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The Algebra of the Energy-Momentum Tensor and the Noether Currents in Classical Non-Linear Sigma Models

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Abstract: The recently derived current algebra of classical non-linear sigma models on arbitrary Riemannian manifolds is extended to include the energy-momentum tensor. It is found that in two dimensions the energy-momentum tensor $\theta_{\mu\nu}$, the Noether current j_μ associated with the global symmetry of the theory and the composite field j appearing as the coefficient of the Schwinger term in the current algebra, together with the derivatives of j_μ and j, generate a closed algebra. The subalgebra generated by the light-cone components of the energy-momentum tensor consists of two commuting copies of the Virasoro algebra, with central charge $c = 0$, reflecting the classical conformal invariance of the theory, but the current algebra part and the semidirect product structure are quite different from the usual Kac-Moody/Sugawara type construction.

In a recent paper [1], we have derived the current algebra for classical non-linear sigma models defined on Riemannian manifolds. This algebra is quite simple to write down and yet does not seem to belong to any of the algebras which are well known in mathematical physics, mainly because it involves non-standard (in particular, non-central) extensions of loop algebras [4].

On the other hand, the classical non-linear sigma model in two dimensions is conformally invariant, so its energy-momentum tensor must satisfy the classical version of the standard commutation relations of conformal field theory, that is, under Poisson brackets its light-cone components must generate two commuting copies of the Witt algebra (the Virasoro algebra with vanishing central charge). We shall verify that this is indeed the case. Moreover, we shall derive the Poisson bracket relations between the energy-momentum tensor on the one hand and the Noether currents on the other hand. The resulting total algebra exhibits, in a concrete field-theoretical model with continuous internal symmetries, the possibility of reconciling conformal

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invariance, expressed through a chiral energy-momentum tensor algebra, with a non-chiral current algebra, at least at the classical level.

Thus consider the classical two-dimensional non-linear sigma model, whose configuration space is the space of (smooth) maps φ from a given two-dimensional Lorentz manifold Σ to a given Riemannian manifold M, with metric g, while the corresponding phase space consists of pairs (φ, π) with φ as before and π a (smooth) section of the pull-back $\varphi^*(T^*M)$ of the cotangent bundle of M to Σ via φ. The action, written in terms of isothermal local coordinates x^μ on Σ and of arbitrary local coordinates u^i on M, reads

$$S = \tfrac{1}{2} \int d^2x \, \eta^{\mu\nu} g_{ij}(\varphi) \, \partial_\mu \varphi^i \partial_\nu \varphi^j \,, \tag{1}$$

where the $\eta_{\mu\nu}$ are the coefficients of the standard Minkowski metric. Thus using a dot to denote the time derivative and a prime to denote the spatial derivative, we have

$$\pi_i = g_{ij}(\varphi) \dot{\varphi}^j \,, \tag{2}$$

and the canonical Poisson brackets are

$$\begin{gathered} \{\varphi^i(x), \varphi^j(y)\} = 0 \,, \qquad \{\pi_i(x), \pi_j(y)\} = 0 \,, \\ \{\varphi^i(x), \pi_j(y)\} = \delta^i_j \delta(x-y) \,. \end{gathered} \tag{3}$$

The energy-momentum tensor $\theta_{\mu\nu}$ of the theory is most conveniently derived by variation of the Lagrangian with respect to the metric on Σ. (For details, see e.g. [3, p. 64ff] or [5, p. 504f].) It reads

$$\theta_{\mu\nu} = g_{ij}(\varphi) \, \partial_\mu \varphi^i \partial_\nu \varphi^j - \tfrac{1}{2} \eta_{\mu\nu} \eta^{\kappa\lambda} g_{ij}(\varphi) \, \partial_\kappa \varphi^i \partial_\lambda \varphi^j \,, \tag{4}$$

and it is obviously traceless:

$$\eta^{\mu\nu} \theta_{\mu\nu} = 0 \,. \tag{5}$$

We also assume that the theory exhibits a global invariance under some internal symmetry group, represented by a (connected) Lie group G acting on M by isometries, and we shall write $\mathfrak{g}$ for the corresponding Lie algebra and $X_M \in \mathfrak{X}(M)$ for the fundamental Killing vector field on M associated with a given generator $X \in \mathfrak{g}$ of G:

$$X_M(m) = \frac{d}{dt} (\exp(tX) \cdot m) \Big|_{t=0} \,. \tag{6}$$

Then the Noether current j_μ, taking values in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$, and the scalar field j, taking values in the second symmetric tensor power of $\mathfrak{g}^*$, are given by [1]

$$\langle j_\mu, X \rangle = -(X_M)_i(\varphi) \, \partial_\mu \varphi^i \tag{7}$$

for $X \in \mathfrak{g}$, and

$$\langle j, X \otimes Y \rangle = g_{ij}(\varphi) X^i_M(\varphi) Y^j_M(\varphi) \tag{8}$$

for $X, Y \in \mathfrak{g}$. ($\langle \cdot, \cdot \rangle$ denotes the natural pairing between a vector space and its dual.)

