ Then $\mathscr{T}$ is an eigenvalue for L_m if and only if there exist a $\mu \in \mathscr{M}(P, Q)$ which is absolutely continuous with respect to m*

Proof Suppose that $\mathscr{T}$ is an eigenvalue for L_m and let $u \in \mathscr{D}_2^m$ be a unit vector such that $L_m\eta = \mathscr{T}u$. From the discussion preceding the theorem we see that $\mu_u \in \mathscr{M}(P, Q)$.

Let us now prove the other implication. Let μ be a measure in $\mathscr{M}(P,Q)$ which is absolutely continuous with respect to m. Suppose that $d\mu = g\,dm$. As in the proof of the previous theorem, we will show that $u = \sqrt{g}$ represents an element in the domain of L_m and that $L_m u = \mathscr{T}u$. The first step is to show that u represents an element in $\mathscr{D}_1^m$.

Denote $h = D_\mu^* 1 \in L^2(X, g\,dm) = L^2(X,\mu)$. Then

$$\int_X f' g\,dm = \int_X fhg\,dm$$

for every $f \in C_\alpha^1(X)$. By the Cauchy-Schwarz inequality, $hg \in L^1(X, dm)$. Since $\frac{d}{dr} f \circ \alpha_r = (f \circ \alpha_r)'$, we have

$$\begin{aligned}\int_X (f\circ\alpha_t - f\circ\alpha_s) g\,dm &= \int_X \left[\int_s^t \frac{d}{d\lambda} f\circ\alpha_\lambda d\lambda\right] g\,dm \\ &= \int_X \left[\int_s^t (f\circ\alpha_\lambda)' d\lambda\right] g\,dm \\ &= \int_t^s \left[\int_X f\circ\alpha_\lambda hg\,dm\right] d\lambda\,. \end{aligned} \tag{3.5}$$

Hence for $\delta \neq 0$ and $f \in C_\alpha^1(X)$,

$$\begin{aligned}&\left|\int_X f\left[\frac{g\circ\alpha_\delta - g}{\delta} + hg\right] dm\right| \\ &= \left|\int_X \left[\frac{f\circ\alpha_{-\delta} - f}{\delta} g + fhg\right] dm\right| = \left|\frac{1}{\delta}\int_0^{-\delta}\left[\int_X (f\circ\alpha_\lambda - f)hg dm\right] d\lambda\right| \\ &= \left|\frac{1}{\delta}\int_0^{-\delta}\int_X f[(hg)\circ\alpha_{-\lambda} - hg]\,dm\,d\lambda\right| \le \|f\|_\infty \sup_{|\lambda|\le|\delta|} \|(hg)\circ\alpha_\lambda - hg\|_{dm,1}\,.\end{aligned}$$

This implies that

$$\lim_{\delta\to 0}\left\|\frac{g\circ\alpha_\delta - g}{\delta} - (-hg)\right\|_{dm,1} = 0. \tag{3.6}$$

For each $\varepsilon > 0$, let $u_\varepsilon = \sqrt{g+\varepsilon}$. For any $f \in C_\alpha^1(X)$,

$$\begin{aligned}\int_X f' u_\varepsilon\,dm &= \lim_{\delta\to 0}\int_X \frac{f\circ\alpha_\delta - f}{\delta} u_\varepsilon\,dm = \lim_{\delta\to 0}\int_X f\frac{u_\varepsilon\circ\alpha_{-\delta} - u_\varepsilon}{\delta}\,dm \\ &= \lim_{\delta\to 0}\int_X \left(f\frac{1}{u_\varepsilon\circ\alpha_{-\delta} + u_\varepsilon}\right)\frac{g\circ\alpha_{-\delta} - g}{\delta}\,dm\,.\end{aligned} \tag{3.7}$$

Upon choosing a sequence $\delta_n \to 0$ such that $u_\varepsilon \circ \alpha_{-\delta_n} \to u_\varepsilon$ a.e., we see from (3.6) and (3.7) that

$$\int_X f' u_\varepsilon \, dm = \int_X (fhg/2u_\varepsilon)\, dm \,. \tag{3.8}$$

