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On Classification of $N = 2$ Supersymmetric Theories

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Abstract: We find a relation between the spectrum of solitons of massive $N = 2$ quantum field theories in $d = 2$ and the scaling dimensions of chiral fields at the conformal point. The condition that the scaling dimensions be real imposes restrictions on the soliton numbers and leads to a classification program for symmetric $N = 2$ conformal theories and their massive deformations in terms of a suitable generalization of Dynkin diagrams (which coincides with the A–D–E Dynkin diagrams for minimal models). The Landau-Ginzburg theories are a proper subset of this classification. In the particular case of LG theories we relate the soliton numbers with intersection of vanishing cycles of the corresponding singularity; the relation between soliton numbers and the scaling dimensions in this particular case is a well known application of Picard-Lefschetz theory.

1. Introduction

Quantum field theories in two dimensions have been under intensive investigation recently in part due to their importance in string theory and in part serving as exactly soluble toy models for quantum field theories in higher dimensions. The interest in studying them for string theory has mostly focused on conformal field theories, i.e., the ones with traceless energy momentum tensor (with only massless excitations). On the other hand, as examples of interesting exactly soluble quantum field theories with interesting S-matrices, the massive ones have been under investigation [1]. In view of the fact that massive QFT's can be viewed as deformation of the conformal theories, it is natural to ask if there is any way to understand properties of conformal theories, by studying the massive analogs. This program has been followed with a spectacular degree of success originating with the work of Zamolodchikov's [2, 3]. The method to relate properties of integrable massive theories to the conformal ones uses thermodynamical Bethe ansatz (TBA). In this way, just by studying the S-matrices of the massive integrable theories one can deduce for example the central charge of the conformal theory.

An interesting class of conformal theories for superstrings is the class with $N = 2$ superconformal symmetry. These can be used to construct string vacua. For instance, σ-models on Calabi-Yau manifolds provide examples of such theories. In view of their importance in constructing string vacua, it is natural to ask if one can classify all $N = 2$ theories. Progress in this direction was made [4] when it was realized that $N = 2$ Landau-Ginzburg theories is an effective way of classifying some of them. In particular all the minimal $N = 2$ models were found to have a simple Landau-Ginzburg description which fitted with the known classification of simple singularities [5]. This program had the following limitation: It is known that not all the $N = 2$ conformal theories can be realized as a LG theory. So this program leads to a partial classification.

Massive integrable deformation of $N = 2$ superconformal theories has also been considered [6–9]. Furthermore the TBA has been applied to these theories (and in particular the central charge and the charge of chiral primary fields at the conformal point has been recovered in this way). In this paper we will consider massive perturbations of $N = 2$ theories in 2 dimensions and show that there is a very simple relation between the $U(1)$ charges of chiral fields at the conformal point (the highest of which is equal to the central change) and the degeneracy of solitons which saturate the Bogomolonyi bound in the massive theory. This relation exists *whether or not the theory is integrable*. Turning this around, we end up with the following classification program: Start with n-vacua, and impose having a certain number of solitons between each pair. Then deduce the structure of the chiral ring at the conformal point. In particular in this way we can compute the charges of primary fields at the conformal point. It turns out that the condition that the charges of chiral fields be *real* puts a strong restriction on the number of solitons allowed. For instance, we show that for a minimal model, defined by the condition that all chiral fields are relevant perturbations, there is at most 1 soliton allowed between vacua. Using the soliton numbers, we can associate a bilinear form (with 2's on the diagonal) to each massive $N = 2$ theory. We also find a relation between the signature of the bilinear form and the charges of chiral fields. We show that for minimal models this bilinear form is positive definite, which with the above restriction leads to the well known ADE classification of the minimal models. This method explains in the most natural way why the A–D–E classification arises while classifying minimal models. For theories with higher central charge more general types of "Dynkin diagrams" arise, which encode the soliton structure of the theory.

The organization of this paper is as follows: In Sect. 2 we describe the soliton structure of the $N = 2$ LG models and relate it to intersection theory of the Homology cycles (as in singularity theory [5]). We will also show how, in this subclass, one can obtain the charges of chiral fields from the number of solitons. In Sect. 3 we discuss how to formulate these results generally *independently* of whether they come from a LG theory. In Sect. 4 we give a proof of the general reformulation. The proof uses the topological-anti-topological equations (tt^*) formulated in [10] which has been reformulated as an isomonodromy deformation of a linear system of equations by Dubrovin [11]. We show that the phase of the eigenvalues of the monodromy of these equations are simply the chiral charges. Relating the monodromy operator to the soliton numbers gives the desired relation between the charges and the soliton numbers. In Sect. 5 we discuss a criterion to select which massive models have a non-degenerate UV limit. In Sect. 6 we show how these ideas lead to a classification program for massive $N = 2$ theories (up to addition of D-terms), or by taking the UV limit to the classification of *conformal* $N = 2$ theories which admit a massive deformation (the D-term being fixed by the conformal condition). These ideas may be useful in classifying $c_1 > 0$ Kähler manifolds (with diagonal Hodge numbers), as to each such manifold (which admit massive deformation) one can associate a particular bilinear form by considering sigma models on them. We give a number of examples where we can use these techniques. In particular in Sect. 7 we rederive the A–D–E classification of minimal models, as well as its "affine" counterpart (including orbifolds of S^2) and (in Sect. 8) supersymmetric sigma models on $\mathbf{C}P^n$ and Grassmannians. Moreover we spell out the classification of theories with up to 3 vacua as well as that of models with a $\mathbf{Z}_n$ symmetry. In Sect. 9 we present our conclusions and suggest some directions for future research. In Appendix A some aspects of the Grassmannian σ-models are worked out. In Appendices B, C some further properties of the classification program are discussed.

We would like to make a historical remark: The order we have decided to present our results does not reflect the order in which we discovered them, but rather the order in which it can be understood most easily. In particular a time ordered sequence of our understanding is roughly Sects. 4, 6, 3, 2, 5, 7, 8.

2. Landau-Ginzburg Solitons and Monodromy

An interesting subclass of $N = 2$ QFT's in two dimensions is given by Landau-Ginzburg theories (see e.g. [12, 13] for the definition). These theories are characterized by a superpotential $W(x^i)$ which is a holomorphic function of n chiral superfields x^i, up to variation in D-terms which is represented by a positive function $K(x^i, \bar{x}^i)$. The bosonic part of the LG action is given by

$$S = \int d^2 z \quad G_{i\bar{j}} \partial_\mu x^i \partial_\mu x^{\bar{j}} + G^{i\bar{j}} \partial_i W \overline{\partial_j W} \,,$$

where $G_{i\bar{j}} = \partial_i \bar{\partial}_j K$ (which is positive definite for a unitary theory). The scalar potential is minimized at $x^j = a^j$ such that

$$\left. \frac{\partial W}{\partial x^i} \right|_{a^j} = 0 \quad \text{for all} \quad i \,,$$

which thus correspond to vacua of this theory. Let us assume that the vacua are non-degenerate, in the sense that near each of them W is quadratic. This can always be arranged, if necessary, by perturbing W. Let us find the number of solitons in this theory. Our argument is a simple generalization of that given in [6] from one variable case to higher n.

Solitons are configurations of fields as a function of space, where on the left $x^i(-\infty) = a^i$ and on the right $x^i(+\infty) = b^i$, where a, b label two distinct critical points of W. Stable solitons are the ones satisfying the above boundary condition which minimize the energy. Let us denote the space variable by σ. The energy of the soliton configuration is given by

$$E_{ab} = \int d\sigma |\partial_\sigma x - \alpha \overline{\partial W}|^2 + 2\,\mathrm{Re}(\alpha^* \Delta W) \,,$$

where $\Delta W = W(b) - W(a)$, and α is some arbitrary phase, and we have hidden all the indices and raising and lowering of indices is done with $G_{i\bar{j}}$. It is easy to see that there is a lower bound for the energy: choose $\alpha = \Delta W/|\Delta W|$, then we see from the above representation of E that

$$E_{ab} \geq 2|\Delta W| .$$

Since W is not renormalized in the quantum theory (due to the existence of topological ring which characterizes it) this is precisely the same as the Bogomolnyi bound in the quantum field theory. So the number of solitons which *saturate* the Bogomolnyi bound are given by solving the equation with

$$\partial_\sigma x^i = \alpha G^{i\bar{j}} \overline{\partial_j W} . \tag{2.1}$$

Note that for any such solution the image of the soliton configuration in the W-plane is a straight line

$$\partial_\sigma W = \partial_i W \cdot \partial_\sigma x^i = \alpha |\partial W|^2 .$$

In other words the image is a straight line connecting $W(a)$ to $W(b)$. Now we come to asking how many solutions are there to (2.1)? For simplicity, and with no loss of generality we take $W(a) = 0$ and $W(b)$ to be a positive real number, which means taking $\alpha = 1$. First let us analyze solutions to (2.1) near a. Again with no loss of generality we take a to correspond to $x^i = 0$, where near it we take $W = \sum_i (x^i)^2$ and $G_{i\bar{j}} = \delta_{i\bar{j}}$. Then the equation for soliton (2.1) near the critical point becomes

$$\partial_\sigma x^i = \bar{x}^i . \tag{2.2}$$

So the solution which at $\sigma = -\infty$ is at the critical point is given by

$$x^i = \alpha^i e^\sigma \quad \text{with} \quad \alpha^i = (\alpha^i)^* . \tag{2.3}$$

Of course it is not clear if for all α^i we get a solution, i.e., if this trajectory ends up on another critical point. In order to analyze how many of these initial conditions would correspond to an actual soliton, we should look at the totality of allowed solutions *near* each critical point, and try to match them with solutions near others. Let us consider the points Δ_a of the totality of all possible solutions (2.3) with a given value of $W = r^2$ (where r is a small real number) near the critical point a. In other words let's look at the intersection of $W^{-1}(r^2)$ with all the potential solutions originating from a. This intersection is given by the condition

$$\Delta_a : \sum_{i=1}^{n} (x^i)^2 = r^2 ,$$

where from (2.3) the only restriction on x_i is that it be *real*. So the "wave front" of all possible solutions originating from a critical point with a given value of W is an $n-1$-dimensional sphere. Note that this sphere vanishes as $r \to 0$. This is precisely the definition of a *vanishing cycle* in singularity theory [5]. In fact near each critical point we get a vanishing cycle which is diffeomorphic to S^{n-1}. Now suppose we consider the vanishing cycle Δ_b near the critical point b. Those points will represent the points which by (2.1) can flow from the critical point b (along the negative real axis), where we need to set $\alpha = -1$ in (2.1). Now consider going on a straight line in the W-plane connecting the two critical values $W(a) = 0$ and $W(b)$. Let us fix a point p on this line, say $W = W(b)/2$. The wave front originating from a over p continues to be an $n-1$ dimensional cycle in $W^{-1}(p)$. It gets deformed from the original shape but it is still an $n-1$ dimensional sphere (as the flow with the vector field given by (2.1) is just a diffeomorphism). Let us still denote this cycle by Δ_a. Also consider the intersection of wave front originating from the point b with $W^{-1}(p)$ and denote the cycle by Δ_b. These two cycles intersect at a discrete number of points [note that each one is *half* the dimension of $W^{-1}(p)$]. *For each point of their intersection we get a soliton.* This is almost obvious: For each point that they intersect the flow of the vector field from a which reaches that point continues to flow to the critical point b. Here it is crucial that (2.1) is a first order equation. So we get a solution to (2.1) with the boundary condition that $x(-\infty) = a$ and $x(+\infty) = b$. Moreover the points on Δ_a that do not intersect any point of Δ_b will not flow to b when evolved with (2.1) as Δ_b is the totality of all such points that flow to b. Therefore the number of solitons is exactly the number of points that Δ_a and Δ_b intersect. This is not necessarily the intersection number of these two cycles, because the intersection number counts each intersection point with ± 1 depending on the orientations. However the intersection number appears naturally for us as follows: The solitons come in pairs, as they are Bogomolnyi saturated states. We have been focusing on the bosonic piece of the soliton, there will also be a fermionic partner obtained by acting on this state with Q^- the supersymmetry charge (which decreases the fermion number by 1). In weighing the solitons with phases the natural thing to consider is $(-1)^F$. However this would cancel for pairs of solitons. Instead as in [14] we consider weighing the soliton pairs with $(-1)^F F$ which is the same as weighing the bosonic components with $(-1)^F$. What we will now show is that the number of bosonic solitons weighted with $(-1)^F$, is just this intersection number, i.e.,

$$|\mu_{ab}| = \left| \sum_{ab \text{ solitons}} (-1)^F F \right| = \left| \sum_{ab \text{ bosonic solitons}} (-1)^F \right| = |\Delta_a \circ \Delta_b| \,. \tag{2.4}$$

From now on whenever we talk about soliton numbers we mean this *weighted* soliton number. Generically all the solitons have the same fermion number and so this is just the counting of the soliton. At any rate this weighted soliton number is more useful for our purposes than the actual soliton number, in case they are not the same. Also we show that the absolute value signs can be taken out of the above equation in the following sense: First note that the fermion number of any state in the ab sector is $f_{ab} + k$, where f_{ab} is in general fractional and can be written as a difference $f_a - f_b$ (see [14]) and k is an integer. In fact in a Landau-Ginzburg theory f_{ab} is given [1] by [7]

$$e^{2\pi i f_{ab}} = \text{phase}\left[\frac{\det H(b)}{\det H(a)}\right],$$

where $H_{ij} = \partial_i \partial_j W$. In the LG case f_a and f_b can be identified with the phases of the determinant of Hessian at the respective critical points. So μ_{ab} will in general carry a phase $\pm \exp(2i\pi f_{ab})$. Viewing μ_{ab} as a matrix we see that we can get rid of

[1] There is a general topological proof of this fact which holds for any $N = 2$ theory not just for LG models. In the general case one has

$$\exp[2\pi i f_{ab}] = \text{phase}\,[\eta_a/\eta_b]\,,$$

where η_a is defined by the corresponding TFT metric as $\eta_{ab} = \delta_{ab}\eta_a$. The proof of this formula follows from comparing three known facts: 1. f_{ab} is defined (mod. 1) by $Q_{ab} = \pm \exp[i\pi f_{ab}]\,|Q_{ab}|$, where Q_{ab} is the new index of [14] computed in the spectral-flow "point basis" $|e_a\rangle$ (see [14]). 2. In the canonical basis $|f_a\rangle$ Q_{ab} is real. See [15]. 3. By definition [15] $|f_a\rangle = (\eta_a)^{-1/2}\,|e_a\rangle$. The statement in the text is obtained by replacing η_a with its explicit expression for a LG model phases up to $\pm$ signs by a redefinition of the basis using f_i as in [14]. Note also from the definition (2.4) that μ is an anti-symmetric matrix in this basis.

To remove the absolute value signs in (2.4) it is more convenient to consider the case when we have an even number of LG fields, i.e., n is even. This can be done with no loss of generality by simply adding, if necessary, a field with x^2 contribution to superpotential. In order to have a consistent definition on the right-hand side the intersection matrix should be anti-symmetric which is the case when n is even, because vanishing cycles are odd dimensional. We will show that with this choice in a suitable basis we have [2]

$$\mu_{ab} = \Delta_a \circ \Delta_b \,. \tag{2.5}$$

Note also the fact that there is no soliton from one vacuum to itself $\mu_{aa} = 0$ is automatic because an odd dimensional sphere has zero self intersection (the Euler character is zero). We still have the freedom of redefining the basis by multiplications by $\pm$. So the invariant quantities are obtained when we consider "cycles" which means that if we consider $\mu_{i_1 i_2}\mu_{i_2 i_3}\cdots\mu_{i_r i_1}$ it is independent of conventions.

Now we come to showing (2.5) which requires a rather long and delicate analysis. Suppose we have two different soliton trajectories from a and b, and we wish to show that their relative contribution to the left- and right-hand side of the above equation is the same. In order to show that we need to show that if these two trajectories correspond to the same sign for intersection between cycles they also have the same fermion number mod 2, and if they have opposite intersection number their fermion number differs by 1 mod 2.

Let us consider a given solution to (2.1) (with $\alpha = 1$) and consider the family of solutions which is near this solution. If we write the perturbation as $x \to x + \delta$ the equation we get for δ is given by

$$\partial_\sigma \delta = H^* \delta^* \,. \tag{2.6}$$

Note that an obvious solution to this variational equation is the "velocity vector" of the soliton trajectory $v^i = \partial_\sigma x^i$. Near a critical point H is a constant, and the above equation can be solved by finding solutions to

$$H^* \delta_k^* = \lambda_k \delta_k \,, \tag{2.7}$$

where $\lambda_k > 0$ for solitons which at $\sigma = -\infty$ start at the critical point. In fact the above equation has n independent solutions. Indeed if we pair (δ, δ^*) as a $2n$ dimensional vector and consider

$$\mathscr{H} = \begin{pmatrix} 0 & H^* \\ H & 0 \end{pmatrix}$$

as a Hermitian hamiltonian, then its eigenvalues come in pairs with opposite signs. If (δ_i, δ_i^*) has eigenvalue $\lambda_i > 0$ then $(i\delta_i, -i\delta_i^*)$ has eigenvalue $-\lambda_i$. The vectors tangent to the vanishing cycle Δ_a near the critical point a are *real* linear combinations of the vectors (2.7) with positive eigenvalues subject to the additional constraint that they are at the preimage of a fixed value of W, i.e.,

$$dW = \partial W \cdot \delta = 0 \,.$$

[2] If we had chosen n to be odd we would need to order the vacua and use one definition of sign when $a > b$ and the other when $b < a$

Using the equation of motion (2.1) this means that

$$a_k \delta_k \cdot \bar{v} = 0$$

(where the field indices are implicit). Thus the vectors tangent to vanishing cycle are always orthogonal to v and iv. This means that tangents to vanishing cycle *near the critical point* span positive eigenspace of $\mathscr{H}$ orthogonal to v (which itself belongs to this subspace). Note that near b the vanishing cycle Δ_b is spanned by the *negative* eigenspace of (2.7) (because $\alpha = -1$) which are orthogonal to v and iv. Note that the v, which near a belonged to the positive subspace of (2.7), near b belongs to the negative subspace of (2.7), while iv which near a belonged to the negative subspace near b belongs to the positive subspace. The tangents to the vanishing cycle Δ_a near the critical point b must however belong again to the positive eigenspace of (2.7) near the critical point b, as they are orthogonal both to v, iv and the tangents to Δ_b.

We wish to compute the fermion number of this trajectory. This is the $N = 2$ version of a similar problem which arose in Witten's considerations of Morse theory [16]. We have two options in finding this sign: either find the number of times the phase of $\det H$ wraps around the origin modulo 2 as we go along soliton trajectories, or more directly relate the sign of the amplitude by relating fermions to the tangent vectors of the vanishing cycle. We will use the second option. First we note that the object we are computing is an index, and thus can be computed by reducing the theory from 2 dimensions to 1, and the question of determining the sign in this set up is the same as determining the sign for an overlap of the vacuum evolved from critical point a transported along the soliton trajectory, with the vacuum at point b. It is important to note that Eq. (2.6) is the same equation which evolves the *fermions* of the theory (this follows from supersymmetry transformation), and each vacuum will correspond for us to an n-form, when we identify the fermions with the tangent vectors (or forms via the metric G). We identify the fermionic degree of freedom for the state evolving from a with the ordered set of vectors $\delta_1, \ldots, \delta_{n-1}, v$, where $\delta_1, \ldots, \delta_{n-1}$ forms a basis for Δ_a near a and in such a way that the orientation of it is compatible with that of Δ_a, which means that $\delta_1, i\delta_1, \ldots, \delta_{n-1}, i\delta_{n-1}, v, iv$ give the standard orientation of x^i which is $\mathbf{C}^n$. The fermionic degree of freedom of the state evolving from b will be identified with vector $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, v$, where γ_i form the tangent to the vanishing cycle at b, ordered in the canonical way. In taking the overlap between these two states, the F insertion (in the definition of μ) removes the zero mode we would have obtained and so using the definition of Grassmann integration of fermions we are just left with the standard definition of the intersection number $\Delta_a \circ \Delta_b$ for the contribution of this path integral. Therefore the sign of this amplitude is the same as the sign of the intersection number.

This completes what was to be shown as far as relative contribution of two trajectories beginning at a critical point a and ending at another critical point b is concerned. By repeating this argument by including another critical point c it is easy to see that the relative sign in the ac, cb and ab sectors are correlated with $\Delta_a \circ \Delta_c$, $\Delta_c \circ \Delta_b$ and $\Delta_a \circ \Delta_b$. This finally shows (2.5) is true for a suitable choice of basis for the vacua.

We now consider what happens to the soliton numbers when we perturb the superpotential W. As we perturb W the critical values move in the W-plane. As long as no critical value crosses the straight line connecting two other critical values, the stability of intersection numbers under continuous deformations guarantee that the soliton numbers do not change. But suppose the critical value of a vacuum j

lies exactly on the straight line from critical value of vacuum i to k (see Fig. 1). Then the wave front originating from i cannot be continued past the point j, as some trajectories originating from i may get absorbed by j or some new trajectories may open up. So in this way the soliton number of the ik sector changes, as some new solitons may appear which previously used to go through j and were not *primitive*, or some primitive solitons in the ik sector may become composite solitons ij, jk. Can we compute this change in soliton number? We should be able to as it just involves understanding what happens to the vanishing cycles as the vacua pass through an aligned configuration.

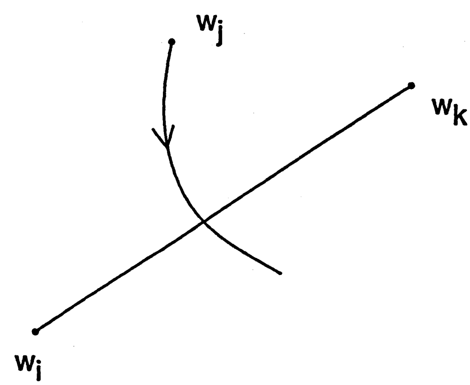

Fig. 1. Vacuum j labeled by its critical value in the W-plane will pass, by perturbing the theory, through a straight line connecting two other vacua i, k

In order to discuss this it is useful to recall some facts about vanishing cycles. If we fix a non-critical value t in the W plane and look at the preimage of that point, we get an $n-1$ complex dimensional space. The (compact) homology cycles are in real dimension $n-1$ and they can be described as follows [5]: Connect the point t to the critical values $w_1, \ldots, w_n$ along some cyclically ordered paths γ_i which do not cross any critical values (see Fig. 2), and consider the $n-1$ cycles that vanish as we

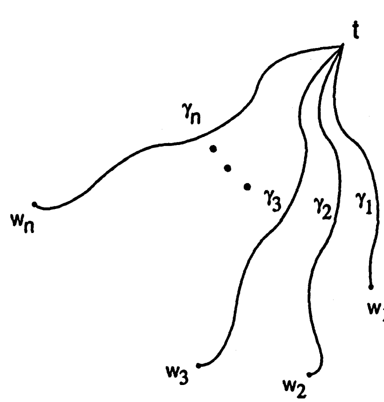

Fig. 2. The critical points in the W-plane are connected to a point t along some cyclically ordered paths $\gamma_1, \ldots, \gamma_n$

go along these paths (by homotopy lifting) to each of the critical values. These form a basis for $n-1$ cycles. Let us denote the i^{th} vanishing cycle, the one that vanishes along γ_i, by Δ_i. The question we wish to address now is how to use the intersection between these cycles to find the soliton numbers μ_{ij}. If the point t is along the straight line connecting i and j and γ_i and γ_j are the straight lines connecting t to i and j respectively then it is clear that $\mu_{ij} = \Delta_i \circ \Delta_j$. Since the intersection numbers are rigid under continuous deformations this means that as long as we can deform t and γ's continuously to the above situation *without having the paths γ_i cross critical values w_r* we can still use these intersections to count the corresponding soliton numbers. However sometimes this cannot be done with a particular choice of the paths γ_i, and we will have to choose a different set of paths connecting t to critical values, and this will give us a different basis for the vanishing cycles. Equivalently for any given choice of paths γ_i, by deforming the critical values by perturbation of the theory, we can arrange so that the intersection numbers of the corresponding cycles do count the soliton numbers of the perturbed theory. So there is a one to one correspondence between the set of paths and the set of possible perturbations of the critical values.

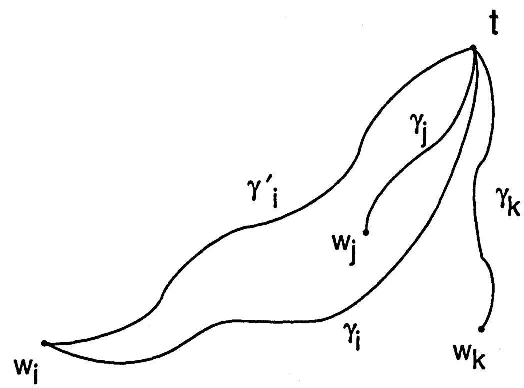

Fig. 3. The vanishing cycles change if we choose a different set of paths. In this case Δ_i changes to Δ_i' as we have deformed path γ_i by passing it through w_j to a new path γ_i'

The theory of how the vanishing cycles change by choosing a different basis of γ_i is known as the Picard-Lefschetz theory [5]. Suppose we wish to change a particular cycle γ_i to a path γ_i' by passing it through the critical value w_j (see Fig. 3). If we know how the cycles change under this particular change of basis, since we can get an arbitrary basis by just repeating such steps over arbitrary critical values we would be done. The Picard-Lefschetz theorem implies that the new vanishing cycle Δ_i' is given by

$$\Delta_i' = \Delta_i \pm (\Delta_i \circ \Delta_j)\Delta_j\,, \tag{2.8}$$

where the $\pm$ sign corresponds respectively to whether the circle $\gamma_i(\gamma_i')^{-1}$ is clockwise or counter-clockwise in the W plane (in the case of Fig. 3 it is $+$ sign). This formula is very much like the formula for "Weyl reflection" and it is indeed exactly that for the example of minimal models that we will discuss later.

Now we are set to compute the change of the soliton number as critical values pass through configurations in which three critical values get aligned. As discussed above as the j^{th} vacuum crosses the ik line this can be equivalently described by a change of basis of path by changing γ_i (see Fig. 1 and Fig. 3). So the new soliton number μ'_{ik} is given by

$$\mu'_{ik} = \Delta'_i \circ \Delta_j = (\Delta_i \pm (\Delta_i \circ \Delta_j)\Delta_j) \circ \Delta_k = \mu_{ik} \pm \mu_{ij} \cdot \mu_{jk}\,, \tag{2.9}$$

where the $\pm$ will correspond respectively to whether the right-hand rule applied to the triangle ijk *before* the j^{th} vacuum crosses the ik line is into or out of the W plane. The formula (2.9) can be intuitively understood by noting that we get new solitons (or lose solitons) by the fact that composite solitons in the ik sector (composed of two solitons in the ij and jk sectors) become primitive solitons (or vice-versa). We will show how to derive Eq. (2.9) from purely physical reasoning in the next section for arbitrary $N = 2$ models, thus generalizing this result for the Landau-Ginzburg theory.

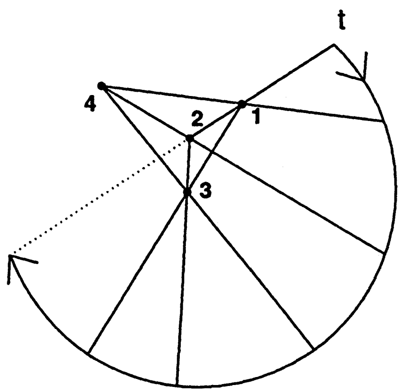

Fig. 4. The monodromy of the vanishing cycles can be computed by taking t on a large circle in the W-plane connected by straight lines to the vacua. As the straight lines overlap lines joining pairs of vacua we pick up contributions to the monodromy

Having set up all the machinery we now come to proving a surprising relation between the monodromy of vanishing cycles and the intersection numbers. Suppose we pick a point t on the W plane very far from the critical points. Furthermore let us choose the paths γ_i to be straight lines connecting t to the critical values. As t goes around a large circle in a clockwise direction with $|t|$ fixed (see Fig. 4), the vanishing cycles undergo a monodromy. We can compute what this monodromy is just by using (2.8) which tels us what happens when γ_i cross any of the vacua. In other words, consider $n(n-1)$ half lines passing through the vacua in pair and originating from one of the vacua. Let us denote the half-line originating at the j^{th} and passing through the i^{th} vacuum by l_{ij}. Then as t cross the line l_{ij} the basis for the vanishing cycles change, using (2.8) by multiplication with the matrix

$$M_{ij} = 1 - A_{ij}\,,$$

where 1 denotes the identity matrix and A_{ij} is a matrix whose only non-vanishing entry is the ij entry and that is equal to $A_{ij} = \mu_{ij} = \Delta_i \circ \Delta_j$.

