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Massive Fields of Arbitrary Spin in Curved Space-Times

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Abstract: A possibility to describe massive fields of spin $s \geqq \frac{1}{2}$ within general relativity theory without auxiliary fields and subsidiary conditions is proposed. Using the 2-component spinor calculus the Lagrangian is given for arbitrary s in an uniform manner. The related Euler–Lagrange equations are the wave equations studied by Buchdahl and Wünsch. The results are specified for fields of integer and half-integer spin: A suitable generalization of Proca's equation and Lagrangian leads to an equivalent tensor description of bosonic fields, whereas a generalization of Dirac's theory allows an equivalent description of fermionic fields by use of bispinors. A $U(1)$-gauge invariance of the Lagrangian is obtained by coupling to an electromagnetic potential. The current vector of the spin-s field is derived.

1. Introduction

Relativistic wave equations for particles of arbitrary spin were first considered by P.A.M. Dirac in 1936 [11]. In the notation of Penrose and Rindler [28], his equations read

$$\partial^{D}_{\dot{X}_0}\varphi_{DA_1\dots A_n\dot{X}_1\dots\dot{X}_k} + \mu\chi_{A_1\dots A_n\dot{X}_0\dot{X}_1\dots\dot{X}_k} = 0\,,$$
$$\partial^{\dot{Z}}_{A_0}\chi_{A_1\dots A_n\dot{Z}\dot{X}_1\dots\dot{X}_k} - \nu\varphi_{A_0A_1\dots A_n\dot{X}_1\dots\dot{X}_K} = 0\,, \tag{1.1}$$

where $n, k = 0, 1, 2, \dots$ and the spinor fields φ and χ are symmetric in their dotted and undotted indices (corresponding to the irreducible representations $D((n+1)/2, k/2)$ and $D(n/2, (k+1)/2)$ of the restricted Lorentz group $SO^+(1,3)$). The particles (quanta) of the field described by (1.1) have the mass $m^2 = -2\mu\nu$ and the spin $s = \frac{1}{2}(n+k+1)$.

The system (1.1) of differential equations allows an *uniform* description of *free* fields of particles with arbitrary spin. Various other field equations can be comprehended as special cases of it. If we write the Dirac [10] and the Rarita-Schwinger [31] equations in terms of 2-component spinors then we obtain (1.1) with $\mu = \nu$ and $n = k$. The equations of Proca [30] and Fierz [16] for bosonic fields can also be derived from (1.1) (see also [2, 15, 24–26, 28, 32, 33] and Chapter 2). If

$m = 0$, then the system (1.1) is "decoupled"; the first equation of (1.1) with $\mu = 0$ and $k = 0$ is the field equation for a massless free field of spin $\frac{1}{2}(n+1)$, especially we obtain Weyl's ($n = 0$) and Maxwell's ($n = 1$) equations (cf. [28, 29, 32]). Massless fields will not be considered in this paper (see [23]).

Several difficulties arise with the generalization of (1.1) to interacting fields. The inclusion of an electromagnetic field given by its vector potential A is affected by the rule

$$\partial_a \to \partial_a - ieA_a\,, \tag{1.2}$$

where e is the charge of the particles. The generalization of (1.1) by (1.2) yields algebraical subsidiary conditions between the electromagnetic spinor and the field spinors φ and χ if the spin is greater than a half [17]. The minimal coupling of the equations (1.1) to a gravitational field, i.e.

$$\text{flat} \to \text{curved space-time}, \quad \partial_a \to \nabla_a \tag{1.3}$$

yields algebraical supplementary conditions between the curvature spinors and the field spinors if $s > 1$ [5, 6]. This occurrence of additional algebraical constraints is usually denoted by inconsistency [17].

There are various attempts to obtain consistent equations for "higher" spin (i.e. $s > 1$). Fierz and Pauli [17] established a theory of interacting fields with arbitrary spin which is based on an action principle. However, the technique is exceedingly difficult even for free fields because it is necessary to introduce an increasing number of auxiliary fields if the spin is greater than two ([33], see also [15] for the massless case).

The extension of general relativity theory, e.g. to space-times with torsion [1, 27] or to complexified space-times [29], is another possibility to look for consistent field equations. However, there result also consistency conditions which restrict the "background geometry." Supergravity theories seem to be unsuitable to describe fields of arbitrary spin, too [26].

After some remarks on higher-spin fields in flat space-times, we present here a possibility to describe massive fields of arbitrary spin s within the framework of Einstein's general relativity without any auxiliary fields and subsidiary conditions in an uniform manner. This approach is based on irreducible representations of type $D(s, 0)$ and $D(s - \frac{1}{2}, \frac{1}{2})$ instead of $D(s/2, s/2)$ in the Fierz [16] and $D(t, t + \frac{1}{2})$ in the Rarita–Schwinger [31] theory, i.e. the field equations are of type (1.1) with $k = 0$ (cf. also [2, 12, 13, 24, 34]).

It was first pointed out by H.A. Buchdahl [7] that this type of field equations can be generalized to an arbitrary $(\mathscr{M}, g)$. V. Wünsch has shown in [37] that Buchdahl's equations can be simplified to

$$\begin{aligned} \nabla^{D}_{\dot{X}}\varphi_{DA_1\ldots A_n} + \mu\chi_{A_1\ldots A_n\dot{X}} &= 0\,, \\ \nabla^{\dot{Z}}_{(A}\chi_{A_1\ldots A_n)\dot{Z}} - \nu\varphi_{AA_1\ldots A_n} &= 0\,. \end{aligned} \tag{1.4}$$

We emphasize that the main point in (1.4) is the symmetrization of the indices in the second equation. In the sense of Buchdahl [7], (1.4) seems to be the "simplest possible" pair of mutually compatible equations which reduces to (1.1) in flat space-times and which allows the inclusion of an electromagnetic field by (1.2) [37].

Chapter 3 of this paper begins with some basic definitions and general properties of the differential operators contained in (1.4). Then we discuss a generalized form of the field equations (1.4) and derive second-order equations for the fields φ and χ. The second-order equation for φ is of normal hyperbolic type and can be comprehended as generalized Klein–Gordon equation (cf. [7, 21]). Using this equation Cauchy's problem for the system (1.4) can be solved. In this respect we refer to the detailed analysis of V. Wünsch [35, 37].

The action principle is fundamental in field theories [32]. In Sect. 3.3 we present an uniform Lagrangian density $L^{(s)}$ for all possible values of $s \geqq \frac{1}{2}$. However, to construct $L^{(s)}$ in that general form, we need four independent spinor fields and we obtain two systems of type (1.4) as Euler–Lagrange equations.