Therefore

$$\left| \int_X f' u_\varepsilon dm \right| \le \left[\int_X |f|^2 dm \right]^{1/2} \left[\int_X (hg/2u_\varepsilon)^2 dm \right]^{1/2} \le \frac{1}{4} \left[\int_X |f|^2 dm \right]^{1/2} \left[\int_X h^2 g \, dm \right]^{1/2} .$$

Letting $\varepsilon \to 0$, we obtain $\left| \int_X f' u \, dm \right| \le \frac{1}{4} \|f\|_{dm,2} \|h\|_{g\,dm,2}$ for every $f \in C^1_\alpha(X)$. This inequality implies that $u \in \mathscr{D}_1^m$.

It follows from (3.8) that

$$\int_X f D_m u \, dm = - \int_X f' u \, dm = -\frac{1}{2} \int f(hg/u)\, dm \,,$$

where $(hg/u)(x)$ is defined to be 0 whenever $g(x) = 0$. Hence

$$\langle P D_m u, D_m u \rangle_m = \frac{1}{4} \int_X P h^2 g \, dm = \frac{1}{4} \langle P D^*_\mu 1, D^*_\mu 1 \rangle_\mu$$

Define

$$H_m(\xi, \eta) = \langle P D_m \xi, D_m \eta \rangle_m + \langle Q\xi, \eta \rangle_m - \mathscr{T} \langle \xi, \eta \rangle_m$$

for ξ, $\eta \in \mathscr{D}_1^m$. Then

$$H_m(u, u) = 0$$

It follows from the paragraph preceding the theorem that $H_m(f, f) = \langle (L_m - \mathscr{T}) f, f \rangle_m \ge 0$ for every $f \in \mathscr{D}_2^m$. For each $f \in \mathscr{D}_1^m$, there is a sequence $\{f_n\} \subset \mathscr{D}_2^m$ such that $\|f_n - f\|_{m,2} \to 0$ and $\|D_m f_n - D_m f\|_{m,2} \to 0$. Hence $H_m(f, f) \ge 0$ for every $f \in \mathscr{D}_1^m$. It follows from the Cauchy-Schwarz inequality for the positive Hermitian form H that

$$\langle (L_m - \mathscr{T}) f, u \rangle = H_m(f, u) = 0$$

whenever $f \in \mathscr{D}_2^m$. Hence $u \in \mathscr{D}_2^m$ and $L_m u = \mathscr{T}$. □

Theorem 3.4. *Suppose that X is metrizable Then $\mathscr{M}(P, Q)$ is a non-empty, convex, weak-* closed subset of $C(X)^*$*

Proof Suppose that $\mu_1, \cdots, \mu_n \in \mathscr{M}(P, Q)$ and $a_1, \cdots, a_n \in (0, 1]$ are such that $a_1 + \cdots + a_n = 1$. Let $\mu = a_1\mu_1 + \cdots + a_n\mu_n$. For $\nu = \mu$ or μ_j, we have

$$\langle P D^*_\nu 1, D^*_\nu 1 \rangle_\nu = \|\sqrt{P} D^*_\nu 1\|^2_{\nu,2} = \sup \left\{ \left| \int_X (\sqrt{P} f)' d\nu \right|^2 : f \in C^1_\alpha(X),\ \|f\|_{\nu,2} = 1 \right\} \tag{3 9}$$

Therefore for any $f \in C^1_\alpha(X)$,

$$\begin{aligned}\left|\int_X (\sqrt{P}f)'\,d\mu\right| &\le \sum_{j=1}^n a_j \left|\int_X (\sqrt{P}f)'\,d\mu_j\right| \le \sum_{j=1}^n a_j \|\sqrt{P}D^*_{\mu_j}1\|_{\mu_j,2}\|f\|_{\mu_j,2} \\ &\le \left[\sum_{j=1}^n a_j\|\sqrt{P}D^*_{\mu_j}1\|^2_{\mu_j,2}\right]^{1/2}\left[\sum_{j=1}^n a_j\|f\|^2_{\mu_j,2}\right]^{1/2} \\ &= \left[\sum_{j=1}^n a_j\|\sqrt{P}D^*_{\mu_j}1\|^2_{\mu_j,2}\right]^{1/2}\|f\|_{\mu,2}\,. \end{aligned} \tag{3.10}$$