Now the energy-momentum tensor algebra reads (note $\theta_{11} = \theta_{00}$)

$$\{\theta_{00}(x), \theta_{00}(y)\} = (\theta_{01}(x) + \theta_{01}(y)) \, \delta'(x-y) \,, \tag{9}$$

$$\{\theta_{00}(x), \theta_{01}(y)\} = (\theta_{00}(x) + \theta_{00}(y)) \, \delta'(x-y) \,, \tag{10}$$

$$\{\theta_{01}(x), \theta_{01}(y)\} = (\theta_{01}(x) + \theta_{01}(y)) \, \delta'(x-y) \,, \tag{11}$$

while the current algebra is [1]

$$
\begin{aligned}
\{\langle j_0(x), X\rangle, \langle j_0(y), Y\rangle\} &= -\langle j_0(x), [X,Y]\rangle \delta(x-y)\,, && (12)\\
\{\langle j_0(x), X\rangle, \langle j_1(y), Y\rangle\} &= -\langle j_1(x), [X,Y]\rangle \delta(x-y) \\
&\quad + \langle j(y), X\otimes Y\rangle \delta'(x-y)\,, && (13)\\
\{\langle j_0(x), X\rangle, \langle j(y), Y\otimes Z\rangle\} &= -\langle j(x), [X,Y]\otimes Z + Y\otimes [X,Z]\rangle \delta(x-y)\,, && (14)
\end{aligned}
$$

(the remaining Poisson brackets vanish), and the mixed Poisson brackets are

$$
\begin{aligned}
\{\theta_{00}(x), \langle j_0(y), X\rangle\} &= \langle j_1(x), X\rangle \delta'(x-y)\,, && (15)\\
\{\theta_{00}(x), \langle j_1(y), X\rangle\} &= \langle j_0(x), X\rangle \delta'(x-y) \\
&\quad - \langle (\partial_0 j_1 - \partial_1 j_0)(x), X\rangle \delta(x-y)\,, && (16)\\
\{\theta_{01}(x), \langle j_0(y), X\rangle\} &= \langle j_0(x), X\rangle \delta'(x-y)\,, && (17)\\
\{\theta_{01}(x), \langle j_1(y), X\rangle\} &= \langle j_1(x), X\rangle \delta'(x-y)\,, && (18)\\
\{\theta_{00}(x), \langle j(y), X\otimes Y\rangle\} &= -\langle \partial_0 j(x), X\otimes Y\rangle \delta(x-y)\,, && (19)\\
\{\theta_{01}(x), \langle j(y), X\otimes Y\rangle\} &= -\langle \partial_1 j(x), X\otimes Y\rangle \delta(x-y)\,. && (20)
\end{aligned}
$$

In higher dimensions, the current algebra [Eqs. (12–14)] and the mixed Poisson brackets [Eqs. (15–20)] remain essentially unchanged, while the energy-momentum tensor algebra [Eqs. (9–11)] is substantially modified and in fact no longer closes.

In order to prove Eqs. (9–20) in $d = 2$ and at the same time explain what goes wrong for $d > 2$, consider the classical non-linear sigma model over a d-dimensional Lorentz manifold Σ, with metric h, where the formulae for the action and for the energy-momentum tensor, Eqs. (1) and (4), are replaced by

$$
S = \tfrac{1}{2}\int d^d x \sqrt{|\det(h)|}\, h^{\mu\nu} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial_\nu \varphi^j\,, \tag{21}
$$

and by

$$
\theta_{\mu\nu} = g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial_\nu \varphi^j - \tfrac{1}{2}\, h_{\mu\nu} h^{\kappa\lambda} g_{ij}(\varphi)\, \partial_\kappa \varphi^i \partial_\lambda \varphi^j\,, \tag{22}
$$

respectively, while most of the other relations given above, namely Eqs. (2, 3) and (7, 8), remain as they stand. For simplicity, we shall carry out our calculations for the case where Σ is d-dimensional Minkowski space, that is, $h = \eta = \mathrm{diag}(1, -1, \ldots, -1)$, which is obviously sufficient when $d = 2$, due to the existence of isothermal local coordinates. The generalization to arbitrary d-dimensional Lorentz manifolds does not present any new features. To further simplify the calculation, we introduce an auxiliary field $\tilde{\theta}_{\mu\nu}$ according to

$$
\tilde{\theta}_{\mu\nu} = g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial_\nu \varphi^j\,, \tag{23}
$$

so that

$$
\theta_{\mu\nu} = \tilde{\theta}_{\mu\nu} - \tfrac{1}{2}\, \eta_{\mu\nu} \tilde{\theta}\,, \tag{24}
$$

where $\tilde{\theta}$ is the trace of $\tilde{\theta}_{\mu\nu}$:

$$
\tilde{\theta} = \eta^{\kappa\lambda} \tilde{\theta}_{\kappa\lambda}\,. \tag{25}
$$

