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Composition of Kinetic Momenta: The $\mathcal{U}_q(sl(2))$ Case

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Abstract. The tensor products of (restricted and unrestricted) finite dimensional irreducible representations of $\mathcal{U}_q(sl(2))$ are considered for q a root of unity. They are decomposed into direct sums of irreducible and/or indecomposable representations.

1. Introduction

When the parameter of deformation q is not a root of unity, the theory of representations of quantum algebras $\mathcal{U}_q(\mathcal{G})$ (with $\mathcal{G}$ a semi-simple Lie algebra) is equivalent to the classical theory [1]. In the following, we consider $\mathcal{U}_q(sl(2))$, with q a root of unity. In this case, the dimension of the finite dimensional irreducible representations (irreps) is bounded, and a new type of representations occurs, depending on continuous parameters [2–5]. Moreover, finite dimensional representations are not always direct sums of irreps: they can contain indecomposable sub-representations. Some kinds of indecomposable representations actually appear in the decomposition of tensor products of irreps.

Another peculiarity with q a root of unity is that the fusion rules are generally not commutative. There exist, however, many sub-fusion-rings that are commutative. The well-known one is the fusion ring generated by the irreps of the finite dimensional quotient of $\mathcal{U}_q(sl(2))$ [6–8)]. Families of larger commutative fusion ring that contain the latter will also be defined later.

The following section is devoted to definitions, to the description of the centre of $\mathcal{U}_q(sl(2))$, and finally recalls the classification of the irreps of $\mathcal{U}_q(sl(2))$. The irreps of $\mathcal{U}_q(sl(2))$ can be classified into two types:

– The first type, called type $\mathcal{A}$ in the following, corresponds to the deformations of representations that exist in the classical case $q = 1$. These representations are also called restricted representations since they are also representations of the finite

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dimensional quotient of $\mathscr{U}_q(sl(2))$. [This quotient consists in imposing classical values to the enlarged centre of $\mathscr{U}_q(sl(2))$]

– The second type, denoted by $\mathscr{B}$, contains finite dimensional irreducible representations that have no finite dimensional classical analogue. They are generically characterized by three continuous complex parameters, which correspond to the values of the generators of the enlarged centre, and they all have the same dimension. (This property is a particularity of $\mathscr{U}_q(sl(2))$. At higher ranks, several dimensions are allowed for irreps. The dimension remains however bounded.)

Section 3 is a review of the fusion rules for type $\mathscr{A}$ or restricted irreps [6–8]. The fusion ring generated by the type $\mathscr{A}$ irreps also contains a class of indecomposable representations of dimension called $\mathscr{I}\mathrm{nd}_{\mathscr{A}}$ representations in the following.

Section 4 deals with the composition of type $\mathscr{A}$ (restricted) with type $\mathscr{B}$ (unrestricted) irreps. These tensor products generically lead to sums of type $\mathscr{B}$ irreps. For non-generic parameters, these fusion rules also lead to a new class of indecomposable representations called $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ representations.

The composition of type $\mathscr{B}$ irreps is the subject of Sect. 5. The tensor product of two type $\mathscr{B}$ irreps is generically reducible into type $\mathscr{B}$ irreps. However, it can also contain $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ representations when the components of the tensor product do not have generic parameters. For sub-sub-generic cases, the indecomposable representations $\mathscr{I}\mathrm{nd}_{\mathscr{A}}$ reappear, together with, in even more particular cases, another type of indecomposable representations denoted by $\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'$.

The results presented in Sects. 3, 4, 5 are summarized in Tables 1, 2, 3.

In Sect. 6, we prove that the fusion ring generated by the irreducible representations closes with the indecomposable representations $\mathscr{I}\mathrm{nd}_{\mathscr{A}}$, $\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'$ and $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$.

The results of Sect. 5 are finally used as an example in Sect. 7 in the decomposition of the regular representation of $\mathscr{U}_q(sl(2))$.

2. Definitions, Centre, and Irreducible Representations

2.1. Definitions

The quantum algebra $\mathscr{U}_q(sl(2))$ is defined by the generators k, k^{-1}, e, f, and the relations

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1\,, \qquad & kek^{-1} &= q^2 e\,, \\ [e,f] &= \frac{k-k^{-1}}{q-q^{-1}}\,, \qquad & kfk^{-1} &= q^{-2} f\,. \end{aligned} \tag{2.1}$$

The coproduct Δ is given by

$$\begin{aligned} \Delta(k) &= k \otimes k\,, \\ \Delta(e) &= e \otimes 1 + k \otimes e\,, \\ \Delta(f) &= f \otimes k^{-1} + 1 \otimes f\,, \end{aligned} \tag{2.2}$$

while the opposite coproduct Δ' is $\Delta' = P\Delta P$, where P is the permutation map $Px \otimes y = y \otimes x$. The result of the composition of two representations ϱ_1 and ϱ_2 of $\mathscr{U}_q(sl(2))$ is the representation $\varrho = (\varrho_1 \otimes \varrho_2) \circ \Delta$, whereas the composition in the reverse order is equivalent to $\varrho' = (\varrho_1 \otimes \varrho_2) \circ \Delta'$.

2.2. Centre of $\mathscr{U}_q(sl(2))$

The usual q-deformed quadratic Casimir

$$C = fe + (q - q^{-1})^{-2}(qk + q^{-1}k^{-1}) \tag{2.3}$$

belongs to the centre of $\mathscr{U}_q(sl(2))$. When q is not a root of unity, C generates this centre.

In the following, the parameter q will be a root of unity. Let m' be the smallest integer such that $q^{m'} = 1$. Let m be equal to m' if m' is odd and to $m'/2$ otherwise.

Then the elements e^m, f^m, and $k^{\pm m}$ of $\mathscr{U}_q(sl(2))$ also belong to the centre [2]. Together with C, they actually generate the centre of $\mathscr{U}_q(sl(2))$, and these generators are related by a polynomial relation [5]. We write here this relation as follows: let P_m be the polynomial in X, of degree m, ($P(X) = X^m + \cdots$), such that

$$P_m(X) = \frac{2}{(q - q^{-1})^{2m}} T_m\left(\frac{1}{2}(q - q^{-1})^2 X\right), \tag{2.4}$$

where T_m is the m^{th} Chebychev polynomial of the first kind

$$T_m(X) = \cos(m \arccos X). \tag{2.5}$$

Then the relation becomes

$$P_m(C) = e^m f^m + q^m \frac{k^m + k^{-m}}{(q - q^{-1})^{2m}}. \tag{2.6}$$

2.3. Finite Dimensional Irreducible Representations of $\mathscr{U}_q(sl(2))$

We now recall the classification [2] of the irreducible representations of $\mathscr{U}_q(sl(2))$. The new facts (with respect to the classical case or to the case q not being a root of unity) are that the dimensions of the finite dimensional irreps are bounded by m, and that the irreps of dimension m depend on three complex continuous parameters. In the following, we will call type $\mathscr{A}$ irreps those that have a classical analogue (restricted representations) and type $\mathscr{B}$ irreps the others. We will mostly use a module notation.

We will denote by x, y, $z^{\pm 1}$, and c the values of e^m, f^m, $k^{\pm m}$, and C on irreducible representations.

The q-deformed classical irreps (type $\mathscr{A}$) are labelled by their half-integer spin j, which is such that $1 \leq 2j + 1 \leq m$, and by another discrete parameter $\omega = \pm 1$ [9]. They are given by the basis $\{\omega_0, \ldots, \omega_{2j}\}$ and, in a notation of module,

$$\begin{cases} kw_p = \omega q^{2j-2p} w_p & \text{for } 0 \leq p \leq 2j \\ fw_p = w_{p+1} & \text{for } 0 \leq p \leq 2j - 1 \\ fw_{2j} = 0 & \\ ew_p = \omega [p][2j - p + 1] w_{p-1} & \text{for } 1 \leq p \leq 2j \\ ew_0 = 0 & \end{cases} \tag{2.7}$$

where as usual

$$[x] \equiv \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2.8}$$

We denote this representation by $\mathscr{S}\mathrm{pin}(j,\omega)$. On it, the central elements e^m, f^m, k^m, and C take the values $x=y=0$, $z=(\omega q^{2j})^m=\pm 1$, and $c=\omega(q-q^{-1})^{-2}(q^{2j+1}+q^{-2j-1})$ respectively.