This implies $\langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu \le \sum_{j=1}^n a_j\|\sqrt{P}D^*_{\mu_j}1\|^2_{\mu_j,2}$. Therefore

$$\begin{aligned}\mathcal{T} &\le \langle Q1,1\rangle_\mu + \frac{1}{4}\langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu \\ &\le \sum_{j=1}^n a_j\langle Q1,1\rangle_{\mu_j} + \frac{1}{4}\sum_{j=1}^n a_j\langle PD^*_{\mu_j}1, D^*_{\mu_j}1\rangle_{\mu_j} = \sum_{j=1}^n a_j\mathcal{T}\,.\end{aligned}$$

Hence $\mu \in \mathcal{M}(P,Q)$. This proves that $\mathcal{M}, (P,Q)$ is convex.

The assumption that X is metrizable implies the unit ball of the dual space $C(X)^*$ is metrizble in the weak-* topology. Hence there is a sequence $\{\mu_n\} \subset \mathcal{M}$ which converges to a probability measure μ on X in the weak-* topology and which has the property $\lim_{n\to\infty}[\langle Q,1\rangle_{\mu_n} + \frac{1}{4}\langle PD^*_{\mu_n}1, D^*_{\mu_n}1\rangle_{\mu_n}] = \mathcal{T}$. Then $\langle Q,1\rangle_{\mu_n} \to \langle Q,1\rangle_\mu$ and, therefore, $\langle PD^*_{\mu_n}1, D^*_{\mu_n}1\rangle_{\mu_n} \to 4(\mathcal{T} - \langle Q,1\rangle_\mu)$. Since $|\int_X(\sqrt{P}f)'d\mu_n|^2 \le \langle PD^*_{\mu_n}1, D^*_{\mu_n}1\rangle_{\mu_n}\ \|f\|^2_{\mu_n,2}$, $\|f\|^2_{\mu_n,2} \to \|f\|^2_{\mu,2}$, and $\int_X(\sqrt{P}f)'d\mu_n \to \int_X(\sqrt{P}f)'d\mu$ for every $f \in C^1_\alpha(X)$, we have

$$\left|\int_X(\sqrt{P}f)'d\mu\right|^2 \le 4(\mathcal{T} - \langle Q,1\rangle_\mu)\|f\|^2_{\mu,2}\,. \tag{3.11}$$

This implies that

$$\begin{aligned}\left|\int_X f'd\mu\right|^2 &= \left|\int_X(\sqrt{P}(f/\sqrt{P}))'d\mu\right|^2 \\ &\le 4(\mathcal{T} - \langle Q,1\rangle_\mu)\|f/\sqrt{P}\|^2_{\mu,2} \le 4(\mathcal{T} - \langle Q,1\rangle_\mu)\|1/\sqrt{P}\|^2_\infty\|f\|^2_{\mu,2}\end{aligned}$$

whenever $f \in C^1_\alpha(X)$. Hence the constant function 1 belongs to the domain of D^*_μ. It follows from (3.9) and (3.11) that $4(\mathcal{T} - \langle Q,1\rangle_\mu) \ge \langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu$. That is,

$$\langle Q,1\rangle_\mu + \frac{1}{4}\langle PD^*_\mu 1, D^*_\mu 1\rangle_\mu \le \mathcal{T}\,.$$

By Theorem 3.1, the reverse inequality always holds. Hence $\mu \in \mathcal{M}(P,Q)$. This proves that $\mathcal{M}(P,Q)$ is a closed non-empty set. □