In Chapter 4 we specify the results of Sect. 3.3 for fields with integer and half-integer spin. For bosonic fields of spin s, the field functions are complex "bivectors and vector-bivectors of rank s" [22] and we obtain $L^{(s)}$ by a suitable generalization of the Lagrangian density of Proca fields. The Euler–Lagrange equations can be splitted into a self dual and anti-self dual part and each of them is related to a system of type (1.4). If the tensor fields are real, then the anti-self dual part is the complex conjugate of the self dual. Consequently, we have only two independent spinor fields and the Lagrangian density $L^{(s)}$ reduces to that of [19, 20].

For fermionic fields of spin $t + \frac{1}{2}$, the field functions are the tensor product of a bivector of rank t and a bispinor (cf. also [34]) and $L^{(t+\frac{1}{2})}$ can be obtained by generalization of the Lagrangian density of Dirac fields. In the Weyl representation, the Euler–Lagrange equations are again two systems of type (1.4). In the case of "lower" spin (i.e. $s = \frac{1}{2}$ and $s = 1$), certain spinor fields coincide and we get the well-known Lagrangians for Dirac and Proca fields (cf. e.g. [4, 8, 32]).

The complex character of the field functions allows the action of the gauge group $U(1)$. In Chapter 5 we show that the Lagrangian density $L^{(s)}$ becomes gauge invariant if a gauge field is coupled by (1.2). From this generalized Lagrangian we derive the current vector j of the spin-s field, which was given for flat space-times (without knowledge of the Lagrangian density!) already by Fierz [16].

The field equations for the metric tensor result from the total Lagrangian, obtained by adding the Einstein–Hilbert action to the "matter part" $L^{(s)}$ [32]. But their derivation is rather complicated because $L^{(s)}$ contains derivatives of g if $s > 1$ (cf. [20] for first results). Therefore, we postpone this problem to a forthcoming paper. Likewise, the question of whether or not the Lagrangian $L^{(s)}$ contains ghost modes is still open.

In the following, we assume that the (four-dimensional) space-time $(\mathscr{M}, g)$ and all spinor and tensor fields are of class C^∞. All considerations are purely local. We denote by $\mathscr{S}_{n,k}$ $(n, k = 0, 1, 2, \ldots)$ the set of all symmetric ("irreducible" [28]) spinor fields with n undotted and k dotted indices:

$$\zeta \in \mathscr{S}_{n,k} \Leftrightarrow \zeta_{A_1 \ldots A_n \dot{X}_1 \ldots \dot{X}_k} = \zeta_{(A_1 \ldots A_n)(\dot{X}_1 \ldots \dot{X}_k)} \,.$$

Moreover, we use the notations and conventions of Penrose and Rindler [28], especially for the curvature spinors $\Psi_{\mathrm{ABCD}} \in \mathscr{S}_{4,0}$, $\Phi_{\mathrm{AB\dot{X}\dot{Y}}} \in \mathscr{S}_{2,2}$, and $\Lambda \in \mathscr{S}_{0,0}$. As usual in the literature, the symbol Ψ is also used for bispinors (4-component or Dirac spinors), but confusion with the Weyl spinor Ψ_{ABCD} is impossible because of the explicit use of indices.

2. Some Remarks on Higher Spin Fields in Flat Space-Times

Before generalizing the system (1.1) to curved space-times we discuss some aspects of theories on free fields with higher spin.

Remark 2.1. The system (1.1) is of normal hyperbolic type. Every solution (φ, χ) of (1.1) is divergence-free

$$\partial^{D\dot{Z}} \varphi_{DA_1 \ldots A_n \dot{Z}\dot{X}_2 \ldots \dot{X}_k} = 0 \quad \text{for } k > 0\,,$$

$$\partial^{D\dot{Z}} \chi_{DA_2 \ldots A_n \dot{Z}\dot{X}_1 \ldots \dot{X}_k} = 0 \quad \text{for } n > 0 \tag{2.1}$$

and satisfies the Klein–Gordon equation

$$(\partial^a \partial_a + m^2)\varphi_{AA_1 \ldots A_n \dot{X}_1 \ldots \dot{X}_k} = (\partial^a \partial_a + m^2)\chi_{A_1 \ldots A_n \dot{X}\dot{X}_1 \ldots \dot{X}_k} = 0\,. \tag{2.2}$$

The Cauchy problem for the system (1.1) is properly posed if the initial data satisfy suitable constraints ([35], see also Sect. 3.2).

Remark 2.2. If the mass m and the spin $s > \frac{1}{2}$ of the field are fixed, then there are various possibilities to realize the condition $s = \frac{1}{2}(n + k + 1)$ in (1.1), namely $n = 0$, $k = 2s - 1$; $n = 1$, $k = 2s - 2$; $\ldots$; $n = 2s - 1$, $k = 0$. Consequently, there are $2s$ possibilities to formulate the system (1.1), but all of them can be converted into each other in flat space-times [9, 16]. In a curved space-time, these transformations are generally impossible for $s > 1$, because a "potential" of the field exists only in particular cases [22].

Remark 2.3. Besides the system (1.1), there are other possibilities to formulate equations for fields of higher spin (see e.g. [2]). Especially we should mention:

a) *Bosonic fields*: Fields with integer spin can be described by tensor fields. In the theories of Proca ([30], $s = 1$) and Fierz ([16], $s \geqq 2$) the "field functions" are symmetric, traceless tensor fields of rank s. The field equations read (see also [2, 12, 17, 25, 33])

$$(\partial^a \partial_a + m^2) U_{a_1 \ldots a_s} = 0, \quad \partial^{a_1} U_{a_1 a_2 \ldots a_s} = 0\,. \tag{2.3}$$

Because the spinor equivalent of U is an element of $\mathscr{S}_{s,s}$, it follows from the above-mentioned remarks that these equations can be considered in a sense as special cases of (1.1) with $n + 1 = k = s$. If $s \geqq 2$ then it is impossible to construct a Lagrangian that will yield (2.3) by using only U, but auxiliary fields are necessary [15, 17, 33].