Note that

$$M_{ji} = 1 - A_{ji} = M_{ij}^{-t}\,, \tag{2.10}$$

where we used the fact that $A_{ji} = -A^t_{ij}$ and that as a matrix $(A_{ij})^k = 0$ for $k > 1$. Let S denote the ordered product (ordered according to which l_{ij} line crosses the circle first) of matrices M_{ij} as we go half the way around the large circle

$$S = \prod_{l_{ji}\text{ cross half circle}}^{\rightarrow} M_{ij}\,. \tag{2.11}$$

Then as we go around the full circle, because of the identity (2.10), and because the order in which the lines cross the second half circle is the same as the order in which they cross the first half circle modulo replacing l_{ij} with l_{ji} we get the full monodromy matrix M to be

$$M = S^{-t}S\,.$$

We will be interested in the eigenvalues of M. We will compute the eigenvalues of the monodromy matrix in another way: We first note that the matrix M is independent of finite deformations of the vacua. So in the limit in which all the critical values become equal, i.e. the conformal case in which W is quasi-homogeneous, the eigenvalues of the monodromy matrix M can be computed by a suitable choice of $n-1$ forms, which form a basis for the dual space to the vanishing $n-1$ cycles. Let ϕ_k be a monomial basis for the chiral ring $\mathscr{R} = \dfrac{C[x_i]}{dW}$. Let q_k be its degree (charge). Consider the n form

$$\omega_k = \phi_k dx^1 \ldots dx^n\,.$$

Since t is not a critical value of W we define an $n-1$-form α_k defined on the preimage of $W = t$ by

$$\omega_k = \alpha_k \wedge dW\,.$$

Then it is known that α_k form a basis for the dual to the vanishing cycles [5]. Now consider deforming $t \rightarrow e^{2\pi i}t$. This can be undone, since W has charge 1 by letting

$$x^i \rightarrow e^{2\pi i q_i} x^i\,.$$

So using the above formula for α_k, we see that it transforms by

$$\alpha_k \rightarrow (-1)^n e^{2\pi i\left(q_k - \frac{\hat{c}}{2}\right)} \alpha_k\,,$$

where $\hat{c} = \sum_i (1 - 2q_i)$ is the (normalized) central charge of the conformal theory (we put $(-1)^n$ in the above to cancel the term involving $\sum_i 1/2$ in the definition of $\hat{c}/2$). Now if we take even number of variables, as we have done, the $(-1)^n$ disappears and we get

$$\text{Eigenvalues}(S^{-t}S) = e^{2\pi i q_k^R}\,, \tag{2.12}$$

where $q_k^R = q_k - \dfrac{\hat{c}}{2}$ denotes the charge of the ground states of the Ramond sector. This is the relation we were after, which connects the information about the soliton spectrum on the left, a property of the massive theory, to the spectrum of the charges of chiral fields of the conformal theory on the right, a property of the massless theory [3].

[3] The reader may wonder how reliable is the argument in the text since we claim to be able to compute UV quantities in the semi-classical limit (which is the IR limit for the LG models). The point is that $\mathrm{tr}\, M^m$ is equal to $\mathrm{Tr}(-1)^F g^m$, where g is the operator generating a $U(1)$ transformation by 2π. Such objects are susy indices and so can be reliably computed. But then the eigenvalues of M can be computed in any regime we please

This theorem for the LG case was known to the mathematicians (in the mathematical way of thinking it is a relation between the intersection numbers of vanishing cycles with the Milnor monodromy of the singularity) [5]. Note that Eq. (2.12) gives the Ramond charges only mod integers. We will find a method, which applies to an arbitrary $N = 2$ QFT in later sections, which also gives the *integral* part of the charges.

The matrix S can be simplified further if we choose a particular deformation of the theory. This certainly should not affect the monodromy as the monodromy is independent of the perturbation. We deform the critical values so that the polygon $w_1, w_2, \ldots, w_n, w_1$ is convex [4]. Moreover we assume that the polygon is such that l_{ij} crosses the half circle for all $i < j$. This configuration of vacua we call *standard* configuration. Then the matrix S given by (2.11) simplifies because the products of A's vanish for this convex geometry and we get

$$S = 1 - A,$$

where

$$A = \sum_{i<j} A_{ij}.$$

Note that A is strictly an *upper triangular* matrix, and thus in this deformed version S is just upper triangular, with 1's on the diagonal and $-\mu_{ij}$, i.e. minus the ij soliton number, on the ij entry with $i < j$.

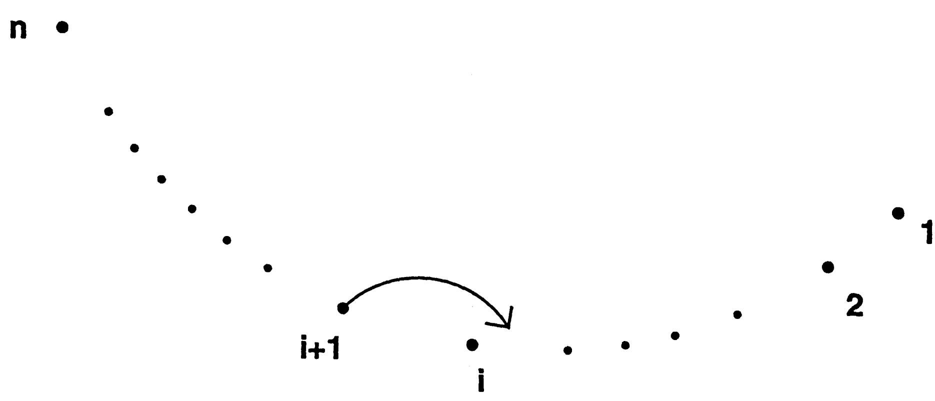

Fig. 5. The exchange of the $i + 1^{\text{th}}$ vacuum with the i^{th} vacuum generate a Braid group

For a given $N = 2$ theory there are many "standard" configurations. Going from one such configuration to another will give a new matrix A, as the number of solitons will change. So even after we restrict to upper triangular matrices we will end up with many matrices A which are *equivalent* modulo perturbations of the original theory. Indeed there is an action of the Braid group on S which corresponds to this equivalence: Consider ordering the vacua according to decreasing value of $\mathrm{Re}(W)$

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[4] It is not clear that we can always do this, because not all the chiral fields are relevant perturbations, and so generally we cannot add all of them to the action. Nevertheless this does not modify the relation we derived for the monodromy, as choosing such configurations correspond to conjugating SS^{-t} by some matrix and does not affect the relation between charges and the monodromy. Therefore it is useful to assume that at least formally we can choose such a configuration

from 1, ..., n. Let us further assume that the vacua are in the form of a convex polygon. Order the w_k's so that $\mathrm{Re}\, w_k > \mathrm{Re}\, w_j$ for $k < j$ and choose the imaginary parts so that they form a "standard" convex polygon where $S = 1 - A$. Let us deform the theory. It is easy to see from the definition (2.11) and (2.9) that a deformation which leaves invariant the real parts of the w_k's does not change S (as long as l_{ij} crosses the half circle for all $i < j$), although in general, the soliton numbers change since some vacua get aligned. Next let us perturb the theory so that the $i + 1^{\text{th}}$ and i^{th} vacua exchange their positions as shown in Fig. 5. We deform the w_{i+1} coupling along a clockwise path making an half turn around w_i in such a way that we end up with the "standard" configuration but now with w_i in the $(i+1)^{\text{th}}$ place. In doing this, w_{i+1} crosses once all soliton lines emanating from the point w_i (except, of course, the line through w_i itself). The effect on S, using our discussion of how soliton numbers change, is

$$S \to PSP^t \qquad P = (1 + A^t_{i,i+1}) P_{i,i+1} , \tag{2.13}$$

where $P_{i,i+1}$ is the matrix permuting i and $i + 1$. It is clear from this geometrical description that repeating this operation for all i forms a braid group. Note that the above transformation on S acts as

$$S^{-t} S \to P^{-t} (S^{-t} S) P^t$$

and thus does not change the eigenvalues of the monodromy matrix SS^{-t} as expected.

3. Generalization

In this section we discuss how the results of the previous section can be *stated* for any $N = 2$ massive quantum field theory in two dimensions. This is not automatic as even the definition of some of the objects in the previous sections seemed to depend on the fact that we were describing the Landau-Ginzburg models. We will show that this is not an obstacle. We prove some of the general statements that we make, but the proof of the main statement relating the S matrix with the $U(1)$ phases is left for the next section.

The first thing to discuss is what we mean by a *massive* $N = 2$ theory. We mean one which has a mass gap with non-degenerate vacua. In particular this means that each of the vacua support local massive excitations. Let us label the vacua by $i = 1, \ldots, n$. In an abstract definition, this "point basis" can be defined by the condition of diagonalizing the chiral ring, i.e., we can choose representatives of the chiral ring labeled by Φ_j such that

$$\Phi_j \, |i\rangle = \delta^i_j \, |i\rangle \, .$$

Note that the condition of having non-degenerate vacua which is needed for a massive theory cannot be satisfied for $N = 2$ theories which have elements in the chiral ring with non-vanishing fermion number F. In particular since fermion number is conserved by the $N = 2$ algebra (even for a massive theory) we will end up having degenerate vacua. So a necessary condition for a conformal theory to admit a non-degenerate massive deformation is that *it have vanishing fermion number for chiral ring elements* [5].

[5] It would be very interesting to study massive theories which do not satisfy this constraint, an example of which is provided by Kähler manifolds with positive c_1 with non-vanishing off-diagonal Hodge numbers

A crucial ingredient in our discussion of the LG case was the understanding of the solitons in the theory. The definition of solitons of interest is as easy in the general case: We consider the ij sector defined by the condition that we start with a vacuum i on spatial infinity at left and end up with vacuum j at spatial infinity to the right. The solitons of interest to us are the ones that saturate the Bogomolnyi bound. What this means is the following: The $N = 2$ algebra in the ij sector has a central extension which we denote by w_{ij} and appears in

$$\{Q^+, \bar{Q}^+\} = 2w_{ij} \,. \tag{3.1}$$

It is easy to show, using the rest of the $N = 2$ alebra, that the mass m of any state in the ij sector satisfies

$$m \geq 2|w_{ij}| \,.$$

As discussed in [14] $\mathrm{Tr}(-1)^F F$ counts the number of Bogomolnyi solitons. So at least this part of the definition which we used in the LG case exists quite naturally in the general set up.

In the previous section we also saw that the critical values in the W-plane played a crucial role in the change of soliton numbers as we perturb the theory. In particular when three vacua passed through a configuration in which they were aligned in the W-plane the number of solitons changed. So if we wish to understand how soliton numbers change we first need to see if we can define the notion of a critical value of a vacuum. This can be done as follows: The central term in the supersymmetry algebra (3.1) is additive, i.e.,

$$w_{ik} = w_{ij} + w_{jk} \,.$$

This together with the fact that $w_{ii} = 0$, implies that we can assign to each vacuum i a critical value w_i, unique up to an overall shift, such that

$$w_{ij} = w_i - w_j \,.$$

So the notion of critical value is also universal and not restricted to LG theories. So we now ask if the number of solitons change as in the LG case when three vacua pass through an aligned configuration. The answer is exactly as in the LG case, but the proof will be different; after all in the general case we do not have the analog of Picard-Lefschetz theory which gave the formula in the LG case. What we have instead is the fact that the tt* equations (topological-anti-topological equations) [10] have continuous solutions. In particular the new supersymmetry index defined in [14] which computes $Q = \mathrm{Tr}(-1)^F F \exp(-\beta H)$ is a continuous function of moduli of the theory. Now the leading contribution, up to two soliton terms, to this index was computed in [14]. Using the results of that paper, it is clear that there will be a jump in the contribution of two particle solitons to Q in the ik sector precisely as the j^{th} critical value passes through the straight line connecting w_i to w_k (see Eq. 4.14 of [14]). This jump is unphysical, as Q should be continuous. Indeed the jump in two soliton contribution is of the same form as the one soliton contribution in the ik sector. So to compensate that jump the number of solitons in the ik sector must have jumped precisely by

$$\mu_{ik} \to \mu_{ik} \pm \mu_{ij}\mu_{jk} \,,$$

where the $\pm$ sign depends again on the orientations of the j^{th} critical value crossing the ik line (as follows from Eq. 4.14 of [14]). This is exactly the same answer as in the LG case and so we have recovered it without using Picard-Lefschetz theory. This suggests that *in the general case* tt* *equations are sufficiently powerful to replace the Picard-Lefschetz theory*. Indeed we will find that not only this is true, but in some sense it is even *stronger* than Picard-Lefschetz theory. In particular we will use tt* equations to derive results which were not known to mathematicians (as far as we know) using Picard-Lefschetz theory.

Since we have translated the number of solitons and the geometry of change of soliton numbers to the abstract "W-plane" even when we are not dealing with LG, it is clear that all the rest of the discussion about the LG case would lead to a natural guess about the relation between the soliton numbers and the chiral charges at the conformal point. Namely the eigenvalues of SS^{-t}, where S is as defined in the previous section, should be related to $\exp(2\pi i q_i)$, where q_i are the (left) charges of Ramond vacua at the conformal point. Also, the choice of a simple vacuum geometry, i.e., the "standard configuration" for critical values simplifies the formula for S to be $S = 1 - A$, where A is strictly upper triangular and counts the soliton numbers. Also the discussion about the action of the Braid group on S at the end of the previous section is equally applicable in the general set up. In other words, we do not need the notion of *vanishing cycles* which does not exist in any obvious sense in the general set up to formulate the main results of the previous section.

We will indeed go one step further in the general set up, which was not done in the LG case: Note that from S it seemed that we have only a way of fixing the chiral charges q_i modulo addition of integers. We will show in the next section that we can also fix its integral part. The idea is to consider

$$S(t) = 1 - A(t)\,,$$

where $A(t)$ is a continuous function of t and is a real strictly upper triangular matrix interpolating from 0 to A as t runs from 0 to 1. We then consider the eigenvalues of SS^{-t} as a function of t. Note that the eigenvalues are never zero and so we can consider how many times a given eigenvalue wraps around the origin as t goes from 0 to 1. This will be the integral part of q_i [6]. As far as we know this is a new result even for the singularity theory [7].

4. Isomonodromic Deformations and the General Solution of tt*

4.1. *The* tt* *Equations*

In this section we give a general proof of (2.12) which does not depend on a particular Lagrangian formulation of the theory, Landau-Ginzburg or otherwise. The idea is to use the differential equations which describe the ground state geometry (tt* equations [10]) to connect the leading IR behaviour (encoded in the soliton spectrum) to the UV one which is specified by the $U(1)$ charges of the Ramond vacua q_k. The basic quantity of interest is the "new index" [14], i.e. the matrix

$$Q_{ij} = \lim_{L\to\infty} \frac{i\beta}{2L}\, \mathrm{Tr}_{(i,j)}[(-1)^F F e^{-\beta H}]\,. \tag{4.1}$$

Here $\mathrm{Tr}_{(i,j)}$ means the trace over the sector (i, j) of the (infinite volume) Hilbert space. This sector is specified by requiring that as $x \to +\infty$ (resp. $-\infty$) the field

[6] The reader may worry about collision of eigenvalues, but this can be avoided by considering a slight perturbation of A

[7] To make it fully rigorous we need one assumption which we have not been able to rigorously prove (see Sect. 4)

configuration approaches the j^{th} (resp. i^{th}) vacuum. By definition Q is related to the axial $U(1)$ charge of the vacua. At the conformal point (UV limit) Q is the same as the left (or right) charges of Ramond ground states, and at the IR the leading contribution to (4.1) counts the number of solitons. So this object knows about both sides of (2.12) and is thus no surprise that studying it would lead to proving (2.12).

For a completely massive theory there is a natural system of coordinates in coupling constant space, i.e. the canonical coordinates[8] $w_k (k = 1, \ldots, n)$ [17, 15, 11]. They are defined as follows. Let $Z = \{Q^+, \bar{Q}^+\}$ be the central charge in the $N = 2$ algebra. Then as discussed before we can set

$$w_i - w_j = \tfrac{1}{2} Z|_{(i,j)} \, .$$

In the canonical coordinates one has [15]

$$Q = -\sum_k w_k g \partial_k g^{-1} = -\tfrac{1}{2}\, g\beta\partial_\beta g^{-1} \, , \tag{4.2}$$

where $g_{i\bar{j}} = \langle \bar{j} \mid i \rangle$, is the ground state-metric in the canonical basis[9]. g satisfies the differential equations (tt* *equations*)

$$\begin{aligned} \bar{\partial}_i (g \partial_j g^{-1}) &= [C_j, \bar{C}_i] \, , \\ [g\partial_j g^{-1}, C_k] &= [g\partial_k g^{-1}, C_j] \, , \end{aligned} \tag{4.3}$$

where C_k are the matrices representing in $\mathscr{R}$ the multiplication by the chiral primary operator ϕ_k such that

$$\delta S = \sum_k \delta w_k \int \phi_k^{(2)} \, .$$

By definition, in the canonical basis we have

$$(C_k)_i^j = \delta_{ki} \delta_i^j \, . \tag{4.4}$$

Then, in this basis, the tt* equations take a universal form [15, 11] which is nothing else than the equations for the Ising n-point functions (see [15] for details). Different models differ *only in the boundary conditions* satisfied by solutions of tt* equations. Thus, finding a universal way to describe the boundary conditions will lead to a classification of different models.

From the thermodynamical interpretation of Q_{ij} [14] it is clear that the general solution to tt* can be written in the form of a soliton expansion, and that the specific boundary conditions for (4.3) are encoded in the soliton spectrum. More precisely, we have $n(n-1)/2$ soliton "fugacities" $\mu_{ij} = -\mu_{ji}$ corresponding to the $n(n-1)/2$ possible soliton masses [6],

$$m_{ij} = |Z|\,|_{(i,j)} = 2|w_i - w_j| \, .$$

The "fugacities" are defined by the asymptotics [10] [14]

$$Q_{ij}|_{\beta \to \infty} \cong -\frac{1}{2\pi} \mu_{ij} m_{ij} \beta K_1(m_{ij}\beta) \, , \tag{4.5}$$

[8] For simplicity, we assume we are in a *generic* situation, i.e. $w_i \neq w_j$ for $i \neq j$

[9] The canonical basis $|i\rangle$ is the topological basis (see [10]) such that: 1) the chiral ring $\mathscr{R}$ is diagonal, and 2) the topological metric is normalized to 1. The canonical basis is unique up to sign

[10] The convention-dependent phases are chosen so that μ_{ij} is real. Here and below $K_\nu(\cdot)$ are modified Bessel functions

or, in terms of g, ([10] App. B)

$$g_{i\bar{j}} \cong \delta_{ij} - \frac{i}{\pi} \mu_{ij} K_0(m_{ij}\beta) . \tag{4.6}$$

Although regular solutions to (4.3) exist for real [11] μ_{ij}, in the physical case $|\mu_{ij}|$ is an integer counting the number of soliton species connecting the i^{th} and j^{th} vacua. On physical grounds one expects that varying μ_{ij} one gets all possible solutions to the tt* equations. The UV asymptotics is

$$Q_{ij}|_{\beta\to 0} = q_{ij} , \tag{4.7}$$

where q_{ij} are the $U(1)$ charges of the Ramond vacua at the UV fixed point [10]. Since the solution depends on the boundary data μ_{ij}, the tt* equations may be seen as a map from the soliton spectrum μ_{ij} to the possible values of the $U(1)$ charges. Below we show that this map is precisely the one predicted by Eq. (2.12).

Since Q_{ij} can be computed from the ground-state metric, we should be able to read the n soliton contribution to the Q matrix from the tt* equations. Indeed the general solution to the tt* equation (for a massive model) has naturally the form of a grand-canonical sum over n-soliton sectors. For the case of two vacua (corresponding to PIII) this has been shown in [15, 14]. This case is particularly easy since there is only one soliton of mass $2|w_1 - w_2|$. In the soliton expansion of (4.1), the n-soliton sector contributes a term of order

$$\exp(-2\beta|w_1 - w_2|n) \quad \text{for } \beta \text{ large.}$$

For PIII the soliton expansion (first obtained in [18]) is in terms of Ising form-factors. By the remark after (4.4) this is true in general.

4.2. The Integral Formulation of tt* [11]

In principle to get the general soliton expansions we could start from the Ising form factors or, equivalently, from the known series for the Ising correlation functions [19]. However it is more convenient to take advantage of the analysis of the tt* equations due to Dubrovin [11]. He was able to reformulate the (massive) tt* equations as a Riemann-Hilbert problem [12] having a very convenient expression in terms of linear integral equations. Here we recall the aspects of his work we need in the following.

Introducing the covariant derivatives

$$\begin{aligned} \nabla_i &= \partial_i + (g\partial_i g^{-1}) - xC_i , \\ \bar{\nabla}_{\bar{i}} &= \bar{\partial}_{\bar{i}} - x^{-1}\bar{C}_{\bar{i}} , \end{aligned} \tag{4.8}$$

where x is a spectral parameter, we can rewrite Eqs. (4.3) as the consistency (zero-curvature) conditions for the system of linear differential equations,

$$\nabla_i \Psi(x, w_k) = \bar{\nabla}_{\bar{i}} \Psi(x, w_k) = 0 , \tag{4.9}$$

[11] Only for μ_{ij} small enough the solutions are expected to be regular. This reflects the fact that there is an upper bound for the UV central charge $\hat{c}$ of a unitary "massive" theory with a given Witten index n. For instance, for $n = 2$ we get $\hat{c} \leq 1$. Stated differently, let us order the eigenvalues q_i of the Q-matrix in increasing order. Then the gaps $(q_{i+1} - q_i)$ cannot be too big

[12] For a review of this problem, see e.g. [20]

where $\Psi(x, w_k)$ is an $n \times n$ matrix. Clearly, in order to solve the tt* equations it is enough to compute [13] $\Psi(x)$. To completely specify the tt* geometry one needs to impose the condition that g is independent of an overall rotation in the value of w_i. In order to incorporate this condition naturally, let us consider the dependence of Ψ on the overall scale β and the overall chiral angle θ. This amounts to a redefinition of the canonical coordinates as

$$w_k \to \beta e^{i\theta} w_k \,. \tag{4.10}$$

From (4.8), (4.9) we get (after the identification $x = e^{i\theta}$, natural in view of (4.8))

$$x \frac{\partial}{\partial x} \Psi = (\beta x C + Q - \beta x^{-1} \bar{C}) \Psi \,, \tag{4.11}$$

$$\beta \frac{\partial}{\partial \beta} \Psi = (\beta x C + Q + \beta x^{-1} \bar{C}) \Psi \,, \tag{4.12}$$

where

$$C = \sum_k w_k C_k \,, \qquad \bar{C} = g C^\dagger g^{-1} \,.$$

Notice that Q, C and $\bar{C}$ are independent of x. Indeed, the w_k's overall phase can be absorbed in the phase of the fermions. The introduction of a spectral parameter x allows us to extend $\Psi(x)$, which originally was defined only for $|x| = 1$, to a *piecewise analytic* function in the whole x plane $\Psi(x)$, whose dependence on x is governed by (4.11). In fact the nice thing about (4.11) is that the compatibility of this equation with (4.9) automatically implies that the solution to tt* are independent of θ (are "self similar"), a condition which was previously imposed by hand. So the compatibility of the above linear system of equations completely captures the tt* geometry.

The differential equation (4.11) has two *irregular* singular points for $x = 0$ and ∞. Then $\Psi(x)$ presents the Stokes phenomenon [21]. This means that $\Psi(x)$ is well defined only in certain angular sectors centered at the origin. In the present case we need (at least) two angular sectors. For convenience we choose these two sectors to be two suitable angular neighborhoods of the upper and lower half-plane, respectively. This means that $\Psi(x)$ should be replaced by a coule of $n \times n$ matrices $(\Psi_+(x), \Psi_-(x))$ which are analytic in the half-planes $\mathrm{Im}\, x > 0$ and $\mathrm{Im}\, x < 0$ respectively. In the overlap between the two angular sectors, Ψ_+ and Ψ_-, being both solutions to the linear equation (4.11), should satisfy a relation $\Psi_- = \Psi_+ M$ for some *constant* matrix M. More precisely, along the real axis they satisfy the following Riemann boundary condition (here $y > 0$)

$$\begin{aligned} \Psi_-(y) &= \Psi_+(y) S \,, \\ \Psi_-(-y) &= \Psi_+(-y) S^t \,. \end{aligned} \tag{4.13}$$

General Stokes theory gives constraints [14] on the matrix S

$$\begin{aligned} S_{ii} &= 1 \,, \\ S_{ij} &= 0 \quad \text{for} \quad \mathrm{Re}(w_i - w_j) < 0 \,. \end{aligned} \tag{4.14}$$

Moreover, PCT requires S to be real.

[13] To save print we usually omit the dependence of Ψ on the couplings w_k and $\bar{w}_k$

[14] We choose the overall phase of the w_k's so that $\mathrm{Re}(w_i - w_j) \neq 0$ for $i \neq j$. Of course, this can be done only locally in coupling space. To get the global solution one has to glue all the local solutions so obtained

The matrix $\Psi(x)$ satisfies the following boundary condition

$$\lim_{x \to \infty} \Psi(x) \exp[\beta(xC + x^{-1}C^{\dagger})] = 1 . \tag{4.15}$$

Using this boundary condition and the well-known identity (P means principal part)

$$\frac{1}{x - y \mp i\varepsilon} = P \frac{1}{x-y} \pm i\pi\delta(x-y) , \tag{4.16}$$

one rewrites the above Riemann boundary problem as the integral equation

$$\begin{aligned} \Phi(x)_{ij} &= \delta_{ij} + \frac{1}{2\pi i} \sum_k \int_0^\infty \frac{dy}{y - (x + i\varepsilon)} \Phi(y)_{ik} A_{kj} e^{-\beta(y\Delta_{kj} + y^{-1}\bar{\Delta}_{kj})} \\ &+ \frac{1}{2\pi i} \sum_k \int_{-\infty}^0 \frac{dy}{y - (x + i\varepsilon)} \Phi(y)_{ik} A^t_{kj} e^{-\beta(y\Delta_{kj} + y^{-1}\bar{\Delta}_{kj})} \end{aligned} \tag{4.17}$$

where

$$\Delta_{kj} = w_k - w_j \equiv \tfrac{1}{2} m_{ij} e^{i\phi_{ij}} ,$$

and

$$\Psi_+(x) = \Phi(x) \exp[-\beta(xC + x^{-1}C^{\dagger})] .$$

In terms of A the Stokes matrix reads

$$S = 1 - A , \tag{4.18}$$

so, in particular, (4.14) gives

$$A_{kj} \neq 0 \quad \text{only if} \quad \operatorname{Re} \Delta_{kj} > 0 ,$$

which is nothing else than the condition needed in order to make sense out of the integrals in (4.17).

The solution to (4.17) is automatically a solution to all Eqs. (4.3). Indeed, the matrix S encodes (with respect to the chosen angular sectors) the monodromy properties of the linear differential equations with rational coefficients (4.11). In particular the monodromy around the singular point $x = 0$ is given by $H = S(S^t)^{-1}$. *A priori* the monodromy data depend on the coefficients in Eq. (4.11), i.e. on w_k, Q and the ground-state metric g. However Eqs. (4.3) just represent the isomonodromic deformations of Eq. (4.11), that is they describe the variations of the coefficients in (4.11) which *do not change* its monodromy data. Said differently, the fact that Ψ is a solution to (4.8) and (4.11) means that the matrix S is a constant independent of both x *and* w_i. In fact, from the general theory of isomonodromic deformations [22] we know that the condition for having isomonodromic deformations is just the zero-curvature condition above [15], i.e. the tt* equations themselves.

Now, the solution to (4.17) for a given (fixed) S is certainly a family of isomonodromic solutions to (4.11) parametrized by the couplings w_k. Indeed the monodromy data S is a constant by construction. Then it must be also a solution to (4.3). (Mathematically oriented people may find complete proofs in [11]; for the special $n = 2$ case see also [22, 24]). This "monodromic" viewpoint also explains

[15] Recall that the massive tt* equations are those for the Ising correlations. It is well known that these equations describe isomonodromic deformation. In fact this is precisely what the Kyoto school mean when they talk of *"holonomic field theory"* [23]

how the Stokes parameters A_{ij} encode the boundary conditions needed to specify a solution of (4.3).

For small temperatures, $\beta \to \infty$, the kernel in (4.17) is exponentially suppressed. Hence for small enough temperature we have a unique solution with given monodromy data A_{ij}. Whether this can be extended to a regular solution for all β's depends on the particular A_{ij}. This should happen for the physical values of the Stokes parameters. From the Riemann problem (4.13) and the uniqueness of the solution we infer that the piecewise analytic function $\Psi \equiv (\Psi_+, \Psi_-)$ satisfies [11]

$$\begin{aligned} \Psi(x)\Psi^t(-x) &= 1\,, \\ \overline{\Psi(1/\bar{x})} &= g^{-1}\Psi(x)\,, \end{aligned} \tag{4.19}$$

where the second equation is nothing else than the statement that complex conjugation acts on the vacuum wave function [16] as the ground state metric g [10].

From (4.15) and (4.19) we get

$$g_{i\bar{j}} \equiv \lim_{x\to 0} \Phi(x)_{ij}\,. \tag{4.20}$$

4.3. The Ultra-Violet Limit: The Q-Matrix

Now we study the large temperature asymptotics (4.17) of the solutions to tt*. This would give us the conformal dimensions of the chiral primary operators at the UV fixed point as a function of the Stokes parameters A_{ij}.

To get the eigenvalues $q_i(A)$ of the matrix $q_{ij}(A)$ we exploit its physical meaning. As $\beta \to 0$ the $U(1)$ invariance is restored and $q_i(A)$ are just the vacuum values of the corresponding conserved charge. Therefore when we increase θ by 2π in Eq. (4.10) the wave functions Ψ pick up phases $\exp[2\pi i q_j(A)]$. This can also be seen from the differential equation (4.11) satisfied by $\Psi(x)$. As $\beta \to 0$, and as long as we restrict ourselves to the region

$$\beta \ll |x| \ll \beta^{-1}\,, \tag{4.21}$$

we can approximate Eq. (4.11) by one with constant coefficients, namely [17]

$$\frac{d}{d\theta}\Psi_i \approx q_{ij}\Psi_j\,.$$

Hence in the region (4.21) we have

$$\Psi(\theta + 2\pi i)_i \approx (e^{2\pi i q})_{ij}\Psi(\theta)_j\,. \tag{4.22}$$

On the other hand from (4.13) we see that

$$\Psi(\theta + 2\pi i) = \Psi(\theta(S(S^{-1})^t\,.$$

Comparing the last two equations we get

$$\exp[2\pi i q_j] = \text{Eigenvalues}[S(S^{-1})^t]\,. \tag{4.23}$$

This is the equation expressing the UV charges q_j in terms of the Stokes parameters we look for, modulo showing the relation between A defined here and the soliton

[16] Indeed for LG models $\Psi(x)$ is related by a linear integral transform to the usual SQM wave function

[17] Recall that $q_{ij} = \lim_{\beta\to 0} Q_{ij}$

numbers μ_{ij} which needs a detailed analysis which we postpone to the next two subsections.

