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On the Parametrisation of Unitary Matrices by the Moduli of their Elements

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Abstract: The parametrisation of an $n \times n$ unitary matrix by the moduli of its elements is not a well posed problem, i.e. there are continuous and discrete ambiguities which naturally appear. We show that the continuous ambiguity is $(n-1)(n-3)$-dimensional in the general case and $\frac{n(n-3)}{2}$-dimensional in the symmetric case $S_{ij} = S_{ji}$. We give also lower bounds on the number of discrete ambiguities, the number of solutions being at least $2^{\frac{n(n-3)}{2}}$ in the first case and $2^{\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]-1}$ for the symmetric one, where $[r]$ denotes the integral part of r.

1. Introduction

There has been much recent interest in the problem of reconstructing the phases of a unitary matrix from the knowledge of the moduli of its matrix elements [1–2, 4–5, 7–8]. Stated in this general form the problem is of broad interest for people working in circuit theory, phase shift analyses, multichannel scattering, standard model, CP violation, etc.

Actually the last two items explicitly raised the problem alluded to in the title. People working in the study of the Cabibbo-Kobayashi-Maskawa (CKM) mass matrix had to take into account the experimental fact that almost all the accessible information we have about the unitary CKM mass matrix is given in terms of the moduli of its matrix elements.

From a pragmatic point of view a parametrisation of a unitary matrix by the moduli of its elements is very appealing. On the other hand, such a parametrisation is not natural. A natural one would be one whose parameters are free, i.e. there are no supplementary constraints upon them to enforce unitarity. Natural parametrisations are the Euler-type parametrisation given by Murnagham [10], or that found by us [6] which generalises to an arbitrary dimension the one given by Watson [11] for 2×2 unitary matrices.

However, due to its experimental implications, the problem has been raised in the form: to what extent the knowledge of the moduli $|S_{ij}|$ of an $n \times n$ unitary matrix $S = (S_{ij})$ determines S. In such a formulation it is implicitly supposed that S_{ij} satisfy all the requirements imposed by unitarity, i.e. conditions like

$$\sum_{k=1}^{n} |S_{jk}|^2 = \sum_{i=1}^{n} |S_{ij}|^2 = 1\,, \qquad j = 1, 2, \ldots, n\,, \tag{1.1}$$

and a set of complicated inequalities [9]. The last inequalities give a very intricate description of the domain of variation for the relevant parameters.

Fortunately this description can be simplified since supplementary information exists. In the case of the CKM matrix, there is a natural constraint in the frame of the standard model, namely that the mass matrix must be invariant under rephasing transformation,

$$S_{ij} \to e^{i(\alpha_i+\beta_j)} S_{ij} \quad (\alpha_i, \beta_j \text{ arbitrary modulo } 2\pi)\,. \tag{1.2}$$

The following is an abstract argument. The multiplication of a row or a column by an arbitrary phase factor does not affect the unitarity property or the values of its moduli, so the "trivial" ambiguity (1.2) is benign and it is of little importance. One consequence is the following: we can fix the phases of elements of a row and a column taken arbitrarily. Customarily one sets to zero or π the phases of elements in the first row and the first column of S. In this way the number of free real parameters is reduced from n^2 to $n^2 - (2n-1) = (n-1)^2$, which is the number of independent moduli and is easily seen if we observe that the number of independent constraints in (1.1) is equal to $2n - 1$. We conclude that the number of the parameters describing an $n \times n$ unitary (rephasing invariant) matrix is equal to $(n-1)^2$, and if they are identified with the moduli, they are lying within the simple domain

$$D = (0,1) \times \ldots \times (0,1) \equiv (0,1)^{(n-1)^2}\,,$$

where the above notation means that the number of factors entering the topological product is $(n-1)^2$. The preceding considerations were more or less supposed in all the previous approaches, but that conceptual clarification is due to Auberson, Martin and Mennessier [2].

In conclusion we can, at least in principle, parametrise an $n \times n$ unitary rephasing invariant matrix by the upper left corner moduli of its matrix elements. In this parametrisation we exclude the moduli of the last row and the last column since they are deduced from unitarity constraints.

However, a question remains and it is the following. To what extent this parametrisation is one-to-one. The answer to the first part of the question is evident since a unitary matrix uniquely determines a set of moduli of its elements. The converse is not always true; given a set of $(n-1)^2$ moduli, there may exist even a continuum of unitary matrices corresponding to them [1, 2].