Note that the representation $\mathscr{S}\mathrm{pin}(j,\omega=-1)$ can be obtained as the tensor product of $\mathscr{S}\mathrm{pin}(j,1)$ by the one-dimensional representation $\mathscr{S}\mathrm{pin}(j=0,\omega)$.

A *type* $\mathscr{B}$ irrep is an irreducible representation that has no finite dimensional analogue when q is equal to one. It has dimension m.

It is characterized by three complex parameters x, y, z corresponding to the values of e^m, f^m, k^m, and by a discrete choice among m values c_l for the quadratic Casimir C. These values are just the roots of

$$P_m(c) - xy - q^m \frac{z+z^{-1}}{(q-q^{-1})^{2m}} = 0\,. \tag{2.9}$$

If we define ζ by

$$xy + q^m \frac{z+z^{-1}}{(q-q^{-1})^{2m}} = \frac{\zeta^m+\zeta^{-m}}{(q-q^{-1})^{2m}}\,, \tag{2.10}$$

then, by virtue of the identity

$$\cos m\psi - \cos m\phi = 2^{m-1}\prod_{k=0}^{m-1}(\cos\psi - cos(\phi+2k\pi/m))\,, \tag{2.11}$$

the c_l's are given by

$$c_l = \frac{\zeta q^{2l}+\zeta^{-1}q^{-2l}}{(q-q^{-1})^2}\,, \qquad l=0,\dots,m-1\,. \tag{2.12}$$

Let λ be an m^{th} root of z and c one of the c_l's. Then the type $\mathscr{B}$-representation, denoted in the following by $\mathscr{B}(x,y,z,c)$, is given in the basis $\{v_0,\dots,v_{m-1}\}$, by

$$\begin{cases} kv_p = \lambda q^{-2p} v_p & \text{for } 0\le p\le m-1 \\ fv_p = v_{p+1} & \text{for } 0\le p\le m-2 \\ fv_{m-1} = yv_0 & \\ ev_p = \left(c - \dfrac{1}{(q-q^{-1})^2}(\lambda q^{-2p+1}+\lambda^{-1}q^{2p-1})\right)v_{p-1} & \text{for } 1\le p\le m-1 \\ ev_0 = \dfrac{1}{y}\left(c - \dfrac{1}{(q-q^{-1})^2}(\lambda q+\lambda^{-1}q^{-1})\right)v_{m-1}\,. & \end{cases} \tag{2.13}$$

Remark 1. In this basis, the generators e and f do not play symmetric roles. The normalizations of the vectors are such that f is extremely simple in this basis. There exist of course more symmetric bases, and bases where e has a simple expression (related to the latter by a simple change of normalization). The advantage of this basis is that it can describe (irreducible) representations with two highest-weight vectors (e vanishes on two vectors of the basis) and a non-vanishing y. For cases where y vanishes but not x, another basis could be preferable. However, the limit $y\to 0$ is well-defined if $c=\dfrac{\lambda q+\lambda^{-1}q^{-1}}{(q-q^{-1})^2}$ and $ev_0=\beta v_{m-1}$, $\beta\in C$.

The representation (2.13) is actually irreducible iff one of the four following conditions is satisfied:

a) $x \neq 0$,
b) $y \neq 0$,
c) $z \neq \pm 1$,
d) $c = \dfrac{2\omega}{(q-q^{-1})^2} \quad (\omega = \pm 1)$.

Remark 2. Note that $\mathscr{B}\left(0,0,\pm 1,\dfrac{2\omega}{(q-q^{-1})^2}\right) = \mathscr{S}\mathrm{pin}((m-1)/2,\omega)$ (fourth case) is actually of type $\mathscr{A}$. This case will not be considered as type $\mathscr{B}$ in the following. So a type $\mathscr{B}$ irrep will have $(x,y,z) \neq (0,0,\pm 1)$.

Remark 3. The representations described by (2.13) with $(x,y,z) = (0,0,\pm 1)$, and one of the other possible values for c (β arbitrary, cf. Remark 1) are indecomposable. These representations, called $\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'$, will appear in the last section as indecomposable parts of some tensor products.

For further use, we define the function $c(\zeta)$ by

$$c(\zeta) \equiv \frac{\zeta+\zeta^{-1}}{(q-q^{-1})^2}\,. \tag{2.14}$$

The representation (2.13) will be called periodic if $xy \neq 0$ In this case it is irreducible and has no highest-weight and no lowest-weight vectors. A semi-periodic representation is a representation for which only one of the parameters x and y vanishes. It is then also irreducible. Following [10], a type $\mathscr{B}$ representation with $x=y=0$, $z \neq \pm 1$ will be called nilpotent.

3. Composition of Type $\mathscr{A}$ Representations

This section will be a brief review of the results of Pasquier and Saleur [6], of Keller [7], and of Kerler [8]. The tensor product of two representations $\mathscr{S}\mathrm{pin}(j_1,\omega_1)$ and $\mathscr{S}\mathrm{pin}(j_2,\omega_2)$ decomposes into irreducible representations of the same type and also, if $2(j_1+j_2)+1$ is greater than m, into some indecomposable spin representations.

An indecomposable spin representation $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j,\omega)$ has dimension $2m$. It is characterized by a half integer j such that $1 \leq 2j+1 < m$ and by $\omega = \pm 1$. In a basis $\{w_0,\ldots,w_{m-1},x_0,\ldots,x_{m-1}\}$ the generators of $\mathscr{U}_q(sl(2))$ act as follows:

$$\begin{cases} kw_p = \omega q^{-2j-2-2p} w_p & \\ fw_p = w_{p+1} & \text{for } 0 \leq p \leq m-2 \\ fw_{m-1} = 0 & \\ ew_p = \omega[p][-2j-p-1]w_{p-1} & \text{for } 0 \leq p \leq m-1 \\ kx_p = \omega q^{2j-2p} x_p & \\ fx_p = x_{p+1} & \text{for } 0 \leq p \leq m-2 \\ fx_{m-1} = 0 & \\ ex_p = f^{p+m-2j-2}w_0 + \omega[p][2j-p+1]x_{p-1} & \text{for } 0 \leq p \leq m-1\,. \end{cases} \tag{3.1}$$

(In particular, $ex_0 = w_{m-2j-2}$ and $ex_{2j+1} = w_{m-1}$, and e^m, f^m are 0 on such a module.)

Table 1. Summary of the fusion rules for type $\mathscr{A}$ irreps

$\mathscr{S}\mathrm{pin}(j_1, \omega_1) \otimes \mathscr{S}\mathrm{pin}(j_2, \omega_2)$	Decomposes into
$2(j_1 + j_2) + 1 \leq m$	$\mathscr{S}\mathrm{pin}(j, \omega_1\omega_2)$
$2(j_1 + j_2) + 1 > m$	$\mathscr{S}\mathrm{pin}(j, \omega_1\omega_2)$ and $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, \omega_1\omega_2)$

This indecomposable representation contains the sub-representation $\mathscr{S}\mathrm{pin}(j, \omega)$. It is a deformation of the sum of the classical $\mathscr{S}\mathrm{pin}(j)$ and $\mathscr{S}\mathrm{pin}(m/2 - j - 1)$ representations.

The fusion rules are

$$\mathscr{S}\mathrm{pin}(j_1, \omega_1) \otimes \mathscr{S}\mathrm{pin}(j_2, \omega_2) = \left(\bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, m-j_1-j_2-2)} \mathscr{S}\mathrm{pin}(j, \omega_1\omega_2) \right) \oplus \left(\bigoplus_{j=m-j_1-j_2-1}^{(m-1)/2} \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, \omega_1\omega_2) \right), \tag{3.2}$$

where the sums are limited to integer values of j if $j_1 + j_2$ is integer, and to half-(odd)-integer values if $j_1 + j_2$ is half-(odd)-integer. In conformal field theories, the fusion rules (3.2) are truncated to the first parenthesis, keeping only those representations that have a q-dimension different from 0.