b) *Fermionic fields*: Fields with half-integer spin cannot be described by linear equations for tensor fields [28]. In the theories of Dirac ([10], $s = \frac{1}{2}$) and Rarita–Schwinger ([31], $s = t + \frac{1}{2}$, $t \geqq 1$) the field equations read (see also [2, 17, 26, 33, 34] and Sects. 4.3, 4.4)

$$(i\gamma^a \partial_a + m\mathbb{1})\Psi_{b_1 \ldots b_t} = 0, \quad \gamma^{b_1} \Psi_{b_1 b_2 \ldots b_t} = 0\,, \tag{2.4}$$

where γ^a are the Dirac matrices and the "field function" Ψ is a tensor product of a symmetric, traceless tensor of rank t and a bispinor. If one considers only the restricted Lorentz group $SO^+(1, 3)$ the representation space splits into two invariant subspaces [2]. Hence we have $\Psi = \Psi^{(+)} \oplus \Psi^{(-)}$; the second equation of (2.4) is then equivalent to $\Psi^{(+)} \in \mathscr{S}_{t+1,t}$, $\Psi^{(-)} \in \mathscr{S}_{t,t+1}$. Using the Weyl representation of the Dirac matrices (see Sect. 4.3) the first (differential) equation of (2.4) yields just the system (1.1) with $n = k = t$. In curved space-times, the Rarita–Schwinger equations are inconsistent unless this one is an Einstein space ($t = 1$) or has constant curvature ($t > 1$) [5, 6, 35].

Remark 2.4. It is easy to state systems of differential equations which are equivalent to (1.1) in flat space-times. If one replaces the derivatives in *both* equations by their symmetric parts and $n > 0$, $k > 0$, then the arising system is no longer of hyperbolic type and does not lead to a wave equation of second order [17]. But there is no loss in information if one replaces in *one* of the equations (1.1) the derivative by its symmetric part. The reason for this is the fact that from *each* equation the vanishing of the divergence of *both* spinor fields φ and χ follows.

Remark 2.5. The facts mentioned in the Remarks 2.2 and 2.4 complicate the generalization of the system (1.1) to curved space-times, because one does not a priori know from which of the (in flat space-times) equivalent systems one has to start (see also the discussion in [7]). In this connection, the knowledge of the Lagrangian density should be helpful, because a generally covariant action (without curvature terms) produces the Euler–Lagrange equations with "minimally coupled" gravitational field (see [32] and Sect. 3.3).

3. Uniform Description of Arbitrary Spin Fields in Curved Space-Times

3.1. Definition of Suitable Differential Operators

Definition 3.1. *Let $B \in \mathscr{S}_{1,1}$ be a given spinor field and $n, k \in \{0, 1, 2, \dots\}$. We define first-order differential operators $M^{(\pm)}\colon \mathscr{S}_{n+1,k} \to \mathscr{S}_{n,k+1}$ by*

$$\varphi_{AA_1\dots A_n\dot{X}_1\dots\dot{X}_k} \mapsto (M^{(\pm)}[\varphi])_{A_1\dots A_n\dot{X}\dot{X}_1\dots\dot{X}_k} := (\nabla^A_{(\dot{X}} \pm B^A_{(\dot{X}})\varphi_{|AA_1\dots A_n|\dot{X}_1\dots\dot{X}_k)}$$

and $N^{(\pm)}\colon \mathscr{S}_{n,k+1} \to \mathscr{S}_{n+1,k}$ by

$$\chi_{A_1\dots A_n\dot{X}\dot{X}_1\dots\dot{X}_k} \mapsto (N^{(\pm)}[\chi])_{AA_1\dots A_n\dot{X}_1\dots\dot{X}_k}$$
$$:= (\nabla^{\dot{X}}_{(A} \pm B^{\dot{X}}_{(A})\chi_{A_1\dots A_n)\dot{X}\dot{X}_1\dots\dot{X}_k}\,.$$

In the case of $B \equiv 0$ we write only M and N.

Remark 3.1. Complex conjugation maps $\mathscr{S}_{r,t}$ into $\mathscr{S}_{t,r}$ and we have

$$\overline{M[\varphi]} = N[\bar{\varphi}], \quad \overline{N[\chi]} = M[\bar{\chi}]\,.$$

Proposition 3.1. *Let $\varphi \in \mathscr{S}_{n+1,k}$ and $\chi \in \mathscr{S}_{n,k+1}$ be some spinor fields. If the metric g undergoes a conformal transformation, i.e.*

$$\hat{g}_{ab} = e^{2\rho} g_{ab}, \quad \hat{\sigma}_a{}^{A\dot{X}} = \sigma_a{}^{A\dot{X}}, \quad \hat{\varepsilon}_{AB} = e^{\rho}\varepsilon_{AB}$$

with a positive scalar function ρ (see [28]), then we have $\hat{B}_a = B_a$ and

$$\hat{M}^{(\pm)}[e^{(k+1)\rho}\varphi] = e^{k\rho} M^{(\pm)}[\varphi], \quad \hat{N}^{(\pm)}[e^{(n+1)\rho}\chi] = e^{n\rho} N^{(\pm)}[\chi]\,. \quad (3.1)$$

Proof. (cf. also [35]). The transformation law of the Christoffel symbols reads [18, 28, 35]

$$\hat{\Gamma}^c_{ab} = \Gamma^c_{ab} + 2\delta^c_{(a}\nabla_{b)}\rho - g_{ab}\nabla^c\rho\,,$$

$$\hat{\Gamma}^B_{aA} = \Gamma^B_{aA} + \sigma_a{}^{B\dot{Y}}\nabla_{A\dot{Y}}\rho\,.$$

Using these formulas one obtains for $\xi \in S_{r,t}$ (cf. [28])

$$\hat{\nabla}_{B\dot{Y}}\xi_{A_1 \dots A_r\dot{X}_1 \dots \dot{X}_t} = \nabla_{B\dot{Y}}\xi_{A_1 \dots A_r\dot{X}_1 \dots \dot{X}_t} - r\nabla_{(A_1\dot{Y}}\rho\,\xi_{A_2 \dots A_r)B\dot{X}_1 \dots \dot{X}_t} - t\nabla_{B(\dot{X}_1}\rho\,\xi_{A_1 \dots A_r\dot{X}_2 \dots \dot{X}_t)\dot{Y}} \,.$$

From this equation we get for $r = n + 1$, $t = k$ after multiplication with $\hat{\varepsilon}^{AB}$,

$$\hat{\nabla}^{A}_{\dot{Y}}\xi_{AA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k} = e^{-\rho}(\nabla^{A}_{\dot{Y}}\xi_{AA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k} - (k+1)\nabla^{A}_{(\dot{Y}}\rho\,\xi_{AA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k)})$$

Setting $\xi = e^{(k+1)\rho}\varphi$ one obtains $\hat{M}[e^{(k+1)\rho}\varphi] = e^{k\rho}M[\varphi]$. Because $\hat{B}_a = B_a$ implies $\hat{B}^{A}_{\dot{X}} = e^{-\rho}B^{A}_{\dot{X}}$, the first equation of (3.1) is already proved. The proof of the second equation is the same.