This shortcoming gives rise to a natural question. It is possible to describe in the $(n-1)^2$-dimensional parameter space the variety upon which the parametrisation is not one-to-one? The answer to this question is positive and one of the aims of the paper is to give an analytical description of this variety in the case $n = 4$.

Our approach to the problem is very simple. We find first a one-to-one parametrisation of a unitary matrix, i.e. we introduced a system of coordinates on the unitary group acting on an n-dimensional vector space. Then we change the coordinates, taking as new coordinates the moduli of the $(n-1)^2$ upper left corner matrix elements and $2n-1$ phases. We use, afterwards, the implicit function theorem to find the points where the new parametrisation fails to be one-to-one. In this way the variety upon which the application is not bijective is given by setting to zero the Jacobian of the transformation.

The conclusion is that for $n \geq 4$ the unitary group $U(n)$ can be fully parametrised, if and only if, the moduli are outside the above variety. When the parameters are lying on this surface the reconstruction of S from its moduli is impossible, and when the parameters are in a sufficiently small neighborhood of it the reconstruction is a very unstable process; in other words the problem is not well posed.

But the most interesting consequence of our approach is the description of the continuous ambiguity which naturally appears. Thus in the symmetric case the maximal dimension can be $\frac{n(n-3)}{2}$ and in the general case it is equal to $(n-1)(n-3)$. For $n=4$ the ambiguity is 2-dimensional and 3-dimensional, respectively. The explicit parametrisation of the ambiguity leads to an interesting problem from algebraic geometry that we did not solve.

Supposing now that we are outside the above variety, a second problem emerges, namely that of the multiplicity of solutions for the inverse problem. This problem appears quite naturally since the implicit function theorem is a local result [3]. Here we obtain that, generically, the multiplicity of solutions is at least $2^{\frac{n(n-3)}{2}}$ in the non-symmetric case and $2^{\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]-1}$ for the symmetric case, where $[r]$ denotes the integral part of r. Thus we give only a lower bound on the number of solutions although there is some evidence that they can be considered as upper bounds also [2].

The paper is organized as follows. In Sect. 2 we find a one-to-one parametrisation of unitary matrices and we use it to find the lower bounds upon the number of discrete multiplicity of the solution for the inverse problem, and in Sect. 3 we give the explicit description of the singular surface in the $n=4$ case. The paper ends with conclusions.

2. Parametrisation of Unitary Matrices

The aim of this section is to present a one-to-one parametrisation of unitary matrices that is useful in solving the above raised problems. We have obtained such a parametrisation some time ago [6] and it is a straightforward generalisation of Watson's parametrisation of unitary matrices [11]. That means that its parameters are a number $p(n)$ of "inelasticities," i.e. positive numbers less than unity and a number $\phi(n)$ of phases, defined modulo 2π, such that $p(n)+\phi(n)=n^2$.

Our algorithm is a recursive one, allowing the parametrisation of $n \times n$ matrices through the parametrisation of lower dimensional matrices. The starting point is the partitioning of the matrix S in blocks

$$S=\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.1}$$

Here we make the most convenient choice for our purpose by taking A the simplest contraction, i.e. a complex number whose modulus is less than unity. With that choice the parametrisation is equivalent to an Euler-type parametrisation à la Murnagham [10]; other choices for the contraction A lead to completely different parametrisations. If the inelasticity parameters are allowed to vary within the open set $(0, 1)$ and the phases inside the set $[0, 2\pi)$, the given parametrisation is one-to-one, i.e. the application Ψ

$$S(S \in \mathbf{M}_n, S^*S = \mathbf{I}_n) \to E = (0,1)^{p(n)} \times [0, 2\pi)^{\phi(n)} \subset \mathbf{R}^{n^2}$$

is bijective. In other words given an $n \times n$ unitary matrix we uniquely find a point inside E, i.e. a set of $p(n)$ inelasticities and $\phi(n)$ phases completely specifying it, and, conversely, given a point inside E, we find a unique unitary matrix corresponding to it.

In the following we shall sketch the main ingredients that are necessary in describing the above application Ψ. For details see our paper [6].