Next we will linearize the extreme problem of finding $\mu \in \mathscr{M}$ for which the equality $\mathscr{T} = \langle Q, 1\rangle_\mu + \frac{1}{4}\langle PD_\mu^* 1, D_\mu^* 1\rangle_\mu$ holds. For each $r \geq 0$ let

$$\mathscr{M}_r(P) = \{\mu \in \mathscr{M} : \langle PD_\mu^* 1, D_\mu 1\rangle_\mu \leq r\}.$$

Proposition 3.5. *Suppose that X is metrizable. Then for each $r \geq 0$, $\mathscr{M}_r(P)$ is a convex subset of $C(X)^*$ which is closed in the weak-* topology.*

Proof. Suppose that $\mu_1, \dots, \mu_n \in \mathscr{M}_r(P)$ and $a_1, \dots, a_n \in (0, 1]$ are such that $a_1 + \cdots + a_n = 1$. Let $\mu = a_1\mu_1 + \cdots + a_n\mu_n$. By (3.10), we have

$$\left|\int_X (\sqrt{P} f)' d\mu\right| \leq \left[\sum_{j=1}^n a_j \|\sqrt{P} D_{\mu_j}^* 1\|_{\mu_j,2}^2\right]^{1/2} \|f\|_{\mu,2}$$

for every $f \in C_\alpha^1(X)$. By (3.9), this implies $\langle PD_\mu^* 1, D_\mu^* 1\rangle_\mu \leq \sum_{j=1}^n a_j \|\sqrt{P} D_{\mu_j}^* 1\|_{\mu_j,2}^2 \leq r$. Therefore $\mathscr{M}_r(P)$ is convex.

Suppose that $\{\mu_n\}$ is a sequence in $\mathscr{M}_r(P)$ which converges to some probability measure μ on X in the weak-* topology. Then for any $f \in C_\alpha^1(X)$,

$$\begin{aligned}\left|\int_X (\sqrt{P} f)' d\mu\right| &= \lim_{n\to\infty} \left|\int_X (\sqrt{P} f)' d\mu_n\right| \\ &\leq \limsup_{n\to\infty} \|\sqrt{P} D_{\mu_n}^* 1\|_{\mu_n,2} \|f\|_{\mu_n,2} \\ &\leq \sqrt{r} \lim_{n\to\infty} \|f\|_{\mu_n,2} = \sqrt{r}\|f\|_{\mu,2}\,.\end{aligned}$$

Hence $\langle PD_\mu^* 1, D_\mu^* 1\rangle_\mu \leq r$. □

Corollary 3.6. *Suppose that X is metrizable. Let $\mu_0 \in \mathscr{M}(P, Q)$ and let $r_0 = \langle PD_{\mu_0}^* 1, D_{\mu_0}^* 1\rangle_{\mu_0}$. Then there is an extreme point μ of the convex set $\mathscr{M}_{r_0}(P)$ such that*

$$\mathscr{T} = \langle Q, 1\rangle_\mu + \frac{1}{4}\langle PD_\mu^* 1, D_\mu^* 1\rangle_\mu\,.$$

Proof. Define $\varrho(\nu) = \mathrm{Re}(\int_X Q\, d\nu)$ on $C(X)^*$. It is a well-known fact that on a convex compact (in the weak-* topology) set such as $\mathscr{M}_{r_0}(P)$, the real functional ϱ attains its extreme values at extreme points. Hence there is an extreme point $\mu \in \mathscr{M}_{r_0}(P)$ such that $\langle Q, 1\rangle_\mu = \min\{\langle Q, 1\rangle_\nu : \nu \in \mathscr{M}_{r_0}(P)\}$. Therefore $\langle Q, 1\rangle_\mu + \frac{1}{4}\langle PD_\mu^* 1, D_\mu^* 1\rangle_\mu \leq \langle Q, 1\rangle_{\mu_0} + r_0/4 = \mathscr{T}$. □