Definition 3.2. *For two spinor fields ξ, ϑ of the same type we define the scalar product $(\cdot,\cdot)$ by*

$$\xi, \vartheta \mapsto (\xi, \vartheta) := \xi_{A_1 \dots A_r\dot{X}_1 \dots \dot{X}_t}\vartheta^{A_1 \dots A_r\dot{X}_1 \dots \dot{X}_t} \,.$$

Proposition 3.2. *The operator $N^{(-)}$ $(N^{(+)})$ is the adjoint of the operator $M^{(+)}$ $(M^{(-)})$.*

Proof. Let $\Omega \subseteq \mathcal{M}$ be a domain with a sufficiently smooth boundary and $\varphi \in \mathcal{S}_{n+1,k}$, $\chi \in \mathcal{S}_{n,k+1}$ with compact support in Ω. Then one obtains by use of the Gaussian theorem

$$\int_{\Omega} \{(M^{(+)}[\varphi], \chi) - (\varphi, N^{(-)}[\chi])\}\,dV = \int_{\Omega} \nabla^{A}_{\dot{X}}(\varphi_{AA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k}\chi^{A_1 \dots A_n\dot{X}\dot{X}_1 \dots \dot{X}_k})\,dV = 0 \,. \tag{3.2}$$

In the same way one proves the statement for $M^{(-)}$ and $N^{(+)}$. We emphasize that the scalar product (3.2) contains only the symmetric parts of the derivatives of φ and χ.

Corollary. *The iterated operator $N^{(+)}M^{(+)}$ $(M^{(-)}N^{(-)})$ is the adjoint of the iterated operator $N^{(-)}M^{(-)}$ $(M^{(+)}N^{(+)})$.*

Proof. Put in (3.2) $\chi = M^{(-)}[\tilde{\varphi}]$ and $\varphi = N^{(+)}[\tilde{\chi}]$, respectively.

Lemma 3.1. *Let $\varphi \in \mathcal{S}_{n+1,k}$ be some spinor field. Then we have*

$$\begin{aligned}
&\nabla^{\dot{X}}_{A}\nabla^{E}_{\dot{X}}\varphi_{EA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k} \\
&= \varepsilon^{EF}(\nabla^{\dot{X}}_{[A}\nabla_{F]\dot{X}}\varphi_{EA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k} + \nabla^{\dot{X}}_{(A}\nabla_{F)\dot{X}}\varphi_{EA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k}) \\
&= -\frac{1}{2}\nabla^{a}\nabla_{a}\varphi_{AA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k} + n\Psi^{DE}{}_{A(A_1}\varphi_{A_2 \dots A_n)DE\dot{X}_1 \dots \dot{X}_k} \\
&\quad + k\Phi^{E}_{A(\dot{X}_1}{}^{\dot{Y}}\varphi_{|EA_1 \dots A_n|\dot{X}_2 \dots \dot{X}_k)\dot{Y}} - (n+3)\Lambda\varphi_{AA_1 \dots A_n\dot{X}_1 \dots \dot{X}_k} \,.
\end{aligned} \tag{3.3}$$

Proof. Use the Ricci identities for spinor fields (cf. also [22, 35]). ∎

Using the relation (3.3) one can show that the iterated operators NM: $\mathcal{S}_{n+1,k} \to \mathcal{S}_{n+1,k}$ and MN: $\mathcal{S}_{n,k+1} \to \mathcal{S}_{n,k+1}$ are generally not of normal hyperbolic type (because of the symmetrization). The case of $k = 0$ is an exception, as we will see in the next section.

3.2. The Field Equations. Using the differential operators $M^{(-)}$ and $N^{(-)}$ we can establish the field equations (1.4) in the generalized form [21, 37]

$$M^{(-)}[\varphi] + \mu\chi = 0$$

$$N^{(-)}[\chi] - \nu\varphi = 0 \tag{3.4}$$

for spinor fields $\varphi \in \mathscr{S}_{n+1,0}$ and $\chi \in \mathscr{S}_{n,1}$. The purpose of adding the spinor field B is its possible physical interpretation as electromagnetic potential (see (1.2) and Chapter 5). The factors μ and ν are complex constants and connected with the mass m of the field quanta by

$$m^2 = -2\mu\nu\,. \tag{3.5}$$

We assume $\mu \neq 0$ in the whole paper.

We can formulate the system (3.4) with $B \equiv 0$ in an alternative manner (cf. [37]). From the first equation of (3.4) and Lemma 3.1. we obtain for $n > 0$,

$$\begin{aligned}(\nabla^{C\dot{X}} - B^{C\dot{X}})\chi_{CA_2\ldots A_n\dot{X}} &= -\frac{1}{\mu}(\nabla^{C\dot{X}} - B^{C\dot{X}})(\nabla^{A}_{\dot{X}} - B^{A}_{\dot{X}})\varphi_{ACA_2\ldots A_n} \\ &= \frac{1}{\mu}(\varphi_{CDA_2\ldots A_n}\nabla^{C\dot{X}}B^{D}_{\dot{X}} - (n-1)\Psi^{DEF}{}_{(A_2}\varphi_{A_3\ldots A_n)DEF})\,.\end{aligned} \tag{3.6}$$

Using the identity

$$\nabla^{\dot{Z}}_{(A}\chi_{A_1\ldots A_n)\dot{Z}} = \nabla^{\dot{Z}}_{A}\chi_{A_1\ldots A_n\dot{Z}} + \frac{n}{n+1}\varepsilon_{A(A_1}\nabla^{C\dot{Z}}\chi_{A_2\ldots A_n)C\dot{Z}} \tag{3.7}$$

and (3.6) we obtain the system (3.4) with $B \equiv 0$ in the form

$$\begin{aligned}&\nabla^{D}_{\dot{X}}\varphi_{DA_1\ldots A_n} + \mu\chi_{A_1\ldots A_n\dot{X}} = 0\,, \\ &\nabla^{\dot{Z}}_{A}\chi_{A_1\ldots A_n\dot{Z}} - \nu\varphi_{AA_1\ldots A_n} = \frac{n(n-1)}{\mu(n+1)}\varepsilon_{A(A_1}\Psi^{DEF}{}_{A_2}\varphi_{A_3\ldots A_n)DEF}\,.\end{aligned} \tag{3.8}$$

These are the equations of Buchdahl [7]. The form (3.8) of the field equations shows explicitly that the symmetrization in the second equation of (3.4) with $B \equiv 0$ can be omitted if $n = 0$, $n = 1$ or the space-time is conformally flat.