In components[1],

$$\tilde{\theta}_{00} = g^{ij}(\varphi)\,\pi_i\pi_j\,, \tag{26}$$
$$\tilde{\theta}_{0\alpha} = \pi_i\partial_\alpha\varphi^i\,, \tag{27}$$
$$\tilde{\theta}_{\alpha\beta} = g_{ij}(\varphi)\,\partial_\alpha\varphi^i\partial_\beta\varphi^j\,, \tag{28}$$

and

$$\theta_{00} = \tfrac{1}{2}\,(\tilde{\theta}_{00} + \tilde{\theta}_{\gamma\gamma})\,, \tag{29}$$
$$\theta_{0\alpha} = \tilde{\theta}_{0\alpha}\,, \tag{30}$$
$$\theta_{\alpha\beta} = \tilde{\theta}_{\alpha\beta} + \tfrac{1}{2}\,\delta_{\alpha\beta}(\tilde{\theta}_{00} - \tilde{\theta}_{\gamma\gamma})\,. \tag{31}$$

In addition, a composite field containing second (covariant) derivatives of φ,

$$D_\mu\partial_\nu\varphi^i = \partial_\mu\partial_\nu\varphi^i + \Gamma^i_{kl}(\varphi)\,\partial_\mu\varphi^k\partial_\nu\varphi^l \tag{32}$$

will appear:

$$\tilde{\theta}_{\kappa(\mu\nu)} = g_{ij}(\varphi)\,\partial_\kappa\varphi^i D_\mu\partial_\nu\varphi^j\,. \tag{33}$$

Note that the $\tilde{\theta}_{\kappa(\mu\nu)}$ with two indices equal can be expressed as derivatives of the $\tilde{\theta}_{\mu\nu}$:

$$\tilde{\theta}_{\mu(\mu\nu)} = \tfrac{1}{2}\,\partial_\nu\tilde{\theta}_{\mu\mu} \qquad \text{(no summation)}\,, \tag{34}$$
$$\tilde{\theta}_{\nu(\mu\mu)} = \partial_\mu\tilde{\theta}_{\mu\nu} - \tfrac{1}{2}\,\partial_\nu\tilde{\theta}_{\mu\mu} \qquad \text{(no summation)}\,. \tag{35}$$

Now we are ready to write down the Poisson brackets involving the auxiliary field $\tilde{\theta}_{\mu\nu}$. The mixed Poisson brackets are

$$\{\tilde{\theta}_{00}(x), \langle j_0(y), X\rangle\} = 0\,, \tag{36}$$
$$\begin{aligned}\{\tilde{\theta}_{00}(x), \langle j_\gamma(y), X\rangle\} &= 2\langle j_0(x), X\rangle\partial_\gamma\delta(x-y)\\ &\quad - 2\langle(\partial_0 j_\gamma - \partial_\gamma j_0)(x), X\rangle\delta(x-y)\,,\end{aligned} \tag{37}$$
$$\{\tilde{\theta}_{0\alpha}(x), \langle j_0(y), X\rangle\} = \langle j_0(x), X\rangle\partial_\alpha\delta(x-y)\,, \tag{38}$$
$$\begin{aligned}\{\tilde{\theta}_{0\alpha}(x), \langle j_\gamma(y), X\rangle\} &= \langle j_\alpha(x), X\rangle\partial_\gamma\delta(x-y)\\ &\quad - \langle(\partial_\alpha j_\gamma - \partial_\gamma j_\alpha)(x), X\rangle\delta(x-y)\,,\end{aligned} \tag{39}$$
$$\{\tilde{\theta}_{\alpha\beta}(x), \langle j_0(y), X\rangle\} = \langle j_\alpha(x), X\rangle\partial_\beta\delta(x-y) + \langle j_\beta(x), X\rangle\partial_\alpha\delta(x-y)\,, \tag{40}$$
$$\{\tilde{\theta}_{\alpha\beta}(x), \langle j_\gamma(y), X\rangle\} = 0\,, \tag{41}$$
$$\{\tilde{\theta}_{00}(x), \langle j(y), X\otimes Y\rangle\} = -2\langle\partial_0 j(x), X\otimes Y\rangle\delta(x-y)\,, \tag{42}$$
$$\{\tilde{\theta}_{0\alpha}(x), \langle j(y), X\otimes Y\rangle\} = -\langle\partial_\alpha j(x), X\otimes Y\rangle\delta(x-y)\,, \tag{43}$$
$$\{\tilde{\theta}_{\alpha\beta}(x), \langle j(y), X\otimes Y\rangle\} = 0\,, \tag{44}$$

[1] We use letters from the beginning of the greek alphabet to denote spatial indices, running from 1 to $d-1$. In this case no distinction is made between upper and lower indices, and the usual summation convention for Euclidean space remains in force