Because of this corollary, the problem of finding $\mu \in \mathscr{M}(P, Q)$ is reduced to the problem of characterizing the extreme points of $\mathscr{M}_r(P)$ for $r \geq 0$. One should think of this as the linearization of the extreme problem of determining $\mathscr{T}$. The set $\mathscr{M}_r(P)$ is certainly more accessible than $\mathscr{M}(P, Q)$. The problem of determining the extreme points of $\mathscr{M}_r(P)$ is completely independent of the study of Schrödinger operators and is interesting in its own right. For example, $\mathscr{M}_0(P)$ consists of all the invariant measures of the flow $(X, \mathbf{R}, \alpha)$ and its extreme points are precisely the ergodic measures. So when we consider the extreme points of $\mathscr{M}_r(P)$, it seems that we are investigating a generalization of the notion of ergodicity. In view of Theorems 3.2 and 3.3, if we know what the extreme points of $\mathscr{M}_r(P)$ are then we can completely answer the question of whether or not $\mathscr{T}$ is an eigenvalue for L_x or for L_m.

4. When the Flow Is Generated by Functions

For a function f defined on $\mathbf{R}$ and an $s \in \mathbf{R}$, denote $f_s(t) = f(t+s)$. Suppose that p and q are real-valued, bounded, uniformly continuous functions on $\mathbf{R}$. In addition, we assume that p' is also bounded and uniformly continuous on $\mathbf{R}$ and that there is a $c > 0$ such that $p(t) \geq c$ for every $t \in \mathbf{R}$. We will now consider the Schrödinger operator

$$L = -\left(\frac{d}{dt}\right) p \left(\frac{d}{dt}\right) + q$$

on $\mathscr{D}_2 \subset L^2(\mathbf{R})$.

Let $\mathscr{A}$ be the C^*-algebra generated by the translations of p, q, p' and the constant functions. Denote the maximal ideal space of $\mathscr{A}$ by Ω. Because $\mathscr{A}$ is separable, Ω is a metrizable space. $\mathbf{R}$ is naturally identified with a dense subset in Ω. The point in Ω corresponding to $t \in \mathbf{R}$ under this identification will be denoted by $\hat{t}$. For each $s \in \mathbf{R}$, the map $f \mapsto f_s$ is a C^*-algebra isomorphism on $\mathscr{A}$ and, therefore, induces a homeomorphism τ_s on Ω. We have $\tau_s(\hat{t}) = (t+s)\hat{}$ for all t, $s \in \mathbf{R}$. The uniform continuity of p, q and p' ensures that the map $(\omega, s) \mapsto \tau_s(\omega)$ is continuous. Hence we obtain a flow $(\Omega, \mathbf{R}, \tau)$. If π denotes the Gelfand transform from $\mathscr{A}$ to $C(\Omega)$, then

$$\pi(f_s)(\omega) = \pi(f)(\tau_s(\omega))$$

for all $f \in \mathscr{A}$, $\omega \in \Omega$, and $s \in \mathbf{R}$. The following is a family of operators which are related to the operator L : For each $\omega \in \Omega$, we have an operator

$$L_\omega = -\left(\frac{d}{dt}\right) \pi(p)_\omega \left(\frac{d}{dt}\right) + \pi(q)_\omega \,.$$

[Recall that $\varphi_\omega(s) = \varphi(\tau_s(\omega))$.] Naturally $L = L_{\hat{0}}$. Similarly, if we denote

$$L_s = -\left(\frac{d}{dt}\right) p_s \left(\frac{d}{dt}\right) + q_s$$

for every $s \in \mathbf{R}$, then $L_s = L_{\hat{s}}$.