We will now see that we can use tt* to also fix the integral part of q_i. To do this note that even though the physical values for the matrix A are integer, as we will relate it to soliton numbers, as far as the tt* equations are concerned we can take them to be arbitrary. Consider $A \to A(t)$ with $A(0) = 0$ and $A(1) = A$. Then at $t = 0$ we get the trivial theory with the charges equal to zero. As we vary t from 0 to 1, we can trace the eigenvalues of $H(t) = S(t)(S(t))^{-t}$ on the complex plane. Since these eigenvalues do correspond to $\exp(2\pi i q)$, where q is the solution of tt* at the UV point [18] (unphysical as they may be), by continuing the eigenvalues until we get to $t = 1$ we can deduce the integral part of the charges by the number of times they have wrapped around the origin in the complex plane. This clearly shows the power of tt* equations as they can be used even in the unphysical regime (non-integral soliton numbers) to give some physical results (with integral soliton numbers). This result applies in particular to the LG case, and as far as we know it was not known to the mathematicians how to fix the integral part of charges purely from the S matrix. In the singularity language the trick we are using is like taking a "continuous real intersection number" which is not easy to see how would one interpret.

Using the idea of "building up the charge" we can also learn something about the signature of the matrix $B = S + S^t$. Note that this matrix is a symmetric integral matrix with 2's on the diagonal. It can be interpreted as the bilinear form for an integral lattice. It is useful to discuss the signature of this form when we begin to classify $N = 2$ quantum field theories. We know that at $t = 0$ and $t = 1$ the eigenvalues of $H(t) = SS^{-t}$ are pure phases (i.e. have norm 1). Let us assume that by a proper choice of t-dependence of S which connects these points we go *only* through phases. Let us consider the signature of $B(t)$. Clearly $B(0)$ is positive definite. For its signature to change we should come across a zero eigenvector of B, which means we must have a vector v with

$$Sv = -\,S^t v$$

which implies

$$H^t v = -\,v\,.$$

In other words the signature changes precisely when one of the eigenvalues crosses the negative real axis. Of course if that eigenvalue crosses the negative real axis another time, it will change back the signature. Now noting the connection between the integral part of charges and the number of times an eigenvalue wraps around the origin we see that the number of positive directions r and negative directions s of B are given by

$$\begin{aligned} r &= \#(2n - \tfrac{1}{2} < q < 2n + \tfrac{1}{2})\,, \\ s &= \#(2n + \tfrac{1}{2} < q < 2n + \tfrac{3}{2})\,. \end{aligned} \tag{4.24}$$

When there are some charges equal to $1/2 \bmod 1$ we also get some null directions. This result for the signature of B agrees with what is known to mathematicians in the context of singularity theory.

We made the assumption that by continuously changing the parameters of S we can vary the eigenvalues of H maintaining the condition that they remain roots of unity. Indeed if the eigenvalues of H end up having norm other than one, then q becomes

[18] See the discussion below on the requirement of the existence of solution to tt*

complex and this implies that for the solution of tt* there is some singularity because otherwise q is given by the eigenvalues of $-g\beta\partial_\beta g^{-1}/2$ and that is real for a solution of tt*. So as long as the regular solution space of tt* is connected, we should be able to go through phases only. In general it is easy to see (as we will argue later) that for small t this is the generic case. Indeed the eigenvalues of H come in groups of four $\lambda, \lambda^*, \lambda^{-1}, \lambda^{*-1}$ and for t near zero (S near one) it is easy to see that they come in pairs because λ is a root of unity. In general if we just take an arbitrary deformation of S like letting $A \to tA$ this condition will not be maintained for larger values of t. However, it is natural to expect that by proper tuning of the coefficients of A (with arbitrary real functions of t), we should be able to maintain the condition that eigenvalues of H be pure phases. It would be nice to prove this highly plausible statement. The fact that in the LG case the result obtained in this way agrees with what mathematicians had obtained lends further support to this statement.

Subtleties with Asymptotic Freedom. At first sight one would also expect that the two matrices $\exp[2\pi i q]$ and $H = S(S^t)^{-1}$ are similar. However it is not so: H many have non-trivial Jordan blocks. This possibility arises because of UV sub-leading terms that we have neglected in the above analysis. Instead of discussing the (well-known) mathematics of this phenomenon, let us explain its deep physical meaning. To make things as simple as possible, we consider a specific model, namely the supersymmetric $\mathbf{C}P^1$ σ-model [25]. This model has a mass-gap [26]. Since it is asymptotically free, its UV fixed point is just free field theory. At this UV fixed point the (unique) non-trivial chiral primary field has dimension $\left(\frac{1}{2}, \frac{1}{2}\right)$. However it is not really a marginal operator, otherwise the σ-model would be conformal for all β's. As it is well-known, this state of affairs leads to *logarithmic violations of scaling*. The non-trivial Jordan blocks are related to these violations. For instance, for $\mathbf{C}P^1$,

$$H = \begin{pmatrix} 1 & -2 \\ 2 & -3 \end{pmatrix} \overset{\text{similarity}}{\longmapsto} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \tag{4.25}$$

and so we expect logarithmic corrections to scaling [19]. The Jordan structure of (4.25) can be extracted directly from the basic equations (4.11), (4.12). It is natural to look for a solution of the form

$$\Psi(x, \beta) = \exp[q(\log x + \log \beta)]\,\Phi(x, \beta).$$

In the limit $\beta \to 0$ the differential equation for $\Phi(x, \beta)$ reduces to

$$x\frac{d}{dx}\Phi = [B - x^{-2}\bar{B}]\Phi + O\left(\frac{1}{\log\beta}\right), \tag{4.26}$$

where

$$\begin{aligned} B &= \lim_{\beta\to 0} \beta x[(x\beta)^{-q} C (x\beta)^q], \\ \bar{B} &= \lim_{\beta\to 0} \beta x[(x\beta)^{-q} \bar{C} (x\beta)^q]. \end{aligned} \tag{4.27}$$

The matrix C represents in $\mathscr{R}$ some chiral operator $\hat{\phi} \equiv \sum_k w_k \phi_k$. Let us decompose ϕ into a sum $\sum_{i\in I} \tilde{\phi}_i$ of operators having definite $U(1)$ charge q_i at the UV fixed point. Let $\bar{q} = \max_{i\in I}\{q_i\}$. Then, for small β $(x\beta)^{-q}C(x\beta)^q$ is of order $\beta^{-\bar{q}}$. Thus, if our

[19] These corrections can be found by an exact computation, see [25]

perturbation ϕ has an UV dimension less than 1 (i.e. it is "super-renormalizable") $B = \bar{B} = 0$ and there is no new subtlety. Instead for an "asymptotically free" (AF) theory $\bar{q} = 1$ and B is finite [20]. For instance, in the $\mathbf{C}P^{n-1}$ case we have (up to similarity)

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & & \dots & & \dots & \\ 0 & 0 & 0 & \dots & \dots & 1 \end{pmatrix}, \tag{4.28}$$

and $B^n = 0$. From (4.26) we see that for x large (but still $x \ll \beta^{-1}$)

$$\Psi(x, \beta) \sim \exp[q(\log x + \log \beta)] \exp[B(\log x + \log \beta)] \Phi_0 ,$$

from which it is manifest that the Jordan structure of H is that of

$$\exp[2\pi i q] \exp[2\pi i B] . \tag{4.29}$$

In particular, for the $\mathbf{C}P^{n-1}$ models H should consist of a single block of length n.

4.4. Infra-Red Asymptotics

To complete our proof of the formula relating q_j to the soliton matrix μ_{ij} we have still to find the relation between the Stokes parameters A_{ij} and the soliton numbers μ_{ij}. In order to do this, we have to find the asymptotic behaviour as $\beta \to \infty$ of the tt* solutions. Here the integral formulation of Sect. 4.2 becomes crucial.

We write symbolically Eq. (4.17) as

$$\Phi = 1 + \Phi \mathscr{K} .$$

For β large enough we can solve this equation by the method of successive approximations. In this way we get a *convergent* (for β large enough) series for the ground-state metric

$$g = 1 + 1 \cdot \sum_{m=1}^{\infty} \mathscr{K}^m \Big|_{x=0} . \tag{4.30}$$

The term $1 \cdot \mathscr{K}^m$ is of order $O(A^m)$. To begin with, let us consider the first order contribution. Using the formula (valid for $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, b < 0$)

$$\int_0^{\infty} x^{\nu - 1} \exp[-ax - bx^{-1}] = 2\left(\frac{b}{a}\right)^{\nu/2} K_\nu(2\sqrt{ab}) , \tag{4.31}$$

one gets

$$g_{i\bar{j}} = \delta_{ij} - i(A_{ij} - A_{ji}) \frac{1}{\pi} K_0(2|w_i - w_j|\beta) + O(A^2) . \tag{4.32}$$

The first order contribution has precisely the form predicted by the large β asymptotics (4.6). This may suggest that the first order saturates the one-soliton contribution and, more generally, that the m^{th} order term $1 \cdot \mathscr{K}^m$ corresponds to m soliton processes.

[20] If $\bar{q} > 1$ it is not clear how to make sense of the corresponding perturbation. Below we shall see that (typically) the non-renormalizable interactions lead to singular solutions to tt* and so they are "pathological"

This is almost *but not quite* true. Explicitly $[1 \cdot \mathscr{K}^m]_{i\bar{j}}$ can be written as a sum of terms, one for each sequence $\alpha(k)\,(k = 1, \ldots, m)$ in $\{1, 2, \ldots, n\}$ with $\alpha(1) = i$, $\alpha(m) = j$. The sequence of $\alpha(k)$ specifies a particular chain of m would be "solitons" connecting the i^{th} vacuum to the j^{th} one. Then

$$1 \cdot \mathscr{K}^m = \sum_{m-\text{chains}} G_\alpha(\beta, A, w_k), \tag{4.33}$$

where $G_\alpha(\beta, A, w_k)$ has the general form (here $\tilde{A} = A - A^t$)

$$\begin{aligned} G_\alpha(\beta, A, w_k) &\\ = \left(\prod_{k=1}^{m} \tilde{A}_{\alpha(k)\,\alpha(k+1)}\right) & \int_0^\infty \prod_k dx_k F_\alpha(x) \\ \times \exp & \left[-\beta \sum_{k=1}^{m} \sigma_k (x_k (w_{\alpha(k)} - w_{\alpha(k+1)}) + x_k^{-1} (\bar{w}_{\alpha(k)} - \bar{w}_{\alpha(k+1)})) \right], \end{aligned} \tag{4.34}$$

where $F_\alpha(x)$ is an universal function independent of the parameters and

$$\sigma_k = \text{sign}[\text{Re}(w_{\alpha(k)} - w_{\alpha(k+1)})] .$$

Now, *where the kernel $\mathscr{K}$ non-singular*, we could evaluate the large β asymptotics of (4.34) by the usual saddle-point method. The relevant saddle point is at

$$x_k = \sigma_k \sqrt{\frac{\bar{w}_{\alpha(k)} - \bar{w}_{\alpha(k+1)}}{w_{\alpha(k)} - w_{\alpha(k+1)}}} ,$$

and then we would have

$$G_\alpha(\beta, A, w_k) \approx \exp\left[-2\beta \sum_{k=1}^{m} |w_{\alpha(k)} - w_{\alpha(k+1)}| \right], \tag{4.35}$$

which is the expected result for a chain of m solitons having masses $2|w_{\alpha(k)} - w_{\alpha(k+1)}|$.

However, since $\mathscr{K}$ is singular, (4.35) is not necessarily correct. Indeed in order to use the saddle point technique [27] we have to deform the integration contour to pass through the saddle point. In this process we may cross poles (resp. cuts) of the integrand and hence we pick up residue (resp. discontinuity) contributions to (4.34). From (4.16) it is clear that these contributions have also the general structure (4.34) but with a *smaller* m. Moreover the presence of these additional terms depends in a crucial way on the angles in W-plane since the number and type of singularities encountered while deforming the path depends on the vacuum geometry in W-space.

Because of this mechanism, the k-soliton processes may get contributions from all terms in (4.30) with $m \geq k$. This, in particular, holds for the one soliton term which defines the soliton matrix μ_{ij}. So[21],

$$\mu_{ij} = A_{ij} - A_{ji} + O(A^2) . \tag{4.36}$$

[21] Note that our definition of A in this section defers from the one used in Sect. 2. There it was defined in terms of soliton numbers. Here it is defined by $S = 1 - A$. In the "standard" configuration the two definitions are the same

Under our genericity assumption, the rhs of (4.36) is a *finite polynomial.* Indeed, without deforming the integration contour, we get the weaker bound (for β large)

$$G_\alpha(\beta, A, w_k) \leq C \exp\left[-2\beta \sum_{k=1}^{m} |\operatorname{Re}(w_{\alpha(k)} - w_{\alpha(k+1)})|\right],$$

and thus only sequences satisfying

$$\sum_{k=1}^{m} |\operatorname{Re}(w_{\alpha(k)} - w_{\alpha(k+1)})| \leq |w_i - w_j| \tag{4.37}$$

may contribute to μ_{ij}. Clearly, there are only finitely many such sequences. Then to compute μ_{ij} we can truncate the expansion after a *finite* number of terms. However, this method is rather impractical since the nubmer of terms needed varies very much from model to model. For this reason, we shall adopt a different strategy based on the known properties of the solution (4.19) rather than on the integral equation itself. In order to do this, we need more details on the analytic properties of the functions which appear in the expansion (4.30). We pause a while to digress on this more technical material. The reader may wish to jump directly to Sect. 4.5.

Some Useful Functions. The purpose of this digression is to describe the functions one gets when the integrals (4.34) are computed along a contour for which the saddle-point analysis is correct. As discussed above, the functions appearing in the expansion (4.30) can be expressed in terms of these ones, the precise relation being determined by their analytic properties as well as the vacuum geometry.

We introduce a function $\mathscr{F}[z,\zeta]$, where z is a real positive variable and ζ is a variable taking value in the complex plane cut along the positive real axis, by

$$\mathscr{F}[z,\zeta] = \frac{1}{2\pi}\int_0^\infty \frac{ds}{s-\zeta}\, e^{-z(s+s^{-1})}. \tag{4.38}$$

For ζ real positive, $\mathscr{F}[z,\zeta]$ is defined to be $\mathscr{F}[z,\zeta+i\varepsilon]$. The discontinuity at the cut along the positive real axis is given by

$$\mathscr{F}[z, x+i\varepsilon] - \mathscr{F}[z, x-i\varepsilon] = i e^{-z(x+x^{-1})}. \tag{4.39}$$

As $z \to \infty$ one has the asymptotic expansion (for ζ *not real positive*)

$$\mathscr{F}[z,\zeta] \approx \frac{1}{2\pi}\sqrt{\frac{\pi}{z}}\, e^{-2z} \sum_{k=0} \frac{g_k(\zeta)}{(2z)^k}, \tag{4.40}$$

where

$$g_k(\zeta) = \frac{\zeta^{k+\frac{1}{2}}}{k!}\left(\frac{d}{d\zeta}\right)^{2k}\left[\frac{\zeta^{\left(k-\frac{1}{2}\right)}}{1-\zeta}\right]. \tag{4.41}$$

These formulae are a consequence of

$$\mathscr{F}[z.\zeta] = \frac{1}{2\pi}\sum_{k=0}^{\infty} \zeta^k \int_0^\infty \frac{ds}{s^{k+1}}\, e^{-z(s+s^{-1})},$$

together with (4.31). Instead for ζ real positive

$$\mathscr{F}[z,x] = \frac{1}{2\pi} P \int_0^\infty \frac{ds}{s-x}\, e^{-z(s+s^{-1})} + \frac{i}{2}\, e^{-z(x+x^{-1})}\,, \tag{4.42}$$

where $P\int$ means principal part and we used (4.16). As $z \to \infty$, the integral in (4.42) has the same asymptotic expansion as above, except for $x = 1$ when it vanishes more rapidly. In this case the leading term is the second one. Hence for $\zeta = 1$ the asymptotics is

$$\mathscr{F}[z,1] \sim \frac{i}{2}\, e^{-2z}\,.$$

Then, for $\zeta = 1$ the large z behaviour differs [22] for a factor $i\sqrt{z\pi}$ with respect to that for $\zeta \neq 1$.

Next we define a function $\mathscr{F}^{(2)}[z_1, z_2, \zeta, \phi]$, (where z_i are real positive, ζ is as above, and ϕ is an angle with $\phi \neq 0 \bmod 2\pi$)

$$\mathscr{F}^{(2)}[z_1, z_2, \zeta, \phi] = \frac{1}{2\pi} \int_0^\infty \frac{ds}{s-\zeta}\, e^{-z_1(s+s^{-1})} \mathscr{F}[z_2, e^{i\phi} s]\,.$$

If $\phi = 0$ we define this function as the limit as $\phi \downarrow 0$. Here again we have a discontinuity for ζ real as well as for $\phi = 0$. In particular,

$$\mathscr{F}^{(2)}[z_1, z_2, \zeta, \varepsilon] - \mathscr{F}^{(2)}[z_1, z_2, \zeta, -\varepsilon] = i\mathscr{F}[z_1 + z_2, \zeta]\,, \tag{4.43}$$

$$\mathscr{F}^{(2)}[z_1, z_2, x + i\varepsilon, \phi] - \mathscr{F}^{(2)}[z_1, z_2, x - i\varepsilon, \phi] = ie^{-z_1(x+x^{-1})} \mathscr{F}[z_2, e^{i\phi} x]\,. \tag{4.44}$$

So one has [23]

$$\begin{aligned}\mathscr{F}^{(2)}[z_1, z_2, \zeta, 0] = {} & \frac{1}{(2\pi)^2} \int_0^\infty \frac{ds}{s-\zeta}\, e^{-z_1(s+s^{-1})} P \\ & \times \int_0^\infty \frac{dt}{t-s}\, e^{-z_2(t+t^{-1})} + \frac{i}{2} \mathscr{F}[z_1 + z_2, \zeta]\,.\end{aligned} \tag{4.45}$$

If $e^{i\phi} \neq 1$ and $\zeta \neq 1$, for $z_i \to \infty$ we have

$$\mathscr{F}^{(2)}[z_1, z_2, \zeta, \phi] \sim \frac{e^{-2(z_1+z_2)}}{4\pi\sqrt{z_1 z_2}} \frac{1}{(1-\zeta)(1-e^{i\phi})}\,. \tag{4.46}$$

If $e^{i\phi} = 1$ one has instead

$$\sim \frac{i}{4\pi} \sqrt{\frac{\pi}{z_1}}\, e^{-2(z_1+z_2)}\,, \tag{4.47}$$

[22] This discontinuity in the low temperature behaviour can be understood physically as due to contact terms

[23] Taken at the face value, this says that two solitons represented by two exactly aligned segments give a contribution which looks like a "half" soliton of mass $m_1 + m_2$. This strange effect is effectively seen in explicitly computable models (see Appendix B for an example). It can also be understood from the S-matrix viewpoint

so we have again the discontinuity in the large z_i behaviour. We have a similar result when $\zeta = 1$ and when both variables are equal to 1. In this last case both factors $z_i^{-1/2}$ cancel.

Clearly, the above analysis may be generalized. Let us define recursively the functions

$$\mathscr{F}^{(k)}[z_1, \ldots, z_k, \zeta, \phi_1, \ldots, \phi_{k-1}] = \frac{1}{2\pi} \int_0^\infty \frac{ds}{s-\zeta} e^{-z_k(s+s^{-1})} \mathscr{F}^{(k-1)}[z_1, \ldots, z_{k-1}, e^{i\phi_{k-1}} s, \phi_1, \ldots, \phi_{k-2}].$$

Here z_i are real positive variables, ζ is a complex parameter taking value in the plane cut along thre real semi-axis (for ζ real, by convention, we define the function as its limit by above), and ϕ_i are angular variables in the range $0 < \phi_i < 2\pi$, and for $\phi_i = 0$ we take as definition the limit by above. As $z_i \to \infty$ one has

$$\mathscr{F}^{(k)}[z_1, \ldots, z_k, \zeta, \phi_1, \ldots, \phi_k] \sim e^{-2\sum_i z_i},$$

up to a power of the z_i which depends on ζ and the ϕ_i's.

From their recursive definition, it is clear that the discontinuity of $\mathscr{F}^{(k)}$ for ζ real positive (resp. for $\phi_i = 0$) can be expressed in terms of $\mathscr{F}^{(h)}$ with $h < k$, possibly multiplied by factors $\exp[-z_i(x + x^{-1})]$.

Sample Integrals. As a preparation to Sect. 4.5, and illustration of the above mechanism, we compute some sample integrals one gets in (4.30). At the first order the typical integral is

$$\frac{1}{2\pi i} \int_0^\infty \frac{dy}{y - (x + i\varepsilon)} A_{ij} e^{-\beta(y\Delta_{ij} + y^{-1}\bar{\Delta}_{ij})}, \tag{4.48}$$

where $\operatorname{Re} \Delta_{ij} > 0$ and

$$\Delta_{ij} = \tfrac{1}{2} m_{ij} e^{i\phi_{ij}}, \quad \text{with} \quad -\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi.$$

Assume $0 \le \phi_{ij} < \frac{1}{2}\pi$, and let C_R be the segment $y = te^{-i\phi_{ij}}$, $0 \le t \le R$, and γ_R the arc $y = \mathrm{Re}^{i\theta}$, $-\phi_{ij} \le \theta \le 0$. Denote by $F(y)$ the integrand in Eq. (4.48). $F(y)$ is holomorphic in the lower half-plane. Then we have

$$-\int_0^R F(y)\,dy + \int_{C_R} F(y)\,dy + \int_{\gamma_R} F(y)\,dy = 0.$$

As $R \to \infty$, the last integral vanishes exponentially. Hence (4.48) reduces to

$$\int_{C_\infty} F(y)\,dy = -iA_{ij} \mathscr{F}\left[\tfrac{1}{2} m_{ij}\beta, e^{i\phi_{ij}} x\right]. \tag{4.49}$$

Consider now the case $-\frac{1}{2}\pi < \phi_{ij} < 0$. This time C_R is in the upper half-plane. Since $F(y)$ has a pole for $y = x + i\varepsilon$, we have

$$\int_0^\infty F(y)\,dy + \int_{\gamma_R} F(y)\,dy - \int_{C_R} F(y)\,dy = \vartheta(x) \mathrm{Res}_{x+i\varepsilon} F(y).$$

Taking $R \to \infty$ we get

$$\int_0^\infty F(y)\,dy = \int_{C_\infty} F(y)\,dy + \vartheta(x) A_{ij} \exp\big\{ -\tfrac{1}{2} m_{ij}\beta[e^{i\phi_{ij}}x + e^{-i\phi_{ij}}x^{-1}]\big\}\,.$$

The integral in the rhs is given by (4.49); but in this second case there is also a contribution from the residue. Notice that this term is present only if the angle ϕ_{ij} belongs to the IV quadrant.

A typical integral appearing in the next order is

$$\frac{1}{2\pi i} A_{ik}A_{kj} \int_0^\infty \frac{dy}{y-(x+i\varepsilon)} \mathscr{F}\big[\tfrac{1}{2} m_{ik}\beta, e^{i\phi_{ik}}y\big] \times \exp\big[-\tfrac{1}{2} m_{kj}\beta(ye^{i\phi_{kj}} + y^{-1}e^{-i\phi_{kj}})\big]\,, \tag{4.50}$$

where $-\frac{1}{2}\pi < \phi_{ik}, \phi_{kj} < \frac{1}{2}\pi$. Again, the idea is to deform the integration contour to the ray $y = e^{-i\phi_{kj}}t$. When deforming the contour we can cross two kind of singularities, i.e. the pole at $y = x + i\varepsilon$ and the cut of the function $\mathscr{F}[z,\zeta]$ for $\zeta = x + i\varepsilon, x$ real positive (i.e. on the ray $y = e^{-i\phi_{ik}}t$). There are four distinct cases

$$\begin{aligned}
&\text{case 1} \quad 0 \le \phi_{kj} < \tfrac{1}{2}\pi \quad \text{and} \quad \phi_{kj} < \phi_{ik} < 2\pi\,,\\
&\text{case 2} \quad 0 \le \phi_{kj} < \tfrac{1}{2}\pi \quad \text{and} \quad 0 \le \phi_{ik} \le \phi_{kj}\,,\\
&\text{case 3} \quad -\tfrac{1}{2}\pi < \phi_{kj} < 0 \quad \text{and} \quad 0 < \phi_{ik} < \phi_{kj}\,,\\
&\text{case 4} \quad -\tfrac{1}{2}\pi < \phi_{kj} < 0 \quad \text{and} \quad \phi_{kj} \le \phi_{ik} \le 0\,.
\end{aligned} \tag{4.51}$$

In case 1 we encounter no singularity when deforming the contour from the real positive semi-axis to the ray $y = se^{-i\phi_{kj}}$, $0 \le s \le \infty$. Instead in case 2 deforming the contour to the ray $y = e^{-i\phi_{kj}}s$ we encounter a cut along the ray $y = e^{-i\phi_{ik}}t$. Using the discontinuity (4.39), we find

$$\begin{aligned}
&\frac{1}{2\pi i} \int_0^\infty \frac{dy}{y-(x+i\varepsilon)} \mathscr{F}\big[\tfrac{1}{2} m_{ik}\beta, e^{i\phi_{ik}}y\big] \exp\big[-\tfrac{1}{2} m_{kj}\beta(ye^{i\phi_{kj}} + y^{-1}e^{-i\phi_{kj}})\big]\\
&= -i\mathscr{F}^{(2)}\big[\tfrac{1}{2} m_{kj}\beta, \tfrac{1}{2} m_{ik}\beta, e^{i\phi_{kj}}x, \phi_{ik} - \phi_{kj}\big] + \int_0^\infty \frac{dt}{2\pi(t - e^{i\phi_{ik}}x)}\\
&\quad \times e^{-m_{ik}\beta(t+t^{-1})/2} \exp\big[-\tfrac{1}{2} m_{kj}\beta(e^{i(\phi_{kj}-\phi_{ik})}t + e^{i(\phi_{ik}-\phi_{kj})}t^{-1}\big]\,.
\end{aligned}$$

Consider the integral

$$\frac{1}{2\pi} \int_0^\infty \frac{dt}{t - e^{i\phi_{ik}}x} e^{-m_{ik}\beta(t+t^{-1})/2} \times \exp\big[-\tfrac{1}{2} m_{kj}\beta(e^{i(\phi_{kj}-\phi_{ik})}t + e^{i(\phi_{ik}-\phi_{kj})}t^{-1})\big]\,. \tag{4.52}$$

Using the identity

$$\begin{aligned}
&m_{ik}e^{i\phi_{ik}} + m_{kj}e^{i\phi_{kj}} = m_{ij}e^{i\phi_{ij}}\,,\\
&0 < \phi_{ik} < \phi_{ij} < \phi_{kj} < \tfrac{1}{2}\pi \quad \text{(case 2)},
\end{aligned} \tag{4.53}$$

(4.52) becomes

$$\frac{1}{2\pi}\int_0^\infty \frac{dt}{t-e^{i\phi_{ik}}x}\exp\left\{-\tfrac{1}{2}m_{ij}\beta[e^{i(\phi_{ij}-\phi_{ik})}t+e^{i(\phi_{ik}-\phi_{ij})}t^{-1}]\right\},$$

i.e. the typical first order. Again we deform the integration contour to the ray $t=e^{-i(\phi_{ij}-\phi_{ik})}s$. From (4.53) we see that we encounter no singularity in this process. Hence (4.52) is equal to

$$\mathscr{F}\left[\tfrac{1}{2}m_{ij}\beta, e^{i\phi_{ij}}x\right],$$

i.e. (4.49), but for the third side of the triangle (w_i, w_k, w_j). Then, in case 2, (4.50) is

$$\begin{aligned}&-iA_{ik}A_{kj}\mathscr{F}^{(2)}\left[\tfrac{1}{2}m_{kj}\beta, \tfrac{1}{2}m_{ik}\beta, e^{i\phi_{kj}}x, \phi_{ik}-\phi_{kj}\right]\\ &+A_{ik}A_{kj}\mathscr{F}\left[\tfrac{1}{2}m_{ij}\beta, e^{i\phi_{ij}}x\right].\end{aligned}\tag{4.54}$$

As $\beta\to\infty$, the first term in the rhs is of order (4.46), whereas the second one is of order

$$\exp[-2m_{ij}\beta]\gg\exp[-2(m_{ik}+m_{kj})\beta],$$

unless the three points w_i, w_j and w_k are aligned, in which case the two sides are roughly of the same order. If no three points are aligned in W-space, the one-soliton contributions are unambiguously determined to be the coefficient of $\mathscr{F}[m_{ij}\beta/2]$; the second term in (4.54) is an explicit example of an $O(A^2)$ contribution to μ_{ij}. In case 3 (resp. 4) we get the same result as in case 1 (resp. 2) except that now when deforming the controur we pick up also a contribution from the residue at $y=x+i\varepsilon$.