and the Poisson brackets between the $\tilde{\theta}_{\mu\nu}$ read

$$\{\tilde{\theta}_{00}(x), \tilde{\theta}_{00}(y)\} = 0\,, \tag{45}$$
$$\{\tilde{\theta}_{00}(x), \tilde{\theta}_{0\alpha}(y)\} = (\tilde{\theta}_{00}(x) + \tilde{\theta}_{00}(y))\partial_\alpha \delta(x-y)\,, \tag{46}$$
$$\begin{aligned}\{\tilde{\theta}_{00}(x), \tilde{\theta}_{\alpha\beta}(y)\} &= 2\tilde{\theta}_{0\alpha}(x)\,\partial_\beta \delta(x-y) + 2\tilde{\theta}_{0\beta}(x)\,\partial_\alpha \delta(x-y)\\ &\quad + 4\tilde{\theta}_{0(\alpha\beta)}(x)\,\delta(x-y)\,,\end{aligned} \tag{47}$$
$$\{\tilde{\theta}_{0\alpha}(x), \tilde{\theta}_{0\beta}(y)\} = \tilde{\theta}_{0\beta}(x)\,\partial_\alpha \delta(x-y) + \tilde{\theta}_{0\alpha}(y)\,\partial_\beta \delta(x-y)\,, \tag{48}$$
$$\begin{aligned}\{\tilde{\theta}_{0\gamma}(x), \tilde{\theta}_{\alpha\beta}(y)\} &= \tilde{\theta}_{\gamma\alpha}(x)\,\partial_\beta \delta(x-y) + \tilde{\theta}_{\gamma\beta}(x)\,\partial_\alpha \delta(x-y)\\ &\quad + 2\tilde{\theta}_{\gamma(\alpha\beta)}(x)\,\delta(x-y)\,,\end{aligned} \tag{49}$$
$$\{\tilde{\theta}_{\alpha\beta}(x), \tilde{\theta}_{\gamma\delta}(y)\} = 0\,. \tag{50}$$

The proof goes by explicit computation, using the formulae

$$\langle \partial_\mu j_\nu, X\rangle = -(X_M)_i(\varphi) D_\mu \partial_\nu \varphi^i - (X_M)_{j|i}(\varphi)\,\partial_\mu \varphi^i \partial_\nu \varphi^j\,, \tag{51}$$

and

$$(f(x) - f(y))\,\delta'(x-y) = -f'(x)\,\delta(x-y)\,,$$

together with the following relations,

$$\begin{aligned}\partial_k g_{ij} Z^k &= -g_{ki}\partial_j Z^k - g_{kj}\partial_i Z^k\,,\\ \partial_k g^{ij} Z^k &= +g^{ki}\partial_k Z^j + g^{kj}\partial_k Z^i\,,\end{aligned}$$

valid for any Killing vector field Z on M (we omit the argument x or y as soon as there is a factor $\delta(x-y)$):

$$\begin{aligned}
\{\tilde{\theta}_{00}(x), \langle j_0(y), X\rangle\} &= -\{g^{ij}(\varphi(x))\,\pi_i(x)\,\pi_j(x), \pi_k(y) X^k_M(\varphi(y))\}\\
&= 2g^{ij}\pi_j\pi_k(\partial_i X^k_M)\,\delta(x-y) - (\partial_k g^{ij})\,\pi_i\pi_j X^k_M \delta(x-y)\\
&= (2g^{ij}\pi_k(\partial_i X^k_M) - (g^{ik}\partial_k X^j_M + g^{jk}\partial_k X^i_M)\,\pi_i)\,\pi_j \delta(x-y)\\
&= 0\,,\\
\{\tilde{\theta}_{00}(x), \langle j_\gamma(y), X\rangle\} &= -\{g^{ij}(\varphi(x))\,\pi_i(x)\,\pi_j(x), (X_M)_k(\varphi(y))\,\partial_\gamma \varphi^k(y)\}\\
&= -2g^{ij}(\varphi(x))\,\pi_j(x)\,(X_M)_i(\varphi(y))\,\partial_\gamma \delta(x-y)\\
&\quad + 2(\partial_i (X_M)_k)\,g^{ij}\pi_j \partial_\gamma \varphi^k \delta(x-y)\\
&= -2\pi_i(x) X^i_M(\varphi(x))\,\partial_\gamma \delta(x-y)\\
&\quad + 2((\partial_i (X_M)_k) - (\partial_k (X_M)_i))\,g^{ij}\pi_j \partial_\gamma \varphi^k \delta(x-y)\\
&= 2\langle j_0(x), X\rangle \partial_\gamma \delta(x-y) - 2((X_M)_{i|j} - (X_M)_{j|i})\,\partial_0 \varphi^i \partial_\gamma \varphi^j \delta(x-y)\\
&= 2\langle j_0(x), X\rangle \partial_\gamma \delta(x-y) + 2\langle(\partial_\gamma j_0 - \partial_0 j_\gamma)(x), X\rangle \delta(x-y)\,,
\end{aligned}$$