The fusion rules for type $\mathscr{A}$ irreps are summarized in Table 1.

The fusion rules of type $\mathscr{A}$ representations close with

$$\begin{aligned} \mathscr{S}\mathrm{pin}(j_1, \omega_1) \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j_2, \omega_2) &= \bigoplus_{\text{some } j, \omega} \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, \omega) \\ \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j_1, \omega_1) \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j_2, \omega_2) &= \bigoplus_{\text{some } j, \omega} \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, \omega) . \end{aligned} \tag{3.3}$$

The $\mathscr{S}\mathrm{pin}$ and $\mathscr{I}\mathrm{nd}_{\mathscr{A}}$ representations thus build a closed fusion ring.

4. Fusion Rules Mixing Type $\mathscr{A}$ and Type $\mathscr{B}$ Representations

Proposition 1. *The tensor product of a type $\mathscr{B}$ representation $\mathscr{B}(x, y, z, c)$ with the spin $1/2$ representation $\mathscr{S}\mathrm{pin}(1/2, 1)$ is completely reducible* iff $c \neq \dfrac{\pm 2}{(q - q^{-1})^2}$. *More precisely, if $c = c(\zeta) = \dfrac{\zeta + \zeta^{-1}}{(q - q^{-1})^2}$,*

$$\begin{aligned} &\mathscr{B}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(1/2, 1) \\ &\quad = \mathscr{B}(x, q^m y, q^m z, c(q\zeta)) \oplus \mathscr{B}(x, q^m y, q^m z, c(q^{-1}\zeta)) . \end{aligned} \tag{4.1}$$

If $c = c(\pm 1) = \dfrac{\pm 2}{(q - q^{-1})^2}$, the tensor product is a type $\mathscr{B}$ indecomposable representation of dimension $2m$, denoted by $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^m y, q^m z, c' = c(\pm q) = \pm \dfrac{q + q^{-1}}{(q - q^{-1})^2}$ and defined below.

Proof. First write $c = \dfrac{\zeta + \zeta^{-1}}{(q-q^{-1})^2}$. The matrix of the quadratic Casimir on a weight space of the tensor product is diagonalizable iff $\zeta \neq \pm 1$ and the eigenvalues are $c(q\zeta) \neq c(q^{-1}\zeta)$. Each eigenvector of C generates a type $\mathscr{B}$ irrep $\mathscr{B}(x, q^m y, q^m z, c)$ since $(x, q^m y, q^m z) \neq (0, 0, \pm 1)$.

When $\zeta = \pm 1$, the eigenvalues $c(q\zeta)$ and $c(q^{-1}\zeta)$ coincide and C is not diagonalizable. It has only one eigenvector (up to a normalization) on each weight space, which generates a type $\mathscr{B}$ irrep $\mathscr{B}(x, q^m y, q^m z, c(\pm q))$. The quotient of the total representation by this subrepresentation is again equivalent to $\mathscr{B}(x, q^m y, q^m z, c(\pm q))$. The tensor product is hence the $2m$ dimensional indecomposable representation $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^m y, q^m z, c' = c(\pm q))$.

Definition. The type $\mathscr{B}$ indecomposable representation $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c)$ is characterized as follows: the central elements f^m and k^m take the scalar values (y, z), and there is a basis $\{v_0^{(i)}, \ldots, v_{m-1}^{(i)}\}$, $(i = 1, 2)$, in which this representation is written

$$\begin{cases} kv_p^{(1)} = \lambda q^{-2p} v_p^{(1)} & \text{for } 0 \le p \le m-1 \\ fv_p^{(1)} = v_{p+1}^{(1)} & \text{for } 0 \le p \le m-2 \\ fv_{m-1}^{(1)} = y v_0^{(1)} & \\ ev_p^{(1)} = \Big(c - \dfrac{(\lambda q^{-2p+1} + \lambda^{-1} q^{2p-1})}{(q-q^{-1})^2}\Big) v_{p-1}^{(1)} & \text{for } 1 \le p \le m-1 \\ ev_0^{(1)} = y^{-1}\Big(c - \dfrac{(\lambda q + \lambda^{-1} q^{-1})}{(q-q^{-1})^2}\Big) v_{m-1}^{(1)} & \\ kv_p^{(2)} = \lambda q^{-2p} v_p^{(2)} & \text{for } 0 \le p \le m-1 \\ fv_p^{(2)} = v_{p+1}^{(2)} & \text{for } 0 \le p \le m-2 \\ fv_{m-1}^{(2)} = y v_0^{(2)} & \\ ev_p^{(2)} = \Big(c - \dfrac{(\lambda q^{-2p+1} + \lambda^{-1} q^{2p-1})}{(q-q^{-1})^2}\Big) v_{p-1}^{(2)} + v_{p-1}^{(1)} & \text{for } 1 \le p \le m-1 \\ ev_0^{(2)} = y^{-1}\Big(\Big(c - \dfrac{(\lambda q + \lambda^{-1} q^{-1})}{(q-q^{-1})^2}\Big) v_{m-1}^{(2)} + v_{m-1}^{(1)}\Big) & \end{cases} \tag{4.2}$$

with $\lambda^m = z$. We call this representation a type $\mathscr{B}$ indecomposable representation, because $(x, y, z) \neq (0, 0, \pm 1)$. It does not belong to the fusion ring generated by the type $\mathscr{A}$ irreps.

The sub-representation generated by the set of $v_p^{(1)}$, as well as the quotient of the whole representation by this sub-representation are equivalent to $\mathscr{B}(x, y, z, c)$.

If $c = c(\zeta)$ with $\zeta^{2m} = 1$ and $\zeta \neq \pm 1$ (which will always be satisfied in the cases we will consider), the central element e^m is scalar with value x on $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c(\zeta))$. Otherwise, we would have

$$e^m v_p^{(1)} = x v_p^{(1)}, \quad e^m v_p^{(2)} = x v_p^{(2)} + \frac{m}{y} \frac{\zeta^m - \zeta^{-m}}{\zeta - \zeta^{-1}} v_p^{(1)}.$$

In the following, we restrict the definition of $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ representations to those representations that have one of the special values for c (i.e $\zeta^{2m} = 1$). The operators e^m, f^m

and k^m hence take scalar values on $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ representations. As we will see in the next section, the property that these operators are scalar on a representation is preserved in the composition of representations. The fusion ring generated by the irreducible representations then contains only representations with diagonal e^m, f^m and k^m.

The case $x = 0$ and $y \neq 0$ (semi-periodic representation $\otimes$ spin $1/2$) is included here. The description of the case $x \neq 0$ and $y = 0$ is simply obtained by considering bases with simple action of e instead of f. The case $x = y = 0$ (nilpotent representation $\otimes$ spin $1/2$) is included in this proposition and it does not lead to indecomposability since the parameter z, c (related to the highest weight λ through $z = \lambda^m$ and $c = c(q\zeta)$) of the type $\mathscr{B}$ nilpotent representation has to be generic (see Remark 2).

Let us again consider $\mathscr{B}(x, y, z, c)$ with $c = c(\zeta)$ (2.14). As a consequence of the previous proposition, we have:

Theorem 1. *The tensor product of the type* $\mathscr{B}$ *representation* $\mathscr{B}(x, y, z, c)$ *with the spin* j *representation* $\mathscr{S}\mathrm{pin}(j, 1)$ *is completely reducible as long as all the values* $c_l = c(q^{2j-2l}\zeta)$ *for* $l = 0, \ldots, 2j$ *are different (which is satisfied in particular if* $\zeta^{2m} \neq 1$*). Moreover,*

$$\mathscr{B}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(j, 1) = \bigoplus_{l=0}^{2j} \mathscr{B}(x, q^{2jm}y, q^{2jm}z, c_l = c(q^{2j-2l}\zeta)) . \tag{4.3}$$

The tensor product is not-completely reducible when some pairs of $c_l = c(q^{2j-2l}\zeta)$ $(l = 0, \ldots, 2j)$ *coincide (since* $2j + 1 \leq m$*, the* $2j + 1$ *values* c_l *can be only doubly degenerate). In this case, the decomposition is obtained from* (4.3) *by simply replacing each pair of* irreps *arising with the same* c_l *by the indecomposable type* $\mathscr{B}$ *sub-representation* $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^{2jm}y, q^{2jm}z, c_l)$ (4.2).