Large β Asymptotics of Φ. We use the following short-hand

$$\begin{aligned}\mathscr{F}_{ij}(x)&=\mathscr{F}\left[\tfrac{1}{2}m_{ij}\beta, e^{i\phi_{ij}}x\right],\\ \mathscr{E}_{ij}(x)&=\exp\left[-\tfrac{1}{2}m_{ij}\beta(e^{i\phi_{ij}}x+e^{-i\phi_{ij}}x^{-1})\right].\end{aligned}$$

Notice the identities

$$\begin{aligned}&\mathscr{E}_{ik}(x)\mathscr{E}_{kj}(x)=\mathscr{E}_{ij}(x)\quad \text{not summed over } k,\\ &\mathscr{F}_{ij}(-x)=\mathscr{F}_{ji}(x),\qquad \mathscr{E}_{ij}(-x)=\mathscr{E}_{ji}(x).\end{aligned}\tag{4.55}$$

Moreover, (t real positive)

$$\mathscr{F}_{ij}(e^{-i\phi_{ij}+i\varepsilon}t)-\mathscr{F}_{ij}(e^{-i\phi_{ij}-i\varepsilon}t)=i\mathscr{E}_{ij}(e^{-i\phi_{ij}}t).\tag{4.56}$$

The previous discussion shows that just above the real positive (resp. negative) axis $\Phi(x)$ has the following IR expansion:

$$\begin{aligned}\Phi_{i\bar{j}}(x)&=\delta_{ij}-i\mu_{ij}\mathscr{F}_{ij}(x)+B_{ij}\mathscr{E}_{ij}(x)-i\sum_k D^k_{ij}\mathscr{F}_{ij}(x)\mathscr{E}_{kj}(x)\\ &\qquad+\text{terms containing higher }\mathscr{F}\text{'s}\quad (x>0)\\ &=\delta_{ij}-i\mu_{ij}\mathscr{F}_{ij}(x)+\tilde{B}_{ij}\mathscr{E}_{ij}(x)-i\sum_k \tilde{D}^k_{ij}\mathscr{F}_{ij}(x)\mathscr{E}_{kk}(x)\\ &\qquad+\text{terms containing higher }\mathscr{F}\text{'s}\quad (x<0).\end{aligned}\tag{4.57}$$

The various coefficients in this expansion are polynomials in the Stokes parameters A_{ij}. As long as no three points w_j are aligned, the omitted terms are subleading in the

IR limit. Notice that B_{ij} (resp. $\tilde{B}_{ij}$) can be non-vanishing only if $\cos(\phi_{ij}) > 0$ (resp. < 0). A similar condition holds for D^k_{ij}, $\tilde{D}^k_{ij}$. Then the third and fourth terms in the rhs of (4.57) vanish exponentially as $x \to 0$, and hence by (4.20) do not contribute to g. The coefficient of $\mathscr{F}_{ij}$ is fixed by the asymptotics (4.6), (4.40) to be $-i\mu_{ij}$.

Equation (4.19) gives strong restrictions on the various coefficients in (4.57). Indeed, (for x real positive) one has

$$\begin{aligned}\delta_{ij} &= (\Phi(x+i\varepsilon)\Phi^t(-x-i\varepsilon))_{ij}\\ &= \sum_{k,l} \Phi_{ik}(x+i\varepsilon)\,(\delta_{kl} - A_{kl}\mathscr{E}_{kl}(x))\,\Phi_{jl}(-x+i\varepsilon)\,,\end{aligned}$$

which, in view of (4.55), gives

$$\mu^t + \mu = 0\,, \tag{4.58}$$

$$D^k_{ij} = \mu_{ik}B_{kj}\,, \qquad \tilde{D}^k_{ij} = \mu_{ik}\tilde{B}_{kj}\,, \tag{4.59}$$

$$(1+\tilde{B}^t)(1+B) = (1-A)^{-1} \equiv S^{-1}\,. \tag{4.60}$$

This last relation allows us to read the Stokes parameters directly from the IR expansion of Φ near the real axis.

For simple situations, the relation between μ_{ij} and A_{ij} can be obtained by inserting this truncated expansion for Φ in the integral equation (4.17). However this does not work in general, since – because of the singularity of $\mathscr{K}$ – the integrals of terms ignored in (4.57) may contribute to the coefficients μ and B. A better approach is presented below.

4.5. Multi-Sector Formulation

The rays $te^{-i\phi_{ij}}$ (t real positive) divide the plane into $n(n-1)$ sectors [24]. We number these sectors according to the anti-clockwise order starting from the one containing the real positive axis which is called sector 1. The ray separating the α^{th} sector from the $(\alpha+1)^{\text{th}}$ one is called the α^{th} ray. To each α there is associated an angle $-\phi_{ij}$. The corresponding indices will be denoted by $i(\alpha), j(\alpha)$, respectively. The sector containing the negative real axis is the $(m+1)^{\text{th}}$ one, where $m = \frac{1}{2}n(n-1)$. If $M = (M_{kl})$ is a $n \times n$ matrix, we denote by $M^{[\alpha]}$ the matrix

$$(M^{[\alpha]})_{kl} = \delta_{ki(\alpha)}\delta_{lj(\alpha)}M_{i(\alpha)j(\alpha)}\,.$$

The analysis of Sect. 4.4 with the Stokes axis rotated by suitable angles in the x-plane shows that in (some angular neighborhood of) the α^{th} sector the function $\Phi(x)_{i\bar{j}}$ has an IR expansion of the form

$$\begin{aligned}\Phi^{(\alpha)}_{i\bar{j}}(x) &= \delta_{ij} - i\mu_{ij}\mathscr{F}_{ij}(x) + B^{(\alpha)}_{ij}\mathscr{E}_{ij}(x)\\ &\quad - i\sum_k D^{(\alpha),k}_{ij}\mathscr{F}_{ik}(x)\mathscr{E}_{kj} + \text{higher } \mathscr{F}\text{'s}\,.\end{aligned} \tag{4.61}$$

[24] To simplify the discussion we assume that the angles ϕ_{ij} are all distinct. Notice that the relevant rays are not the soliton lines but their mirror images with respect to the real axis. Indeed $\Psi(x)$ is the momentum space wave function and the above condition corresponds to alignment in position space

Comparing with (4.57), we have

$$B^{(1)}_{ij} = B_{ij}\,, \qquad B^{(m+1)}_{ij} = \tilde{B}_{ij}\,. \tag{4.62}$$

As before, $B^{(\alpha)}_{ij}$ may be non-vanishing only if

$$\mathrm{Re}(e^{i\phi_{ij}}x) > 0 \quad \text{for } x \text{ in the } \alpha^{\text{th}} \text{ sector}. \tag{4.63}$$

The crucial point is the identity

$$D^{(\alpha),k}_{ij} = \mu_{ik} B^{(\alpha)}_{kj} \tag{4.64}$$

(cf. (4.59)). This can be seen as before. Let $\tilde{\alpha}$ be the sector opposite to α (i.e. x belongs to the $\tilde{\alpha}^{\text{th}}$ sector if $-x$ belongs to the α^{th} one). Then inserting (4.61) into the identity

$$\Phi^{(\alpha)}(x)\,[\Phi^{(\tilde{\alpha})}(-x)]^t = 1\,, \tag{4.65}$$

we get

$$B^{(\alpha)} = -[(1 + B^{(\tilde{\alpha})})^{-1} B^{(\tilde{\alpha})}]^t\,, \tag{4.66}$$

$$[D^{(\alpha),k}(1 + B^{(\tilde{\alpha})})^t]_{ij} = -\,\mu_{ik} B^{(\tilde{\alpha})}_{jk}\,. \tag{4.67}$$

Plugging (4.66) into (4.67) yields (4.64).

The function $\Phi_{i\bar{j}}(x)$ is globally defined in the upper half-plane. Then $\Phi^{(\alpha+1)}$ and Φ^{α} should agree on the α^{th} ray. On the other hand, the single terms in (4.61) are discontinuous as we cross the α-ray because of (4.56). Then the continuity of the sum gives relations between the coefficients in (4.61). Notice that (assuming no three vacua get aligned) the discontinuity of terms omitted in (4.61) cannot contribute to $B^{(\alpha+1)}$. On the contrary, they do contribute to $D^{(\alpha+1)}$. Luckily there is no need to control these terms: Their net effect is just to produce the right discontinuity so that Eq. (4.64) remains true as we cross the α-ray. Equation (4.56) yields

$$\mathscr{F}_{ij}(x)|_{\alpha+1} - \mathscr{F}_{ij}(x)|_{\alpha} = -\,i\delta_{ii(\alpha)}\delta_{jj(\alpha)}\mathscr{E}_{ij}(x)\,.$$

Then, comparing the coefficients of $\mathscr{E}_{ij}(x)$ in $\Phi^{(\alpha+1)}$ and $\Phi^{(\alpha)}$, with the help of (4.55) we get

$$B^{(\alpha+1)} = -\,\mu^{[\alpha]} + \beta^{(\alpha)} - \mu^{[\alpha]} B^{(\alpha)}\,,$$

or

$$(1 + B^{(\alpha+1)}) = (1 - \mu^{[\alpha]})\,(a + B^{(\alpha)})\,.$$

In view of (4.62) this implies

$$(1 + \tilde{B}) = (1 + B^{(m+1)}) = \overrightarrow{\prod_{1 \le \alpha \le m}} (1 - \mu^{[\alpha]})\,(1 + B)\,,$$

where the overarrow means that the product is taken in the anti-clockwise order.

Finally, let

$$L := (1 + \tilde{B})\,(1 + B)^{-1} = \overrightarrow{\prod_{1 \le \alpha \le m}} (1 - \mu^{[\alpha]})\,. \tag{4.68}$$

Using (4.60), the monodromy reads

$$\begin{aligned} S(S^t)^{-1} &= (1 + B)^{-1}(1 + \tilde{B}^t)^{-1}(1 + B^t)\,(1 + B) \\ &= (1 + B)^{-1}(L^t)^{-1} L(1 + B)\,, \end{aligned}$$

i.e. (up to a unimodular change of bases) the monodromy is given by $(L^t)^{-1}L$, or explicitly [25, 26]

$$\prod_{1\leq\alpha\leq 2m}^{\longrightarrow} (1-\mu^{[\alpha]})\,. \tag{4.69}$$

Equation (4.69), together with (4.23), is our relation between the soliton numbers and the UV $U(1)$ charges.

Let us consider (4.68) in more detail. We know that B_{ij} (resp. $\tilde{B}_{ij}$) can be non-vanishing only if $\cos(\phi_{ij}) > 0$ (resp. < 0). In view of this remark, Lazzari's lemma [28] aplied to (4.68) gives

$$(1+B)^{-1} = \prod_{\text{I quadrant}}^{\longrightarrow} (1-\mu^{[\alpha]})\,,$$

$$(1+\tilde{B}) = \prod_{\text{II quadrant}}^{\longrightarrow} (1-\mu^{[\alpha]})\,,$$

where the ordered products are on the α's whose corresponding angles $-\phi_{ih}$ belong to the first (resp. second) quadrant. Then

$$(1+\tilde{B}^t)^{-1} = \prod_{\text{IV quadrant}}^{\longrightarrow} (1-\mu^{[\alpha]})\,.$$

Finally, from (4.60) we have

$$S = (1+B)^{-1}(1+\tilde{B})^{-1} = \prod_{\text{right half-plane}}^{\longrightarrow} (1-\mu^{[\alpha]})\,. \tag{4.70}$$

This shows that the formula we derived for the relation between S and soliton numbers in the context of LG theories in Sect. 2 is generally valid for any massive $N=2$ quantum field theory.

5. More on Degenerate UV Critical Theories

5.1. The "Strong" Monodromy Theorem

In this section we wish to study in slightly more detail the critical theories one gets as the ultra-violet limit of a given massive $N=2$ model. In general one may get a degenerate superconformal theory, i.e. a model with a continuous spectrum of dimensions. For instance, in the $\mathbf{C}P^n$ case the UV limit corresponds to free field theory and this limit is reached up to logarithmic deviations. Typically a degenerate limit looks like a σ-model with a *non-compact* target space. In this case L_0 has a

[25] This equation has been obtained under the condition that all ϕ_{ij} are distinct. However it is valid even if this condition does not hold (provided no three vacua are aligned). Indeed if, say, $\phi_{ij} = \phi_{kl}$ then there is an α for which the α^{th} and $(\alpha+1)^{\text{th}}$ rays coincide. Then the order of the corresponding matrices $(1-\mu^{[\alpha]})$ and $(1-\mu^{[\alpha+1]})$ is ambiguous. But (since the four points w_i, w_j, w_k, w_l are all distinct) these two matrices commute and hence the order ambiguity is totally immaterial

[26] We can reinterpret this equation in terms of the original soliton lines $te^{i\phi_{ij}}$. We have just to take the product in the clockwise order

continuous spectrum and hence the states of definite dimension are not normalizable. In particular $|1\rangle$ is not a normalizable state, as it is obvious from classical geometry (harmonic forms in non-compact manifolds are usually non-normalizable).

One of the purposes of this section is to characterize the massive theories having "nice" UV limits. If a model has a nice UV limit, we can find a basis $\mathscr{O}_i$ of $\mathscr{R}$ such that as $\beta \to 0$,

$$\frac{\langle \bar{\mathscr{O}_i} \mid \mathscr{O}_i \rangle}{\langle \bar{1} \mid 1 \rangle} \cong C_i \beta^{-q_i}(1 + O(\beta^a)), \qquad a > 0,$$

for some constants C_i. Equivalently,

$$Q_{ij}(\beta) = q_{ij} + O(\beta^a).$$

This is just the statement that the UV theory has a positive gap a in the spectrum of dimensions. In particular, this implies the normaizability of $|1\rangle$

$$\langle \bar{1} \mid 1 \rangle < \infty,$$

where in the lhs we mean the state obtained by spectral-flow of 1 in a special field representation [27]. This no-degeneracy criterion fails, say, for the $\mathbf{C}P^1$ σ-model, where [25]

$$\langle \bar{1} \mid 1 \rangle \cong -4 \log \beta, \qquad \beta \to 0.$$

The first remark is that the UV limit cannot be non-degenerate if the monodromy $H = SS^{-t}$ has non-trivial Jordan blocks. This was shown in Sect. 4.3, see Eq. (4.29). Then we have the natural question: *is the triviality of the Jordan structure of H enough to ensure the non-degeneracy of the* UV *limit* (assuming that the original massive theory is regular)?

To begin with, let us consider the Landau-Ginzburg models with a polynomial superpotential. In this case the UV limit is "nice" if and only if in the limit the superpotential $W(X_i)$ becomes a quasi-homogeneous function. Indeed, this is precisely the condition needed in order for $W(X_i)$ to be $U(1)$-invariant. If $W(X_i)$ is quasi-homogeneous, i.e. if there are rational numbers q_i such that

$$W(\lambda^{q_i} X_i) = \lambda W(X_i) \quad \forall \lambda \in \mathbf{C},$$

then

$$W(X_j) = \sum_j q_i X_i \partial_i W(X_j) \cong 0 \quad \text{in} \quad \mathscr{R}. \tag{5.1}$$

Conversely, let $W(X_i)_{\mathrm{uv}}$ be the superpotential in the UV limit, and C_{uv} the matrix representing multiplication by the chiral operator $W(X_i)_{\mathrm{uv}}$ in $\mathscr{R}$. We claim that we can choose the additive constant in $W(X_i)$ so that all the eigenvalues of C_{uv} vanishes. Indeed, for all $\beta \neq 0$, multiplication by the superpotential is represented by the matrix βC and hence [28]

$$\det[z - C_{\mathrm{uv}}] = \lim_{\beta \to 0} \det[z - \beta C] = z^n.$$

Then C_{uv} is nilpotent and therefore it is fully determined by the dimensions of its Jordan blocks. The UV limit superpotential is quasi-homogeneous if and only if these

[27] I.e. in a basis such that η_{ij} is constant

[28] Notice that this argument does not imply that C_{uv} is the matrix 0. Indeed, the transformation relating the canonical basis of $\mathscr{R}$ to the operator one becomes singular as $\beta \to 0$. Otherwise, the ring $\mathscr{R}$ itself would trivialize at the UV fixed point, which is obviously not the case. The characteristics polynomial is invariant under changes of bases and hence we are allowed to take its limit as $\beta \to 0$

blocks are all trivial. For $W(X_j)$ a polynomial, this is an easy consequence of (5.1). So in the LG case the UV limit is "nice" iff $C_{\text{uv}} = 0$.

The "strong" monodromy theorem of Singularity Theory (first proven by Varčenko [29]) states that the Jordan structures of C_{uv} and of the Milnor monodromy H are equal. Then for LG models (with polynomial superpotentials) the answer to our question is *yes*.

Now let us go to the general case. By analogy with the LG case, to answer *yes* we have to show that: 1. the "strong" monodromy theorem holds in general, and 2. that the UV limit is nice iff $C_{\text{uv}} = 0$. We have already shown 1. Indeed from (4.27) we see that

$$C_{\text{uv}} = B|_{|x|=1},$$

and so the "strong" monodromy theorem is equivalent to the remark just after Eq. (4.29). Instead 2 is a well-known consequence of the tt* equations. In fact, these equations imply (here the matrix C' is βC rewritten in the operator basis)

$$\bar{\partial}_i Q = [C', \bar{C}_i],$$

we see that Q has a constant limit if and only if the UV limit of the rhs vanishes for all i's, i.e.

$$[C_{\text{uv}}, g_{\text{uv}} C_i^\dagger g_{\text{uv}}^{-1}] = 0 \quad \forall i. \tag{5.2}$$

Assume that the UV limit is a non-degenerate conformal theory. Then the metric g_{uv} is a non-singular positive-definite inner product on $\mathscr{R}$. In this case (5.2) implies [29] $C_{\text{uc}} = 0$.

The fact that a "strong" version of the monodromy theorem holds allows us to borrow manay results from Algebraic Geometry which are consequences of this theorem. Some of these results were developed in the context of the degeneration theory for complex structures over algebraic manifolds and eventually evolved in Deligne's theory of *mixed Hodge structures* [30]. Physically they are related to "mirror symmetry." It is not appropriate to discuss further these developments here, so we limit ourselves to the simplest result in this direction (Schmid's orbit theorems [31]) that we need below.

The basic idea is that from the Jordan structure of H we can cook up an $SU(2)$ action on $\mathscr{R}$. In fact, given a nilpotent matrix L acting on a vector space $\mathscr{V}$, we can always find (by Jacobson-Morosov) an $sl(2)$ representation on $\mathscr{V}$ such that the generator J_+ is mapped into L. Applying this remark to the nilpotent matrix B (acting on $\mathscr{R}$), we see that we can use $SU(2)$ representation theory to "measure" the degeneration of the UV critical theory. The bigger the "angular momentum" the more degenerate the UV limit is. In particular the theory is non-degenerate if and only if the corresponding $SU(2)$ representation is trivial. More generally, we get logarithmic corrections of the form $(\log \beta)^k$, where $k/2$ is the larger "spin" appearing in the above $SU(2)$ representation. We illustrate the physical applications of this viewpoint in the special case of σ-models.

5.2. AF σ-Models

We consider an AF σ-model with action

$$S = \sum_{(1,1)\ \text{classes}} t_a \int \omega_1^{(2)} + D\text{-term}.$$

[29] Indeed a nilpotent matrix which commutes with its own adjoint, vanishes

Asymptotic freedom requires the Ricci tensor $R_{i\bar{j}}$ to be (strictly) positive-definite. The corresponding $(1,1)$ form R can be decomposed as

$$R = \sum_{(1,1)\ \text{classes}} s_q \omega_a^{(2)} .$$

By definition, the matrix βC represents in $\mathscr{R}$ multiplication by the operator $\hat{\phi}$ such that

$$\delta_{\text{RG}} S = \int d^2 z d^2 \theta \hat{\phi} + \text{h.c} + D\text{-terms} ,$$

(here δ_{RG} is the infinitesimal Renormalization Group flow). In the present case $\hat{\phi}$ is just the chiral field associated to the Ricci form, and thus βC is the matrix representing multiplication by the Ricci class in the *quantum* cohomology ring.

As $\beta \to 0$, the target space metric $G_{i\bar{j}}$ flows towards one cohomologous to a Kähler-Einstein metric of infinite volume.[30] Moreover, this is the weak coupling (= semiclassical) limit. In this limit the chiral ring reduces to the classical cohomology ring.

Hence B is proportional to the matrix representing multiplication by the (asymptotic) Kähler class in the (classical) cohomology ring. For instance, in the $\mathbf{C}P^n$ case B is given by Eq. (4.28). From that equation it is obvious that B represents the multiplication by the Kähler class in the cohomology ring.

Then for an AF σ-models having as target space a (compact Kähler) manifold $\mathscr{M}$ of complex dimension d, the Jordan structure of H is completely specified in terms of the geometry of $\mathscr{M}$. Indeed the set of all harmonic forms on $\mathscr{M}$ can be decomposed into irreducible representations of $SU(2)$ (*Lefschetz decomposition* [33]). Comparing the hard Lefschetz theorem with the our construction above, we see that the Lefschetz $SU(2)$ coincides with the one measuring the degeneracy of the UV theory. Let $\{s_j\}$ be the set of "spins" appearing in the Lefschetz decomposition (counted with multiplicity). Then the length $(k_j + 1)$ of the j^{th} Jordan block is equal to $(2s_j + 1)$. In particular in the (AF) σ-model case H has *one and only one* Jordan block of length $d+1$, and no Jordan block has length $l > d+1$. Moreover for all blocks $k_j \equiv d \bmod 2$. These geometrical facts are easily recovered from the general classification of $N = 2$ superconformal models discussed in the present paper.

This example also "explains" in which sense the Jordan structure measures the failure of the UV fixed theory to be a nice superconformal theory. The Ricci tensor is the β-function of the model, and its topological class (i.e. the first Chern class) measures the obstruction to find a fixed point, i.e. a point where the β-function really vanishes. But B encodes exactly this topological information.

From the above formulae one can also extract the leading UV behaviour for the ground-state metric g. Again we illustrate this in the $\mathbf{C}P^{n-1}$ case. Since $B^n = 0$, we have

$$\Phi \sim \sum_{r=0}^{n-1} \frac{1}{r!} (\log \beta)^r B^r \Phi_0 .$$

Let X be the chiral primary operator dual to the hyperplane section. In the UV limit it acts on the ring as the matrix $c^{-1} B$, for some normalization coefficient[31] c. Then

[30] Indeed, by [32] we have $G_{i\bar{j}}(\beta) \cong G_{i\bar{j}}(1) - R_{i\bar{j}} \log \beta$, where $\cong$ mean equality in cohomology

[31] One has $c^{-1} = 2n$. This can be seen as follows. In our conventions, C represents on $\mathscr{R}$ the chiral field $2\mu\partial_\mu \omega$, where ω is the Kähler class and μ is the RG scale. For $\mathbf{C}P^{n-1}$ we have $\omega = -\log(\alpha/\mu^n) X$, and thus $C = 2nX$

as $\beta \to 0$,

$$X^k \Phi = c^{-k} B^k \Phi \sim \frac{1}{(n-1-k)!} (c \log \beta)^{(n-1-k)} \frac{1}{c^{n-1}} B^{n-1} \Phi_0 .$$

On the other hand, by definition [32]

$$X^k \Phi = \overline{X^{n-1-k} \Phi} \langle \bar{X}^k \mid X^k \rangle ,$$

and thus

$$\langle \bar{X}^k \mid X^k \rangle \sim \frac{k!}{(n-1-k)!} (-|c| \log \beta)^{(n-1-2k)} , \tag{5.3}$$

in agreement with [25]. Of course, this is just the result predicted by classical geometry [34].

6. The Classification Program

We have seen in the previous sections that the number of vacua and the number of solitons between them is enough to give the full solution to tt* equations. This means that the geometry of ground states of the supersymmetric theory are completely determined by the IR data which is the counting of the soliton numbers. Note that the geometry of the ground state is sensitive only to F-term perturbations and are insensitive to D-terms. Therefore two theories which differ only by a variation of the D-term will have the same ground state geometry and soliton numbers. However as we have seen the soliton numbers fully capture the F-term perturbations of the theory. As an example, if we consider CP^1 σ-model, the Kähler class of the metric is the information contained in the F-term, whereas the precise form of the Kähler metric is determined by the D-term. In particular there are infinitely many ways to vary the D-term which is equivalent to the space of all Kähler metrics with a fixed Kähler class. So what we will be able to do is therefore to begin classifying massive $N = 2$ quantum field theories up to variation of D-terms. Indeed this turns out to be *equivalent* to classifying all $N = 2$ CFT's which admit a massive deformation. The reason is that the condition of conformal invariance automatically picks a D-term for a given F-term. This can be proven rigorously in the SCFT by noting that the only supersymmetric perturbations which preserve conformal invariance is via chiral fields, which are F-term, i.e., there is no continuous variation of the D-term which preserves conformal invariance. So the UV limit of any of the theories we consider will automatically label a *conformal* theory, the D-term of which is adjusted to make the theory conformal! In this way we get a mapping between soliton numbers and $N = 2$ superconformal models. As we discussed before this will not give all superconformal models, but only those which admit a non-degenerate massive deformation, a precondition of which is that the left (q_L) and right (q_R) charges of chiral fields be equal (i.e., chiral fields have zero fermion number). It is not clear that all conformal theories satisfying $q_L = q_R$ automatically admit a massive deformation but we know of no counterexample to such an expectation. Assuming this is generally true, our method thus classifies all the $N = 2$ CFT's with left-right symmetric $U(1)$ charges for Ramond ground states.

Note that a particularly interesting class of conformal theories for constructing string vacua, i.e., Calabi-Yau case, admit no massive deformation (there are no

[32] In writing this equation we used the fact that q commutes with g for $\beta \sim 0$

relevant operators). However we know that one can obtain examples of Calabi-Yau by considering orbifolds of LG models. The same is true for the left-right symmetric theories under consideration here [33] from which we can obtain Calabi-Yau manifolds by taking orbifolds (it is an interesting question to see if for every Calabi-Yau there exists a point on moduli space which is related to a symmetric theory by orbifoldizing).

We may be interested in classifying non-degenerate (or "compact") $N = 2$ CFT's, in which case we have to impose the condition that $H = SS^{-t}$ has a trivial Jordan block structure, as discussed before. As an example the UV limit of CP^1 is R^2 which is degenerate (in the sense that it has a continuous spectrum). In this section we consider both degenerate and non-degenerate theories.

We may also be interested in uncovering the allowed perturbations of our theories which send some vacua to infinity. This would for example be interesting in understanding the RG-flows among the theories. From the classification program all the perturbations which send some vacua to infinity are allowed as long as we end up with *real* $U(1)$ charges for the theory with fewer vacua. In other words if the reduced S matrix gives rise to real $U(1)$ charges then it presumably is an allowed perturbation of the theory.

To begin with classifying the theories, we first fix the Witten index of the theory to be n. Then we take an arbitrary strictly upper-triangular integral $n \times n$ matrix A which is taken to count the soliton numbers (taking into account $(-1)^F$) between these vacua (where we assume the vacua to be in a "standard" configuration in the W-plane) [34]. For a general triangular matrix A, the eigenvalues of $H \equiv (1 - A)(1 - A^t)^{-1}$ need not have norm 1. However in the physical case they should, since q is Hermitian. This gives a severe restriction on the entries of physically allowed Stokes parameters [35] A which count soliton numbers. Thus $H \in SL(n, \mathbf{Z})$ is a *modular* matrix. From Lazzari's lemma [28], A can be recovered uniquely from H.

Then the classification of $N = 2$ superconformal models having a totally massive perturbation is reduced to the following Diophantine problem [36]. Find all *integral* strictly upper-triangular $n \times n$ matrices A such that all the eigenvalues λ_i of the modular matrix

$$H = SS^{-t} = (1 - A)(1 - A^t)^{-1}$$

belong to the unit circle $|\lambda_i| = 1$. Two solutions A and A' are "equivalent" if they are related by a braiding transformation and a change of sign in the canonical basis discussed in Sect. 2. The very same number-theoretical problem arises in Algebraic-Geometry [35] and Singularity Theory [5]. Unitarity gives further restrictions on the physically allowed solutions. In particular in any (irreducible) unitary theory we have only one chiral primary with vanishing charge, i.e. 1. Then the smallest value of q should be non-degenerate.

Here we discuss some general facts about this classification program. In the following subsections we apply these methods to obtain the complete classification for the case of small Witten indices $n < 4$. In the next section we rederive the ADE

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33 The reader should be careful to distinguish the usage of "left-right symmetric $N = 2$ theories" here from that used in the context of RCFT's

34 As usual, we order the vacua such that $\mathrm{Re}(w_i - w_j) > 0$ for $i < j$

35 It is conceivable that the condition that eigenvalue be a phase is already implied by the regularity of the solution for all β's. Though the *integrality* of the matrix A is not guaranteed by regularity alone, as there are counterexamples, and should be viewed as an additional physical constraint

36 Strictly speaking, those discussed in the text are only *necessary* conditions. However, experience suggests that these conditions are very close to being also sufficient

classification of minimal models using our methods. It turns out to be extremely simple to obtain this classification with these methods.