$$
\begin{aligned}
\{\tilde{\theta}_{0\alpha}(x), \langle j_0(y), X\rangle\} &= -\{\pi_i(x)\,\partial_\alpha\varphi^i(x), \pi_k(y)X_M^k(\varphi(y))\} \\
&= -\pi_i(x)X_M^i(\varphi(y))\,\partial_\alpha\delta(x-y) + (\partial_i X_M^k)\,\partial_\alpha\varphi^i\pi_k\delta(x-y) \\
&= -\pi_i(x)X_M^i(\varphi(x))\,\partial_\alpha\delta(x-y) \\
&= \langle j_0(x), X\rangle\partial_\alpha\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{0\alpha}(x), \langle j_\gamma(y), X\rangle\} &= -\{\pi_i(x)\,\partial_\alpha\varphi^i(x), (X_M)_k(\varphi(y))\,\partial_\gamma\varphi^k(y)\} \\
&= -\partial_\alpha\varphi^i(x)\,(X_M)_i(\varphi(y))\,\partial_\gamma\delta(x-y) \\
&\quad + (\partial_i(X_M)_k)\,\partial_\alpha\varphi^i\partial_\gamma\varphi^k\delta(x-y) \\
&= \langle j_\alpha(x), X\rangle\partial_\gamma\delta(x-y) - ((X_M)_{i|j} - (X_M)_{j|i})\,\partial_\alpha\varphi^i\partial_\gamma\varphi^j\delta(x-y) \\
&= \langle j_\alpha(x), X\rangle\partial_\gamma\delta(x-y) + \langle(\partial_\gamma j_\alpha - \partial_\alpha j_\gamma)(x), X\rangle\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{\alpha\beta}(x), \langle j_0(y), X\rangle\} &= -\{g_{ij}(\varphi(x))\,\partial_\alpha\varphi^i(x)\,\partial_\beta\varphi^j(x), \pi_k(y)X_M^k(\varphi(y))\} \\
&= -(\partial_k g_{ij})\,\partial_\alpha\varphi^i\partial_\beta\varphi^j X_M^k\delta(x-y) \\
&\quad - g_{ij}(\varphi(x))\,\partial_\beta\varphi^j(x)X_M^i(\varphi(y))\,\partial_\alpha\delta(x-y) \\
&\quad - g_{ij}(\varphi(x))\,\partial_\alpha\varphi^i(x)X_M^j(\varphi(y))\,\partial_\beta\delta(x-y) \\
&= -g_{ij}(\varphi(x))\,\partial_\beta\varphi^j(x)X_M^i(\varphi(x))\,\partial_\alpha\delta(x-y) \\
&\quad - g_{ij}(\varphi(x))\,\partial_\alpha\varphi^i(x)X_M^j(\varphi(x))\,\partial_\beta\delta(x-y) \\
&\quad - (g_{ij}\partial_\alpha\varphi^k\partial_\beta\varphi^j(\partial_k X_M^i) + g_{ij}\partial_\alpha\varphi^i\partial_\beta\varphi^k(\partial_k X_M^j))\,\delta(x-y) \\
&\quad - (\partial_k g_{ij})\,\partial_\alpha\varphi^i\partial_\beta\varphi^j X_M^k\delta(x-y) \\
&= \langle j_\alpha(x), X\rangle\partial_\beta\delta(x-y) + \langle j_\beta(x), X\rangle\partial_\alpha\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\{\tilde{\theta}_{\alpha\beta}(x), \langle j_\gamma(y), X\rangle\} \\
&= -\{g_{ij}(\varphi(x))\,\partial_\alpha\varphi^i(x)\,\partial_\beta\varphi^j(x), (X_M)_k(\varphi(y))\,\partial_\gamma\varphi^k(y)\} \\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{00}(x), \langle j(y), X\otimes Y\rangle\} &= \{g^{ij}(\varphi(x))\,\pi_i(x)\,\pi_j(x), g_{kl}(\varphi(y))X_M^k(\varphi(y))Y_M^l(\varphi(y))\} \\
&= -2(\partial_i(g_{kl}X_M^k Y_M^l))\,\partial_0\varphi^i\delta(x-y) \\
&= -2\langle\partial_0 j(x), X\otimes Y\rangle\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{0\alpha}(x), \langle j(y), X\otimes Y\rangle\} &= \{\pi_i(x)\,\partial_\alpha\varphi^i(x), g_{kl}(\varphi(y))X_M^k(\varphi(y))Y_M^l(\varphi(y))\} \\
&= -(\partial_i(g_{kl}X_M^k Y_M^l))\,\partial_\alpha\varphi^i\delta(x-y) \\
&= -\langle\partial_\alpha j(x), X\otimes Y\rangle\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\{\tilde{\theta}_{\alpha\beta}(x), \langle j(y), X\otimes Y\rangle\} \\
&= \{g_{ij}(\varphi(x))\,\partial_\alpha\varphi^i(x)\,\partial_\beta\varphi^j(x), g_{kl}(\varphi(y))X_M^k(\varphi(y))Y_M^l(\varphi(y))\} \\
&= 0\,.
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{00}(x), \tilde{\theta}_{00}(y)\} &= \{g^{ij}(\varphi(x))\,\pi_i(x)\,\pi_j(x), g^{kl}(\varphi(y))\,\pi_k(y)\,\pi_l(y)\} \\
&= -2(g^{ij}(\partial_i g^{kl})\,\pi_j\pi_k\pi_l - g^{kl}(\partial_k g^{ij})\,\pi_i\pi_j\pi_l)\,\delta(x-y) \\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{00}(x), \tilde{\theta}_{0\alpha}(y)\} &= \{g^{ij}(\varphi(x))\,\pi_i(x)\,\pi_j(x), \pi_k(y)\,\partial_\alpha\varphi^k(y)\} \\