Proof. The previous proposition with the coassociativity of Δ is the basic tool. The representation $\mathscr{B}(x, y, z, c)$ is composed with $(\mathscr{S}\mathrm{pin}(1/2, 1))^{\otimes 2j}$, which contains $\mathscr{B}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(j, 1)$. We however still need to know the result of the composition of $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c)$ with $\mathscr{S}\mathrm{pin}(1/2, 1)$, since $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c)$ can appear in intermediate stages.

Let $c = \dfrac{\zeta + \zeta^{-1}}{(q - q^{-1})^2}$. We look at the matrix of $\Delta(C)$ on a weight space of the tensor product

$$\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(1/2, 1) .$$

This matrix is a 4×4 matrix. It can be decomposed into two 2×2 non-diagonalizable blocks with eigenvalues $c(q\zeta)$ and $c(q^{-1}\zeta)$ if ζ is different from $\pm q$ and $\pm q^{-1}$. If $\zeta = \pm q^{\pm 1}$, it can be decomposed into one 2×2 non-diagonalizable block with eigenvalue $c(\pm q^2)$ and two 1×1 blocks containing $c(\pm 1)$. So the tensor product of $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c)$ with $\mathscr{S}\mathrm{pin}(1/2, 1)$ reduces to

$$\begin{aligned}&\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(1/2, 1)\\ &\quad = \mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^m y, q^m z, c(q\zeta)) \otimes \mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^m y, q^m z, c(q^{-1}\zeta))\end{aligned} \tag{4.4}$$

if ζ is different from $\pm q$ and $\pm q^{-1}$, and

$$\begin{aligned}&\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(1/2, 1)\\ &\quad = \mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^m y, q^m z, c(\pm q^2)) \oplus 2\mathscr{B}(x, q^m y, q^m z, c(\pm 1))\end{aligned} \tag{4.5}$$

if $\zeta = \pm q^{\pm 1}$. The factor 2 means a multiplicity of 2 of the representation in the decomposition, i.e. $C^2 \otimes \ldots$.

Proposition 2. *If $\zeta^{2m} \neq 1$ the tensor product of the type $\mathscr{B}$ representation $\mathscr{B}(x, y, z, c)$ with a type $\mathscr{A}$ indecomposable representation $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, 1)$ is completely reducible and*

$$\mathscr{B}(x, y, z, c) \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, 1) = \bigoplus_{l=0}^{m-1} 2\mathscr{B}\left(x, q^{2jm}y, q^{2jm}z, \frac{q^{2j-2l}\zeta + q^{-2j+2l}\zeta^{-1}}{(q-q^{-1})^2}\right). \tag{4.6}$$

If $\zeta^{2m} = 1$, we have

$$\mathscr{B}(x, y, z, c) \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, 1) = \bigoplus_{l=0}^{m-1} \mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^{2jm}y, q^{2jm}z, c(q^{2j-2l}\zeta)), \tag{4.7}$$

with the prescription that $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, q^{2jm}y, q^{2jm}z, c(\pm 1))$, if it appears, has to be replaced by $2\mathscr{B}(x, q^{2jm}y, q^{2jm}z, c(\pm 1))$. (Such a prescription is of much easier use than an exploration of all the cases: the parity of m', $2j$ and the value of ζ enter in the game.)

Proof. The proof follows from the fact that $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, 1)$ enters in the decomposition of tensor products of some ordinary spin irreps, as explained in the previous section. This result is then obtained as the previous theorem by further composition with the $\mathscr{S}\mathrm{pin}(1/2, 1)$ representation and using the coassociativity of Δ. (Note that the reducibility obtained for $\zeta^{2m} \neq 1$ holds although each root of the characteristic polynomial of the quadratic Casimir is doubly degenerate, whereas in the case of non-complete reducibility we do not get $4m$-dimensional indecomposable representations.)

The same technique leads to the decompositions of the tensor products $\mathscr{I}\mathrm{nd}_{\mathscr{B}} \otimes \mathscr{S}\mathrm{pin}$ and $\mathscr{I}\mathrm{nd}_{\mathscr{B}} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}$. We can actually replace $\mathscr{B}$ by $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ in (4.3) and (4.6), (4.7), always using the prescription given for (4.7). (The representations $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(\,.\,,\,.\,,\,.\,, c(\pm 1))$ never appear in our fusion rules, which is a key point for the closure of the fusion ring.)

We have only considered $\omega = 1$ in the type $\mathscr{A}$ representations entering in the fusion rules. We complete the fusion rules of type $\mathscr{A}$ with type $\mathscr{B}$ representations by adding

$$\mathscr{B}(x, y, z, c) \otimes \mathscr{S}\mathrm{pin}(0, -1) = \mathscr{B}(x, (-1)^m y, (-1)^m z, -c). \tag{4.8}$$

These fusion rules were already considered in [11], in the cases involving generic semi-periodic representations. The sub-cases leading to indecomposability were however not considered.

The decomposition of tensor products of type $\mathscr{B}$ irreps with type $\mathscr{A}$ irreps is summarized in Table 2. The cases involving the $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ and $\mathscr{I}\mathrm{nd}_{\mathscr{A}}$ representations are also summarized.

One could remark here that the "logarithm" of the parameter ζ used in the expression of c extends the role of the spin to the case of type $\mathscr{B}$ representations: the value of ζ for $\mathscr{S}\mathrm{pin}(j, 1)$ is q^{2j+1}, whereas the tensor product by the spin $1/2$ representation changes ζ to $q^{\pm 1}\zeta$. This is however not so simple in the following.

Table 2. Summary of the results of fusion of $\mathscr{B}$ or $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ representations with type $\mathscr{A}$ representations

$\mathscr{B}_1$	$\mathscr{A}_2$	Decomposes into
$\mathscr{B}$ irrep with $\zeta_1^{2m} \neq 1$	$\mathscr{S}\mathrm{pin}(j_2, \omega_2)$	$\mathscr{B}$
$\mathscr{B}$ irrep with $\zeta_1^{2m} \neq 1$	$\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j_2, \omega_2)$	$\mathscr{B}$
$\mathscr{B}$ irrep with $\zeta_1^{2m} = 1$	$\mathscr{S}\mathrm{pin}(j_2, \omega_2)$	$\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ and/or $\mathscr{B}(.,.,.,c(\pm 1))$
$\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ rep (with $\zeta_1^{2m} = 1$)	$\mathscr{S}\mathrm{pin}(j_2, \omega_2)$	$\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ and/or $\mathscr{B}(.,.,.,c(\pm 1))$
$\mathscr{B}$ irrep with $\zeta_1^{2m} = 1$	$\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j_2, \omega_2)$	$\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ and/or $\mathscr{B}(.,.,.,c(\pm 1))$
$\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ rep (with $\zeta_1^{2m} = 1$)	$\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j_2, \omega_2)$	$\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ and/or $\mathscr{B}(.,.,.,c(\pm 1))$

5. Fusion of Type $\mathscr{B}$ Irreducible Representations

This section has many subsections. A summary of its content, including the subsection numbers, is given in Table 3.

Consider two irreps of type $\mathscr{B}$: $\varrho_1 = \mathscr{B}(x_1, y_1, z_1, c_1)$ and $\varrho_2 = \mathscr{B}(x_2, y_2, z_2, c_2)$.

Then the central elements e^m, f^m, k^m are scalar on the tensor product $\varrho = (\varrho_1 \otimes \varrho_2) \circ \Delta$ and take the values

$$\begin{aligned} x &= x_1 + z_1 x_2 \,, \\ y &= y_1 z_2^{-1} + y_2 \,, \\ z &= z_1 z_2 \,. \end{aligned} \tag{5.1}$$

They are also scalar on $\varrho' = (\varrho_1 \otimes \varrho_2) \circ \Delta'$ and take the values $(x' = x_2 + z_2 x_1, y' = y_2 z_1^{-1} + y_1, z' = z_1 z_2)$.