From the first-order system (3.4) we can deduce second-order equations for the fields φ and χ. Substituting

$$\chi = -\frac{1}{\mu}M^{(-)}[\varphi] \tag{3.9}$$

into the second equation of (3.4) we get

$$N^{(-)}M^{(-)}[\varphi] + \mu\nu\varphi = 0\,. \tag{3.10}$$

Using Definition 3.1, Lemma 3.1 and (3.5) we obtain after multiplication by -2 the following explicit form of (3.10) (cf. [21] as well as [7, 37] for $B \equiv 0$)

$$\nabla^a\nabla_a\varphi - 2B^a\nabla_a\varphi + \mathscr{C}\varphi = 0\,, \tag{3.11}$$

where the linear operator $\mathscr{C}: \mathscr{S}_{n+1,0} \to \mathscr{S}_{n+1,0}$ has the coordinates

$$\mathscr{C}^{BB_1 \ldots B_n}_{AA_1 \ldots A_n} = (-2n\Psi^{BB_1}{}_{AA_1} + 2\nabla^{\dot{X}}_{A} B^{B}_{\dot{X}} \delta^{B_1}_{A_1} + [B^a B_a + 2(n+3)\Lambda + m^2]\delta^{B}_{A}\delta^{B_1}_{A_1})\delta^{B_2}_{A_2} \ldots \delta^{B_n}_{A_n}. \tag{3.12}$$

On the right-hand side of (3.12) one has to symmetrize the indices $AA_1 \ldots A_n$ and $BB_1 \ldots B_n$, respectively. The second-order equation (3.11) for the spinor field $\varphi \in \mathscr{S}_{n+1,0}$ is linear and of normal hyperbolic type. With the help of it we can solve Cauchy's problem for the system (3.4) (see below).

Remark 3.2. Whereas the system (3.4) makes sense only for $n \geqq 0$, i.e. for $s \geqq \frac{1}{2}$, the second-order equation (3.11) can be generalized to scalar fields, i.e. to $s = 0$, too. If one replaces the operator $\mathscr{C}$ by its trace and puts formally $n = -1$, then (3.11) reduces to

$$\nabla^a \nabla_a \varphi - 2B^a \nabla_a \varphi + \left(B^a B_a - \nabla^a B_a + \frac{R}{6} + m^2 \right) \varphi = 0 .$$

This is just the Klein–Gordon equation with a minimally coupled electromagnetic field and a conformally coupled gravitational field [3]. Therefore, we can comprehend (3.11) as generalized Klein–Gordon equation. ■

The second-order equation for the field χ

$$M^{(-)}N^{(-)}[\chi] + \mu\nu\chi = 0$$

can be transformed with the help of the relation (3.7). We obtain

$$\begin{aligned}(\nabla^{A}_{\dot{X}} - B^{A}_{\dot{X}})(\nabla^{\dot{Z}}_{A} - B^{\dot{Z}}_{A})\chi_{A_1 \ldots A_n \dot{Z}} + \mu\nu\chi_{A_1 \ldots A_n \dot{X}} \\ = -\frac{n}{n+1}(\nabla_{(A_1 \dot{X}} - B_{(A_1 \dot{X}})(\nabla^{C\dot{Z}} - B^{C\dot{Z}})\chi_{A_2 \ldots A_n)C\dot{Z}} \\ =: -\frac{n}{n+1}(DD^{(-)}[\chi])_{A_1 \ldots A_n \dot{X}} .\end{aligned} \tag{3.13}$$

Using Lemma 3.1 we get after multiplication by -2

$$\nabla^a \nabla_a \chi - 2B^a \nabla_a \chi + \mathscr{D}\chi = \frac{2n}{n+1} DD^{(-)}[\chi] , \tag{3.14}$$

where the linear operator $\mathscr{D}: \mathscr{S}_{n,1} \to \mathscr{S}_{n,1}$ has the coordinates

$$\mathscr{D}^{B_1 \ldots B_n \dot{Z}}_{A_1 \ldots A_n \dot{X}} = (-2n\Phi_{A_1}{}^{B_1}{}_{\dot{X}}{}^{\dot{Z}} + 2\nabla^{D}_{\dot{X}} B^{\dot{Z}}_{D} \delta^{B_1}_{A_1} + [B^a B_a + 6\Lambda + m^2]\delta^{\dot{Z}}_{\dot{X}}\delta^{B_1}_{A_1})\delta^{B_2}_{A_2} \ldots \delta^{B_n}_{A_n}. \tag{3.15}$$

If $n > 0$, then (3.14) is not of normal hyperbolic type because the right-hand side of it contains second derivatives of χ, too. But these can be eliminated with the help of Eq. (3.6). Therefore, if the spinor field φ is already known by solving (3.11) (see Proposition 3.3) then (3.14) is a normal hyperbolic equation for the spinor field χ. But it is generally inhomogen (in contrast to (3.11) for φ). A wave equation which contains the spinor field χ alone follows from the system (3.4) for $n > 0$ only if $\nabla^{\dot{Z}}{}_{(C}B_{D)\dot{Z}} = 0$ and $(n-1)\Psi_{ABCD} = 0$.

Before solving Cauchy's problem for the system (3.4) some investigations on the geometry on the initial hypersurface $\mathscr{F}$ are necessary. In this respect we refer to the detailed analysis in [35, 37] and present only the main ideas here.

Let a spacelike hypersurface $\mathscr{F}$ of class C^∞ with parametric representation

$$x^a = x^a(t^\alpha) \quad (\alpha = 1, 2, 3) \tag{3.16}$$

be given. Let $\mathfrak{n}^a$ denote the future-directed unit normal vector on $\mathscr{F}$ and $\nabla_\mathfrak{n} := \mathfrak{n}^a \nabla_a$. For an arbitrary spinor field ξ let $\xi_{|\mathscr{F}}$ arise from ξ by substituting (3.16) into the coordinates of ξ. We denote the spinor fields constructed in this manner from $\mathscr{S}_{n,k}$ by $\mathscr{S}_{n,k|\mathscr{F}}$.