The blocks A, B, C and D entering (2.1) are, in general, arbitrary contractions, this property being a consequence of the unitarity relation $S^*S = SS^* = \mathbf{I}_n$. Here $\mathbf{I}_n$ denotes the unit matrix of $\mathbf{M}_n$.

If we choose A the simplest contraction, $A = ae^{i\phi}$ with $a \in (0, 1)$ and $\phi \in [0, 2\pi)$, (2.1) can be written as [6]

$$S = \begin{pmatrix} ae^{i\phi} & (1-a^2)^{1/2}U \\ (1-a^2)^{1/2}V & -ae^{-i\phi}VU + XMY^* \end{pmatrix}, \tag{2.2}$$

where $U, V \in \mathbf{C}^{n-1}$ are row and column vectors, respectively, lying on the complex unit sphere, i.e. their components satisfy

$$\sum_{i=1}^{n-1} |u_i|^2 = \sum_{j=1}^{n-1} |v_j|^2 = 1 \tag{2.3}$$

and X and Y are those unitary matrices which bring the operators $D_{V^*} = (\mathbf{I}_{n-1} - VV^*)^{1/2}$ and $D_U = (\mathbf{I}_{n-1} - U^*U)^{1/2}$ respectively, to a diagonal form, i.e.

$$X^* D_{V^*} X = P, \qquad Y^* D_U Y = P,$$

where P is the projection

$$P = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

and the matrix M entering (2.2) has the form

$$M = \begin{pmatrix} 0 & 0 \\ 0 & S_{n-2} \end{pmatrix}.$$

Here S_{n-2} denotes an arbitrary $(n-2) \times (n-2)$ unitary matrix.

With the above restrictions on U and V, D_U and D_{V^*} are orthogonal projections and in writing the above formulae we have supposed that the eigenvectors entering matrices X and Y are ordered such that in the first columns the eigenvectors corresponding to the eigenvalue $\lambda = 0$ enter, whose multiplicity is equal to unity. The multiplicity of $\lambda = 1$ eigenvalues is $n - 2$. The most difficult problem in obtaining a complete description of the application Ψ is finding the explicit form for the matrices X and Y. We did not find their explicit form for arbitrary n although we believe this to be possible. Since the eigenvector which corresponds to $\lambda = 0$ eigenvalue is easily obtained, we have suggested in our paper [6] the use of the Gram-Schmidt procedure for finding X and Y. However, the resulting parametrisation becomes unnecessarily complicated. Here we suggest another route which leads to simpler calculations. The new procedure will be applied to the case $n = 4$ which is of interest for us.

The new route is the following. Since X and Y are unitary matrices of dimension $n-1$ whose first columns are known, we can use again the procedure (2.2) to find their missing elements, thus the procedure works recursively and gives quickly the desired results.

Now we want to exploit the recursive feature of the parametrisation (2.2) in order to find the still unknown functions $p(n)$ and $\phi(n)$.

From Eq. (2.2) it is easy seen that these functions satisfy the equations [6]

$$\begin{aligned} p(n) &= 2n-3+p(n-2)\,, \\ \phi(n) &= 2n-1+\phi(n-2)\,, \end{aligned} \tag{2.4}$$

with the initial conditions

$$p(1)=0\,, \qquad p(2)=1\,; \qquad \phi(1)=1\,, \qquad \phi(2)=3\,. \tag{2.5}$$

The solution is

$$p(n)=\frac{n(n-1)}{2}\,; \qquad \phi(n)=\frac{n(n+1)}{2}\,, \qquad n=1,2,\,\ldots\,. \tag{2.6}$$

For the symmetric case $S_{ij}=S_{ji}$, Eqs. (2.4) have the form

$$p(n)=n-1+p(n-2)\,, \qquad \phi(n)=n+\phi(n-2)\,. \tag{2.4'}$$

The initial conditions for $p(n)$ are unchanged, but for $\phi(n)$ they are

$$\phi(1)=1\,, \qquad \phi(2)=2\,. \tag{2.5'}$$

The solution is

$$p(n)=\left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right], \qquad \phi(n)=\left[\frac{n+1}{2}\right]\left[\frac{n+2}{2}\right], \qquad n=1,2,\,..\;, \tag{2.6'}$$

where $[r]$ denotes the integral part of r.

These last results are useful in solving the multiplicity problem of discrete solutions for the reconstruction of unitary matrices from the knowledge of the moduli of their elements.