Standard number-theoretical argument (based on Kronecher's theorem [36]) shows that H is *quasi-idempotent*, i.e. there exist integers m and k such that

$$(H^m \pm 1)^{k+1} = 0 \quad \text{weak monodromy theorem,} \tag{6.1}$$

(here m, k are assumed to be the smallest integers for which (6.1) is true; k is known as the *index* of H). In particular the q_k's are *rational numbers*. This is our first general conclusion. In the geometrical case one has also a strong form of the monodromy theorem stating that the index k is always less or equal to the (complex) dimension d. In the physical context d should be replaced by the UV central charge $\hat{c}$. The known $N = 2$ theories satisfy this stronger statement (in particular the σ-models, as we saw in Sect. 5). It is tempting to conjecture that the strong form of the monodromy theory is always true. Indeed, in the LG case, the theorem is a simple consequence of the "strong" monodromy theorem we discussed in Sect. 5.2. Since this "strong" theorem holds in full generality, it is reasonable to expect that also the bound $k \leq \hat{c}$ is always valid. Here we limit ourselves to a sketch of the proof for the general case, under the additional assumption that in the (degenerate) UV critical theory the only primary chiral field with vanishing charge is the identity operator 1. In this case, all nilpotent chiral operators are linear combinations of fields of *positive* charge. This remark, in particular, applies to the field $\hat{\phi}$ corresponding to the matrix B. Consider then the subset of operators $\phi'_1(1 = 1, \ldots, k+1)$ belonging to a Jordan block of H of maximal index k, and let q'_i be their $U(1)$ charges. By definition, $q'_i = q'_j$ *modulo one*. On the other hand, the arguments of Sect. 5 imply

$$\phi'_r \approx \hat{\phi}^{r-1}\phi'_1 .$$

Since $\hat{\phi}$ has positive charge, we have $q'_r > q'_{r-1}$, which implies $1'_r \geq q'_{r-1} + 1$. Then $q'_r \geq (r-1) + q'_1$, which gives

$$\hat{c} = q_{\max} - q_{\min} \geq q'_{k+1} - q'_1 \geq k , \tag{6.2}$$

which is the strong form of the monodromy theorem.

Consider the characteristic polynomial $P(z) = \det[z - H]$. It satisfies $P(0) = (-1)^n$ and

$$P(z) = (z-)^n P(1/z) .$$

Then A is a solution to (6.1) if and only if all roots of $P(z)$ are m-roots of 1, i.e. if $P(z)$ has the form

$$P(z) = \prod_{d|m} \Phi_d(z)^{k_d} , \tag{6.3}$$

where $\Phi_d(z)$ are the cyclotomic polynomials [38]. Since $\deg P(z) = n$, we get a relation between the Witten index and the possible $U(1)$ charges q_k of a Ramond ground state [37]. Indeed let $q_k = r/2$ with $(r, s) = 1$. Then $\phi(s) \leq n$, where $\phi(s)$ is Euler's totient function. Moreover, if we have n_s Ramond vacua with charge $r/s \bmod 1$ and $(r, s) = 1$, corresponding to a set of Jordan blocks of lengths $(k_{j_s} + 1)$,

[37] Some of the following restrictions can also be derived using integrality of the number of states in twisted sectors of the orbifolds of the corresponding conformal theory

then for all $l \in (\mathbf{Z}/s\mathbf{Z})^{\times} \cong \mathrm{Gal}(\mathbf{Q}(e^{2\pi i/s})/\mathbf{Q})$ there are precisely n_s Ramond vacua with charge

$$q = \frac{l}{s} \bmod 1\,, \tag{6.4}$$

and they are organized in Jordan blocks of the same lengths.[38]

Not all products of cyclotomic polynomial can appear in (6.3). Let

$$P(z) = \prod_{m \in \mathbf{N}} (\Phi_m(z))^{\nu(m)}\,, \tag{6.5}$$

where $\nu(m) \in \mathbf{N}$ are almost all vanishing. Then one has the following constraints on the possible $\nu(m)$'s (physically they are selection rules on the allowed $U(1)$ charges):

1. $\sum_m \nu(m)\,\phi(m) = n$.
2. $\nu(1) = n \bmod 2$.
3. For n even, *either* $\nu(1) > 0$ or $\sum_{k \geq 1} \nu(p^k) = 0 \bmod 2$ for all primes p.

For instance, in degree 2 there are 6 polynomials of the form (6.5). Only three of these satisfy the selection rules, namely Φ_1^2, Φ_2^2, and Φ_6. In the same way, in degree 3 only 5 out of 10 possibilities are allowed, and in degree 4 only 12 out of 24 (e.g. Φ_8, Φ_5 and $\Phi_3\Phi_2^2$ cannot appear).

This is shown as follows: 1 is obtained by equating the degree of both sides of (6.5) (recall that $\deg \Phi_m = \phi(m)$). 2 follows from the fact that $P(0) = \det[-H] = (-1)^n$, whereas $\Phi_m(0) = 1$ for all m's but for $m = 1$, where $\Phi_1(0) = -1$.

To get 3, notice the identity

$$P(1) \equiv \det[1 - H] = \det[S^t - S]\det[S^t] = (\mathrm{pf}[S^t - S])^2\,,$$

where pf[·] is the Pfaffian. Hence

$$\prod_m (\Phi_m(1))^{\nu(m)} = (\mathrm{pf}[S^t - S])^2\,. \tag{6.6}$$

Moreover, one has

$$\Phi_m(1) = \begin{cases} 0 & \text{if } m = 1 \\ p & \text{if } m = p^k,\ p \text{ prime},\ k \geq 1 \\ 1 & \text{otherwise.} \end{cases} \tag{6.7}$$

Now, if n is odd, $\mathrm{pf}[S^t - S] \equiv 0$ in agreement with 2. Instead, if n is even, either $\mathrm{pf}[S^t - S] = 0$ or, by (6.6), $\prod_m (\Phi_m(1))^{\nu(m)}$ is a non-trivial square, and hence its order at each prime should be even. The order at the various primes is easily computed with the help of (6.7). This gives 3.

For small n it is easy to solve the above Diophantine problem thus getting a complete classification. Here we limit ourselves to $n = 1, 2, 3$.

For $n = 1$ there are no solitons, and then we have only the trivial solution to our Diophantine problem. This solution corresponds to the free massive model. This model has an unbroken $U(1)$ symmetry whose charge q counts the number of Bose particles. On the vacuum $Q = q = 0$, as required by PCT.

[38] This follows from the fact that the minimal polynomial of H belongs to $\mathbf{Z}[z]$

6.1. Complete Solution for $n = 2$

For $n = 2$ we have

$$S = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} .$$

Notice that the sign of a is physically irrelevant since it can be flipped by redefining the canonical basis. Then we can assume $a \leq 0$. Moreover, $|a|$ is equal to the number $|\mu|$ of solitons connecting the two vacua. The characteristic polynomial of H reads

$$P(z) = z^2 + (a^2 - 2)z + 1 . \tag{6.8}$$

There are three values of a for which $P(z)$ has the form (6.5). These correspond to the three possibilities allowed by the selection rule. They are

1. $a = 0$. This gives $P(z) = \Phi_1(z)^2$, i.e. $q = 0$. This case corresponds to the trivial model (no solitons at all).
2. $a = -1$ This gives $P(z) = \Phi_6(z)$, i.e.

$$q_i = \left(-\tfrac{1}{6}, \tfrac{1}{6}\right) .$$

The integral part of the charge is fixed here and in what follows by taking $a \to ta$ and letting t go from 0 to 1, as discussed before. This solution corresponds to the Landau-Ginzburg model with superpotential

$$W(X) = X^3 - X .$$

The uniqueness of solution for (4.17) also fixes the integral part of q. Thus $|a| = 1$ implies $q = \pm 1/6$.

3. $a = -2$, which gives $P(z) = \Phi_2(z)^2$. In this case we have

$$q_i = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right) ,$$

and H has a non-trivial Jordan block. Indeed, this is precisely the solution (4.25) we have discussed in detail in Sect. 4.3. From that analysis we see that this model corresponds to the $\mathbf{P}^1$ σ-model (or, equivalenty, the Ising two-point function). Again this also fixes the integral part of the charges.

In cases 2 and 3 the matrix $B = S + S^t$ is the Cartan matrix for A_2 and $\hat{A}_1$, respectively. In fact, the model 3 can also be realized as the $N = 2$ $\hat{A}_1$ Toda theory, i.e. the LG model with superpotential

$$W(X) = \lambda(e^X + e^{-X}) , \tag{6.9}$$

and the identification $X \sim X + 2\pi i$.

The above number-theoretical result should be compared with the known regularity theorems for Painlevé III (PIII) [18]. (For a massive model with two vacua the tt* equations can be always recast in the PIII form, see [10].) In the $n = 2$ case, the eigenvalues of the Q-matrix are [10, 14]

$$Q(z) = \pm \frac{1}{4} z \frac{d}{dz} u(z) , \tag{6.10}$$

where $z = m\beta$ and $u(z)$ satisfies special PIII, i.e. the radial sinh-Gordon equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} = \sinh u . \tag{6.11}$$

As discussed in Sect. 4, the boundary condition for (6.11) is encoded in the Stokes parameter a. In terms of the more usual boundary datum r, defined by the behaviour of $u(z)$ as $z \to 0$,

$$\begin{aligned} u(z) &\cong r \log \frac{z}{2} + s + \dots && \text{for } |r| < 2 \\ &\cong \pm 2 \log \frac{z}{2} \pm \log \left[-\log \left(\frac{z}{4} + \gamma \right) \right] + \dots && \text{for } r = \pm 2 , \end{aligned} \tag{6.12}$$

a is given by [18]

$$a = 2 \sin \left(\frac{\pi r}{4} \right) . \tag{6.13}$$

In view of Eq. (6.10), the datum r is essentially the $U(1)$ charge at the UV fixed point. Indeed,

$$q = \lim_{z \to 0} Q(z) = \pm \tfrac{1}{4} r ,$$

so, in physical terms, (6.13) reads

$$|\mu| = 2 \sin(\pi |q|) . \tag{6.14}$$

In fact this result can be derived directly by continuously turning on the soliton number and considering the eigenvalues of SS^{-t}, which gives a nice illustration of what we mean by continuously deforming the soliton number in order to recapture the integral part of q. It is known [18] that PIII has *one* regular solution $u(z)$ for reach r with $|r| \leq 2$, and that all regular real solutions (bounded as $z \to \infty$) have the UV asymptotics (6.12) for some r, $|r| \leq 2$. In view of (6.13), we have a regular solution for all (real) a with $|a| \leq 2$. Comparing with (6.8), we see that this is just the condition

$$-2 \leq \operatorname{tr} H \leq 2 ,$$

i.e. $u(z)$ is regular iff the eigenvalues of the monodromy have norm 1. So, in this case, *unitarity implies regularity*. In particular, the three possible integral values of $|a|$ do correspond to regular solutions having the expected UV behaviour. Thus for $n = 2$ all solutions to the Diophantine problem are realized by physical systems. (In fact even the non-integral *unitary* solutions play a role for *non-generic* $n > 2$ models, see e.g. [10, 25, 14]).

6.2. Complete Solution for $n = 3$

Consider next $n = 3$. We put

$$S = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} .$$

Two triples (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) correspond to (massive perturbations of) the same superconformal model if we can pass from one to the other by a repeated application of the following transformations:

a) flipping the sign of *two* x_j's;
b) a permutation of (x_1, x_2, x_3);
c) replacing x_1 by $x_2 x_3 - x_1$ *or* x_2 by $x_3 x_1 - x_2$ *or* x_3 by $x_1 x_2 - x_3$.

Indeed, a) just corresponds to a redefinition of the signs for the canonical basis, while b) and c) can be obtained by suitable combinations of i) rotations of the Stokes axis, and ii) deformation of one vacuum w_i across the line connecting the other two vacua w_j, w_k (see Sects. 2 and 3).

The characteristic polynomial of the monodromy is

$$P(z) = \det[z - S(S^t)^{-1}] = z^3 + \alpha(x_i)z^2 - \alpha(x_i)z - 1\,,$$

where

$$\alpha(x_i) \equiv x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - 3\,.$$

The requirement that $P(z)$ has the form (6.5) leads to the following Diophantine equation

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = b\,, \tag{6.15}$$

where b depends on the particular product of cyclotomic polynomials. Explicitly one has

$P(z)$	$U(1)$ charges	b
$\Phi_1(z)^3$	$(-1, 0, 1)$	0
$\Phi_1(z)\Phi_2(z)^2$	$\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$	4
$\Phi_1(z)\Phi_3(z)$	$\left(-\frac{1}{3}, 0, \frac{1}{3}\right)$	3
$\Phi_1(z)\Phi_4(z)$	$\left(-\frac{1}{4}, 0, \frac{1}{4}\right)$	2
$\Phi_1(z)\Phi_6(z)$	$\left(-\frac{1}{6}, 0, \frac{1}{6}\right)$	1

Luckily enough, (6.15) is a well-studied Markoff-type Diophantine equation [39, 40]. All solutions are explicitly known. Let us summarize the main results for (6.15) [40].

i) All *non-trivial* solutions[39] (6.15) can be obtained from a fundamental solution[40] by a repeated application of the transformations a), b), and c). *So the physically distinct solutions are in one-to-one correspondence with the fundamental ones.*

ii) For $b = 0$ the only fundamental solution is $(3, 3, 3)$. Using a), b), c) this generates an infinite number of "equivalent" solutions.

iii) For $b = 4$ the fundamental solutions are $(1, 1, 2)$ and $(2, y, y)$ for $y \geq 2$. Each of these generates an infinite number of non-fundamental solutions.

iv) For $b = 3$ there are no fundamental solutions.

v) For $b = 2$ the only fundamental solution is $(1, 1, 1)$. There is only a *finite* number of solutions generated by this fundamental one. Up to permutations and changes of signs there are just two: $(1, 1, 1)$ and $(1, 0, 1)$.

iv) For $b = 1$ there is no fundamental solution.

The absence of solutions for b odd is an additional number-theoretical "selection rule" on the possible UV charges.

Regarding the Jordan structure, one checks that, except for the trivial solution $(0, 0, 0)$, and for $(2, 2, 2)$, H is non-derogatory, i.e. it has Jordan blocks of dimension equal to the multiplicity of the corresponding eigenvalue. This claim is equivalent to

[39] A solution is trivial if at least *two* of the x_i vanish. The only trivial solutions are (up to permutations) $(0, 0, 0)$ for $b = 0$; $(\pm 2, 0, 0)$ for $b = 4$; and $(\pm 1, 0, 0)$ for $b = 1$. Clearly, these solutions correspond to physically trivial models

[40] A solution is fundamental if $0 < x_1 \leq x_2 \leq x_3$ and $x_1 + x_2 + x_3$ is minimal

the statement that, unless $H = 1$, the minimal polynomial for H is the characteristic one. Since the minimal polynomial $P(z)_{\min}$ is given by the formula [41],

$$P(z)_{\min} = \frac{\det[z - H]}{\gcd[\text{Minors}(z - H)]},$$

we see that $P(z)_{\min} \neq P(z)$ is possible only if (6.5) contains a factor with $\nu(m) > 1$. From the table we see that this can happen only for $b = 0$, or 4. Then the above claim is obtained by direct inspection.

Comparison with Degenerate Painlevé III. As in Sect. 5.3, the above classification should be compared with the known results [42] about the regularity of solutions to the *degenerate* PIII (i.e. radial Bullough-Dodd)

$$\frac{d}{d\tau}\left(\tau \frac{d}{d\tau} u\right) = e^{u} - e^{-2u}. \tag{6.16}$$

The tt* equations for an $n = 3$ massive model can always be recast in this form provided we have a $\mathbf{Z}_3$ symmetry. Hence (6.16) is connected to the special cases in our classification with $x_1 = x_2 = x_3 = s$. Here s is the Stokes parameter for the degenerate PIII, see Eg. (13) of [42]. A solution of degenerate PIII is a solution to our integral equation provided the other parameters in [42] take the value

$$g_1 = g_2 = 0, \qquad g_3 = 1.$$

Then, for large τ (= large β) the asymptotic behaviour of the solution is

$$\exp[u(\tau)] \cong 1 + \frac{s}{2}\sqrt{\frac{3}{\pi}}\,(3\tau)^{-1/4} e^{-2\sqrt{3\tau}} + \dots .$$

from which it is obvious that s counts the number of solitons connecting any two vacua, in agreement with out general discussion (see also [10, 25]). In terms of $u(\tau)$ the two non-trivial eigenvalues of the Q-index are

$$Q(\tau) = \pm\tau \frac{\partial}{\partial\tau} u(\tau). \tag{6.17}$$

To compute the UV charges q, we need the asymptotic expansion of $u(\tau)$ for small τ. One has [42]

$$\begin{aligned}
e^{u(\tau)} &\cong -\frac{\sigma^2}{2\tau \sin^2\left\{\frac{i}{2}[\sigma \log\tau + \log a]\right\}} && s > 0,\ s \neq 3, \\
&\cong \frac{2}{\tau[\log\tau - (2\log 3 + 2\gamma)]^2} && s = 3, \\
&\cong \frac{2}{i\nu}\tau^{1/2} \sin\left[\frac{i}{2}\nu \log\tau - \log b\right] && s < 0,\ s \neq -1, \\
&\cong -\tau^{1/2}\log\tau + \tau^{1/2}\left(2\log 3 - \gamma - \tfrac{2}{3}\log 2\right) && s = -1,
\end{aligned} \tag{6.18}$$

where σ, ν, a, and b are

$$\begin{aligned}\mu &= \frac{3}{2\pi i}\log\left(\frac{s-1}{2}+\sqrt{\left(\frac{s-1}{2}\right)^2-1}\right) \qquad \text{with } |\operatorname{Re}\mu|<1\\ \nu &= 3-\frac{3}{\pi i}\log\left(\frac{s-1}{2}+\sqrt{\left(\frac{s-1}{2}\right)^2-1}\right) \qquad \text{with } |\operatorname{Re}\nu|<1\end{aligned} \tag{6.19}$$

$$\begin{aligned}a &= 3^{-2\mu}\Gamma(1-\mu/3)\Gamma(1-2\mu/3)\Gamma(1+\mu/3)^{-1}\Gamma(1+2\mu/3)^{-1}\\ b &= 3^{\nu}\Gamma(1/2+\nu/6)\Gamma(\nu/3)\Gamma(1/2-\nu/6)^{-1}\Gamma(\nu/3)^{-1}.\end{aligned} \tag{6.20}$$

From (6.17) and (6.18) we see that $\lim_{\tau\to 0} Q(\tau)$ exists and is real only if σ (resp. ν) is real. In view of (6.19) this is equivalent to

$$\left|\frac{s-1}{2}+\sqrt{\left(\frac{s-1}{2}\right)^2-1}\right| = 1\,;$$

this condition is satisfied iff the expression inside the square-root is non-positive, i.e. for

$$-1 \le s \le 3\,.$$

Then there are precisely five regular solutions to (6.16) with integral soliton number s, namely $s = 3, 2, 1, 0, -1$. These correspond to the five $\mathbf{Z}_3$-symmetric solutions we got for our Diophantine problem [41]. So, "unitarity" implies regularity in this case too. From (6.19) one finds

$$\operatorname{tr} H \equiv 1+\cos(2\pi q) = 3-3s^2+s^3\,,$$

in agreement with the result of our Diophantine analysis. As for the $n = 2$ case, the regularity theorem for degenerate Painlevé III can be stated as the condition

$$-1 \le \operatorname{tr} H \le 3\,.$$

6.3. Identification of the $n = 3$ Models

Now we discuss the physical realizations of the $N = 2$ models corresponding to the non-trivial solutions of the $n = 3$ Diophantine problem. For some models more than one Lagrangian formulation is known.

The $(1,1,1)$ *Model.* The solution $(x_1, x_2, x_3) = (1,1,1)$ corresponds to the A_3 minimal model, i.e. to the LG model with superpotential,

$$W(X) = \tfrac{1}{4}X^4 + \text{lower order.}$$

This identification is confirmed by the value of the UV charges, see the table in Sect. 5.3. The two basic solutions in this class, $(1,1,1)$ and $(1,0,1)$ correspond to the soliton multiplicities μ_{ij} for the two inequivalent geometries in W-space. These two geometries are realized, e.g. by the $\mathbf{Z}_3$-symmetric model $W(X) = X^4 - X$ and

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[41] Notice that the solution $(1,1,2)$ is equivalent to $(-1,-1,-1)$ by perturbation

by the $\mathbf{Z}_2$ invariant one $W(X) = X^4 - X^2$, respectively. In [10] the tt* equations for these two models have been solved in terms of PIII transcendents. The first case leads to *degenerate* PIII (6.16), while the second to *special* PIII (6.11).

The $(3,3,3)$ *Model.* The unique class of solutions for $b = 0$, i.e. $(3,3,3)$ is also an old friend – the $\mathbf{C}P^2$ σ-model. This identification is consistent with the $U(1)$ charges (see table). Moreover, from the explicit solution of the $\mathbf{C}P^2$ model [26] we know that there are precisely 3 solitons (transforming according the fundamental representation of $SU(3)$ connecting any two vacua. Thus the mass-spectrum extracted from the S-matrix agrees with the one predicted by the Diophantine analysis.

This solution corresponds to a sensible physical theory only for special geometries in W-space. Indeed, if we send one of the three vacua to infinity (in W-space) we end up with a model with only two vacua connected by 3 solitons: but this is impossible in view of the classification for $n = 2$. The usual $\mathbf{C}P^2$ σ-model corresponds to the three vacua at the vertices of an equilateral triangle in W-space (its size being related to the Kähler class of $\mathbf{C}P^2$, and its orientation to the θ-angle). Then the model must not make sense if the vacuum triangle is squeezed more than a certain amount. Note that there are three chiral operators in the $\mathbf{C}P^2$ model, $1, k, k^2$, where k denotes the Kähler class chiral field. The operator corresponding to k^2 has dimension bigger than 1 and is non-renormalizable. Addition of this term to the action is not allowed. The corresponding coupling controls the shape of the vacuum triangle in W-space. So stretching the vacuum triangle corresponds to adding non-renormalizable interactions to the action, leading to a pathological field theory. Indeed we see here that if we insist in adding terms which are not renormalizable we should either sacrifice unitarity, as the Hermitian charges are becoming complex, or the decoupling of the infinitely massive states (i.e., somehow the vacua that we move to infinity should still be contributing somehow). At any rate we see that tt* equations allow us to address the question of adding non-renormalizable terms to the action in a simple way. These pathologies must manifest themselves as singularities in the solutions of the tt* equations for certain critical values of β (cf. the discussion in Sects. 4.4, 4.5).

From the result of this section we can also infer the classification of the (compact) complex surfaces with Betti numbers $b_1 = 0$ and $b_2 = 1$ having positive first Chern class [42] (i.e. admitting a Kähler-Einstein metric with positive cosmological constant). Any such manifold will lead to a σ-model having $n = 3$. Its monodromy H should have a Jordan block of order 3. Our classification says that there is only one such manifold, namely $\mathbf{C}P^2$. This is in agreement with the known classification of complex surfaces, see [43]. Indeed consider a Kähler manifold of complex dimension d with $c_1 > 0$ which has only diagonal Hodge structure ($h^{p,q} \neq 0$ only if $p = q$), i.e. with chiral fields which have zero fermion number ($p - q = 0$). σ-model on such manifolds should correspond to a massive $N = 2$ theory which should thus be showing up in our classification. Let

$$n = \sum_{p=1}^{d} h^{p,p}\,.$$

Then in our classification with n vacua these σ-models will show up with $U(1)$ charges ranging from $-d/2$ to $d/2$ in integer steps, for which $h^{p,p}$ is the number of

[42] This condition is needed to ensure AF, see e.g. [25]

charges equal to $-\frac{d}{2}+p$. In particular if two manifolds lead to the same solution in our classification, then they are "mirror" in the sense that the σ-model on the two are isomorphic (at least as far as the ground states are concerned). It would be interesting to see if there are any examples of mirror phenomena of this type. At any rate our soliton diagrams give a new invariant for these Kähler manifolds (up to braiding action discussed before). We expect that this $N=2$ view of diagonal Kähler manifolds with $c_1>0$ should lead to their complete classification.

The $(1,1,2)$ *Model.* This solution is equivalent to $(-1,-1,-1)$ by perturbation (as follows from how the soliton numbers change under perturbation (2.9)). The last one is easier to realize since it is $\mathbf{Z}_3$ symmetric. In this case the matrix

$$B=S+S^t\,, \tag{6.21}$$

is just the Cartan matrix for the *affine* $\hat{A}_2$ Lie algebra. By analogy with (6.9) it is natural to realize the $(-1,-1,-1)$ model as an $N=2$ Toda theory related to the $\hat{A}_2$ root system. In fact, we claim that it is the $N=2$ Bullough-Dodd model, i.e. the LG model with superpotential

$$W(X)=t\big(e^X+\tfrac{1}{2}\,e^{-2X}\big)\,, \tag{6.22}$$

again with the identification $X\sim X+2\pi i$. The simpler way to see this is to compare with the usual A_3 minimal model, i.e. $(1,1,1)$. Clearly the only difference between the two models is the sign of the basic $\mathbf{Z}_2$ soliton cycle (we label the vacua according to the anti-clockwise order in W space)

$$\mu_{12}\mu_{23}\mu_{31}=\begin{cases}-1 & \text{for } A_3\\ \ \ 1 & \text{for } \hat{A}_2\,.\end{cases} \tag{6.23}$$

Let f_{ij} be the Fermi number of the soliton connecting the $i^{\rm th}$ vacuum to the $j^{\rm th}$ one. Then (6.23) holds provided that [14]

$$\exp[i\pi(f_{12}+f_{23}+f_{31})]|_{\hat{A}_2}=-\exp[i\pi(f_{12}+f_{23}+f_{31})]|_{A_3}\,. \tag{6.24}$$

Indeed $f_{ij}=f_i-f_j$, where (here $X_k=e^{2\pi ik/3}$, $k=0,1,2$, are the classical vacua)

$$f_k=-\frac{1}{2\pi}\,\mathrm{Im}\log W''(X_k)=\begin{cases}-\frac{1}{3}k & \hat{A}_2\\ \ \ \frac{1}{3}k & A_3\end{cases}\qquad (\mathrm{mod}\,1)\,,$$

which gives (6.24).

This identification is also consistent with the UV behaviour. The Diophantine analysis shows the the UV central charge is $\hat{c}=1$. Since the UV limit of (6.22) is just (massless) free field theory, this is the correct result.

If one is not satisfied with the above argument, we can do much better, i.e. we can solve explicitly the tt* equation for (6.22) in terms of (degenerate) Painlevé transcendents. To do this, we take the natural vacuum basis generated by spectral flow, i.e.

$$|1\rangle\,,\qquad |e^X\rangle\,,\qquad |e^{2X}\rangle\,.$$

Because of the $\mathbf{Z}_3$ symmetry $X \to X + 2\pi i/3$, in this basis the ground state metric g is diagonal. The diagonal entries of g are further restricted by the reality constraint [10]. This gives[43]

$$\langle \bar{1} \mid 1 \rangle \langle \overline{e^X} \mid e^X \rangle = \frac{1}{|t|^2}\,,$$
$$\langle \overline{e^{2X}} \mid e^{2X} \rangle = \frac{1}{|t|}\,.$$

Using this and setting

$$\log\langle \bar{1} \mid 1 \rangle = u(\tau) - \tfrac{1}{2} \log |t|^2\,, \qquad \tau | \tfrac{9}{4} \, |t|^2\,, \tag{6.25}$$

the tt* equations reduce to (6.16) with $Q(\tau)$ given by (6.17). In view of our discussion at the end of Sect. 2.3, to prove that (6.22) is the $(-1,-1,-1)$ model it is enough to show that $u(\tau)$ in (6.25) is the solution to (6.16) with boundary data $g_1 = g_2 = 0$, $g_3 = 1$, and $s = -1$. This follows from "regularity" [10]. Regularity requires [25] that $\langle \bar{1} \mid 1 \rangle$ is regular as $t \to 0$, possibly up to logarithmic violation of scaling (as predicted by the non-trivial Jordan structure of the monodromy). Comparing (6.25) and (6.18) we see that this condition is satisfied only for $s = -1$.

The $(2,2,2)$ *Model.* This solution corresponds to the Ising 3-point function [15]. Equivalently, we can identify it as the LG model with superpotential the Weierstrass function [15]

$$W(X) = \wp(X)\,,$$

and the identifications

$$X \sim X + n_1\omega_1 + n_2\omega_2\,, \qquad n_i \in \mathbf{N}\,,$$

where ω_i are the two periods of $\wp(X)$. This model has UV central charge $\hat{c} = 1$ as predicted by the number-theoretical viewpoint. Since the Jordan structure is trivial, the UV limit of this model is a viable candidate for a new $\hat{c} = 1$ non-degenerate superconformal model. Indeed the fact that the solution for the metric is non-degenerate at the UV point follows from the explicit solution of degenerate Painlevé III discussed before, which is thus a confirmation of our general arguments. Whether or not this is *sufficient* to obtain a non-degenerate conformal theory remains to be seen. Further details on this model can be found in [15]. More generally the solution with n vacua with all soliton numbers equal to 2 is related to the massive Ising model n spin correlation functions. Note that this is the *only* non-trivial theory for which the number of solitons (in absolute value) does not change by perturbations [see Eq. (2.9)].

The $(2,y,y)$ *Models for* $y > 2$. To our knowledge, no physical realization of these models is known. On the other hand, the consistency of these models requires properties which sound so magical that one wonders if they exist at all (as sensible QFT's). This fact, together with the absence of $\mathbf{Z}_3$ symmetry, would make it very difficult to guess an explicit Lagrangian realization for them, even if they exist. Anyhow, the positive result is that any yet-to-be-discovered $n = 3$ model should

[43] Notice that these reality conditions are quite different from those of the A_3 minimal model. In fact this is the *only* difference between the two models, i.e. they correspond to two inequivalent foldings of the $\hat{A}_2$ Toda equation

belong to this class, and hence have UV charges $\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$ and soliton spectra (related by a), b) and c)) to $(2, y, y)$!