&= 2g^{ij}(\varphi(x))\,\pi_j(x)\,\pi_i(y)\,\partial_\alpha\delta(x-y) \\
&\quad + (\partial_k g^{ij})\,\pi_i\pi_j\partial_\alpha\varphi^k\delta(x-y) \\
&= +g^{ij}(\varphi(x))\,\pi_j(x)\,\pi_i(x)\,\partial_\alpha\delta(x-y) + g^{ij}\pi_j\partial_\alpha\pi_i\,\delta(x-y) \\
&\quad + g^{ij}(\varphi(y))\,\pi_j(y)\,\pi_i(y)\,\partial_\alpha\delta(x-y) \\
&\quad - (\partial_k g^{ij})\,\partial_\alpha\varphi^k\pi_j\pi_i\,\delta(x-y) - g^{ij}\partial_\alpha\pi_j\pi_i\,\delta(x-y) \\
&\quad + (\partial_k g^{ij})\,\pi_i\pi_j\partial_\alpha\varphi^k\delta(x-y) \\
&= (\tilde{\theta}_{00}(x) + \tilde{\theta}_{00}(y))\,\partial_\alpha\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{00}(x), \tilde{\theta}_{\alpha\beta}(y)\} &= \{g^{ij}(\varphi(x))\,\pi_i(x)\,\pi_j(x), g_{kl}(\varphi(y))\,\partial_\alpha\varphi^k(y)\,\partial_\beta\varphi^l(y)\} \\
&= -2g^{ij}(\partial_i g_{kl})\,\pi_j\partial_\alpha\varphi^k\partial_\beta\varphi^l\delta(x-y) \\
&\quad + 2g^{ij}(\varphi(x))\,g_{il}(\varphi(y))\,\pi_j(x)\,\partial_\beta\varphi^l(y)\,\partial_\alpha\delta(x-y) \\
&\quad + 2g^{ij}(\varphi(x))\,g_{ki}(\varphi(y))\,\pi_j(x)\,\partial_\alpha\varphi^k(y)\,\partial_\beta\delta(x-y) \\
&= 2\pi_i(x)\,\partial_\alpha\varphi^i(x)\,\partial_\beta\delta(x-y) + 2\pi_i(x)\,\partial_\beta\varphi^i(x)\,\partial_\alpha\delta(x-y) \\
&\quad + 4\pi_i\partial_\alpha\partial_\beta\varphi^i\delta(x-y) + 4\Gamma^i_{kl}\partial_\alpha\varphi^k\partial_\beta\varphi^l\delta(x-y) \\
&= 2\tilde{\theta}_{0\alpha}(x)\,\partial_\beta\delta(x-y) + 2\tilde{\theta}_{0\beta}(x)\,\partial_\alpha\delta(x-y) \\
&\quad + 4\tilde{\theta}_{0(\alpha\beta)}(x)\,\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{0\alpha}(x), \tilde{\theta}_{0\beta}(y)\} &= \{\pi_i(x)\,\partial_\alpha\varphi^i(x), \pi_k(y)\,\partial_\beta\varphi^k(y)\} \\
&= \partial_\alpha\varphi^i(x)\,\pi_i(y)\,\partial_\beta\delta(x-y) + \pi_i(x)\,\partial_\beta\varphi^i(y)\,\partial_\alpha\delta(x-y) \\
&= +\partial_\alpha\varphi^i(y)\,\pi_i(y)\,\partial_\beta\delta(x-y) - \pi_i\partial_\beta\partial_\alpha\varphi^i\delta(x-y) \\
&\quad + \pi_i(x)\,\partial_\beta\varphi^i(x)\,\partial_\alpha\delta(x-y) + \pi_i\partial_\alpha\partial_\beta\varphi^i\delta(x-y) \\
&= \tilde{\theta}_{0\beta}(x)\,\partial_\alpha\delta(x-y) + \tilde{\theta}_{0\alpha}(y)\,\partial_\beta\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
\{\tilde{\theta}_{0\gamma}(x), \tilde{\theta}_{\alpha\beta}(y)\} &= \{\pi_i(x)\,\partial_\gamma\varphi^i(x), g_{kl}(\varphi(y))\,\partial_\alpha\varphi^k(y)\,\partial_\beta\varphi^l(y)\} \\
&= -(\partial_i g_{kl})\,\partial_\gamma\varphi^i\partial_\alpha\varphi^k\partial_\beta\varphi^l\delta(x-y) \\
&\quad + \partial_\gamma\varphi^i(x)\,g_{il}(\varphi(y))\,\partial_\beta\varphi^l(y)\,\partial_\alpha\delta(x-y) \\
&\quad + \partial_\gamma\varphi^i(x)\,g_{ki}(\varphi(y))\,\partial_\alpha\varphi^k(y)\,\partial_\beta\delta(x-y) \\
&= +\partial_\gamma\varphi^i(x)\,g_{il}(\varphi(x))\,\partial_\beta\varphi^l(x)\,\partial_\alpha\delta(x-y) \\
&\quad + \partial_\gamma\varphi^i(x)\,g_{ki}(\varphi(x))\,\partial_\alpha\varphi^k(x)\,\partial_\beta\delta(x-y) \\
&\quad + 2g_{ij}\partial_\gamma\varphi^i D_\alpha\partial_\beta\varphi^j\delta(x-y) \\
&= \tilde{\theta}_{\gamma\alpha}(x)\,\partial_\beta\delta(x-y) + \tilde{\theta}_{\gamma\beta}(x)\,\partial_\alpha\delta(x-y) + 2\tilde{\theta}_{\gamma(\alpha\beta)}(x)\,\delta(x-y)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\{\tilde{\theta}_{\alpha\beta}(x), \tilde{\theta}_{\gamma\delta}(y)\} \\
&\quad = \{g_{ij}(\varphi(x))\,\partial_\alpha\varphi^i(x)\,\partial_\beta\varphi^j(x), g_{kl}(\varphi(y))\,\partial_\gamma\varphi^k(y)\,\partial_\delta\varphi^l(y)\} \\
&\quad = 0\,.
\end{aligned}
$$