The rephasing invariance condition (1.2) implies that the vectors U and V entering (2.2) are real and $\phi=0$. When U and V are real vectors we can choose the matrices X and Y to be real also. Thus the complex quantities enter our parametrisation (2.2) through the matrix M which contains an arbitrary $(n-2)\times(n-2)$ unitary matrix. More precisely they enter through the phases whose number is equal to $\phi(n-2)$, the inelasticity parameters being positive numbers. The phases appear through the exponential functions $e^{i\psi_j}$, $j=1,2,\,\ldots,\,\phi(n-2)$. Although the exponential function $e^{i\psi}$ is univalent over $[0,2\pi)$, and consequently a given point inside the domain E gives a unique unitary matrix S, the application $S_{ij}\to|S_{ij}|$ is no more unique. The functions $|S_{ij}|$ are expressed as explicit (and sometimes complicated) functions of $\sin\psi_i$ and $\cos\psi_i$, $i=1,2,\,\ldots,\,\phi(n-2)$. But the functions sin and cos are both double-valued over $[0,2\pi)$. Thus the number of discrete solutions for a generic situation is at least $2^{\phi(n-2)}$. If we enforce also the trivial ambiguity $S\to S^*$ we find that the number of discrete solutions is greater than $2^{\phi(n-2)-1}$ and taking into account the results (2.6) and (2.6′) we find that these lower bounds are given by $2^{\frac{n(n-3)}{2}}$ for the non-symmetric case and $2^{\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]-1}$ for the symmetric one.

There are some arguments showing that the above numbers may be taken as upper bounds also, but we have no rigorous proof.

The above results show that the parametrisation of a unitary matrix by the moduli of its matrix elements is not very convenient and we can do it only when we are forced by "physical" constraints.

3. Singular Surface

Now we shall describe the application Ψ in the case of the unitary group $U(4)$ and as a consequence we shall find the singular surface upon which it is not possible to parametrise a unitary matrix through the moduli of its elements.

A is taken as $A = ae^{i\phi_{11}}$, $a \in (0,1)$ and $\phi_{11} \in [0, 2\pi)$.

The vectors U and V entering Eq. (2.2), when $n = 4$, are given by

$$\begin{aligned} U &= (be^{i\phi_{12}}, c(1-b^2)^{1/2}e^{i\phi_{13}}, [(1-b^2)(1-c^2)]^{1/2}e^{i\phi_{14}}), \\ V^T &= (de^{i\phi_{21}}, f(1-d^2)^{1/2}e^{i\phi_{31}}, [(1-d^2)(1-f^2)]^{1/2}e^{i\phi_{41}}), \end{aligned} \tag{3.1}$$

where T denotes transposed, and $*$ in the following will denote the adjoint.

For any contraction C one can define two operators

$$D_C = (\mathbf{I} - C^*C)^{1/2} \quad \text{and} \quad D_{C^*} = (\mathbf{I} - CC^*)^{1/2},$$

where $\mathbf{I}$ is the unit operator on the corresponding space.

They have the properties

$$CD_C = D_{C^*}C, \qquad C^*D_{C^*} = D_C C^*. \tag{3.2}$$

The relations (2.3) can be written as

$$UU^* = 1, \qquad V^*V = 1. \tag{3.3}$$

The relations (3.2) and (3.3) are useful in finding the eigenvalues of D_V and D_{V^*} for the eigenvalue $\lambda = 0$. Indeed we have $D_U U^* = U^* D_{U^*} = 0\ U^* = 0$ since $D_{U^*} = 0$. Thus U^* is the eigenvector of D_U which correspond to $\lambda = 0$. Similarly $D_{V^*}V = VD_V = 0\ V = 0$ because $D_V = 0$.