In fact, since

$$\begin{aligned} \Delta(e)^m &= e^m \otimes 1 + k^m \otimes e^m \,, \\ \Delta(f)^m &= f^m \otimes k^{-m} + 1 \otimes f^m \,, \\ \Delta(k)^m &= k^m \otimes k^m \,, \end{aligned}$$

the fact that the operators e^m, f^m and k^m are scalar is preserved by the tensor product operation. Hence, since they are scalar on irreps, they remain diagonal on the whole fusion ring generated by the irreps.

We also see from (5.1) that ϱ and ϱ' can be equivalent only if their parameters belong to the same algebraic curve [12]:

$$\frac{x_1}{1 - z_1} = \frac{x_2}{1 - z_2} \,, \quad \frac{y_1}{1 - z_1^{-1}} = \frac{y_2}{1 - z_2^{-1}} \,, \tag{5.2}$$

and that in this case $x = x'$, $y = y'$, $z = z'$ also satisfy these relations. In other words since the coproduct is not co-commutative, the fusion rules of representations are not commutative. If the values of the parameters are restricted to belong to the same algebraic curve, the corresponding restricted fusion rules are commutative.

For physical purposes, this condition will probably always be required. However, for more generality, we now consider the composition of ϱ_1 and ϱ_2 with Δ, without imposing the condition (5.2).

The set of tensor products that we consider in this paper can be restricted in such a way that the representations belong to a given subset defined by

$$x = \text{const}\,(1 - z) \ \text{ and/or } \ y = \text{const}'\,(1 - z^{-1})\,. \tag{5.3}$$

This subset of representations is stable under fusion. Restriction of the fusion rules to this subset defines a sub-fusion-ring that is commutative (when both conditions are imposed). The sub-ring generated by the type $\mathscr{A}$ irreps is contained in these commutative sub-rings. (The question of the closure of the fusion rings will be considered at the end.)

Each weight space of $\mathscr{B}(x_1, y_1, z_1, c_1) \otimes \mathscr{B}(x_2, y_2, z_2, c_2)$ has dimension m. The weights are all the m^{th} roots of $z = z_1 z_2$.

The following lemma is the main tool for all the further decompositions:

Lemma 1. *On a weight space of the tensor product*

$$\mathscr{B}(x_1, y_1, z_1, c_1) \otimes \mathscr{B}(x_2, y_2, z_2, c_2)\,,$$

the characteristic polynomial of $\Delta(C)$ is equal to the polynomial

$$P_m(X) - xy - q^m \frac{z + z^{-1}}{(q - q^{-1})^{2m}} \tag{5.4}$$

where x, y and z are given by (5.1).

Proof. The matrix of

$$\Delta(C) = e \otimes f + fk \otimes k^{-1} e + C \otimes k^{-1} + k \otimes C - \frac{q + q^{-1}}{(q - q^{-1})^2} k \otimes k^{-1} \tag{5.5}$$

on a weight space is an $m \times m$ tridiagonal matrix (with three full diagonals, including two terms in the corners). The characteristic polynomial of this matrix is then of degree m, and it contains basically two types of terms:

– The first type consists of the product of the elements of the upper diagonal (respectively lower diagonal) elements. These two terms do not involve the indeterminate X. They correspond to the values of $(e \otimes f)^m$ and $(fk \otimes k^{-1}e)^m$, i.e. $x_1 y_2$ and $x_2 y_1 z_1 z_2^{-1}$.

– The terms that involve at least one diagonal element of the matrix of $\Delta(C) - X \cdot 1 \otimes 1$. These consist in fact of products of diagonal elements with pairs of symmetric off-diagonal ones. The diagonal elements, which are evaluations of the last three terms of (5.5), depend on c_i and z_i only ($i = 1, 2$). The products of symmetric off-diagonal elements have the same property, since the products ef and fe are involved in their evaluation, not e and f individually.

So, one part of the constant term of the characteristic polynomial of $\Delta(C)$ is $(-1)^{m+1}(x_1 y_2 + x_2 y_1 z_1 z_2^{-1})$ whereas the remaining terms only depend on c_i and z_i. The values c_i are related with the products $x_i y_i$ through (2.9), but it is clear that we can vary x_i and y_i in such a way that their products (and c_i) remain constant. This proves that we can vary continuously the constant term of the polynomial, keeping the other terms constant. So this polynomial has m distinct roots for generic values of the parameters. These roots are then the m distinct values for c allowed by (2.9) with the corresponding generic (x, y, z). The characteristic polynomial of $\Delta(C)$ is then equal to (5.4) for generic (x, y, z). Since the characteristic polynomial of $\Delta(C)$

on the tensor product is continuous in the parameters, it is equal to the polynomial (2.9) for all the values of the parameters of the representations.

We know that the roots of (5.4) are either simple, or doubly degenerate. The tensor product will then always be decomposable into a sum of representations of dimension m or $2m$, corresponding to the characteristic spaces of C (each of them being either irreducible indecomposable or again decomposable).

5.1. Generic case

Theorem 2. *Consider two type $\mathscr{B}$ irreps $\mathscr{B}(x_1, y_1, z_1, c_1)$ and $\mathscr{B}(x_2, y_2, z_2, c_2)$. Let (x, y, z) be defined by* (5.1), *and ζ by* (2.10). *If ζ is not a $2m$-root of* 1 *(generic case), the tensor product $\mathscr{B}(x_1, y_1, z_1, c_1) \otimes \mathscr{B}(x_2, y_2, z_2, c_2)$ is reducible and*

$$\begin{aligned}&\mathscr{B}(x_1, y_1, z_1, c_1) \otimes \mathscr{B}(x_2, y_2, z_2, c_2)\\&= \bigoplus_{l=0}^{m-1} \mathscr{B}\left(x, y, z, c_l = c(\zeta q^{2l}) = \frac{\zeta q^{2l}+\zeta^{-1}q^{-2l}}{(q-q^{-1})^2}\right).\end{aligned} \tag{5.6}$$

Proof. We first note that the assumption on ζ forbids $(x, y, z) = (0, 0, \pm 1)$. So the tensor product cannot contain type $\mathscr{A}$ irreps. The type $\mathscr{B}$ irreps involved in the decomposition will be related to eigenvalues of the quadratic Casimir C (2.3) (by the way, today is St. Casimir's day!). The previous Lemma identifies the characteristic polynomial of $\Delta(C)$ with the polynomial (5.4), which has only simple roots if $\zeta^{2m} \neq 1$. The eigenspaces of C then have dimension m and they correspond to the type $\mathscr{B}$ irreps of (5.6), which are the only m-dimensional representations of $\mathscr{U}_q(sl(2))$ with parameters (x, y, z, c_l).

Remark 4. This theorem shows that two tensor products of type $\mathscr{B}$ representations leading to the same (x, y, z) with $\zeta^{2m} \neq 1$ are equivalent, since their decompositions are identical.

The generic case of composition of type $\mathscr{B}$ irreps is then reducibility into type $\mathscr{B}$ irreps.

Remark 5. In [12], the underlying quantum Lie algebra is the affine $\mathscr{U}_q(\widehat{SL}(N))$. Analogous tensor products are in this case irreducible, in contrast with the present results. Remember that in our case the dimension of irreps is bounded by m.

5.2. Sub-generic cases

We consider in this subsection the tensor product

$$\begin{aligned}&\mathscr{B}\left(x_1, y_1, z_1, c_1 = c(\zeta_1) = \frac{\zeta_1 + \zeta_1^{-1}}{(q-q^{-1})^2}\right)\\&\otimes \mathscr{B}\left(x_2, y_2, z_2, c_2 = c(\zeta_2) = \frac{\zeta_2 + \zeta_2^{-1}}{(q-q^{-1})^2}\right)\end{aligned} \tag{5.7}$$

leading to (x, y, z) with $\zeta^{2m} = 1$ (2.10). (The generic case was $\zeta^{2m} \neq 1$.)

5.2.1. $(x, y, z) \neq (0, 0, \pm 1)$. We first assume $(x, y, z) \neq (0, 0, \pm 1)$. All the values c_l (2.12) are now doubly degenerate roots of the characteristic polynomial of $\Delta(C)$ on any weight space, except $c(\pm 1) = \dfrac{\pm 2}{(q - q^{-1})^2}$, which can occur at most once.