Using the relations (29–31), the mixed Poisson brackets involving the $\theta_{\mu\nu}$ can now be easily derived from those involving the $\tilde{\theta}_{\mu\nu}$:

$$\{\theta_{00}(x), \langle j_0(y), X\rangle\} = \langle j_\alpha(x), X\rangle \partial_\alpha \delta(x-y)\,, \tag{52}$$

$$\begin{aligned}\{\theta_{00}(x), \langle j_\gamma(y), X\rangle\} &= \langle j_0(x), X\rangle \partial_\gamma \delta(x-y) \\ &\quad - \langle (\partial_0 j_\gamma - \partial_\gamma j_0)(x), X\rangle \delta(x-y)\,,\end{aligned} \tag{53}$$

$$\{\theta_{0\alpha}(x), \langle j_0(y), X\rangle\} = \langle j_0(x), X\rangle \partial_\alpha \delta(x-y)\,, \tag{54}$$

$$\begin{aligned}\{\theta_{0\alpha}(x), \langle j_\gamma(y), X\rangle\} &= \langle j_\alpha(x), X\rangle \partial_\gamma \delta(x-y) \\ &\quad - \langle (\partial_\alpha j_\gamma - \partial_\gamma j_\alpha)(x), X\rangle \delta(x-y)\,,\end{aligned} \tag{55}$$

$$\begin{aligned}\{\theta_{\alpha\beta}(x), \langle j_0(y), X\rangle\} &= \langle j_\alpha(x), X\rangle \partial_\beta \delta(x-y) + \langle j_\beta(x), X\rangle \partial_\alpha \delta(x-y) \\ &\quad - \delta_{\alpha\beta} \langle j_\gamma(x), X\rangle \partial_\gamma \delta(x-y)\,,\end{aligned} \tag{56}$$

$$\begin{aligned}\{\theta_{\alpha\beta}(x), \langle j_\gamma(y), X\rangle\} &= \delta_{\alpha\beta} \langle j_0(x), X\rangle \partial_\gamma \delta(x-y) \\ &\quad - \delta_{\alpha\beta} \langle (\partial_0 j_\gamma - \partial_\gamma j_0)(x), X\rangle \delta(x-y)\,,\end{aligned} \tag{57}$$

$$\{\theta_{00}(x), \langle j(y), X\otimes Y\rangle\} = -\langle \partial_0 j(x), X\otimes Y\rangle \delta(x-y)\,, \tag{58}$$

$$\{\theta_{0\alpha}(x), \langle j(y), X\otimes Y\rangle\} = -\langle \partial_\alpha j(x), X\otimes Y\rangle \delta(x-y)\,, \tag{59}$$

$$\{\theta_{\alpha\beta}(x), \langle j(y), X\otimes Y\rangle\} = -\delta_{\alpha\beta} \langle \partial_0 j(x), X\otimes Y\rangle \delta(x-y)\,. \tag{60}$$

To compute the pure energy momentum tensor algebra, we use

$$\begin{aligned}\{\theta_{00}(x), \theta_{00}(y)\} &= \tfrac{1}{4}\{\tilde{\theta}_{00}(x), \tilde{\theta}_{\gamma\gamma}(y)\} + \tfrac{1}{4}\{\tilde{\theta}_{\gamma\gamma}(x), \tilde{\theta}_{00}(y)\} \\ &= +\tilde{\theta}_{0\gamma}(x)\, \partial_\gamma \delta(x-y) + \tilde{\theta}_{0(\gamma\gamma)}(x)\, \delta(x-y) \\ &\quad - \tilde{\theta}_{0\gamma}(y)\, \partial_\gamma \delta(y-x) - \tilde{\theta}_{0(\gamma\gamma)}(y)\, \delta(y-x) \\ &= (\theta_{0\gamma}(x) + \theta_{0\gamma}(y))\, \partial_\gamma \delta(x-y)\,, \\ \{\theta_{00}(x), \theta_{0\alpha}(y)\} &= \tfrac{1}{2}\{\tilde{\theta}_{00}(x), \tilde{\theta}_{0\alpha}(y)\} + \tfrac{1}{2}\{\tilde{\theta}_{\gamma\gamma}(x), \tilde{\theta}_{0\alpha}(y)\} \\ &= \tfrac{1}{2}(\tilde{\theta}_{00}(x) + \tilde{\theta}_{00}(y))\, \partial_\alpha \delta(x-y) \\ &\quad - \tilde{\theta}_{\alpha\gamma}(y)\, \partial_\gamma \delta(y-x) - \tilde{\theta}_{\alpha(\gamma\gamma)}(y)\, \delta(y-x) \\ &= \tfrac{1}{2}(\tilde{\theta}_{00}(x) + \tilde{\theta}_{00}(y))\, \partial_\alpha \delta(x-y) \\ &\quad + \tilde{\theta}_{\alpha\gamma}(x)\, \partial_\gamma \delta(x-y) + \tfrac{1}{2}\, \partial_\alpha \tilde{\theta}_{\gamma\gamma}(x)\, \delta(x-y) \\ &= \tfrac{1}{2}(\tilde{\theta}_{00}(x) + \tilde{\theta}_{00}(y) - \tilde{\theta}_{\gamma\gamma}(x) + \tilde{\theta}_{\gamma\gamma}(y))\, \partial_\alpha \delta(x-y) \\ &\quad + \tilde{\theta}_{\alpha\gamma}(x)\, \partial_\gamma \delta(x-y) \\ &= \theta_{00}(y)\, \partial_\alpha \delta(x-y) + \theta_{\alpha\gamma}(x)\, \partial_\gamma \delta(x-y)\,,\end{aligned}$$