In this way we find the first column of the matrix Y which is the vector

$$(be^{-i\phi_{12}}, c(1-b^2)^{1/2}e^{-i\phi_{13}}, [(1-b^2)(1-c^2)]^{1/2}e^{-i\phi_{14}})^T.$$

If we use the route suggested in the preceding section we find the following form of the matrix Y:

$$Y = \begin{pmatrix} be^{-i\phi_{12}} & (1-b^2)^{1/2} & 0 \\ c(1-b^2)^{1/2}e^{-i\phi_{13}} & -bce^{i(\phi_{12}-\phi_{13})} & -(1-c^2)^{1/2}e^{-i\phi_{13}} \\ [(1-b^2)(1-c^2)]^{1/2}e^{-i\phi_{13}} & -b(1-c^2)^{1/2}e^{i(\phi_{12}-\phi_{14})} & ce^{-i\phi_{14}} \end{pmatrix}, \tag{3.4a}$$

and similarly for the matrix X

$$X = \begin{pmatrix} de^{i\phi_{21}} & (1-d^2)^{1/2} & 0 \\ f(1-d^2)^{1/2}e^{i\phi_{31}} & -df e^{i(\phi_{31}-\phi_{21})} & -(1-f^2)^{1/2}e^{i\phi_{31}} \\ [(1-d^2)(1-f^2)]^{1/2}e^{i\phi_{41}} & -d(1-f^2)^{1/2}e^{i(\phi_{41}-\phi_{21})} & fe^{i\phi_{41}} \end{pmatrix}, \tag{3.4b}$$

and both have a simpler form than that given in our paper [6].

We take the matrix M entering Eq. (2.2) in the form

$$M \begin{pmatrix} 0 & 0 & 0 \\ 0 & xe^{iy} & (1-x^2)^{1/2}e^{iz} \\ 0 & (1-x^2)^{1/2}e^{iw} & -xe^{i(z+w-y)} \end{pmatrix}, \tag{3.5}$$

where the range of the parameters is $x \in (0,1)$, $y, w, z \in [0, 2\pi)$.

By using the formulae (3.1), (3.4) and (3.5) in (2.2) we obtain the parametrisation of a 4×4 unitary matrix in full generality.

We shall now impose the rephasing invariance and we make the choice

$$\phi_{1i} = \phi_{i1} = 0\,, \qquad i = 1, \ldots, 4\,.$$

Thus the relevant matrix elements from the first row and the first column are positive numbers and are given by

$$\begin{aligned} S_{11} &= a\,, \\ S_{12} &= b(1-a^2)^{1/2}\,, \\ S_{13} &= c[(1-a^2)(a-b^2)]^{1/2}\,, \\ S_{21} &= d(1-a^2)^{1/2}\,, \\ S_{31} &= f[(1-a^2)(1-d^2)]^{1/2}\,. \end{aligned} \tag{3.6}$$

The four other relevant matrix elements have the form

$$\begin{aligned} S_{22} &= -abd + x\sqrt{(1-b^2)(1-d^2)}\,e^{iy}\,, \\ S_{23} &= -acd\sqrt{1-b^2} - bcx\sqrt{1-d^2}\,e^{iy} \\ &\quad - \sqrt{(1-c^2)(1-d^2)(1-x^2)}\,e^{iz}\,, \\ S_{32} &= -abf\sqrt{1-d^2} - dfx\sqrt{1-d^2}\,e^{iy} \\ &\quad - \sqrt{(1-b^2)(1-f^2)(1-x^2)}\,e^{iz}\,, \\ S_{33} &= -acf\sqrt{(1-b^2)(1-d^2)} \\ &\quad + bcdfxe^{iy} + df\sqrt{(1-c^2)(1-x^2)}\,e^{iz} \\ &\quad + bc\sqrt{(1-f^2)(1-x^2)}\,e^{iw} - x\sqrt{(1-c^2)(1-f^2)}\,e^{i(z+w-y)}\,. \end{aligned} \tag{3.7}$$