We give a discussion of their properties: Certainly they cannot be well-behaved for arbitrary vacuum geometries, since sending an appropriate vacuum to infinity we end up with a model containing just two vacua connected by $y > 2$ solitons, a situation ruled out by the $n = 2$ classification.

The UV charges of the three chiral fields are $\left(0, \frac{1}{2}, 1\right)$. This follows by continuously turning on the soliton number as discussed before. However, in this case we find that *there is no way to turn the soliton numbers and go through phases as the eigenvalues of H*. This suggests to us that indeed these theories are pathological. At any rate, if these theories exist, then $q = 1$ field cannot be a marginal operator, since otherwise all vacuum geometry will be allowed, contrary to the above remark. By the same argument it cannot be an AF coupling. Then the only possibility is that the leading term in their β-function is positive, i.e. that the coupling is infra-red stable. In particular the $q = 1$ field may be the Kähler class of a dimension 1 complex manifold with negative curvature. The fact that there is a non-trivial Jordan block and that B has a negative eigenvalue in this case supports this picture. The field with $q = 1/2$ may in this set up be related to a Z_2 twist field for an orbifold of this σ-model.

7. The A–D–E Minimal Models Revisited

If we restrict our general classification to the models with $\hat{c} < 1$ we should recover the well-known A–D–E classification. Note that since minimal models by definition have chiral charges less than 1, and since the left and right chiral charges differ by an integer, this implies that for minimal models the left and right chiral charges are equal. Moreover since the charges are all less than one, perturbation with all of them are relevant and so we should get a massive theory. Therefore *all* the minimal models *must* appear in our classification. In this section we show how nicely this particular case fits in our general framework. From the discussion below it will be evident how our methods for classification of $N = 2$ theories are the natural generalization of the ones which were successful for the $\hat{c} < 1$ case. From one point of view, our discussion of the minimal models is more detailed than the usual one. In fact as an extra bonus we get the classification of the solitonic spectra which may appear for a given minimal model perturbed in a generic way. To get the usual A–D–E result we just have to "forget about" this extra information.

7.1. Positive Inner Products and Root Systems

Let $B = S + S^t$. We will now show that if B is positive definite, then the integral matrix S is automatically a solution to our Diophantine problem. In fact from the identity

$$HBH^t = S(S^t)^{-1}(S + S^t)S^{-1}S^t = S + S^t = B\,,$$

we see that H is orthogonal with respect to the inner product B. If B is positive its orthogonal group is compct, and hence H is simple with $|\lambda_j| = 1$. Then S solves our Diophantine equation. Note also that if S is close to the identity matrix then $S + S^t$ is positive definite and so the eigenvalues of H are always phases. So in our argument in previous sections for "building up" the charges, at least near $t = 0$ we are guaranteed that the charges are real.

We pause a while to digress on the classification of the positive definite (integral matrices B. A first remark is that $B_{ij} = 0$ or ± 1 (for $i \neq j$). Indeed let (say) $B_{12} = s$. Then consider the vector $V = (v_1, v_2, 0, \ldots, 0)$. Then VBV^t, as a quadratic form in v_1, v_2 is positive definite iff $|s| < 2$. Since s is integral, $s = 0, \pm 1$. To any such B we associate a (generalized) Dynkin diagram by the following rule: the i^{th} and j^{th} vertices are connected by a solid (resp. dashed) line iff $B_{ij} = -1$ (resp. $B_{ij} = +1$).

Since $\det[B] \neq 0$, we can introduce a basis $\vec{e}_i (i = 1, \ldots, n)$ of unit vectors in $\mathbf{R}^n$ such that

$$(\vec{e}_i, \vec{e}_j) = \tfrac{1}{2} B_{ij} .$$

Consider the group W generated by the reflections R_k

$$\vec{e}^{\,\prime}_j = (\vec{e} R_k)_j = \vec{e}_j - B_{jk} \vec{e}_k . \tag{7.1}$$

W, being a discrete subgroup of the compact orthogonal group, is finite. If we take the union of all the images under W of the vectors $\vec{e}_i$ we get a finite set of vectors in $\mathbf{R}^n$ which satisfies the axioms for a (reduced) root system [44]. In fact it is a simply-laced root system since all elements have length 1. Therefore (assuming irreducibility) it belongs to the A–D–E series [44]. There is a simple rule to get the root system associated to a given B. Since the $\vec{e}_i$ generate the root lattice, $\det B$ is the volume of the fundamental cell. Then $\det B$ is $n + 1$ for A_n, 4 for D_n and $9 - n$ for E_n. The general solution to our Diophantine problem with B positive is obtained as follows. Take a simply-laced Lie algebra of rank n, and choose n linearly independent vectors $\vec{e}_i$ belonging to its root system.[44] Then put

$$-A_{ij} = \begin{cases} (\vec{e}_i, \vec{e}_j) & \text{for } i < j \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute the monodromy H of this solution. Consider the matrix

$$R = -H^t = -S^{-1} S^t . \tag{7.2}$$

The matrix R satisfies a remarkable identity due to Coxeter. One has [45]

$$R = R_1 R_2 R_3 \ldots R_n . \tag{7.3}$$

Let us consider first a "standard" solution, i.e. the vectors $\vec{e}_i$ are normal to the walls of a Weyl chamber. In this case the generalized Dynkin diagram reduces to the usual one, and B is just the Cartan matrix. For this "standard" situation R is known as the Coxeter element of the finite reflection group W (= the Weyl group). The Coxeter element is independent of choices (up to conjugation) as long as the Dynkin diagram contains only *solid* lines (which in particular means that it is a tree). The order h of R is called the Coxeter number of the associated Lie algebra. Its eigenvalues are of the form $\exp[2\pi i m_j / h]$, where the integers m_j are the *exponents* of the corresponding Lie algebra [45]. Comparing with (7.2) we see that the UV charges for a "standard" solution are

$$q_j = \frac{m_j}{h} - \frac{1}{2} \pmod{1} . \tag{7.4}$$

Of course, this is precisely the answer for the corresponding A–D–E minimal model.

[44] Notice that two choices differing only for the order of the elements $\vec{e}_i$ should be considered as distinct since they lead to different S's

The next step is to find the solutions which are equivalent to the "standard" one, in the sense of corresponding to different perturbations of the same basic superconformal model. This is the same as asking which sets of roots $\vec{e}_i$ can be obtained from a given one by a continuous deformation of the couplings w_k. These are those obtained from a standard solution by a repeated application of the following "moves." First of all, we can replace a root $\vec{e}_i$ by the opposite root $-\vec{e}_i$ since this is just a redefinition of the sign of the corresponding canonical vacuum. Then we can replace the ordered pair of roots $(\vec{e}_j, \vec{e}_{j+1})$ by the pair $(\vec{e}_{j+1}R_j, \vec{e}_j)$. Indeed, the transformation

$$\begin{aligned} \vec{e}_i' &= \vec{e}_i \quad \text{for } i \neq j, j+1\,, \\ \vec{e}_j' &= \vec{e}_{j+1}R_j \equiv \vec{e}_{j+1} - B_{j+1,j}e_j\,, \\ \vec{e}_{j+1}' &= \vec{e}_j\,. \end{aligned}$$

induces the following transformation on $B_{ij}(i \neq j, j+1)$:

$$B'_{ij} = B_{i,j+1} - B_{j+1,j}B_{ij}\,, \tag{7.5}$$

i.e. the transformation T_j – the braiding action taking the j^{th} vacuum in the anti-clockwise direction replacing the $j+1^{\text{th}}$ vacuum (see Sects. 2 and 3). We know that this transformation can be realized *via* a deformation of the couplings w_k as all the perturbations are relevant and thus allowed. Finally we can cyclically permute the $\vec{e}_i$'s or take them in the inverse order.

In this way many solutions are reduced to the standard ones. The corresponding $N = 2$ model is known to be realizable as a Landau-Ginzburg model [4]. From these explicit LG realizations, we see that their UV limits are just the corresponding minimal models. Since in minimal models all formal perturbations are physically allowed, the solution obtained by braiding the standard ones can be realized as a suitable perturbation of the corresponding minimal model.[45] Then these solutions are just (massive perturbations of) A–D–E minimal models. For example the A-series can be realized as Chebyshev perturbations of the x^n minimal model [7].

However not all solutions with B positive are equivalent to the standard ones. This reflects the fact that the notion of irreducibility for the field algebra of a QFT is a much stronger constraint than the analog notion for a reflection group. Consider e.g. the 4×4 matrix

$$S = \begin{pmatrix} 1 & \sigma_1 + \sigma_2 \\ 0 & 1 \end{pmatrix}\,, \tag{7.6}$$

where σ_i are the Pauli matrices. The corresponding $\vec{e}_i$ generate the D_4 root lattice. However its monodromy H satisfies

$$\det[z - H] = (z^2 + 1)^2\,, \tag{7.7}$$

and thus has nothing to do with the charges of the D_4 minimal model. In fact, $-H^t = R_1R_2R_3R_4$, whereas the Coxeter element (for this unorthodox choice of roots) reads $R_3R_1R_4R_2$, and these two elements are not conjugate in W. Now the point is that (7.7) cannot correspond to an *irreducible* regular critical theory. There are three good physical reasons to discard the solution (7.6): i) The minimal value of q is doubly degenerate (that is 1 is not the only chiral primary with $q = 0$). Then (7.6) is reducible. ii) The solution has $\operatorname{tr} H = 0$; this cannot be for an irreducible theory.

[45] Of course, this is just the usual description of the deformations of a minimal singularity [5]

Indeed the $N = 2$ superconformal algebra together with modular invariance shows[46] that $\operatorname{tr} H$ is the susy index counting *with signs* the number of chiral primaries with $q = 0$. The requirement that the only such object is the standard identity operator gives[47]

$$\operatorname{tr} H = 1 \,. \tag{7.8}$$

iii) The solution to the tt* equations defined by the Stokes matrix (7.6) cannot be *both* regular for all w_k's and irreducible. Indeed, let us send $\operatorname{Im} w_3$, $\operatorname{Im} w_4 \to \infty$ while keeping $\operatorname{Im} w_1 = \operatorname{Im} w_2 = 0$. We end up with a model with just two vacua and *no solitons*. Again this is a reducible situation. For a general model this argument shows that, in order to have irreducibility, we need[48]

$$B_{i,i+1} \neq 0 \,. \tag{7.9}$$

Without loss of generality we can take $B_{i,i+1} = -1$. This condition is not fulfilled by (7.6). The same reasoning shows that, if we have n vacua and take the limit $\operatorname{Im} w_n \to \infty$, the "reduced" Stokes matrix obtained by deleting the last row and the last column should be such as to correspond to a regular irreducible solution of tt*.

Thus we have three necessary criteria for irreducibility. Now we give an argument to the effect that if B is positive definite *and* satisfies these criteria[49] then it is equivalent to a standard solution. The idea is to argue by induction on the number of vacua. Assume we know that this is true for n vacua. Then in the $n+1$ vacuum case we can use criterion iii) to put the "reduced" S in a standard form for the given "reduced" root system.[50] Having fixed the "reduced" S in a standard form we are reduced to a much simpler Diophantine problem for the unknowns $a_i = S_{i,n+1}$. By (7.9) $a_n = -1$, so we have just $n-1$ unknowns which can only take the values 0 and ± 1. At this stage we impose the index restriction $\operatorname{tr} H = 1$ which greatly reduces the allowed values for the a_i's (and kills the "spurious" solutions like (7.6)). Finally one shows that the few surviving possibilities either lead to a non-positive[51] B, or they are equivalent to standard solutions. To illustrate this last step of the process we consider the simpler case in which the ''reduced'' S is associated to the A_n root system. In this case one finds

$$\operatorname{tr} H = 1 + \sum_i (a_i - a_i^2) - \sum_{i<j} a_i a_j \,.$$

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46 In fact comparing with [46] we see that $\operatorname{tr} H = \operatorname{Tr}(-1)^F g$, where $g = \exp[2\pi i J_0]$. The modular tranformation $\tau \to -1/\tau$ transforms the character-valued susy index $\operatorname{Tr}(-1)^F g$ into the Witten index for the sector twisted by g. The ground states in this sector are the harmonic representatives of a certain cohomology (the group $H(1,0)$ in the notation of [46]). Chiral primaries of minimal $U(1)$ charge are always non-trivial elements of $H(0,1)$ [46]. Moreover for $\hat{c} < 1$ (at least) positivity implies that these chiral primaries exhausts $H(0,1)$

47 Notice that this restriction is automatic in LG models. This is an easy consequence of the results of [46] as well as a known fact from singularity theory [5]

48 At first sight, it may seem that this argument implies $B_{ij} \neq 0$ for all i,j. However it is not so since in the limit $\operatorname{Im} w_k \to \infty$ (for $k \neq i,j$) we still get contributions to the soliton number μ_{ij} from the "vacua at infinity" because the singularity mechanism discussed in Sect. 4 spoils the naive decoupling. Instead $-\mu_{i,i+1} = B_{i,i+1}$ because of the bound (4.37) and thus for these particular entries the arguments is correct

49 In fact we can forgot about i) since it is a consequence of the other two

50 Of course this process screws up the elements $S_{i,n+1}$

51 Generically they are not even solutions to our problem. However some are "affine" solutions, see next subsection

Let $m(0 \leq m \leq n-1)$ be the number of a_i's with $|a_i| = 1$ and $r(0 \leq r \leq m)$ the number of those with $a_i = 1$. Then $\operatorname{tr} H$ becomes

$$1 - (m - r) - \tfrac{1}{2}\left[(m - 2r)^2 + (m - 2r)\right].$$

The last two terms in this expression are non-positive. Hence $\operatorname{tr} H = 1$ iff they both vanish. This happens in just two cases: i) all $a_i = 0$; or ii) one $a_i = 1$ while all the others vanish. So we have only n cases to check. The case $a_i = 0$ for all i's and the one with $a_{n-1} = 1$ give the standard A_{n+1} solution. The cases $a_1 = 1$ or $a_{n-2} = 1$ give the usual D_{n+1} solution. Finally the cases $a_2 = 1$ or $a_{n-3} = 1$ give the E_{n+1} "solution" (it is a $\hat{c} < 1$ model only if $(n+1) \leq 8$; for $(n+1) < 6$ it coincides with a D solution, as one sees from the corresponding Stokes matrix). The cases with $a_i = 1$ for an i in the range $2 < i < n - 3$ have never B positive definite. If the "reduced" S is of the D_n or E_n type the analysis is similar although more involved.

It remains to show that the A–D–E series exhausts the models with $\hat{c} < 1$, i.e. that $\hat{c} < 1$ implies B positive definite. One argument was mentioned before, i.e., the fact that for the minimal model $|q| < 1/2$ implies this. We will now give another argument: In a model with $\hat{c} < 1$ all chiral primaries are relevant, and hence all deformations of the theory lead to regular solutions of tt*. Moreover, the property that all UV charges are less than 1 should survive perturbation.

If $\hat{c} < 1$ then all Ramond $U(1)$ charges satisfy $|q| < \frac{1}{2}$. Hence -1 is not an eigenvalue of the monodromy $H = S(S^{-1})^t$. Then

$$0 \neq \det[-1 - H] = (-1)^n \det[S^t + S]. \tag{7.10}$$

Assume that a model with $\hat{c} < 1$ has B non-positive definite. By (7.10), B has (at least) a negative eigenvalue. Consider the pseudo-reflection group W generated by the corresponding R_k's (7.1). Repeating word-for-word the previous argument, we see that the elements of this group can be realized as *formal* perturbations of the model. But in a minimal model all formal perturbations should be good deformations of the theory. So all the Stokes matrices generated by these reflections should correspond to regular solutions of tt* for all w_k's. But now W is infinite [44] and hence in the equivalence class we can find arbitrarily big Stokes parameters A_{ij}. But this is absurd. Indeed the solution of tt* cannot be regular for all w_k's for A_{ij} very large as we can see by considering suitable geometries in W-space and taking some vacua to infinity. Then minimality requires B to be positive.

7.2. "Affine" Models and Their Lagrangian Realizations

$\hat{c} = 1$ *Degenerate Models.* In our context the A–D–E classification has a natural "affine" generalization. The main purpose of this subsection is to provide explicit examples of such "affine" models. We make no attempt at completeness.

Suppose our (integral) Stokes matrix S is such that the associated symmetric form $B = S + S^t$ is *singular* (i.e. $\det B = 0$), while all its (proper) principal minors are *positive definite*. We claim that any such S is also a solution to our Diophantine problem. Indeed, since B is singular, there is a vector v such that $Bv = 0$. This vector is unique (up to normalization) since, by assumption, rank $B = n - 1$. In particular v is *real*. Then v is the *unique* eigenvector of $H^t = S^{-1}S^t$ associated to the eigenvalue $\lambda_0 = -1$; indeed,

$$H^t v = S^{-1}(B - S)v = -v. \tag{7.11}$$

Consider next the quadratic form $\hat{B}$ induced by B on the quotient space $\mathbf{R}^n/\mathbf{R}v$. By assumption, $\hat{B}$ is positive definite. From the argument of Sect. 7.2, we see that the induced monodromy $\hat{H}^t$ acts orthogonally on $\mathbf{R}^n/\mathbf{R}v$. This, together with (7.11), shows that the eigenvalues of H^t satisfy $|\lambda| = 1$, as claimed. However the Jordan structure of H^t is non-trivial. In fact H^t has just one eigenvector associated to this eigenvalue, whereas (-1) is a root of the characteristic polynomial of *even* multiplicity.[52] Since $\hat{H}^t$ is simple, H^t has just a 2×2 block associated to the eigenvalue (-1).

From our general discussion in Sect. 5 we know that these solutions lead to *degenerate* superconformal models with $\hat{c}_{uv} = 1$. Conversely all such degenerate models are associated to an S having the above properties. If we assume that the off-diagonal entries of S are non-positive, then (up to permutations) B is nothing else than the Cartan matrix for a simply laced affine Lie algebra, i.e. $\widehat{A_{n-1}}$, $\widehat{D_{n-1}}$, or $\widehat{E_6}\widehat{E_7}$, and $\widehat{E_8}$. The general solution is obtained as follows. We consider a (finite) simply laced root system of rank $(n-1)$ and take n roots $\vec{e}_i (i = 0, 1, \dots, n-1)$ such that $\vec{e}_i (i = 1, \dots, n-1)$ span $\mathbf{R}^{n-1}$ whereas

$$\vec{e}_0 = \sum_{i=1}^{n-1} k_i \vec{e}_i \,, \qquad k_i \text{ non-vanishing integers.}$$

Then

$$B_{ij} = 2(\vec{e}_i, \vec{e}_j) \qquad i, j = 0, \dots, n-1 \,.$$

The Cartan matrix corresponds to the special solution with $\vec{e}_i (i \neq 0)$ the simple roots and $\vec{e}_0$ *minus* the highest root. Many solutions can be obtained one from the other by "formal" perturbations (i.e. braiding transformations). Since the chiral fields ϕ_i are either soft perturbations or asymptotically free renormalizable interactions, we expect that all the "formal" perturbations make perfect physical sense.

LG *Models with Exponential Interactions.* In Sect. 6 we saw that the $\hat{A}_1$ model can be realized as the $N = 2$ Sinh-Gordon provided we make the identification $X \sim X + 2\pi i$. Other "affine" models are obtained by changing this identification to

$$X \sim X + 2\pi n i \,.$$

In this way we get a $\hat{A}_{2n-1}$ model. The easiest way to see this is to solve the corresponding tt* equations in terms of Painlevé transcendents. As in [10] we introduce the transformation T which shifts X by $2\pi i$. One has $T^n = 1$. Then we consider the θ-vacua, i.e. the ground states such that

$$T\,|a, \theta\rangle = e^{i\theta}\,|a, \theta\rangle \,, \qquad a = 1, 2 \,.$$

For a fixed value of θ there are just two ground states, and hence the tt* equations take the PIII form (see [10] for details). Then the ground state metric is (here $z = m\beta$, with m the mass of the basic soliton)

$$g(z, \theta) = \frac{4}{z} \exp[i\theta\sigma_3/4] \exp[\sigma_1 L(z, \theta)] \exp[-i\theta\sigma_3/4] \,,$$

where $L(z, \theta)$ is the regular solution to PIII with

$$r(\theta) = 2\left(1 - \frac{\theta}{\pi}\right), \qquad (0 \le \theta < 2\pi) \,.$$

[52] Because $\det H = 1$, and if λ is an eigenvalue of H so is λ^{-1}

Returning to the canonical basis, the ground state metric becomes (here $|k\pi\rangle$ denotes the canonical vacuum associated to the critical points $X_k = k\pi i$)

$$\langle \overline{k\pi} \mid j\pi\rangle = \frac{i^{(j-k)}}{2n} \sum_{s=0}^{n-1} e^{i\pi(j-k)s/n} [e^{L(z,2\pi s/n)} + (-1)^{(j-k)} e^{-L(z,2\pi s/n)}] , \tag{7.12}$$

where $(k, j = 0, 1, \ldots, 2n-1)$.

Since as $z \to \infty$,

$$\exp[L(z,\theta)] \cong 1 - 2\cos(\theta/2) \frac{1}{\pi} K_0(z) ,$$

we have the IR asymptotics,

$$\langle \overline{k\pi} \mid j\pi\rangle \cong \delta^{(2n)}_{i,j} - i^{(j-k)}(\delta^{(2n)}_{k,j+1} + \delta^{(2n)}_{j,k+1}) \frac{K_0(z)}{\pi} ,$$

where

$$\delta^{(m)}_{i,j} = \begin{cases} 1 & \text{if } i \equiv j \bmod m \\ 0 & \text{otherwise.} \end{cases}$$

Then the soliton matrix μ_{ij} reads

$$\mu_{jk} = i^{(j-k-1)}(\delta^{(2n)}_{k,j+1} + \delta^{(2n)}_{j,k+1}) . \tag{7.13}$$

The $U(1)$ charges in a given θ-sector are equal to $\pm r(\theta)/4$ (see [10]). Thus

$$\{\text{UV } U(1) \text{ charges}\} = \pm \frac{s}{n} \mp \frac{1}{2} \qquad (s = 0, 1, \ldots, n-1) . \tag{7.14}$$

In the present case there are just two critical values, and so we have $S = 1 - A$ with

$$A_{ij} = \begin{cases} \mu_{ij} & \text{if } i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

After a relabeling of the basis (and a suitable sign redefinition) the Stokes matrix reads

$$S = \begin{pmatrix} 1 & 1+R \\ 0 & 1 \end{pmatrix} ,$$

where the $n \times n$ matrix R corresponds to a cyclic permutation. In particular, $R^n = 1$, and $R^t = R^{-1}$. Then the characteristic polynomial of H is

$$\begin{aligned} P(-z) &= \det[zS^t + S] = \det[(z+1)^2 - z(1+R^{-1})(1+R)] \\ &= [\det(z-R)]^2 = (z^n - 1)^2 , \end{aligned}$$

in agreement with (7.14).

In Sect. 6 we also saw that the $N = 2$ Bullough-Dodd model, i.e. the LG model with superpotential

$$W(X) = \frac{2t}{3} \left(e^X + \tfrac{1}{2} e^{-2X}\right) ,$$

and field identification $X \sim X + 2\pi i$, leads to a model related to the $\hat{A}_2$ root system. Our general discussion above implies that the models obtained by the more general identification $X \sim X + 2\pi n i$ also correspond to degenerate $\hat{c} = 1$ theories. We expect that the corresponding solution to our Diophantine problem is related to the $\widehat{A_{3n-1}}$

root system. However this time it is not possible to check this expectation by writing explicitly the ground-state metric in terms of known transcendents. Luckily there is one special case, namely $n = 2$, in which g can be still written in terms of Painlevé transcendents.

We define the ground state

$$|m\rangle = \sum_{k=0}^{5} e^{\pi imk/3} \left| \frac{2\pi k}{2} \right\rangle ,$$

where $|2\pi k/3\rangle$ is the "point basis" vacuum at $X_k = 2\pi k/3 (k = 0, 1, \ldots, 5)$. Then the $\mathbf{Z}_6$-symmetry implies $\langle \bar{m} \mid l \rangle = 0$ for $m \neq 1$. Using the reality constraint the tt* equations decompose into two decoupled degenerate PIIIs. Then the ground state metric reads

$$\begin{aligned} 2|t|\langle \bar{2} \mid 2 \rangle &= 2|t|\langle \bar{5} \mid 5 \rangle = 1 , \\ 2|t|\langle \bar{4} \mid 4 \rangle &= (2|t|\langle \bar{0} \mid 0 \rangle)^{-1} = e^{u_1(\tau)} , \\ 2|t|\langle \bar{1} \mid 1 \rangle &= (2|t|\langle \bar{3} \mid 3 \rangle)^{-1} = e^{u_2(\tau)} , \end{aligned}$$

where $u_i(\tau)$ are (regular) solutions to Eq. (6.16). In terms of the canonical basis, the ground state metric reads

$$\begin{aligned} g_{k\bar{j}} &= \tfrac{1}{6} [1 + (-1)^{(k-j)}] + \tfrac{1}{6} [e^{\pi i(k-j)/3} e^{u_1} + e^{-\pi i(k-j)/3} e^{u_1}] \\ &\quad + \tfrac{1}{6} [e^{2\pi i(k-j)/3} e^{u_2} + e^{-2\pi i(k-j)/3} e^{u_2}] . \end{aligned}$$

Then the soliton matrix is

$$\mu_{kj} = \frac{1}{\sqrt{3}} s_1 \sin \left[\frac{\pi}{3} (k - j) \right] - \frac{1}{\sqrt{3}} s_2 \sin \left[\frac{2\pi}{3} (k - j) \right] ,$$

where s_i are the Stokes parameters specifying the boundary conditions for $u_i(\tau)$ (see Sect. 6.2). Consistency with the $m = 1$ case (which can be identified with a subsector of the present model) fixes $s_2 = -1$. Then the requirement that μ_{ij} are integers implies

$$s_1 = 1 \bmod 2 .$$

Using regularity (Sect. 6.2) we get $s_1 = -1$, 1, or 3. But 3 is not possible because it gives $\hat{c}$ too big. On the other hand, -1 is ruled out on the basis of the uniqueness of the chiral field with the smallest $U(1)$ charge. Thus $s_1 = 1$. Then

$$\mu_{ij} = \delta^{(6)}_{i,j+1} - \delta^{(6)}_{i,j-1} ,$$

which is the expected result. (Notice that, although the $N = 2$ soliton multiplicities $|\mu_{ij}|$ are the same as in the Sinh-Gordon with $n = 3$, the signs assignments are physically inequivalent). The UV charges are easily computed from the above explicit solutions. They are

$$\left(-\tfrac{1}{2}, -\tfrac{1}{4}, 0, 0, \tfrac{1}{4}, \tfrac{1}{2} \right) .$$

Nedless to say, this is in agreement with the values obtained from the monodromy H.

On general grounds one expects that similar phenomena also happens for more general exponential interactions, in particular the $\mathbf{Z}_n$ invariant one

$$W(X) = e^X + \frac{1}{n-1} e^{-(n-1)X} . \tag{7.15}$$

However explicit computations are not as elementary as in the above cases.

$\mathbf{C}P^1$ *Orbifolds.* We saw in Sect. 6.1 that $\hat{A}_1$ model has a second Lagrangian realization [53] as the σ-model with target space $\mathbf{C}P^1$. Then our general arguments show that any (sensible) orbifold of this σ-model should also be a *degenerate* $\hat{c} = 1$ and hence related to a simply laced root system as above. It is tempting to make the following conjecture on the nature of this correspondence. An orbifold is obtained by modding out a discrete subgroup G of the (double cover of) the $\mathbf{C}P^1$ isometry group $SU(2)$. As is well known, these subgroups are again classified by A–D–E. Then it is natural to expect that the G orbifold is related to the root system associated to the subgroup G. More precisely, the correspondence between a subgroup and a root system is obtained by considering the eigenvectors of the Â–D̂–Ê Cartan matrices: They are the columns of the character table for the corresponding group G [47]. In particular,

$$\hat{A}_r \leftrightarrow \mathbf{Z}_{r+1} \,. \tag{7.16}$$

It is also natural to expect that not all subgroups can appear, since the center $\mathbf{Z}_2$ of $SU(2)$ must belong to G because the physical states are automatically invariant under this subgroup. (That is the original σ-model, having the isometry group $SO(3) = SU(2)/\mathbf{Z}_2$, corresponds to $G = \mathbf{Z}_2$ not $G = 1$).

That these massive orbifolds are *bona fide* quantum field theories was shown in [25]. There the special case $G = \mathbf{Z}_{2n}$ (corresponding to the orbifold $\mathbf{C}P^1/\mathbf{Z}_n$) was studied in great detail, and the corresponding tt* equations were explicitly solved in terms of PIII transcendents. The results of [25] implies that the ground state metric for the $\mathbf{Z}_n$-orbifold is just that for the $N = 2$ Sinh-Gordon with the identification [54] $X \sim X + 2\pi n i$, see Eq. (7.12). Again one can compute the soliton spectrum out of this tt* solution. The computations are a word-for-word repetition of those leading to Eq. (7.13). Then Eq. (7.13) gives the mass-spectrum of the $\mathbf{C}P^1/\mathbf{Z}_n$ orbifold model. In agreement with the guess (7.16) μ_{ij}, as a solution to our classification program, is indeed related to the $\widehat{A_{2n-1}}$ root system (e.g. $2\delta_{ij} - |\mu_{ij}|$ is the $\widehat{A_{2n-1}}$ Cartan matrix). Non-abelian orbifolds of $\mathbf{C}P^1$ have been recently considered in [48] with results in the direction of connecting them to affine D and E series.