and obtain

$$\{\theta_{00}(x), \theta_{00}(y)\} = (\theta_{0\gamma}(x) + \theta_{0\gamma}(y))\, \partial_\gamma \delta(x-y)\,, \tag{61}$$

$$\{\theta_{00}(x), \theta_{0\alpha}(y)\} = \theta_{00}(y)\, \partial_\alpha \delta(x-y) + \theta_{\alpha\gamma}(x)\, \partial_\gamma \delta(x-y)\,, \tag{62}$$

$$\{\theta_{0\alpha}(x), \theta_{0\beta}(y)\} = \theta_{0\beta}(x)\, \partial_\alpha \delta(x-y) + \theta_{0\alpha}(y)\, \partial_\beta \delta(x-y)\,, \tag{63}$$

plus three other Poisson bracket relations which contain the composite field $\tilde{\theta}_{\kappa(\mu\nu)}$: these are rather complicated and not very enlightening, so we shall not write them down explicitly. When $d = 2$, Eqs. (52–60) and (61–63) clearly reduce to Eqs. (15–20) and (9–11), respectively; moreover, at least two of the three indices of the composite field $\tilde{\theta}_{\kappa(\mu\nu)}$ are necessarily equal, to that, according to Eqs. (34) and (35), it can be completely eliminated in favor of derivatives of the energy-momentum tensor, and the resulting formulas turn out to carry no information beyond Eqs. (61–63) given above. When $d > 2$, however, there is no way to eliminate this composite field, so the algebra does not close. Moreover, repeating the process of computing Poisson brackets will give rise to new composite fields built from products of higher (covariant) derivatives of φ. We suspect that a finite number of such additional fields will suffice to close the algebra (when derivatives are included), but we have not analyzed the question any further.

Returning to the two-dimensional case, it remains to be shown that Eqs. (9–11) give rise to a chiral Witt algebra. To do so, we first switch to light-cone components,

$$\theta_{++} = \tfrac{1}{2}\,(\theta_{00} + \theta_{01})\,, \tag{64}$$

$$\theta_{--} = \tfrac{1}{2}\,(\theta_{00} - \theta_{01})\,, \tag{65}$$

(note that $\theta_{+-} = 0$), and then convert the equal-time Poisson bracket relations (9–11) to commutation relations valid on the light cone: this requires invoking the equations of motion for the energy-momentum tensor. These are nothing but the conservation law

$$\partial_-\theta_{++} = 0\,, \qquad \partial_+\theta_{--} = 0\,, \tag{66}$$

so θ_{++} depends only on x^+ and θ_{--} depends only on x^-; then Eqs. (9–11) become

$$\{\theta_{++}(x^+), \theta_{++}(y^+)\} = +(\theta_{++}(x^+) + \theta_{++}(y^+))\,\delta'(x^+ - y^+)\,, \tag{67}$$

$$\{\theta_{--}(x^-), \theta_{--}(y^-)\} = -(\theta_{--}(x^-) + \theta_{--}(y^-))\,\delta'(x^- - y^-)\,, \tag{68}$$

$$\{\theta_{++}(x^+), \theta_{--}(y^-)\} = 0\,. \tag{69}$$

When expressed in Fourier components this becomes the well-known Witt algebra for θ_{++} and for θ_{--}.

For the currents, on the other hand, this procedure cannot be carried out in the same way, because the conservation law for the currents is, in light-cone components,

$$\partial_+ j_- + \partial_- j_+ = 0\,,$$

which by itself is not sufficient to convert equal-time Poisson brackets to light-cone Poisson brackets.

The net result is that the total algebra is a semidirect product of a chiral Witt algebra with a non-chiral (non-Kac-Moody) current algebra.

The most important question to be answered next is how the algebraic structure derived above changes when passing from the classical theory to the quantum theory. Some changes are to be expected due to the phenomenon of dynamical mass generation, which will give the energy-momentum tensor a non-vanishing trace and destroy the chiral nature of the energy-momentum tensor algebra. Our hope is that the non-chiral current algebra should give a clue as to what this non-chiral energy-momentum tensor algebra should be, and ideally that there should exist some non-chiral analogue of the Sugawara construction – which, as is well known, e.g., for the Wess-Zumino-Novikov-Witten models or for certain massless fermionic theories [2], actually derives the chiral energy-momentum tensor algebra of conformal field theory from the corresponding chiral current algebra (two commuting copies of the relevant (untwisted) affine Kac-Moody algebra).

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