From the relations (3.7) we obtain

$$
\begin{aligned}
|S_{22}|^2 &= (abd)^2 + x^2(1-b^2)(1-d^2) \\
&\quad - 2abdx\sqrt{(1-b^2)(1-d^2)}\cos y\,, \\
|S_{23}|^2 &= a^2c^2d^2(1-b^2) + b^2c^2x^2(1-d^2) + (1-c^2)(1-d^2)(1-x^2) \\
&\quad + 2abc^2dx\sqrt{(1-b^2)(1-d^2)}\cos y \\
&\quad + 2acd\sqrt{(1-b^2)(1-c^2)(1-d^2)(1-x^2)}\cos z \\
&\quad + 2bcx(1-d^2)\sqrt{(1-c^2)(1-x^2)}\cos(y-z)\,, \\
|S_{32}|^2 &= a^2b^2f^2(1-d^2) + d^2f^2x^2(1-b^2) + (1-b^2)(1-f^2)(1-x^2) \\
&\quad + 2abdf^2x\sqrt{(1-b^2)(1-d^2)}\cos y \\
&\quad + 2abf\sqrt{(a-b^2)(1-d^2)(1-f^2)(1-x^2)}\cos w \\
&\quad + 2dfx(1-b^2)\sqrt{(1-f^2)(1-x^2)}\cos(y-w)\,, \\
|S_{33}|^2 &= a^2c^2f^2(1-b^2)(1-d^2) + b^2c^2d^2f^2x^2 + d^2f^2(1-c^2)(1-x^2) \\
&\quad + b^2c^2(1-f^2)(1-x^2) + x^2(1-c^2)(1-f^2) \\
&\quad - 2abc^2df^2x\sqrt{(1-b^2)(1-d^2)}\cos y \\
&\quad - 2acdf^2\sqrt{(1-b^2)(1-c^2)(1-d^2)(1-x^2)}\cos z \\
&\quad - 2abc^2f\sqrt{(1-b^2)(1-d^2)(1-f^2)(1-x^2)}\cos w \\
&\quad + 2acfx\sqrt{(1-b^2)(1-c^2)(1-d^2)(1-f^2)}\cos(z+w-y) \\
&\quad + 2bcx(d^2f^2+f^2-1)\sqrt{(1-c^2)(1-x^2)}\cos(y-z) \\
&\quad + 2dfx(b^2c^2+c^2-1)\sqrt{(1-f^2)(1-x^2)}\cos(y-w) \\
&\quad - 2bcdfx^2\sqrt{(1-c^2)(1-f^2)}\cos(2y-z-w) \\
&\quad + 2bcdf(1-x^2)\sqrt{(1-c^2)(1-f^2)}\cos(z-w)\,.
\end{aligned}
\tag{3.8}
$$

As we said before the rephasing invariance implies that the number of relevant parameters is $(n-1)^2$ which coincides with the number of independent moduli.

In our case these $(4-1)^2 = 9$ parameters are given by six positive numbers less than unity $a, b, c, d, f, x \in (0, 1)$ and three phases $y, w, z \in [0, 2\pi)$. The parameters which cause the trouble are x, y, w and z and they are those which enter in the parametrisation of the (arbitrary) unitary part of the matrix (3.5).

Indeed from the formulae (3.6) it is easily seen that the application

$$(|S_{1i}|, |S_{i1}|\, i = 1, 2, 3) \to (a, b, c, d, f \in (0, 1))$$

is one-to-one and further if the moduli are fixed so are these parameters.

We first treat the symmetric case since the formulae are a little simpler. This case is equivalent to the following identities:

$$d \equiv b\,, \qquad b \equiv c\,, \quad \text{and} \quad w \equiv z\,,$$

and the number of independent parameters is reduced to six: four inelasticities $a, b, c, x \in (0,1)$ and two phases $y, z \in [0, 2\pi)$. The formulae (3.8) take the form

$$
\begin{aligned}
|S_{22}|^2 &= a^2b^4 + x^2(1-b^2)^2 - 2ab^2x(1-b^2)\cos y\,,\\
|S_{23}|^2 &= |S_{32}|^2 = (1-b^2)\,[a^2b^2c^2 + b^2c^2x^2 + (1-c^2)(1-x^2)\\
&\quad + 2ab^2c^2x\cos y + 2abc\sqrt{(1-c^2)(1-x^2)}\cos z\\
&\quad + 2bcx\sqrt{(1-c^2)(1-x^2)}\cos(y-z)]\,,\\
|S_{33}|^2 &= a^2c^4(1-b^2)^2 + b^4c^4x^2 + 4b^2c^2(1-c^2)(1-x^2) + x^2(1-c^2)^2\\
&\quad - 2ab^2c^4x(1-b^2)\cos y - 4abc^3(1-b^2)\sqrt{(1-c^2)(1-x^2)}\cos z\\
&\quad + 2ac^2x(1-b^2)(1-c^2)\cos(2z-y) - 2b^2c^2x^2(1-c^2)\cos 2(y-z)\\
&\quad + 4bcx(b^2c^2+c^2-1)\sqrt{(1-c^2)(1-x^2)}\cos(y-z)\,.
\end{aligned}
\tag{3.9}
$$