The characteristic spaces of $\Delta(C)$, which are sub-representations of the tensor product, can have the following structure:

- If related to the eigenvalue $c(\pm 1)$, it has dimension m and is equivalent to $\mathscr{B}(x, y, z, c(\pm 1))$. In this case,there is only one possibility.
- If related to the eigenvalue $c_l \neq c(\pm 1)$, it has dimension $2m$. The only possibilities in this case are
- the corresponding representation is equivalent to the indecomposable representation $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c_l)$.
- it is reducible into a sum of two representations equivalent to $\mathscr{B}(x, y, z, c_l)$.

The study of some cases shows that the first possibility is generic, whereas the second also exists for special values of the parameters.

Conjecture. We conjecture that the tensor product (5.7), in the case $\zeta^{2m} = 1$ (2.10) and $(x, y, z) \neq (0, 0, \pm 1)$ (5.1), is obtained from the decomposition (5.6) by coupling the pairs of type $\mathscr{B}$ irreps $\mathscr{B}(x, y, z, c_l)$ whose values of c_l coincide into type $\mathscr{B}$ indecomposable representations $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c_l)$ (4.2). For special values of the parameters, however, they can remain decoupled. A necessary condition for this decoupling is that ζ_1 and ζ_2 are also $2m$-roots of 1.

5.2.2. $(x, y, z) = (0, 0, \pm 1)$. Consider now $\mathscr{B}(x_1, y_1, z_1, c_1 = c(\zeta_1)) \otimes \mathscr{B}(x_2, y_2, z_2, c_2 = c(\zeta_2))$ leading to $(x, y, z) = (0, 0, \pm 1)$. We choose $z = +1$, the other case being similar. Thus $x_2 = -z_1^{-1} x_1$, $y_2 = -z_1 y_1$, $z_2 = z_1^{-1}$. Applying Eq. (2.10) to each set of variables (x_1, y_1, z_1, c_1) and (x_2, y_2, z_2, c_2), we can fix $\zeta_2 = q^{2j_1}\zeta_1$ with $2j_1$ integer ($\leq m$).

5.2.2.1. $x_1 y_1 \neq 0$. In this case, $\Delta(e)$ and $\Delta(f)$ have a rank equal to $m - 1$ on each weight space of the tensor product. In other words, each weight space contains one and only one highest-weight vector, and also one and only one lowest-weight vector (up to normalization).

Each highest-weight or lowest-weight vector is an eigenvector of $\Delta(C)$ (since it is an eigenvector of $\Delta(k)$).

Lemma 2. *The* $\mathscr{S}\mathrm{pin}(j, \omega)$ irrep *is a sub-representation of the tensor product* $\mathscr{B}(x_1, y_1, z_1, c_1) \otimes \mathscr{B}(x_2, y_2, z_2, c_2)$ *if and only if* ζ_1/ζ_2 *or* $\zeta_1\zeta_2$ *is a weight of* $\mathscr{S}\mathrm{pin}(j, \omega)$.

Proof. Consider a vector of weight ωq^{2j} in the tensor product, annihilated by $\Delta(e)$ (unique up to a normalization; its computation is straightforward). This vector is the only candidate as highest weight of $\mathscr{S}\mathrm{pin}(j, \omega)$. From the relations satisfied by the generators of $\mathscr{U}(sl(2))$, we know that the first power of $\Delta(f)$ that can annihilate this vector is either $2j + 1$ or m. In the first case (and in this case only), the representation $\mathscr{S}\mathrm{pin}(j, \omega)$ is a sub-representation of the tensor product. An explicit calculation proves that the condition for $\Delta(f)^{2j+1}$ to cancel our highest-weight vector is then exactly

$$\prod_{l=-j,-j+1,\ldots,j} \{\zeta_1 + \zeta_1^{-1} - \omega(\zeta_2 q^{2l} + \zeta_2^{-1} q^{-2l})\} = 0\,. \tag{5.8}$$

In this subsection, we already fixed $\zeta_2 = q^{2j_1}\zeta_1$, but Lemma 2 forces us to consider again two cases:

5.2.2.1.1. $\zeta_1^{2m} \neq 1$. Consider (j,ω) such that $q^{2jm}\omega^m = z$. In the case $\zeta_1^{2m} \neq 1$, the preceding lemma proves that the tensor product contains either $\mathscr{S}\mathrm{pin}(j,\omega)$ or $\mathscr{S}\mathrm{pin}\left(\frac{m}{2} - j - 1, q^m\omega\right)$ (not both).

Each characteristic space of $\Delta(C)$ (of dimension $2m$) then contains one, and only one, irreducible sub-representation, which is of course of type $\mathscr{S}\mathrm{pin}$ since $x = y = z^2 - 1 = 0$. The only representation of $\mathscr{U}(sl(2))$ of dimension $2m$, with weights of multiplicity 2, with two highest-weight vectors, two lowest-weight vectors and only one sub-irrep $\mathscr{S}\mathrm{pin}(j,\omega)$ (or $\mathscr{S}\mathrm{pin}\left(\frac{m}{2} - j - 1, q^m\omega\right)$ respectively) is $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j,\omega)$ (or $\mathscr{I}\mathrm{nd}_{\mathscr{A}}\left(\frac{m}{2} - j - 1, q^m\omega\right)$ respectively).

We then have the following proposition:

Proposition 3. *The tensor product $\mathscr{B}(x_1, y_1, z_1, c_1) \otimes \mathscr{B}(x_2, y_2, z_2, c_2)$, with*

$$\begin{aligned}
x_1 + z_1 x_2 = y_1 z_2^{-1} + y_2 &= 0\,, \\
z_1 z_2 &= 1\,, \\
x_1 y_1 &\neq 0\,, \\
\zeta_2 &= q^{2j_1}\zeta_1\,, \\
\zeta_1^{2m} &\neq 1\,,
\end{aligned}$$

is equivalent to the sum

$$\bigoplus_{\substack{j=j_1, j_1+1, \ldots \\ j \leq \frac{m-1}{2}}} \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, 1) \oplus \bigoplus_{\substack{j=\frac{m}{2}-j_1, \frac{m}{2}-j_1+1, \ldots \\ j \leq \frac{m-1}{2}}} \mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, q^m)\,, \tag{5.9}$$

with by convention $\mathscr{I}\mathrm{nd}_{\mathscr{A}}((m-1)/2, \omega) \equiv \mathscr{S}\mathrm{pin}((m-1)/2, \omega)$.

Only type $\mathscr{A}$ representations appear in this decomposition. No continuous parameter survives in the result.

5.2.2.1.2. $\zeta_1^{2m} = 1$. In this limit, some Clebsch-Gordan coefficients related to the decomposition (5.9) diverge and the equivalence does not hold. The previous lemma shows that more type $\mathscr{A}$ irreps ($\mathscr{S}\mathrm{pin}(j,\omega)$) (than in (5.9)) are sub-representations of the tensor product. For some (j,ω), the irreps $\mathscr{S}\mathrm{pin}(j,\omega)$ and $\mathscr{S}\mathrm{pin}\left(\frac{m}{2} - j - 1, q^m\omega\right)$ can both be sub-representations of our tensor product. They appear in this case as sub-representations of the same characteristic space of $\Delta(C)$. In this case, the only possibility for the corresponding characteristic space of $\Delta(C)$ is neither $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j,\omega)$ nor $\mathscr{I}\mathrm{nd}_{\mathscr{A}}\left(\frac{m}{2} - j - 1, q^m\omega\right)$, which contain only one sub-irrep, but the direct sum

$$\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'(j,\omega,\beta) \oplus \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'\left(\frac{m}{2} - j - 1, q^m\omega, \beta\right), \tag{5.10}$$

where $\mathcal{I}\mathrm{nd}_{\mathcal{A}}{}'(j,\omega,\beta)$ is an m-dimensional indecomposable representation[1] containing $\mathcal{S}\mathrm{pin}(j,\omega)$ as sub-irrep, and described by (2.13) with

$$(x=0, y=0, z=(\omega q^{2j})^m, c=c(\omega q^{2j+1}))\,, \quad \lambda=\omega q^{2j}\,,$$

(respectively $\lambda=\omega q^{m-2j-2}$), but $\beta\neq 0$ (see Remarks 1 and 3). These representations never appear in the fusion rules of type $\mathcal{A}$ irreps for the following reason: although they are not periodic (they correspond to $x=y=0$), they share with periodic representations the fact that e^p and f^{m-p} can have non-vanishing matrix elements between the same vectors, in the basis of (2.13), which diagonalizes k. Moreover, unlike the previous case, a continuous parameter (β in Remark 1) remains in these representations, which depends on the parameters of the initial representations. (After all our constraints are taken into account, two parameters remain, e.g. y_1 and z_1.)