8. More on σ-Models

The primary aim of this section is to supply examples which cannot be realized as LG models. Here we focus mainly on σ-models over symmetric spaces (with $c_1 > 0$). The main issue is to compute their (solitonic) mass spectra as we did in Sect. 7.2 for the $\mathbf{C}P^1$ orbifolds. Typically these σ-models are confining theories whose physics is quite similar to that of $4d$ gauge theories. So the possibility of getting (part of) their exact mass spectrum by back-of-an-envelope computation is a very dramatic consequence of our methods.

It is more convenient to start with a more general problem, i.e. the classification of the massive models with a $\mathbf{Z}_n$-symmetry acting transitively on the n ground states. Indeed, besides the cases of small n and small $\hat{c}$, there is a third situation in which a

[53] The structure of the exact S-matrices for the $\mathbf{C}P^1$ σ-model and the $N = 2$ sine-Gordon suggests [7] that these two models are equivalent as QFT's for a special choice of the D-terms

[54] Physically this is due to the fact that (for a special choice of the D-terms) both models are orbifold of the *same* QFT

complete classification is possible, namely in the presence of a "big" symmetry: One looks for all models having a symmetry group $\mathscr{G}$ which is big enough to restrict the μ_{ij}'s in a significant way. In [14] it was shown that the $\mathbf{Z}_n$ invariant models organize themselves in a "family" and that it is somewhat easier to study all the models in the family at once than one at the time.

8.1. The Classification of $\mathbf{Z}_n$-Invariant Models

If our system has a cyclic symmetry, then the matrix C should transform according to some irreducible representation of $\mathbf{Z}_n$, i.e. like $\zeta_r(a) = \exp[2\pi i a r/n]$ for some $r \in \mathbf{Z}_n$. We set $m = n/(n,r)$. Then, without loss of generality, we can assume that the m distinct "critical values" w_k are given by

$$w_k = \exp[2\pi i k/m] \qquad k = 0, \ldots, m-1 . \tag{8.1}$$

All these values have multiplicity (n,r). For convenience, we label the vacua with two indices $(k,a)\,(k = 0,1,\ldots,m-1, a = 1,2,\ldots,(n,r))$ in such a way that $k + ma$ is increased by 1 under a basic $\mathbf{Z}_n$ rotation while k labels the corresponding critical value as in (8.1). In this basis S will not be upper triangular.

The $\mathbf{Z}_n$ symmetry restricts the $(n,r)\times(n,r)$ matrices $(\mu_{kh})_{ab}$. First of all, one has

$$\mu_{ij} = \mu(i-j) \quad \text{with} \quad \mu(k+m) = J\mu(k)\,,$$

where the $(n,r)\times(n,r)$ matrix J is

$$J_{ab} = \delta_{a,b-1} + \varepsilon\delta_{a,1}\delta_{b,(n,r)}\,.$$

ε is fixed by the following identity

$$\mu(k+n) = J^{(n,r)}\mu(k) = \varepsilon\mu(k)\,.$$

At first sight it may seem that this together with the $\mathbf{Z}_n$ symmetry predicts $\varepsilon = 1$. However it is not so. The point is that the canonical basis is well defined *up to sign*, and it is quite possible that acting n times with the basic $\mathbf{Z}_n$ transformation we end up with the *opposite* sign. In this case $\varepsilon = -1$ (in fact most models work this way). The sign assignments for the canonical vacua are specified by the phases of the topological metric η. Again η should belong to a definite representation of $\mathbf{Z}_n$. If it transforms as $\eta \mapsto e^{-2\pi i q/n}\eta$ then $\varepsilon = (-1)^q$. Thus $J^{(n,r)} = (-1)^q$. On the other hand μ is antisymmetric, so $\mu(k) = -\mu(-k)^t$, or

$$\mu(m-k) = -J\mu(k)^t\,. \tag{8.2}$$

Finally, the symmetry implies that $\mu(k)$ commutes with J. But J is non-derogatory and thus

$$\mu(k) = \sum_{l=1}^{(n,r)} \varrho(k,l)\,J^l\,,$$

for some (integral) coefficients $\varrho(k,l)$. In view of (8.2) we have

$$\varrho(m-k,l) = -\varrho(k,(n,r)-l+1)\,,$$

(we used that $J^t = J^{-1}$).

The angles ϕ_{kh} of Sect. 4 are given by

$$e^{i\phi_{kh}} = \begin{cases} -i\exp[\pi i(k+h)/m] & \text{for } k > h \\ +i\exp[\pi i(k+h)/m] & \text{for } k < h\,. \end{cases}$$

The (reflected) soliton rays belonging to the right half-plane are at angles

$$\psi_s = \frac{\pi}{2} - \frac{s}{m}, \qquad s = 0, \ldots, m-1\,.$$

As we have discussed in Sect. 4.5 the matrices $\mu^{[\cdot]}$ associated with a given ray commute. Let $\mu^{\langle s\rangle}$ be the sum of all the matrices associated with the s^{th} ray. From the above formulae one gets $(i, j = 0, \ldots, m-1)$

$$(\mu^{\langle s\rangle})_{ia,jb} = \delta_{i+j,m-s}\lambda_{ia,jb} - \delta_{i+j,2m-s}\lambda_{jb,ia}\,,$$

where

$$\lambda_{ia,jb} = \begin{cases} (\mu_{ij})_{ab} & \text{if } i > j \\ 0 & \text{otherwise.} \end{cases}$$

From the definition it is easy to get the identity

$$\mu^{\langle s+1\rangle}\mu^{\langle s\rangle} = 0\,, \tag{8.3}$$

then the Stokes matrix reads

$$\begin{aligned} S &= (1-\mu^{\langle 2l+1\rangle}-\mu^{\langle 2l\rangle})(1-\mu^{\langle 2l-1\rangle}-\mu^{\langle 2l-2\rangle})\ldots(1-\mu^{\langle 1\rangle}-\mu^{\langle 0\rangle}) \quad & m = 2l+2 \\ &= (1-\mu^{\langle 2l+2\rangle})(1-\mu^{\langle 2l+1\rangle}-\mu^{\langle 2l\rangle})\ldots(1-\mu^{\langle 1\rangle}-\mu^{\langle 0\rangle}) & m = 2l+3\,. \end{aligned}$$

Let R be the orthogonal $n \times n$ matrix

$$R = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & J^{-1} \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

which satisfies $R^m = \mathbf{1} \otimes J^{-1}$. Then the cyclic symmetry of λ_{ij} yields

$$\mu^{\langle s+2\rangle} = R^{-1}\mu^{\langle s\rangle}R\,.$$

So, if (say) $m = sl + 2$

$$S = R^{-1}[(1-M)R]^{l+1}R^{-1}\,,$$

where $M = \mu^{\langle 1\rangle} + \mu^{\langle 0\rangle}$. This procedure can be continued through the other half-plane. Finally we get for the monodromy

$$H = R^{-(m-1)}[(1-M)R]^m R^{-1} \equiv (\mathbf{1} \otimes J)R[(1-M)R]^m R^{-1}\,.$$

So, up to similarity, $(\mathbf{1} \otimes J^{-1})H$ is just $[(1-M)R]^m$. Since $(\mathbf{1} \otimes J^{-1})$ commutes with H, the monodromy eigenvalues are phases iff the matrix $(1-M)R$ satisfies the same condition, i.e. if $\det[z-(1-M)R]$ is a product of cyclotomic polynomials.

Let us assume for the moment that $(n, r) = 1$, so $n = m$. In this case $J = (-1)^q$. Then if

$$\mathcal{Q}(z) \equiv \det[z - (1 - M)R] = \prod_{s \in \mathbf{N}} \Phi_s(z)^{\nu(s)},$$

the characteristic polynomial $P(z)$ of H reads

$$P((-1)^q z) = \prod_{s \in \mathbf{N}} [\Phi_{s/(n,s)}(z)]^{\nu(s)\, \alpha_n(s)}, \tag{8.4}$$

where

$$\alpha_n(s) = \phi(s) \Big/ \phi\left(\frac{s}{(s, n)}\right).$$

One finds

$$\begin{aligned}\mathcal{Q}(z) &\equiv \det[z - (1 - M)R] \\ &= z^n + (-1)^{q+1} + \sum_{k=1}^{(n-1)/2} \mu(k)\,(z^{n-k} + (-1)^{q+1} z^k) \qquad (n \text{ odd}) \\ &= z^n + (-1)^{q+1} + (-1)^{q+1} + \mu\left(\frac{n}{2}\right) z^{n/2} \\ &\quad + \sum_{k=1}^{n/2-1} \mu(k)\,(z^{n-k} + (-1)^{q+1} z^k) \qquad (n \text{ even}),\end{aligned} \tag{8.5}$$

i.e. the coefficients of the polynomial $\mathcal{Q}(z)$ are precisely the soliton numbers $\mu(k) \equiv \mu_{i,i+k}$ [with signs as specified by Eq. (8.5)]. Then for, say, q odd the solution of our Diophantine problem take a very elegant form: *a set of soliton numbers $\mu_{i,j}$ is a $\mathbf{Z}_n$-symmetric solution of our problem if and only if the polynomial*

$$z^n + 1 + \sum_k \mu_{i,i+k} z^k, \tag{8.6}$$

is a product of cyclotomic polynomials. In particular we have the bound

$$|\mu_{i,i+k}| \leq \binom{n}{k}. \tag{8.7}$$

Let us give a few simple examples.

1. The basic example is the perturbed A_n minimal model, i.e. the LG theory with superpotential $W(X) = X^{n+1} - tX$. In this case $q = -1$ and the soliton numbers $\mu(k)$ are all equal to 1. Then

$$\mathcal{Q}(z) = \frac{z^{n+1} - 1}{z - 1} = \prod_{\substack{d|(n+1) \\ d \neq 1}} \Phi_d(z).$$

Using the rule (8.4)

$$P(z) = \prod_{\substack{d|(n+1) \\ d \neq 1}} \Phi_d(-z),$$

which gives the usual result for the UV charges of the A_m model.

2. A second example is the Ising n-point function[55] with the spins located at the vertices of a regular n-gon. In this case the soliton numbers $\mu(k)$ are equal 2 for all k's. Then

$$\mathcal{Q}(z) = (z+1)\frac{(z^n-1)}{(z-1)} = \Phi_2(z)\prod_{\substack{d|n \\ d\neq 1}} \Phi_d(z),$$

and (8.4) gives

$$P(z) = \begin{cases} \Phi_1(-z)^n & \text{for } n \text{ even} \\ \Phi_1(-z)^{(n-1)}\Phi_2(-z) & \text{for } n \text{ odd}. \end{cases}$$

Needless to say, the corresponding $U(1)$ charges agree with the physical picture of the Ising correlators. These two examples (and the trivial case $\mu(k)=0$) exhaust the solutions with all soliton numbers equal for $n \geq 4$.
3. $\mu(k)$ is 1 (resp. -1) for $k=1$ and 0 otherwise. Then

$$\begin{aligned} \mathcal{Q}(z) &= (z+1)(z^{n-1}+1) = \Phi_2(z) \prod_{d|(n-1)} \Phi_{2d}(z) \quad (\mu(1)=+1) \\ &= (z-1)(z^{n-1}-1) = \Phi_1(z) \prod_{d|(n-1)} \Phi_d(z) \quad (\mu(1)=-1). \end{aligned}$$

Then

$$\begin{aligned} P((-1)^q z) &= \Phi_1(z) \prod_{d|(n-1)} \Phi_d(z) \quad \mu(1) = -1 \text{ or } n \text{ even} \\ &= \Phi_2(z) \prod_{d|(n-1)} \Phi_{2d}(z) \quad \text{otherwise}. \end{aligned}$$

The case in the first line (and q odd) leads to the charges

$$q_k = \frac{k}{n-1} - \frac{1}{2} \quad (k = 0, 1, \ldots, n-1).$$

It is conceivable that these solutions correspond to the models in (7.15). This is the case for $n = 2, 3$.
4. This last example can be generalized to $\mu(k) = +1$ (resp. -1) for $k = k_0$ and zero otherwise. Then

$$\begin{aligned} \mathcal{Q}(z) &= \prod_{d|k_0} \Phi_{2d}(z) \prod_{l|(n-k_0)} \Phi_{2l}(z) \quad (\mu(k_0) = +1) \\ &= \prod_{d|k_0} \Phi_d(z) \prod_{l|(n-1)} \Phi_l(z) \quad (\mu(k_0) = -1). \end{aligned}$$

Let us return to the general case, i.e. $(n,r) \neq 1$. In each eigenspace for J the situation is exactly as before. Then, in a sector in which J acts by multiplication, the eigenvalues λ of H are

$$\lambda = Jz^m, \tag{8.8}$$

[55] For a discussion of the associated "hyperelliptic" LG models see [15]

where z is a solution to

$$\begin{aligned}
&z^m - J^{-1} + \sum_{k=1}^{(m-1)/2} \sum_{l=1}^{(n,r)} \varrho(k,l)\, J^l (z^{m-k} - J^{-1} z^k) = 0 \quad (m \text{ odd}), \\
&z^m - J^{-1} - \sum_{l=1}^{(n,r)} \varrho\left(\frac{m}{2}, l\right) J^{l-1} z^{m/2} \\
&\quad + \sum_{k=1}^{m/2-1} \sum_{l=1}^{(n,r)} \varrho(k,l)\, J^l (z^{m-k} - J^{-1} z^k) = 0 \quad (m \text{ even}),
\end{aligned} \tag{8.9}$$

and a set $\varrho(k,l) \equiv \mu_{i,i+k+ml}$ of soliton numbers gives a solution to our Diophantine problem iff the roots of the (n,r) polynomials obtained from (8.9) by replacing J with its eigenvalues

$$\exp\left[\frac{2\pi i}{(n,r)} k + \frac{\pi i}{(n,k)} q\right] \qquad (k = 1, 2, \ldots, (n,r)), \tag{8.10}$$

are phases. As an example, take the $\mathbf{C}P^1/\mathbf{Z}_h$ orbifolds in Sect. 7.2. There $n = 2h$, $r = h$, $m = 2$, and $q = -h$. Moreover,

$$\mu(1) = \mathbf{1} - J\,.$$

Then (8.9) becomes

$$(z - J^{-1})(z + \mathbf{1}) = 0\,,$$

so $z^2 = J^{-2}$ *or* $\mathbf{1}$. Then Eq. (8.8) gives

$$\lambda = \begin{cases} J^{-1} \\ J\,, \end{cases}$$

which, in view of (8.10), is what we got by a direct computation in (7.14).

8.2. The $\mathbf{C}P^{n-1}$ σ-Model

The $\mathbf{C}P^{n-1}$ σ-model has Witten index n as it is obvious from its Hodge diamond $h^{p,q} = \delta_{p,q} (p, q = 0, \ldots, n-1)$. They are AF and hence the UV limit is described purely in terms of classical geometry. The UV $U(1)$ charges are equal to the degree of the corresponding harmonic form shifted by minus one half the complex dimension. Then

$$q_j = j - \tfrac{1}{2}(n-1) \qquad j = 0, \ldots, n-1\,,$$

and $\exp[2\pi i q_j] = (-1)^{(n-1)}$. From the arguments in Sect. 5.2 we know that H consists of a single Jordan block associated to this eigenvalue.

Since the Chern class is n times the hyperplane class, a chiral rotation by $2\pi/n$ is anomaly-free and we have a discrete $\mathbf{Z}_n$ symmetry (spontaneously broken by the vacuum). Then the above discussion applies. The same anomaly argument shows that $q = -1$ and that $r = 1$, i.e. the "critical values" w_k are at the vertices of a regular n-gon.

From the above geometrical considerations we see that the characteristic polynomial of H is

$$P(-z) = \begin{cases} (z+1)^n = \Phi_2(z)^n & n \text{ odd} \\ (z-1)^n = \Phi_1(z)^n & n \text{ even}\,. \end{cases}$$

Using (8.4) we get

$$\mathcal{Q}(z) = \Phi_2(z)^n + (z+1)^n = \sum_{k=0}^{n} \binom{n}{k} z^k .$$

Comparing with (8.6) we get the (solitonic) mass spectrum

$$\mu_{i,i+k} = \binom{n}{k} \qquad (k = 1, \ldots, n-1), \tag{8.11}$$

which saturates the bound (8.7). The value of the masses of each kind of solitons can be easily computed from the vacuum geometry in W-space getting [25]

$$m_{i,i+k} = 4n|t|^{1/n} \sin(\pi k/n) ,$$

where the coupling t is defined by the chiral ring relation $X^n = t$ (here X is the chiral primary associated to the hyperplane class).

The result (8.11) can be understood as follows. Since nothing depends on the D-terms, we can take them to correspond to the usual symmetric metric on $\mathbf{C}P^{n-1}$ (i.e. the Fubini metric). Then the isometry group $SU(n)$ is realized as a symmetry of the mass spectrum. Then the k-solitons belong to the k-fold antisymmetric product of the defining $SU(n)$ representation.

With this symmetric choice of the D-term the model becomes exactly solvable and the S matrix has been computed [26]. The mass spectrum extracted from the exact solutions is just (8.11). (Notice that the solitons give the full particle spectrum for these theories. This is typical in solvable models). We stress that the $\mathbf{C}P^{n-1}$ models are confining theories with a very subtle IR structure, see [25] for a discussion.

8.3. *Grassmannian σ-Models*

Next we consider the σ-models with target space the Grassmannian

$$G(N,M) = \frac{U(N+M)}{U(N) \otimes U(M)} .$$

$G(N,M)$ is an NM dimensional complex manifold. Its Poincaré polynomial reads

$$\mathcal{P}_{t,\bar{t}}(G(N,M)) = \sum_{p,q} h_{p,q} t^p \bar{t}^q = \prod_{k=1}^{N} \frac{[1-(t\bar{t})^{M+k}]}{[1-(t\bar{t})^k]} .$$

Since the corresponding σ-model is AF, the classical cohomology fixes the UV behaviour. Hence

$$\mathrm{Tr}_R[t^{J_0}\bar{t}^{\bar{J}_0}]|_{\text{uv limit}} = (t\bar{t})^{-NM/2} \mathcal{P}_{t,\bar{t}}(G(N,M)) .$$

By construction, $\mathrm{tr}\, H^m$ is the limit of this quantity as$(t\bar{t}) \to e^{2\pi i m}$. Then

$$\mathrm{tr}\, H^m = (-1)^{mNM} \frac{(M+N)!}{N!M!} .$$

Then the Witten index is $\binom{M+N}{M}$ and the characteristic polynomial of H reads

$$P(z) = (z - (-1)^{NM})^{(M+N)!/M!N!} .$$

However, the situation is much subtler than in the $\mathbf{C}P^{n-1}$ case. First of all, in this case we have not a $\mathbf{Z}_{(N+M)!/(N!M!)}$ symmetry as above. Even worse, the general theory discussed in this paper does not apply as it stands. In fact we deduced our main formulae under the "genericity" assumption that no three vacua are aligned in W-space. Usually we can choose a suitable arbitrarily small perturbation such that any alignment is destroyed. However there are special "rigid" cases in which the alignment cannot be undone – since all the formal perturbations which would do the job correspond to non-renormalizable interactions which just make no sense in the quantum case. The Grassmannian σ-models are such a "rigid" case. This is not at all a surprise. It is just the physical counterpart of the fact that the $G(N, M)$ are rigid as complex manifolds – i.e. the moduli space is just a point. This rigidity phenomenon may, in principle, lead to a non-completeness of our classification scheme. However it is not a real problem. In fact, on one hand we can extend our theory to these "aligned" situations just by taking into account a few more terms in the IR expansions of Sect. 4. On the other, the rigidly aligned models have a tendency of being so magical that they can be discussed by direct means, as we do below for the Grassmannian case.

From a direct path integral analysis (summarized in Appendix A) one learns that

$$G(N, M) \doteq (\mathbf{C}P^{N+M-1})^N // S_N\,, \tag{8.12}$$

where $\doteq$ means equivalence [56] as QFT's for the corresponding σ-model. The RHS of (8.12) is a tensor product of N copies of the $\mathbf{C}P^{N+M-1}$ σ-model reduced by the action of the replica symmetry S_N. The double slash in (8.12) is there to remind the reader that *it is not* the S_N orbifold. Rather (topologically speaking) it is the set of *maximal* S_N orbits, i.e. orbits whose elements are *not* fixed by any non-trivial subgroup of S_N. This construction is what was called the "change of variable trick" in [10] (unfortunately this name is appropriate only for the LG case). Morally speaking, (8.12) is the QFT counterpart of the standard description [49–51] of the quantum cohomology ring of $G(N, M)$ in terms of N copies of the (perturbed) A_{N+M} minimal model.

Let us recall how the "change of variable trick" works. One has a map f

$$f : \mathscr{R} \hookrightarrow \mathscr{R}_*$$
$$\phi_i \mapsto f(\phi_i)\det[\partial f] \in \mathscr{R}_*\,,$$

which identifies isomorphically (as $\mathscr{R}$-modules) $\mathscr{R}$ with its image. Then the tt* metric for $\mathscr{R}$ is the pull back *via* f of that for $\mathscr{R}_*$. This construction differs in many respects from an orbifold. In particular it changes the central charge $\hat{c}$. One has [10]

$$\hat{c} = \hat{c}_* - 2q_*(J)\,, \tag{8.13}$$

where $q_*(J)$ is the $U(1)$ charge of the Jacobian $J = \det[\partial f]$ computed in the $*$ theory. In the present case the $*$ theory is just N copies of the $\mathbf{C}P^{N+M-1}$ σ-model.

Let $X_\alpha (\alpha = 1, \ldots, N)$ be the chiral primary associated to the hyperplane class of the α^{th} copy of $\mathbf{C}P^{N+M-1}$. Then the map f reads

$$f_i(X_\alpha) = \sigma_i(X_\alpha) \qquad i = 1, 2, \ldots, N\,,$$

where σ_i is the i^{th} elementary symmetric polynomial. Its Jacobian is

$$J = \Delta(X_\alpha) \equiv \prod_{\alpha>\beta} (X_\alpha - X_\beta)\,.$$

[56] More precisely, equivalence up to a deformation of the D-term

We know from the previous subsection that the UV charge of the $\mathbf{C}P^{N+M-1}$ operator X_α is 1. Then

$$q_*(J) = \tfrac{1}{2}N(N-1).$$

Using (8.13), the UV central charge of the rhs of (8.12) is

$$\hat{c}_{\text{uv}} = N(N+M-1) - N(N-1) = NM,$$

which is the correct result for $G(N,M)$ (i.e. its complex dimension).

Let $|k_\alpha\rangle_\alpha$ be the canonical vacuum for the α^{th} copy of $\mathbf{C}P^{N+M-1}$ at the $k_\alpha{}^{\text{th}}$ critical point, i.e.

$$X_\alpha(k_\alpha) = t^{1/(N+M)}\exp[2\pi i k_\alpha/(N+M)] \qquad (k_\alpha = 0, 1, \ldots, N+M-1).$$

Then the canonical vacua for the σ-model on $(\mathbf{C}P^{N+M-1})^N$ are just

$$\bigotimes_{\alpha=1}^{N} |k_\alpha\rangle_\alpha. \tag{8.14}$$

To get the canonical vacua for the model in the rhs of (8.12) out of those in (8.14) we have to perform three elementary operations [10]:

i) Kill the states (8.14) which are in the kernel of the chiral field J. Since in the present case J is just the Vandermonde determinant, this means that we must keep only the states (8.14) such that the N numbers k_α *are all distinct*,

ii) Project into the appropriate subsector projectively-invariant under S_N. This is done just by summing over all permutations with signs as prescribed by the Jacobian

$$\sum_{s\in S_N}\bigotimes_{\alpha=1}^{N}\frac{\Delta(k_{s(\alpha)})}{\Delta(k_\alpha)}|k_{s(\alpha)}\rangle_\alpha = \pm\sum_{s\in S_N}\sigma(s)\bigotimes_{\alpha=1}^{N}|k_{s(\alpha)}\rangle_\alpha,$$

where $\sigma(s)$ is the signature of the permutation s. Such a state is determined by the *unordered* N-tuple k_i. Since the k_i's can take $N+M$ values, in this way we get $\binom{N+M}{N}$ states, i.e. as many as the Witten index for the $G(N,M)$ σ-model.

iii) Normalize the states so obtained. Then the canonical $G(N,M)$ vacua are

$$|\{k_1,k_2,\ldots,k_N\}\rangle = \frac{1}{\sqrt{N!}}\sum_{s\in S_N}\sigma(s)\bigotimes_{\alpha=1}^{N}|k_{s(\alpha)}\rangle_\alpha, \tag{8.15}$$

with $0 \le k_1 < k_2 < k_2 < \ldots < k_N \le N+M-1$.

Then the $G(N,M)$ tt* metric reads

$$\begin{aligned}\langle\overline{\{h_1,h_2,\ldots,h_N\}} \mid \{k_1,k_2,\ldots,k_N\}\rangle &= \frac{1}{N!}\sum_{s,t\in S_N}\sigma(s)\sigma(t)\prod_{\alpha=1}^{N}\langle\overline{h_{t(\alpha)}} \mid k_{s(\alpha)}\rangle_\alpha \\ &= \det{}_{\{\overline{h_\alpha}\},\{k_\beta\}}[\langle\overline{h_\alpha} \mid k_\beta\rangle],\end{aligned} \tag{8.16}$$

where $\langle\bar{h} \mid k\rangle$ is the $(N+M)\times(N+M)$ matrix giving the ground state metric for the $\mathbf{C}P^{N+M-1}$ σ-model in a canonical basis, and $\det_{\{\overline{h_\alpha}\},\{k_\beta\}}$ means the determinant of the $N\times N$ minor obtained by selecting the rows $(h_1,h_2,\ldots,h_N)$ and the columns $(k_1,k_2,\ldots,k_N)$. Of course $G(M,N) = G(N,M)$ but the rhs of (8.12) is not manifestly invariant under $N \leftrightarrow M$. Instead the final answer (8.16) is manifestly "duality" invariant. We begin to show this in the simpler case $G(1, M) = \mathbf{C}P^M$. Consider then $G(M, 1)$. Its tt* metric is given by the $M \times M$ minors of the usual $\mathbf{C}P^M$ metric $g_{i\bar{j}}$. Let $|\hat{k}_j\rangle = |\{k_1, \ldots, \hat{k}_j, \ldots, k_n\}\rangle$ (where hat means omitted). Rewriting the minors in terms of the inverse metric $g^{\bar{j}i}$, one gets

$$\langle \overline{\hat{h}_j} \mid \hat{k}_i \rangle = (-1)^{i+j} \det[g] g^{\bar{j}i} \equiv (-1)^{i+j} g_{i\bar{j}} \, ,$$

where we used that g is orthogonal. This shows duality invariance for $N = 1$ (the signs $(-1)^{i+j}$ can be absorbed in the definition of the states). The general case $G(M, N)$ is handled analogously using well known properties of minors.

To get the IR (resp. UV) behaviour of (8.16) we have just to insert the known asymptotics for the $\mathbf{C}P^*$ case. For instance, for large β we have

$$\langle \bar{h} \mid k \rangle \cong \delta_{kh} - i \,\mathrm{sign}(k-h) \binom{N+M}{|k-h|} \frac{1}{\pi} K_0(m_{kh}\beta) + \ldots \, ,$$

where

$$m_{kh} = 4(N+M)\, |t|^{1/(N+M)} \sin\left(\frac{\pi|k-h|}{N+M}\right) ,$$

which inserted into (8.16) gives the $G(N, M)$ mass spectrum.

8.4. Applications to "Polytopic" Models

One of the nicest aspects of the tt* equations is that, once you solved a model you easily generalize your result to a whole family of models having the same vacuum geometry in W-space. This strategy was exploited in [14] for the "$\mathbf{Z}_n$-models," i.e. theories whose critical values form the vertices of a regular n-gon. In the same way, the solution (8.16) generalizes to a family of models with a certain "polytopic" vacuum geometry [52]. The general model in the family is obtained by replacing in (8.12) the $\mathbf{C}P^{N+M-1}$ σ-model by another member of the same $(N+M)$-gon family. The tt* metric is still given by (8.16) but with $\langle \overline{h_\alpha} \mid k_\beta \rangle$ replaced by the metric for the given $\mathbf{Z}_{N+M}$ model.

The simplest model in this family is the Kazama-Suzuki Grassmannian coset at level 1 [53] perturbed by the most relevant operator. One has [49, 51]

$$\frac{U(N+M)_1}{U(N) \otimes U(M)} \doteq (A_{N+M})^N // S_N \, ,$$

where A_{N+M} denotes the minimal model deformed by the most relevant operator. Then the mass-spectrum for this model is obtained by inserting example 1 of Sect. 8.1 in the rhs of (8.16). The result has the properties expected on various grounds (see e.g. [52]).

9. Conclusions

We have initiated a program to classify massive $N = 2$ supersymmetric theories in two dimensions. This classification is up to variation in D-terms, and may be viewed, by considering the UV limit, as a classification program for $N = 2$ SCFT's (which admit massive deformation). The central object in this classification program is a generalization of "Dynkin diagram" each node of which represents a non-degenerate $N = 2$ vacuum, and the number of lines between the nodes just counts the number of solitons (which saturate the Bogomolnyi bound) between the vacua.[57] We saw

[57] As we discussed before, there is an additional sign which is important

that perturbations of the theory change the soliton number and modify the Dynkin diagram by the action of Braid group (which is generated by the generalized "Weyl reflections"). We discussed what are the restrictions on these generalized Dynkin diagrams in particular by the condition of reality of $U(1)$ charges of Ramond ground states, which is computable from the Dynkin diagram. We classified all massive $N=2$ theories with up to three vacua. We also rederived the classification of $N=2$ minimal models. We saw that the Dynkin diagram corresponding to the minimal $N=2$ models turns out to be just the usual A–D–E Dynkin diagram.