Cauchy's problem: Let a spacelike hypersurface $\mathscr{F}$ and initial data $\mathring{\varphi} \in \mathscr{S}_{n+1,0|\mathscr{F}}$ and $\mathring{\chi} \in \mathscr{S}_{n,1|\mathscr{F}}$ be given. Find a solution of (3.4) with $\varphi \in \mathscr{S}_{n+1,0}$, $\chi \in \mathscr{S}_{n,1}$ and $\varphi_{|\mathscr{F}} = \mathring{\varphi}$, $\chi_{|\mathscr{F}} = \mathring{\chi}$.

In general the Cauchy data $\mathring{\varphi}$ and $\mathring{\chi}$ cannot be prescribed arbitrarily. If a solution (φ, χ) of Cauchy's problem does exist then the differential equations (3.4) have to be satisfied on $\mathscr{F}$, too. From the first equation of (3.4) we obtain

$$\nabla_\mathfrak{n} \varphi_{AA_1 \ldots A_n|\mathscr{F}} = 2\mathfrak{n}^{\dot{X}}_{A} [(\tilde{\nabla}^{D}_{\dot{X}} - B^{D}_{\dot{X}}) \varphi_{DA_1 \ldots A_n} + \mu \chi_{A_1 \ldots A_n \dot{X}}]_{|\mathscr{F}}\,, \tag{3.17}$$

where the differential operator $\tilde{\nabla}_{A\dot{X}} = \nabla_{A\dot{X}} - \mathfrak{n}_{A\dot{X}} \nabla_\mathfrak{n}$ is just the tangential part of $\nabla_{A\dot{X}}$ with respect to the hypersurface $\mathscr{F}$ [35, 37]. Therefore, the right-hand side of (3.17) is uniquely determined by the initial data. The symmetry of the solution φ implies $\nabla_\mathfrak{n} \varphi \in \mathscr{S}_{n+1,0|\mathscr{F}}$, consequently, the right-hand side of (3.17) must be symmetrical with respect to the undotted indices and we obtain for $n \geqq 1$ the following constraints for the Cauchy data [35, 37]:

$$\mathfrak{n}^{A_1 \dot{X}} [(\tilde{\nabla}^{D}_{\dot{X}} - B^{D}_{\dot{X}}) \mathring{\varphi}_{DA_1 \ldots A_n} + \mu \mathring{\chi}_{A_1 \ldots A_n \dot{X}}]_{|\mathscr{F}} = 0\,. \tag{3.18}$$

Remark 3.3. The constraints (3.18) mean that the spinor field enclosed in the square bracket has to be spatial. They do not express an essential restriction of the Cauchy data. For example, if one chooses $\mathscr{F}$ and $\mathring{\varphi}$ arbitrarily, then (3.18) yields only an algebraic condition on $\mathring{\chi}$. This can be satisfied easily because of $\mu \neq 0$.

Proposition 3.3. *If the Cauchy data $\mathring{\varphi}$ and $\mathring{\chi}$ satisfy the constraints* (3.18) *then there exists a neighbourhood of $\mathscr{F}$ in which Cauchy's problem has a unique solution.*

Sketch of a proof. (cf. [21]). From the initial data we can calculate the normal derivative $\nabla_\mathfrak{n} \varphi_{|\mathscr{F}}$ according to formula (3.17). Since the constraints (3.18) are satisfied we have $\nabla_\mathfrak{n} \mathring{\varphi} \in \mathscr{S}_{n+1,0|\mathscr{F}}$. Now we have a Cauchy problem for the generalized Klein–Gordon equation (3.11) with initial data $\mathring{\varphi}$ and $\nabla_\mathfrak{n} \mathring{\varphi}$ which has a unique solution $\varphi \in \mathscr{S}_{n+1,0}$ (see e.g. [18]). Then we define the spinor field $\chi \in \mathscr{S}_{n,1}$ by (3.9).

The construction of φ implies $\varphi_{|\mathscr{F}} = \mathring{\varphi}$, $\nabla_\mathfrak{n} \varphi_{|\mathscr{F}} = \nabla_\mathfrak{n} \mathring{\varphi}$; consequently we have by (3.17) $\chi_{|\mathscr{F}} = \mathring{\chi}$. The pair (φ, χ) obviously satisfies the first equation of (3.4). Further we obtain

$$N^{(-)}[\chi] - \nu\varphi = N^{(-)}\left[-\frac{1}{\mu} M^{(-)}[\varphi] \right] - \nu\varphi = -\frac{1}{\mu}(N^{(-)} M^{(-)}[\varphi] + \mu\nu\varphi) = 0\,,$$

because φ is a solution of (3.10). Hence, (φ, χ) is a solution of (3.4).

To prove the uniqueness we note that $\nabla_\mathfrak{n} \varphi_{|\mathscr{F}}$ is uniquely determined by $\mathring{\varphi}$ and $\mathring{\chi}$ according to (3.17). Because the spinor field φ of every solution (φ, χ) of (3.4) has to satisfy Eq. (3.11), the uniqueness of φ follows from the uniqueness of the solution of Cauchy's problem for second-order hyperbolic equations (see [18]). The uniqueness of χ is then obvious.

Remark 3.4. The above discussion of the field equations (3.4) shows that the condition $\mu \neq 0$ is an essential supposition for their consistency if $n > 1$. On the contrary, $\nu = 0$ is possible. This gives rise to consider $N[\chi] = 0$, $\chi \in \mathscr{S}_{n,1}$, as possible equations for massless fields with arbitrary spin. These field equations are studied in [23]. The main results are:

i) Let $\zeta \in \mathscr{S}_{n+1,0}$ be given, then the equations $N[\chi] = \zeta$ for $\chi \in \mathscr{S}_{n,1}$ are consistent on a generally curved space-time for all $n \geqq 0$.
ii) The Cauchy problem is properly posed for these equations [22].
iii) The equations $N[\chi] = 0$ are conformally invariant (cf. Proposition 3.1).
iv) The field χ is gauge invariant in the following sense: Let $\omega \in \mathscr{S}_{n-1,0}$ be given, then there exists a solution of $N[\chi] = \zeta$ with

$$\nabla^{C\dot{Z}} \chi_{CA_2 \ldots A_n \dot{Z}} = \omega_{A_2 \ldots A_n} .$$

A Lagrangian that produces the field equations $N[\chi] = 0$ is still unknown (except for $n = 0$, which is just the Weyl equation).