The parameter β in $\mathcal{I}\mathrm{nd}_{\mathcal{A}}{}'(j,\omega,\beta)$, which is the ratio of the action of e and f^{m-1} on $e^{-1}\{\ker f\}$, can be considered as intrinsic and basis-independent. The limit $\beta=0$ is well-defined and appears in the following. The limit $\beta\to\infty$, which is the symmetric of $\beta\to 0$ when the roles of e and f are exchanged, is also well-defined but the representation has first to be written in the basis where e, instead of f, has a simple expression.

Let $\zeta_1=q^{l_1}$, $\zeta_2=q^{l_2}$, $0\le l_i\le m-1$, $2j_1=|l_2-l_1|$. Denote by $2j_2$ either l_1+l_2 if $l_1+l_2\le m$, or $2m-l_1-l_2$ otherwise.

Proposition 4. *With the data given above, the decomposition is*

$$\bigoplus_{\substack{j=\sup(j_1,j_2),\sup(j_1,j_2)+1,\ldots\\ j\le\frac{m-1}{2}}} \mathcal{I}\mathrm{nd}_{\mathcal{A}}(j,1)\oplus \bigoplus_{\substack{j=\frac{m}{2}-\inf(j_1,j_2),\frac{m}{2}-\inf(j_1,j_2)+1,\ldots\\ j\le\frac{m-1}{2}}} \mathcal{I}\mathrm{nd}_{\mathcal{A}}(j,q^m)$$

$$\oplus \bigoplus_{j=\inf(j_1,j_2),\inf(j_1,j_2)+1,\ldots}^{\sup(j_1,j_2)-1} \left(\mathcal{I}\mathrm{nd}_{\mathcal{A}}{}'(j,1,\beta)\oplus\mathcal{I}\mathrm{nd}_{\mathcal{A}}{}'\Big(\frac{m}{2}-j-1,q^m,\beta\Big)\right), \qquad (5.11)$$

for some β's.

5.2.2.2. $x_1y_1=0$. The results in this case are essentially the same as when $x_1y_1\neq 0$. However, they can be obtained through different proofs, using simpler expressions for the highest-weight and lowest-weight vectors of tensor products.

The representations involved in the tensor product (5.7) are now semi-periodic or nilpotent. In the case of a tensor product of semi-periodic representations, we consider $x_1=x_2=0$, the case of lowest-weight semi-periodic representations ($y_1=y_2=0$) being symmetric of the latter. In this case, their parameter ζ can be related to their highest-weight λ through $\zeta_1=q\lambda_1$ and $\zeta_2=q^{-1}\lambda_2^{-1}$.

As for periodic representations, we have to distinguish two cases:

5.2.2.2.1. $\zeta_1^{2m}\neq 1$ (and hence $\zeta_2^{2m}\neq 1$). In this case, the ranks of $\Delta(e)$ and $\Delta(f)$ are still $m-1$ on each weight space of the tensor product (5.7). So the number of highest- and lowest-weight vectors on each characteristic space of $\Delta(C)$ is the same as when $x_1y_1\neq 0$. Lemma 2 is still valid, and the decomposition (5.9) still holds.

[1] These indecomposable representations were denoted by $\mathcal{I}\mathrm{nd}_{\mathcal{B}}{}'$ in a previous version of this paper. We apologize for this change of notation (note that there is no possible confusion) motivated by the fact that $\mathcal{I}\mathrm{nd}_{\mathcal{A}}{}'$ representations are representations of the finite dimensional quotient of $\mathcal{U}_q(sl(2))$, like type $\mathcal{A}$ irreps and $\mathcal{I}\mathrm{nd}_{\mathcal{A}}$ representations. They are actually quotients of $\mathcal{I}\mathrm{nd}_{\mathcal{A}}$ representations

Table 3. Summary of the results of type $\mathscr{B}$ irreps

Section	$\mathscr{B}_1$	$\mathscr{B}_2$	Such that	Decomposes into
5.1			$\zeta^{2m} \neq 1$	$\mathscr{B}$
5.2			$\zeta^{2m} = 1$	
5.2.1			$(x, y, z) \neq (0, 0, \pm 1)$	$\mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B}$
5.2.2			$(x, y, z) = (0, 0, \pm 1)$	
5.2.2.1	$x_1 y_1 \neq 0$	$x_2 y_2 \neq 0$		
5.2.2.1.1	$\zeta_1^{2m} \neq 1$	$\zeta_2^{2m} \neq 1$		$\mathscr{I}\mathrm{nd}_{\mathscr{A}}$
5.2.2.1.2	$\zeta_1^{2m} = 1$	$\zeta_2^{2m} = 1$		$\mathscr{I}\mathrm{nd}_{\mathscr{A}}, \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'$
5.2.2.2	$x_1 y_1 = 0$	$x_2 y_2 = 0$		
5.2.2.2.1	$\zeta_1^{2m} \neq 1$	$\zeta_2^{2m} \neq 1$		$\mathscr{I}\mathrm{nd}_{\mathscr{A}}$
5.2.2.2.2	$\zeta_1^{2m} = 1$	$\zeta_2^{2m} = 1$		$\mathscr{I}\mathrm{nd}_{\mathscr{A}}, \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'$

5.2.2.2.2. $\zeta_1^{2m} = 1$ (and hence $\zeta_2^{2m} = 1$). In this case, each representation entering in the tensor product has two highest-weight vectors, since the weights are $2m$-roots of 1. We consider only tensor products of irreps, so we must have no lowest-weight vectors and hence $y_1 y_2 \neq 0$. (The representations are semi-periodic, not nilpotent.)

The rank of $\Delta(e)$ can now be $m-1$ or $m-2$ on each weight space, depending on the weight, whereas the rank of $\Delta(f)$ remains $m-1$ on each weight space. If a highest-weight q^{2j} is degenerate, we can check that the weight q^{-2j-2} also corresponds to two highest-weights. Consequently, the characteristic space of $\Delta(C)$ that contains them is equivalent to $\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'(j, 1, 0) \oplus \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'\left(\frac{m}{2} - j - 1, -1, 0\right)$. For the pairs of highest-weights q^{2j} and q^{-2j-2} which are not degenerate, it is easy to see, from their explicit expression, that one only is the highest-weight of a $\mathscr{S}$pin sub-representation. This leads then to the same decomposition as for periodic representations, i.e. formula (5.11) with now vanishing β's.

The results of this section are summarized in Table 3.

Some of the fusion rules of type $\mathscr{B}$ irreps have already been considered in the literature. In [7,8,10,13], the fusion of nilpotent representations was studied. The generic case of fusion of semi-periodic irreps was considered in [10]. The fusion of generic periodic irreps for $q = i$ was described in [14]. Generic fusion rules were also presented in [15]. General results on fusion rules and $\mathscr{R}$-matrices for $\mathscr{U}_q(sl(2))$ were given in [16], and developed in [17].

6. Fusion Ring Generated by all the Irreps of $\mathscr{U}_q(sl(2))$

Theorem 3. *The fusion ring generated by all the irreducible representations of $\mathscr{U}_q(sl(2))$ consists in*

- *the irreducible representations of type $\mathscr{A}$ and $\mathscr{B}$,*
- *the type $\mathscr{A}$ indecomposable representations $\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, \omega)$,*
- *the type $\mathscr{B}$ indecomposable representations $\mathscr{I}\mathrm{nd}_{\mathscr{B}}(x, y, z, c(\zeta))$ (with $\zeta^{2m} = 1$ and $\zeta \neq \pm 1$),*
- *the indecomposable representations of type $\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'(j, \omega, \beta)$.*

This fusion ring contains sub-fusion-rings defined by imposing one or both of the relations (5.3) *on the parameters* (x, y, z). *When both conditions are imposed, these sub-rings are commutative.*

Proof. previous results show that these four types of representations are involved in the fusion ring. We still have to prove that it closes without other types of representations.