As a sub-classification we can use these methods to classify (up to mirror symmetry) Kähler manifolds with diagonal Hodge numbers with $c_1 > 0$. We discussed how this works in a particular example (which leads to a known mathematical theorem). It would be very interesting to continue this line of thought and obtain a complete classification of such Kähler manifolds.

We can also use the above models to construct new string vacua. All we have to do is to make sure that $\hat{c} =$ integer and use an orbifold method [46].

The most important open question is "reconstruction." In other words for each of the generalized Dynkin diagrams which are allowed for us can we construct a quantum field theory with that solitonic spectra? Some of the examples we discussed in the main text suggests that this may be possible in the form of "generalized" Toda models constructed from the corresponding generalized Dynkin diagrams. This may also suggest that there is always an integrable deformation of the $N=2$ theory, with a particular choice of D- *and* F-terms. This would be very interesting to develop further. In particular it would be interesting to see if these models are related to (supersymmetric version of) RSOS-like models which have our Dynkin diagram as target space.

Another direction worth investigating is the study of tt* equations directly in the conformal case (in the case of three-folds this is known as *special geometry* [54]). One generically studies the moduli space of these theories, which is the analog of w_i for us here. In our case the natural degeneration point of moduli space are the UV and IR limits, whereas in the conformal case we will have a number of degenerate points (or submanifolds) on moduli space. The solution to the tt* equation will undergo a monodromy around each of these degeneration points. Then what we should do is *to classify all possible representations of the monodromy group* that are consistent with the existence of global regular solutions to tt*. This would mean that we begin to classify all the Calabi-Yau manifolds at once by studying all the possible consistent monodromies on the degeneration points of their moduli spaces. This would be the massless analog of the classification program we have initiated in the massive case here. We intend to return to this idea to classify $N=2$ SCFT's (and thus Calabi-Yau manifolds up to mirror symmetry) in future work.

Appendix A. Exact Path-Integral Computations for σ-Models

In this appendix we want to show Eq. (8.12) by exact path-integral computations. Since the "change of variable trick" has a simple meaning in the LG case, it will be helpful if we could write down "LG" models [58] which are *exactly* equivalent (as QFT's) to our σ-models. For the $\mathbf{C}P^{n-1}$ case this was done long ago by the authors

[58] We put LG in quotes because it is not really a Landau-Ginzburg model. For the purposes of the present appendix the naive interpretation of the effective theory as a LG model is good enough and we shall stick to this naive viewpoint

of [55]. In this appendix we extend their resul to the Grassmannian case. As an aside, this will give us a rigorous path-integral proof of the quantum cohomology ring [59] for Grassmannians as predicted on general grounds in [49–51]. However here – as well as in [55] – one looks for an equivalence at the full QFT level, not just for its topological sector.

In the old days when the authors of [55] obtained their result little was known about $N = 2$ field theories and computations were quite hard. In those days [55] was quite an analytic triumph. Luckily enough, nowadays $N = 2$ theory is so developed that even more sophisticated models can be analyzed without real effort.

Let us begin by a (modern) review of their work. Then we shall generalize to the Grassmannians $G(N, M)$.

The $\mathbf{C}P^{n-1}$ *Model.* The starting point [55] is the "homogeneous coordinates" formulation of the model. The Lagrangian reads

$$\int d^4\theta \left[\sum_{i=1}^{n} \bar{S}_i e^{-V} S_i + \frac{A}{2\pi} V \right], \tag{A.1}$$

where S_i are chiral superfields which map into the homogeneous coordinates on $\mathbf{C}P^{n-1}$ and V is a Legendre multiplier real superfield. In (A.1) we have denoted the coupling constant by A because it has the geometrical interpretation of the area of the basic 2-cycle generating the homology of $\mathbf{C}P^{n-1}$. The field V gauges the $\mathbf{C}^\times$ acting diagonally on $\mathbf{C}^n$, so the physical degrees of freedom are $\mathbf{C}^n/\mathbf{C}^\times \cong \mathbf{C}P^{n-1}$. Explicitly, eliminating V using its equations of motion one gets

$$\frac{A}{2\pi} \int d^4\theta \log \left[\sum_{i=1}^{n} \bar{S}_i S_i \right],$$

that is the usual formulation of the σ-model (for the Fubini metric). From its equations of motion we see that V is nothing else than the susy version of the pull back of the $U(1)$ part of the Fubini spin-connection. Then its field-strength superfield

$$nX = D^+ \bar{D}^- V\,, \tag{A.2}$$

(which is a (c, a) field in the notation of [49]) is the susy analog of the (pull-back of the) trace part of the $\mathbf{C}P^{n-1}$ Riemann tensor, i.e. it is the (c, a) primary operator associated with the first Chern class. Since $c_1(\mathbf{C}P^{n-1}) = n$, the observable X in (A.2) is the basic chiral primary associated with the hyperplane class. This is most easily seen by looking at the last (i.e. auxiliary) component of the superfield X. Up to a normalization coefficient this is

$$(D + i * \phi^* R)\,, \tag{A.3}$$

where D is the auxiliary field of the real superfield V, and $\phi^* R$ is the pull-back to the world-sheet fo the Ricci form.

Now the idea [55] is to perform the Gaussian integral over the $\bar{S}_i$'s *exactly*. Denoting the result by $\exp\{-S[V]\}$, this gives an equivalent formulation of the quantum model in terms of the (super)field V with action $S[V]$. We stress that this

[59] Similar results have been obtained from a more mathematical standpoint by F. Franco and C. Reina (to appear)

procedure is exact. By gauge-invariance the action should depend on the field-strength superfield only. Then it should have the general form

$$S[X,\bar{X}] = \int d^2\theta W(X) + \int d^2\bar{\theta}\bar{W}(\bar{X}) + D\text{-term}. \tag{A.4}$$

A priori we are not guaranteed that $S[X,\bar{X}]$ is *local*. However any non-locality is in the D-term.[60] Since for the purposes of this paper we can change the D-term at will, we can forget about any problem the action (A.4) may have. By the same token, we do not need to compute every detail of the rhs of (A.4) either. Computing $W(X)$ is good enough. In order to extract $W(X)$ from $S[X,\bar{X}]$ notice that the D-terms either contain higher powers of the auxiliary field D or derivatives of D. Instead the term linear in D (at vanishing momentum) reads

$$D\left(\frac{\partial W}{\partial X} + \frac{\partial \bar{W}}{\partial \bar{X}}\right),$$

so to extract W it is enough to get the term linear in D in (A.4). By the same argument, we can as well assume that all fields are constant (and the fermions vanish). Then the computation reduces to that of the determinants of differential operators *with constant coefficients*. Despite this dramatic simplification, the computation is still exact!

Expanding out the action (A.1) in components, we get

$$\exp[-S[X,\bar{X}]]\big|_{\substack{\text{vanishing}\\ \text{momentum}}} = e^{-A\int d^2\theta X/2\pi}\left(\frac{\operatorname{Det}\left[\partial\!\!\!/ + \begin{pmatrix} 0 & X\\ \bar{X} & 0\end{pmatrix}\right]}{\operatorname{Det}[-\partial^2 + (D + X\bar{X})]}\right)^n, \tag{A.5}$$

[the exponential prefactor is the classical value of the action at $S_i = 0$; see the last term in (A.1)]. Taking the derivative of the rhs with respect D and setting $D = 0$, we get

$$\left(\frac{\partial W}{\partial X} + \frac{\partial \bar{W}}{\partial \bar{X}}\right) = \frac{A}{2\pi} + n\operatorname{Tr}\left[\frac{1}{-\partial^2 + X\bar{X}}\right]. \tag{A.6}$$

The trace in the rhs is easily evaluated by ζ-regularization

$$\begin{aligned}\operatorname{Tr}[(-\partial^2 + X\bar{X})^{-s}] &= \int \frac{d^2p}{(2\pi)^2}\frac{1}{(p^2 + X\bar{X})^s} = \frac{1}{2\pi}\frac{1}{(1-s)}(X\bar{X})^{1-s}\\ &= \frac{1}{2\pi(1-s)} + \frac{1}{2\pi}\log X + \frac{1}{2\pi}\log\bar{X} + O(1-s).\end{aligned}$$

As $s \to 1$ this has a pole; comparing with (A.5) we see that the only effect of this infinity is to renormalize the coupling A. We can just forget about this infinity provided we replace A by its running counterpart $A(\mu)$. After having subtracted the infinity, take $s \to 1$. Notice that the rhs of (A.6) is a harmonic function of X as it should; this is a nice consistency check. Integrating (A.6) we get [55]

$$2\pi W(X) = X(\log X^n - n + A(\mu) - i\vartheta), \tag{A.7}$$

where ϑ is a real parameter. Comparing with (A.3) we see that ϑ is the usual instanton angle. The quantum cohomology ring of $\mathbf{C}P^{n-1}$ is just

$$\mathscr{R} = \mathbf{C}[X]/\partial W = \mathbf{C}[X]/(X^n - e^{-A(\mu)+i\vartheta}),$$

which is Witten's result [56].

[60] Why? Because if you do the same analysis in the TFT case you do not have any problem with non-locality

Grassmannian σ-Model. Now we generalize the above approach to the Grassmannians $G(N, M)$. Again we have a "homogeneous" formulation. Now the chiral fields S_{ia} have two indices, a "gauge" $U(N)$ index i, and a "flavour" $SU(N+M)$ index a. The Lagrangian reads

$$\int d^4\theta \Big(\sum_a \bar{S}_a e^{-V} S_a + \alpha \operatorname{tr} V \Big),$$

where now V is a $N \times N$ matrix of superfields which gauge $U(N)$. Also the field-strength superfield X belongs to the adjoint rep. of $U(N)$, and then is gauge *covariant* rather than invariant as before. The basic gauge-invariant objects are the Ad-invariant polynomials in the field-strengths X. Their ring is generated by the superfields $Y_i (i = 1, 2, \ldots, N)$ defined by

$$\det[t - X] = t^N + \sum_{k=1}^{N} (-1)^t t^{N-k} Y_k .$$

Contrary to the X's, the Y's are *bona fide* (c, a) superfields. They generate the (quantum) cohomology ring $\mathscr{R}$ for $G(N, M)$ (as it can be shown by going to the classical limit). A priori computing the determinants is now quite a mess, since everything is non-Abelian. Anyhow we shall use the same strategy as before, i.e. to use our non-perturbative knowledge of the $N = 2$ theories to replace the actual computation with a trivial – but still exact – one.

Again we can take all the background fields constant (and fermions vanishing). Then we make the following observation: at the TFT level the X's and the $\bar{X}$'s do not talk to each other (in fact the $\bar{X}$'s are just gauge-fixing devices) and we can assume, with no loss of generality, that X and $\bar{X}$ (as matrices) commute. Then X is diagonalizable

$$X = \operatorname{diag}(X_1, X_2, \ldots, X_N) . \tag{A.8}$$

Moreover, with probability 1, all X_i are distinct. Consider the superfield $Z_a \equiv \partial W(X)/\partial X_a$ (a is an adjoint rep. index for $U(N)$). Obviously it belongs to the adjoint rep. of $U(N)$. By gauge invariance, $W(X)$ is an Ad-invariant function of the X's. But then, in a Cartan background like (A.8), also Z_a belongs to the Cartan subalgebra (by invariance under the corresponding maximal torus). Given that the terms in $S[X, \bar{X}]$ which are linear in the auxiliary fields D_a should have the form $(D_a Z_a + \text{h.c.})$, we see that no information is lost if we restrict D_a too to the Cartan subalgebra ($= U(1)^N$). But then the full background is Abelian and the functional determinants are just the same as in the $\mathbf{C}P^{n-1}$ case.

Therefore, in a Cartan background, $W(X)$ is just the sum of N copies of what we got for the $\mathbf{C}P^{N+M-1}$ model. But the Cartan background fixes the theory completely. Then

$$2\pi W(X_1, X_2, \ldots, X_N) = \sum_{k=1}^{N} X_k (\log X_k^{N+M} - n + A(\mu) - i\vartheta) .$$

Again we have the relations $X^{N+M} = \text{const}$. However, this time the good gauge-invariant fields are the Ad-invariant polynomials in the X_i, which are generated by the elementary symmetric functions, that is the fields Y_i. Thus the quantum cohomology ring of $G(N, M)$ is the ring generated by the symmetric functions in N indeterminates X_i subject to the relations $X_i^{N+M} = 1$.

This result is not new (at least as far as the classical part is concerned) and was obtained (or found to be very plausible in the quantum case) in [49–51] from quite different considerations. However here we have shown a much stronger result than just computing $\mathscr{R}$. In fact, we have considered a topological truncation of a computation (that of $S[X, \bar{X}]$) which makes perfect sense in the *full* QFT. To put it differently, our "topological" map from $\mathscr{R}_{\text{Grass.}}$ to $\otimes\mathscr{R}_{CP}//S_N$ is induced by a map between the corresponding QFT's. Of course, we have no explicit form for the parent QFT map. However, the pure fact that this map exists and restricts nicely to the topological one, has quite dramatic implications. It shows that both the tt* differential equation *and* their boundary data agree on the two sides of the "dotted"-equality (8.12). In view of the theory we have developed in the main body of the paper this is enough to fix the (solitonic) mass-spectrum of the $G(N, M)$ σ-model.

Appendix B. Subtleties with Collinear Vacua: An Explicit Example

In Sect. 4 we saw that special phenomena take place when three vacua are aligned:
a) We have the "half-soliton" mechanism of Eq. (4.45). When we deform slightly the picture by putting the middle vacuum on one side of the line connecting the other two vacua the number of solitons connecting these two vacua change. For the exactly aligned situation the large β asymptotics looks as if we had

$$\tfrac{1}{2}\,(\mu_{13} + \mu'_{13})\,,$$

where μ_{13} and μ'_{13} refer to soliton numbers *if* the middle vacuum was perturbed one way or another. Of course, there is no such a thing as a "half-soliton." This discontinuity in the IR asymptotics just signals that the IR asymptotic series is not uniform (as always). This is clear from the analysis of Sect. 4.
b) We have various possibilities for the power of β in front of the Boltzmann exponential, see e.g. Eqs. (4.46), (4.47). Physically this is a consequence of the fact that there are states with different numbers of solitons and the same energy. These states have the same Boltzmann exponential but a different phase-factor.

The purpose of this appendix is to illustrate these two points in a concrete example where explicit computations are possible. Consider the LG model with superpotential

$$W(X) = \frac{X^6}{6} - \frac{tX^2}{2}\,. \tag{B.1}$$

In X space the critical points are $X_0 = 0$ and $X_k = i^k t^{k/4}$, ($k = 1, \ldots, 2$). They are at the vertices and center of a quadrate. The critical values are

$$W_0 = 0\,, \qquad W_k = (-1)^{k+1}\,\tfrac{1}{3}\,t^{3/2}\,.$$

We have only three distinct critical values, and these three points are collinear. The naive picture of solitons would suggest that the "fundamental" solitons are the inverse images of the segments connecting the W_k's to W_0, i.e. the half-diagonals of the square in X-space. All other pairs of vacua are connected by a multi-soliton process only. Is this naive picture correct? It better be wrong, since it leads to paradoxes when compared to our general theory. Luckily we can do exact computations to see what is going on.

The ground-state metric for (B.1) was computed in [10]. There it was also checked that this is the only regular solution and that it reproduces the known results in the

UV limit. The non-vanishing elements are [10]

$$
\begin{aligned}
\langle \bar{1} \mid 1 \rangle &= \frac{1}{|t|}\, e^{-u(z)}\,, \\
\langle \bar{1} \mid X^4 \rangle &= \frac{t}{|t|}\, e^{-u(z)}\,; \qquad \langle \overline{X^4} \mid 1 \rangle = \frac{\bar{t}}{|t|}\, e^{-u(z)}\,, \\
\langle \bar{X} \mid X \rangle &= \frac{1}{|t|^{1/2}} \exp\left[-\tfrac{1}{2}\, u(2z)\right], \\
\langle \overline{X^2} \mid X^2 \rangle &= 1\,, \\
\langle \overline{X^3} \mid X^3 \rangle &= |t|^{1/2} \exp\left[\tfrac{1}{2}\, u(2z)\right], \\
\langle \overline{X^4} \mid X^4 \rangle &= |t| \cosh[u(z)]\,,
\end{aligned}
\tag{B.2}
$$

where $u(z)$ is the regular PIII transcendent (cf. Sect. 6.1) with $r = -2/3$ and

$$z = \tfrac{2}{3}\, |t|^{3/2}\beta = 2|W_k - W_0|\beta\,.$$

Let $|f_k\rangle$ ($k = 0, 1, \dots, 4$) be the canonical vacuum associated with each critical point X_k. Then in the canonical basis (B.2) becomes ($r, s \neq 0$)

$$
\begin{aligned}
\langle \bar{f}_0 \mid f_0 \rangle &= \cosh[u(z)]\,, \\
\langle \bar{f}_0 \mid f_r \rangle &= \frac{i}{2}\, \sinh[u(z)]\,, \\
\langle \bar{f}_s \mid f_r \rangle &= \tfrac{1}{4}\, \Big[i^{(s-r)} \exp\left[-\tfrac{1}{2}\, u(2z)\right] + (-i)^{(s-r)} \exp\left[\tfrac{1}{2}\, u(2z)\right] \\
&\quad + (-1)^{(s-r)} + \cosh[u(z)] \Big]\,.
\end{aligned}
$$

Let us study the large z asymptotics of this solution. One has ($r, s \neq 0$)

$$u(z) = -\frac{2}{\pi}\, K_0(z) + O(e^{-2z})\,.$$

Then one has ($r, s \neq 0$)

$$
\begin{aligned}
\langle \bar{f}_0 \mid f_0 \rangle &= 1 + \frac{2}{(\pi)^2}\, K_0(2z)^2 + O(e^{-3z})\,, \\
\langle \bar{f}_0 \mid f_r \rangle &= -i\, \frac{1}{\pi}\, K_0(z) + O(e^{-3z})\,, \\
\langle \bar{f}_s \mid f_r \rangle &= \frac{1}{4} \left\{ i^{(s-r)} \left[1 + \frac{1}{\pi}\, K_0(2z) \right] + (-i)^{(s-r)} \left[1 + \frac{1}{\pi}\, K_0(2z) \right] \right. \\
&\quad \left. + \left[1 + (-1)^{(s-r)}\right] + \frac{2}{(\pi)^2}\, K_0(z)^2 \right\} + O(e^{-3z})\,.
\end{aligned}
\tag{B.3}
$$

So,

$$\langle \bar{f}_i \mid f_j \rangle|_{z=\infty} = \delta_{ij}\,.$$

From (B.3) we see that in the IR expansion of $\langle \bar{f}_s \mid f_r \rangle$ with $r \neq s$ and $r, s \neq 0$ there are two kinds of contributions of order $O(\exp[-2z])$, those of the form $K_0(2z)$ and those of the form $K_0(z)^2$. The power-law in front of the exponential is $\beta^{-1/2}$ and β^{-1} respectively as expected on the basis of Eqs. (4.45), (4.47). This is phenomenon b).

To get the phenomenona a), just extract "μ_{ij}" as the coefficient of the leading terms with a $\beta^{-1/2}$ power-law. Then

$$\mu_{0r} = 1\,,$$
$$\text{“}\mu_{rs}\text{”} = \frac{i}{4}\,[i^{(s-r)} - (-i)^{(s-r)}] = \begin{cases} -\frac{1}{2} & \text{for } s = r+1 \bmod 4\,, \\ 0 & \text{for } s = r+2 \bmod 4\,, \\ \frac{1}{2} & \text{for } s = r+3 \bmod 4\,. \end{cases}$$

This shows how the a) phenomenon appears.

Appendix C. Conjectures on the OPE Coefficients

From one point of view our work here may be seen as a generalization of the connection formula for PIII as discussed in Sect. 6.1. However for PIII the authors of [18, 22] did a better job, since their result not only allows us to compute the UV $U(1)$ charges but also the UV ground state metric, or equivalently the absolute normalizations for the OPE coefficients (see [10] for a number of explicit examples). Then it is natural to ask what we can do in the direction of computing OPE coefficients for the general case.

In this appendix we show how the number-theoretical nature of our classification program may reduce the computation of these normalization factors to the so-called *standard* conjectures of number theory and algebraic geometry. Here we present some preliminary thoughts in this direction. It may seem that there is not much point in building conjectures over "facts" which are themselves conjectures. However sometimes conjectures may be deeper than established facts!

In order to formulate our fancies we should rephrase our Diophantine problem in more abstract terms.

The crucial point is to realize that we have a lattice $\mathscr{L} \in \mathscr{R}$. A chiral primary operator $\mathscr{O}$ belongs to $\mathscr{L}$ iff

$$\mathscr{O} = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n\,, \qquad a_i \in \mathbf{Z}\,,$$

where e_i are the idempotents of $\mathscr{R}$, i.e. the elements of the "point basis." Then the integral elements in the topological Hilbert space $\mathscr{H}$ are those of the form

$$|\mathscr{O}\rangle \equiv a_1\,|e_1\rangle + a_2\,|e_2\rangle + \ldots + a_N\,|e_n\rangle\,, \qquad a_i \in \mathbf{Z}\,,$$

where the map $e_i \to |e_i\rangle$ is the spectral-flow *as realized by the topological path-integral.* We denote this $\mathbf{Z}$-module as $\mathscr{H}_{\mathbf{Z}}$ and consider the $\mathbf{Q}$-space $\mathscr{H}_{\mathbf{Q}} = \mathscr{H}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$. The most important fact about the lattice $\mathscr{H}_{\mathbf{Z}}$ is that *it is preserved by the monodromy* H as a consequence of the integrality of the number of soliton species. We can introduce a natural "Hodge decomposition" of the space $\mathscr{H}$. In general it is a *mixed* one. To make things as easy as possible, here we assume that this additional complication is not present in the model of interest. More concretely, we assume that the characteristic polynomial $P(z)$ has the form

$$P(z) = \prod_{m_i \text{ distinct}} \phi_{m_i}(z)\,. \tag{C.1}$$

The Hodge decompositon is defined by declaring that the subspace $H^{p,\hat{c}-p} \subset \mathscr{H}$ consists of the states $|\mathscr{O}\rangle$ with UV behaviour

$$\langle \bar{\mathscr{O}} \mid \mathscr{O}\rangle \sim \beta^{-2p} \quad \text{as} \quad \beta \to 0\,.$$

For a σ-model on a CY space this definition of "type" (p, q) corresponds to the usual one (up to "mirror symmetry") but in general p and q are not even integral (however they are always in $\mathbf{Q}$). As it is well known, the data $X = \{\mathscr{H}_{\mathbf{Q}}, \otimes_p H^{p,\hat{c}-p}\}$ is a Hodge structure (specified up to isogeny).

Among all Hodge structures there are special ones having peculiar number theoretical properties. Let $\mathscr{F}$ be an Abelian extension[61] of the field $\mathbf{Q}$, and let f be its transcendency degree. We say that a Hodge sub-structure $M \subset X$ has *complex multiplication* by $\mathscr{F}$ if it has rank f as a $\mathbf{Z}$-module and there is an injection of $\mathscr{F}$ into $\mathrm{End}(\mathscr{H}_{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Q}$.

In this language the statements around Eq. (6.3) can be rephrased by saying that to each cyclotomic factor there corresponds a Hodge sub-structure M_{m_j} of rank $\phi(m_j)$ with complex multiplication by the cyclotomic field $\mathbf{Q}(e^{2\pi i/m_j})$. Over $\mathbf{Q}$ (i.e. modulo isogeny) the subspace M_{m_j} is defined by

$$\Phi_{m_j}(H) M_{m_j} = 0 .$$

The product is defined as follows: The element (here $\zeta_j = e^{2\pi i/m_j}$ and $z_i \in \mathbf{Q}$)

$$z_0 + z_1\zeta_j + z_2\zeta_j^2 + \dots z_{m_j-1}\zeta_j^{m_j-1} \in \mathbf{Q}(\zeta)$$

acts on M_{m_j} as the linear operator

$$z_0 + z_1 H + z_2 H^2 + \dots z_{m_j-1} H^{m_j-1} \in \mathrm{End}(\mathscr{H}_{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Q} . \tag{C.2}$$

The interest of this point of view for physics stems from the fact that in the presence of complex multiplication there are standard results (conjectures) for the corresponding period maps. In the $N = 2$ language this means that we can predict the *normalized* UV OPE coefficients in terms of characters for the cyclotomic fields.

This is done as follows. Consider[62] an operator $\mathscr{O}_1$,

$$\mathscr{O}_1 = \sum_i \kappa_i e_i , \qquad \kappa_i \in \bar{\mathbf{Q}} ,$$

satisfying (to save print we write m for m_j)

$$H\mathscr{O}_1 = e^{2\pi i r/m}\mathscr{O}_1 , \qquad (r, m) = 1 .$$

Such an operator always exists as discussed in the main body of the paper. Let $p(1)$ be the "type" of $|\mathscr{O}_1\rangle$. From (C.2) we see that the one-dimensional subspace spanned by $|\mathscr{O}_1\rangle$ carries a representation φ_1 of the cyclotomic field $\mathbf{Q}(\zeta_m)$. Let $l \in (\mathbf{Z}/m\mathbf{Z})^\times \cong \mathrm{Gal}(\mathbf{Q}(\zeta_m)/\mathbf{Q})$ be an element of the Galois group of this cyclotomic extension. Since l corresponds to an automorphism of $\mathbf{Q}(\zeta_m)$, $\phi_l = l \circ \phi_1$ is also a one-dimensional representation of $\mathbf{Q}(\zeta_m)$ on M_m. Let $|\mathscr{O}_l\rangle$ be a state spanning the corresponding representation. This state has also a definite "type" $p(l)$.

For $a \in \mathbf{Z}$, we write $\langle a \rangle$ for unique number $0, 1, 2, \dots, m-1$ congruent to $a \bmod m$. Then we introduce a function $f(a): \mathbf{Z}/m\mathbf{Z} \to \mathbf{Q}$ by[63]

$$p(l) = \frac{1}{m} \sum_{a=0}^{m-1} f(a) \langle la \rangle . \tag{C.3}$$

[61] I.e. a Galois extension whose Galois group is Abelian

[62] As usual $\bar{\mathbf{Q}}$ denotes the algebraic closure of $\mathbf{Q}$

[63] This definition does not fix $f(a)$ uniquely but the ambiguity is immaterial. The existence of $f(a)$ is a consequence of PCT together with a lemma by Deligne

Then the general period conjecture can be restated as follows. We fix the moduli w_i in such a way that $w_i - w_j \in \bar{\mathbf{Q}}$ for all i, j. Then as $\beta \to 0$,

$$\langle \overline{\mathscr{O}_1} \mid \mathscr{O} \rangle|_{\beta \sim 0} = \frac{z}{\beta^{2p(1)}} \prod_{a=0}^{m-1} \Gamma\left(1 - \frac{a}{m}\right)^{[f(a)-f(-a)]}, \tag{C.4}$$

where z is a "trivial kinematical factor" belonging to $\bar{\mathbf{Q}}$.

It is tempting to conjecture the validity of this statement in general. As evidence for this we discuss the A_n minimal models.

Example: The A_n Minimal Models. As an example consider the LG models $W = X^{n+1}$ + lower degree, where the coefficients are assumed (for convenience) to be rational numbers. To make things even easier, we assume $n+1$ to be an odd prime [64] p. Since $H^{2p} = 1$, (C.2) will give a complex multiplication by $\mathbf{Q}(\zeta_{2p})$. Of course this is the same as $\mathbf{Q}(\zeta_p)$. To rewrite the action in a canonical $\mathbf{Q}(\zeta_p)$ form it is sufficient to change sign to H, since $\Phi_p(-H) = 0$. Then $\mathscr{R} = \{X^k \mid k = 0, 1, \ldots, p-2\}$ and the $U(1)$ charge of the Ramond state $|X^k\rangle$ is

$$q_k = \frac{k+1}{p} - \frac{1}{2}.$$

The "type" of this state is

$$p_k = q_k + \frac{1}{2} \equiv \frac{k+1}{p},$$

where the extra $\frac{1}{2}$ arises because of the chiral anomaly. [65] Now we apply the above conjecture to this situation. One has $m = p$. As $\mathscr{O}_1$ we take the operator X^{k-1}, which is associated to the eigenvalue $(\zeta_p)^k$ of $-H$. Under the action of the Galois group $\mathbf{F}_p$ this element generates the full ring [66] $\mathscr{R}$. The corresponding operators $\mathscr{O}_l$ are just $X^{\langle lk \rangle - 1}$. Then

$$p_k(l) = \frac{\langle lk \rangle}{p},$$

and (C.3) becomes

$$p_k(l) = \frac{1}{p} \sum_{a=0}^{p-1} f(a) \langle al \rangle \Rightarrow f(a) = \begin{cases} 1 & \text{if } a = k \\ 0 & \text{if } a \neq k\,. \end{cases}$$

64 In fact, this is the only case one needs. Assume that $W = X^{ab} + \ldots$. Then the change of variables $Y = X^a$ reduces to $W = Y^b + \ldots$. In this way we can always (choosing special submanifolds of moduli space) restrict to odd prime powers

65 The "anomalous" combination $q_k + \frac{n}{2}$ is the natural one from the singularity viewpoint too

66 The element -1 of the Galois group corresponds to spectral flow. So complex multiplication can be seen as a fancy generalization of spectral flow

Thus from (C.4) we have for the UV OPE coefficients

$$\langle \overline{X^{k-1}} \mid X^{k-1} \rangle = z_k \frac{\Gamma\left(\frac{k}{p}\right)}{\Gamma\left(\frac{p-k}{p}\right)}, \tag{C.5}$$

for some *algebraic* numbers z_k. (C.5), with $z_k = 1$, is the well known answer for these coefficients.

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