With the notation $X = |S_{22}|$, $Y = |S_{23}|$, $Z = |S_{33}|$ the points where the transformation (3.9) is not one-to-one are those where the Jacobian of the transformation vanishes, i.e.

$$
\Delta(x,y,z) \equiv \left|\frac{\partial(X,Y,Z)}{\partial(x,y,z)}\right| = 0\,,
$$

which can be written as

$$
\Delta(x,y,z) = \frac{(1-b^2)^2ac}{|S_{22}S_{23}S_{33}|}\,D(x,y,z)\,,
$$

where

$$
D(x,y,z) = ADH + BEF - AEG - BCH\,,
$$

and

$$
\begin{aligned}
A &= x(1-b^2) - ab^2\cos y\,,\\
B &= b^2x\sin y\,,\\
C &= (2c^2-1)x - bc\sqrt{\frac{1-c^2}{1-x^2}}\,[ax\cos z + (2x^2-1)\cos(y-z)]\,,\\
D &= -\,bc\sqrt{(1-c^2)(1-x^2)}\sin z\,,\\
E &= b\sqrt{(1-c^2)(1-x^2)}\,[x\sin(y-z) - a\sin z]\,,\\
G &= -\,c^2x(1-b^2)(1-c^2)\sin(2z-y)\,,\\
F &= [(2c^2-1)^2 + 2b^2c^2(c^2-1)]x + ac^2(1-b^2)(1-c^2)\cos(2z-y)\\
&\quad - 2b^2c^2x(1-c^2)\cos 2(y-z)\\
&\quad + 2bc(2c^2-1)(1-2x^2)\sqrt{\frac{1-c^2}{1-x^2}}\cos(y-z)\,,\\
H &= 2bx(2c^2-1)\sqrt{(1-c^2)(1-x^2)}\sin(y-z)\\
&\quad - 2acx(1-b^2)(1-c^2)\sin(2z-y) - 2b^2cx^2(1-c^2)\sin 2(y-z)\,.
\end{aligned}
$$

In conclusion the reconstruction of S from the moduli of its elements is not possible on the hypersurface

$$D(x, y, z) = 0\,. \tag{3.10}$$

Because we have a relation between three independent parameters we are left with two free parameters which can be taken to be the phases y and z, i.e. the maximal dimension of this variety is two. Thus our results complete those ones obtained by Auberson et al. in their beautiful paper [2].

The non-symmetric case is treated similarly, the only difference being the number of relevant parameters, four instead of three. Equation (3.10) has the form

$$D(x, y, w, z) = 0 \tag{3.10$'$}$$

and the ambiguity is now three-dimensional. Again the independent parameters can be taken to be the three phases y, w, z.

An interesting parametrisation of a unitary matrix would be that which will explicitly exhibit this ambiguity. Unfortunately we did not find it. This leads to an interesting problem from algebraic geometry. The hypersurface (3.10$'$) can be applied onto a hypersurface from the affine space $\mathbf{A}^4$ by the change of variables

$$x = \frac{2s}{1+s^2}\,, \qquad \sin y = \frac{2t}{1+t^2}\,,$$
$$\sin w = \frac{2u}{1+u^2}\,, \qquad \sin z = \frac{2v}{1+v^2}\,,$$

and the problem is reduced to a standard problem in algebraic geometry, finding a birational isomorphism of that surface onto an affine space.

4. Conclusion

The preceding considerations show that the dimension of the variety on which the reconstruction problem has a continuum of solutions is given by the unitary S_{n-2} submatrix entering the matrix M in formula (2.2). The number of its independent parameters is $(n-2)^2$ and thus the maximal dimension of the variety is equal to $(n-2)^2 - 1 = (n-3)(n-1)$ in the general case and $\dfrac{n(n-3)}{2}$ in the symmetric case.

An interesting problem would be the complete description of this variety and specially its decomposition into irreducible parts. A prerequisite for solving the last problem is the factorisation of the Jacobian Δ. In our approach to this factorisation it is difficult to find the number of independent terms being large. For $n = 4$ this number is bigger than one hundred in the general case and equal to sixteen for symmetric matrices, but even in this simplest case there is no a priori hint how to do it. A possible approach would be a clever use of the symmetry of the problem beginning with the explicit determination of the matrices X and Y for arbitrary n.

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