The tensor products that have already been considered are

– irrep $\otimes$ irrep
– $\mathscr{I}\mathrm{nd}_{\mathscr{A}} \otimes \mathscr{S}\mathrm{pin} \longrightarrow \mathscr{I}\mathrm{nd}_{\mathscr{A}}$ (Sect. 3)
– $\mathscr{I}\mathrm{nd}\mathscr{A} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}} \longrightarrow \mathscr{I}\mathrm{nd}_{\mathscr{A}}$ (Sect. 3)
– $\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}} \longrightarrow \mathscr{B}$ or $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ (Sect. 4)
– $\mathscr{I}\mathrm{nd}_{\mathscr{B}} \otimes \mathscr{S}\mathrm{pin} \longrightarrow \mathscr{I}\mathrm{nd}_{\mathscr{B}}$ or $\mathscr{B}(\,.\,,\,.\,,\,.\,, c(\pm 1))$ (Sect. 4)
– $\mathscr{I}\mathrm{nd}_{\mathscr{B}} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}} \longrightarrow \mathscr{I}\mathrm{nd}_{\mathscr{B}}$ or $\mathscr{B}(\,.\,,\,.\,,\,.\,, c(\pm 1))$ (Sect. 4)

(Reversed tensor products are similar, although not always equivalent.)

For the remaining tensor products, we will apply the following procedure: we consider the indecomposable representations involved in the tensor product as a term of the decomposition of a tensor product of irreps. These irreps will always be chosen with the most generic allowed parameters. The decomposition of the original tensor product will then be a part of the decomposition of a tensor product of three or four irreps, on which we will use the coassociativity of Δ (associativity of the fusion rules) and the previous results on the composition of irreps. The first case will be treated in detail, the other being sketched.

– $\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{B}}$ with, on the result, (x, y, z) and ζ, depending, as usual, on the original parameters. Then

$$\begin{aligned} \mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{B}} &\subset \mathscr{B} \otimes (\mathscr{B}_1 \otimes \mathscr{B}_2) \\ &\subset (\mathscr{B} \otimes \mathscr{B}_1) \otimes \mathscr{B}_2 \,. \end{aligned}$$

$\mathscr{B}_1$ is considered as generic and the parameters of $\mathscr{B}_2$ are related to those of $\mathscr{B}_1$ in order to contain $\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ in their fusion. Then $\mathscr{B} \otimes \mathscr{B}_1 = \bigoplus \mathscr{B}_3$, the irreps $\mathscr{B}_3$ being as generic as $\mathscr{B}_1$.

– If $\zeta^{2m} \neq 1$, then $\mathscr{B}_3 \otimes \mathscr{B}_2 = \bigoplus \mathscr{B}_4$, so that

$$\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{B}} \longrightarrow \bigoplus \mathscr{B} \,.$$

– If $\zeta^{2m} = 1$ and $(x, y, z) \neq (0, 0, \pm 1)$, then

$$\mathscr{B}_3 \otimes \mathscr{B}_2 = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{B}} \text{ and/or } \mathscr{B}_4 \,,$$

so that

$$\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{B}} \longrightarrow \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B} \,.$$

– If $\zeta^{2m} = 1$ and $(x, y, z) = (0, 0, \pm 1)$, then $\mathscr{B}_3 \otimes \mathscr{B}_2 = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{A}}$, so that

$$\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{B}} \longrightarrow \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{A}} \,.$$

– $\mathscr{B}(x, y, z, c(\zeta)) \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}' \subset \mathscr{B}(x, y, z, c) \otimes \mathscr{B}_1 \otimes \mathscr{B}_2$. The parameters (x, y, z) of the result are those of the type $\mathscr{B}$ irrep of the tensor product since the $\mathscr{I}\mathrm{nd}_{\mathscr{A}}'$ representations carries $(0, 0, \pm 1)$.

– If $\zeta^{2m} \neq 1$, then $\mathscr{B} \otimes \mathscr{B}_1 = \bigoplus \mathscr{B}_3$ and $\mathscr{B}_3 \otimes \mathscr{B}_2 = \bigoplus \mathscr{B}_4$, so that

$$\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}' \longrightarrow \bigoplus \mathscr{B}\,.$$

– $\zeta^{2m} = 1$ (and $(x, y, z) \neq (0, 0, \pm 1)$ otherwise the first irrep is of type $\mathscr{A}$), then $\mathscr{B} \otimes \mathscr{B}_1 = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B}_3$ and ($\mathscr{I}\mathrm{nd}_{\mathscr{B}}$ or $\mathscr{B}_3$) $\otimes \mathscr{B}_2 = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B}_4$, so that

$$\mathscr{B} \otimes \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}' \longrightarrow \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B}\,.$$

– $\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}' \otimes \mathscr{S}\mathrm{pin} \subset \mathscr{B}_1 \otimes \mathscr{B}_2 \otimes \mathscr{S}\mathrm{pin}$. Since $\mathscr{B}_2 \otimes \mathscr{S}\mathrm{pin} = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B}$, and $\mathscr{B}_1 \otimes (\mathscr{I}\mathrm{nd}_{\mathscr{B}}, \mathscr{B}) = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{A}}, \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'$, we have

$$\mathscr{I}\mathrm{nd}_{\mathscr{A}}{}' \otimes \mathscr{S}\mathrm{pin} = \bigoplus \mathscr{I}\mathrm{nd}_{\mathscr{A}}, \mathscr{I}\mathrm{nd}_{\mathscr{A}}{}'\,.$$

– The remaining cases, $\mathscr{I}\mathrm{nd} \otimes \mathscr{I}\mathrm{nd}$, with at most one $\mathscr{I}\mathrm{nd}_{\mathscr{A}}$ in the tensor product, can be seen as included in $\mathscr{B}_1 \otimes \mathscr{B}_2 \otimes \mathscr{I}\mathrm{nd}$, for which we use the previous cases. The conditions (5.3) define sub-rings of the whole ring of representations. Taking the intersection of the fusion ring generated by irreps with these sub-rings provides interesting commutative sub-fusion-rings.

7. Decomposition of the Regular Representation of $\mathscr{U}_q(sl(2))$

Using (5.9) for nilpotent representations, we can achieve the decomposition of the regular representation.

The regular representation of $\mathscr{U}_q(sl(2))$ is the finite dimensional module defined by the left action of $\mathscr{U}_q(sl(2))$ on itself, with the further relations $e^m = f^m = 0$ and $k^{m'} = 1$.

A natural basis is given by $\{f^{r_1} e^{r_2} k^{r_3}\}$ with r_1, $r_2 \in \{0, \ldots, m-1\}$ and $r_3 \in \{0, \ldots, m'-1\}$. Using the basis $\{v_{r_1, r_2, p} = \sum_{r_3=0}^{m'-1} q^{-r_3 p} f^{r_1} e^{r_2} k^{r_3}\}$ which diagonalizes the action of k, the regular representation was decomposed in [16] into the sum

$$\bigoplus_{p=0}^{m'-1} \mathscr{B}(0, 0, \lambda^m, c(q\lambda)) \otimes \mathscr{B}(0, 0, \lambda^{-m}, c(q^{p+1}\lambda^{-1}))\,, \tag{7.1}$$

which is then equivalent to

$$\bigoplus_{j=0}^{(m-1)/2} (2j+1)\,\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, 1) \underset{(\text{if } m' \text{ is even})}{\oplus} \left(\bigoplus_{j=0}^{(m-1)/2} (2j+1)\,\mathscr{I}\mathrm{nd}_{\mathscr{A}}(j, -1) \right). \tag{7.2}$$

We see that the multiplicity of each indecomposable representation is equal to the dimension of its irreducible part. Although (7.1) is valid for arbitrary λ, it is not a surprise to find that the regular representation is of type $\mathscr{A}$. This result agrees with the decomposition obtained in [18].

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