3.3. The Lagrangian Density. We start with some general remarks on the Lagrangian formalism for field theories (cf. [32]). Consider a physical system which is described by spinor fields $\xi^{(1)}, \ldots, \xi^{(N)}$. Let $\Omega \subseteq \mathscr{M}$ be a domain with sufficiently smooth boundary. With a Lagrangian density L we define the action S of the system in Ω by the integral

$$S = \int_{\Omega} L \, dV .$$

Because we are especially interested in the field equations we consider only the matter part of the Lagrangian density.

The mathematical form of the Hamilton principle leads to the variational problem

$$\frac{\delta S}{\delta \xi^{(r)}} = 0, \quad \delta \xi^{(r)}_{|\partial\Omega} = 0 \quad (r = 1, \ldots, N) . \tag{3.19}$$

Suppose the Lagrangian density L contains at most first derivatives of the field spinors. One obtains by use of the Gaussian theorem the following Euler-Lagrange equations of the variational problem (3.19):

$$\frac{\partial L}{\partial \xi^{(r)}} - \nabla_{B\dot{Y}} \frac{\partial L}{\partial (\nabla_{B\dot{Y}} \xi^{(r)})} = 0 \quad (r = 1, \ldots, N) . \tag{3.20}$$

Now we consider the problem of whether or not there exists a Lagrangian density L in such a manner that the field equations (3.4) are just the Euler–Lagrange equations (3.20). We begin with the case of $B \equiv 0$. The coupling to an electromagnetic field will be discussed in Chapter 5.

Theorem 1. *Let n be a nonnegative integer and $\varphi \in \mathscr{S}_{n+1,0}$, $\chi \in \mathscr{S}_{n,1}$, $\xi \in \mathscr{S}_{0,n+1}$, $\vartheta \in \mathscr{S}_{1,n}$. If the Lagrangian density reads*

$$\begin{aligned} L^{(\frac{n+1}{2})} = {} & a\{(M[\varphi], \bar{\vartheta}) + (\chi, M[\bar{\xi}])\} + b\{(\varphi, N[\bar{\vartheta}]) + (N[\chi], \bar{\xi})\} \\ & + \bar{a}\{(N[\bar{\varphi}], \vartheta) + (\bar{\chi}, N[\xi])\} + \bar{b}\{(\bar{\varphi}, M[\vartheta]) + (M[\bar{\chi}], \xi)\} \\ & + (a+b)\{\mu(\chi, \bar{\vartheta}) - \nu(\varphi, \bar{\xi})\} + (\bar{a}+\bar{b})\{\bar{\mu}(\bar{\chi}, \vartheta) - \bar{\nu}(\bar{\varphi}, \xi)\} \end{aligned} \tag{3.21}$$

with $a = \text{const.}$, $b = \text{const.}$, $a + b \neq 0$, *then the Euler–Lagrange equations are*

$$M[\varphi] + \mu\chi = 0, \quad N[\chi] - \nu\varphi = 0, \tag{3.22}$$

$$N[\xi] + \bar{\mu}\vartheta = 0, \quad M[\vartheta] - \bar{\nu}\xi = 0. \tag{3.23}$$

Proof. Using Definitions 3.1 and 3.2 one obtains the following functional derivatives of L with respect to the spinor field $\varphi \in \mathscr{S}_{n+1,0}$ (note: The fields φ and $\bar{\varphi}$ have to be regarded as independent [32]):

$$\frac{\partial L}{\partial \varphi_{AA_1 \dots A_n}} = b\nabla^{(A|\dot{X}|}\bar{\vartheta}^{A_1 \dots A_n)}{}_{\dot{X}} - (a+b)\nu\bar{\xi}^{AA_1 \dots A_n}$$

$$\frac{\partial L}{\partial(\nabla_{B\dot{Y}}\varphi_{AA_1 \dots A_n})} = a\varepsilon^{(A|B|}\bar{\vartheta}^{A_1 \dots A_n)\dot{Y}}.$$

Hence, the Euler–Lagrange equation (3.20) reads

$$\begin{aligned} &b\nabla^{(A|\dot{X}|}\bar{\vartheta}^{A_1 \dots A_n)}{}_{\dot{X}} - (a+b)\nu\bar{\xi}^{AA_1 \dots A_n} - a\nabla_{B\dot{Y}}\varepsilon^{(A|B|}\bar{\vartheta}^{A_1 \dots A_n)\dot{Y}} \\ &\quad = (a+b)\{(\nabla^{(A|\dot{X}|}\bar{\vartheta}^{A_1 \dots A_n)}{}_{\dot{X}} - \nu\bar{\xi}^{AA_1 \dots A_n}\} = 0. \end{aligned}$$

If this relation is divided by $a + b \neq 0$ then we obtain the last equation of (3.23) by complex conjugation (note Remark 3.1). The other equations of (3.22) and (3.23) arise in the same manner by variation of the fields χ, ξ and ϑ, respectively.

Remark 3.5. For bosonic fields, we can state a Lagrangian density using only two fields φ and χ [19]. This possibility is contained in Theorem 1 as the special case $\xi = \bar{\varphi}$ and $\vartheta = \bar{\chi}$ and appears if the related tensor fields are real (see [20] and Sects. 4.1, 4.2).

Remark 3.6. By Proposition 3.2, the differences $(M[\varphi], \bar{\vartheta}) - (\varphi, N[\bar{\vartheta}])$ and $(\chi, M[\bar{\xi}]) - (N[\chi], \bar{\xi})$ yield a total differential. Adding a total differential to the Lagrangian density does not change the field equations [32]. Of course, the field equations are also unaltered if one multiplies L by a constant factor. Therefore, the constants a and b in (3.21) can be replaced by arbitrary other constants a_1 and b_1 so far as the condition $a_1 + b_1 \neq 0$ is satisfied.

Theorem 1 shows that four independent fields are needed to construct the Lagrangian density in the general case. The system of differential equations (3.23) is of the same type as (3.22); it is just the complex-conjugate system. Therefore, the Lagrangian formalism produces the field equations (3.4) twice, namely for the pairs (φ, χ) and (ξ, ϑ). Why this is the case and how former theories are related to it will be dealt with in the next chapter.