Proposition 4.1. *Suppose that $f \in \mathscr{A}$ Then $\pi(f) \in C^1_\tau(\Omega)$ if and only if $f' \in \mathscr{A}$ Furthermore, if $f' \in \mathscr{A}$, then $\pi(f)' = \pi(f')$.*

Proof. Suppose that $\varphi(\omega) = d\pi(f)(\tau_s(\omega))/ds|_{s=0} \in C(\Omega)$. Then

$$f(t) - f(s) = \pi(f)(\tau_t(\hat{0})) - \pi(f)(\tau_s(\hat{0})) = \int_s^t \varphi(\tau_\lambda(\hat{0}))\, d\lambda \,.$$

Hence $f'(t) = \varphi(\tau_t(\hat{0}))$. On the other hand, if $f' \in \mathscr{A}$, then $f(a+t) - f(a+s) = \int_{a+s}^{a+t} f'(\lambda)\, d\lambda = \int_{a+s}^{a+t} \pi(f')(\tau_\lambda(\hat{0}))\, d\lambda = \int_s^t \pi(f')(\tau_\lambda(\hat{a}))\, d\lambda$. Hence

$$\pi(f)(\tau_t(\omega)) - \pi(f)(\tau_s(\omega)) = \int_s^t \pi(f')(\tau_\lambda(\omega))\, d\lambda$$

for every $\omega \in \Omega$. □

Proposition 4.2. *For every $\omega \in \Omega$, $\sigma(L_\omega) \subset \sigma(L)$*

Proof For each $\omega \in \Omega$, let $\mathscr{B}_\omega$ be the C^*-algebra generated by $\{(L_\omega - z)^{-1} : z \in \mathbf{C}\backslash\mathbf{R}\}$ and the identity operator. To prove the proposition, it suffices to show that the map $\varrho_\omega : (L - z)^{-1} \mapsto (L_\omega - z)^{-1}$ extends to a C^*-algebra homomorphism from $\mathscr{B}_{\hat{0}}$ to $\mathscr{B}_\omega$. This assertion is obviously true when $\omega = \hat{s}$. For in this case, we have $U_s^* L U_s = L_s = L_{\hat{s}}$, where U_s is the unitary operator $(U_s f)(t) = f(t+s)$ on $L^2(\mathbf{R})$. For an arbitrary $\omega \in \Omega$, choose a sequence $\{t_n\} \subset \mathbf{R}$ such that $\hat{t}_n \to \omega$ in the topology of Ω. To complete the proof, it suffices to show that $\varrho_{\hat{t}_n}((L-z)^{-1}) = (L_{\hat{t}_n} - z)^{-1}$ converges to $(L_\omega - z)^{-1}$ in the strong operator topology. For each $u \in L^2(\mathbf{R})$, there is a $v \in \mathscr{D}_2$ such that $u = (L_\omega - z)v$. Therefore

$$\begin{aligned}[(L_\omega - z)^{-1} - (L_{\hat{t}_n} - z)^{-1}]u &= v - (L_{\hat{t}_n} - z)^{-1}(L_{\hat{t}_n} - z + L_\omega - L_{\hat{t}_n})v \\ &= (L_{\hat{t}_n} - z)^{-1}(L_{\hat{t}_n} - L_\omega)v \\ &= (L_{\hat{t}_n} - z)^{-1}[-(\pi(p)_{\hat{t}_n} - \pi(p)_\omega)v'' \\ &\quad - (\pi(p')_{\hat{t}_n} - \pi(p')_\omega)v' + (\pi(q)_{\hat{t}_n} - \pi(q)_\omega)v]\,.\end{aligned}$$

The proposition follows from the fact that for any $\varphi \in C(\Omega)$ and $\eta \in L^2(\mathbf{R})$, $(\varphi_{\hat{t}_n} - \varphi_\omega)\eta \to 0$ in the norm topology of $L^2(\mathbf{R})$. $\square$

Let $\mathscr{T} = \mathscr{T}(p, q) = \inf\{\lambda : \lambda \in \sigma(L)\}$. The preceding proposition tells us that $\mathscr{T} = \inf\{\lambda : \lambda \in \sigma(L_\omega), \omega \in \Omega\}$. Hence the theorems stated in the Introduction are obtained by applying the results in Sects. 1, 2 and 3 to the setting $(X, \mathbf{R}, \alpha) = (\Omega, \mathbf{R}, \tau)$, $P = \pi(p)$ and $Q = \pi(q)$.

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