4. Specification for Bosonic and Fermionic Fields

4.1. Proca Fields. In the theory of complex spin 1-fields, the "field functions" are a complex vector field U and a complex antisymmetric tensor field H (a "bivector"). The Lagrangian density reads

$$L = -\frac{1}{2}\bar{H}_{ab}H^{ab} + m^2\bar{U}_aU^a \tag{4.1}$$

and the field equations are

$$H_{ab} = \nabla_a U_b - \nabla_b U_a, \quad \nabla^c H_{ac} - m^2 U_a = 0 \tag{4.2}$$

(cf. [2, 4, 8, 20, 30]). We will show that Theorem 1 with $n = 1$ yields just (4.1) and (4.2) if the spinor fields φ, χ, ξ and ϑ are related to the tensor fields H and U in a suitable manner. We are going to explain this in some detail because this approach will be generalized to higher spins in the next section.

The spinor equivalents of the tensors U and H have the form [28]

$$U_a \leftrightarrow \chi_{A\dot{X}}, \quad H_{ab} \leftrightarrow \varphi_{AB}\varepsilon_{\dot{X}\dot{Y}} + \xi_{\dot{X}\dot{Y}}\varepsilon_{AB} \tag{4.3}$$

with $\varphi \in \mathscr{S}_{2,0}$, $\xi \in \mathscr{S}_{0,2}$. We define differential operators d and δ by

$$(dU)_{ab} := \nabla_{[a}U_{b]}, \quad (\delta H)_a := \nabla^c H_{ac} \tag{4.4}$$

and obtain after a simple calculation using (4.3) and Definition 3.1 ,

$$\begin{aligned}(dU)_{ab} &\leftrightarrow -\frac{1}{2}\{N[\chi]_{AB}\varepsilon_{\dot{X}\dot{Y}} + M[\chi]_{\dot{X}\dot{Y}}\varepsilon_{AB}\}, \\ (\delta H)_a &\leftrightarrow -\{M[\varphi]_{A\dot{X}} + N[\xi]_{A\dot{X}}\} .\end{aligned} \tag{4.5}$$

From (4.3) and (4.5) we obtain the scalar products (see Definition 3.2, note Remark 3.1)

$$\begin{aligned}(\bar{H}, dU) &= -(\bar{\varphi}, M[\chi]) - (\bar{\xi}, N[\chi]) , \\ (\delta\bar{H}, U) &= -(N[\bar{\varphi}], \chi) - (M[\bar{\xi}], \chi) , \\ (\bar{H}, H) &= 2((\bar{\varphi}, \xi) + (\varphi, \bar{\xi})), \quad (\bar{U}, U) = (\bar{\chi}, \chi) .\end{aligned} \tag{4.6}$$

From the first and second equation of (4.6) and Proposition 3.2 it follows that the operator δ is the adjoint of d. Of course, one can derive this assertion directly from (4.4).

After these preliminaries we notice that we obtained only three spinor fields χ, φ and ξ for $s = 1$ instead of four in the general case of Theorem 1. However, suggested by $\mathscr{S}_{n,1} \equiv \mathscr{S}_{1,n}$ in the special case of $n = 1$, we can identify the fields χ and ϑ and obtain

Proposition 4.1. *Let* $n = 1, \chi \equiv \vartheta, a \in \mathbb{R}, b \in \mathbb{R}, \mu = \frac{1}{2}m^2, \nu = -1$ *and the connection between the spinor and tensor fields be given by* (4.3). *Then the Lagrangian density* (3.21) *reads*

$$\begin{aligned}L^{(1)} = &-a\{(\delta\bar{H}, U) + (\delta H, \bar{U})\} - b\{(\bar{H}, dU) + (H, d\bar{U})\} \\ &+ (a + b)\left\{\frac{1}{2}(H, \bar{H}) + m^2(U, \bar{U})\right\}\end{aligned} \tag{4.7}$$

and the field equations (3.22), (3.23) *are*

$$H = 2dU, \quad \delta H - m^2 U = 0 . \tag{4.8}$$

Proof. Using the relations (4.3), (4.5) and (4.6) one easily shows that the Lagrangian density (4.7) is equivalent to (3.21) with $n = 1$ if the constants are specified in the given manner. By (4.3) and (4.5) the spinor equivalent of the first equation of (4.8) is

$$\varphi_{AB}\varepsilon_{\dot{X}\dot{Y}} + \xi_{\dot{X}\dot{Y}}\varepsilon_{AB} = -N[\chi]_{AB}\varepsilon_{\dot{X}\dot{Y}} - M[\chi]_{\dot{X}\dot{Y}}\varepsilon_{AB}$$

which splits into

$$N[\chi] + \varphi = 0, \quad M[\chi] + \xi = 0 . \tag{4.9}$$

The spinor equivalent of the second equation of (4.8) reads

$$M[\varphi] + N[\xi] + m^2\chi = 0 . \tag{4.10}$$

Because we have the identity

$$MN[\eta] = NM[\eta]$$

for arbitrary $\eta \in \mathscr{S}_{1,1}$ [23] we obtain from (4.9) $N[\xi] = M[\varphi]$. Whence (4.10) splits into

$$M[\varphi] + \frac{m^2}{2}\chi = 0, \quad N[\xi] + \frac{m^2}{2}\chi = 0 . \tag{4.11}$$

If $\chi \equiv \vartheta$ and the constants are specified as mentioned above the four equations (4.9), (4.11) are identical with (3.22), (3.23) and the proposition is proved.

Corollary. *Using the field equations* (4.8) *the Lagrangian density* (4.7) *can be simplified to*

$$L^{(1)} = (b - a)\left(-\frac{1}{2}(\bar{H}, H) + m^2(\bar{U}, U)\right), \tag{4.7$'$}$$

which is just a multiple of (4.1) (cf. *Remark* 3.6).

The second-order equations for U and H can be obtained either by iteration of (4.8) or as Euler–Lagrange equations from (4.7)′ by substituting $H = 2dU$ and varying U as well as substituting $U = m^{-2}\delta H$ and varying H, respectively. The equation for the field U

$$\delta dU - \frac{m^2}{2}U = 0$$

reads in coordinate form

$$\nabla^c\nabla_c U_a - \nabla_a\nabla^c U_c + R_a{}^c U_c + m^2 U_a = 0 \tag{4.12}$$

(cf. e.g. [8]) and is of normal hyperbolic type because it implies $\nabla^a U_a = 0$. Equation (4.12) is just the tensor equivalent of (3.14) (note $B_a \equiv 0$ and $DD^{(-)}[\chi] = 0$ if $n = 1$ by (3.13) and (3.6)). The second-order equation of H

$$d\delta H - \frac{m^2}{2}H = 0$$

reads in coordinate form

$$\nabla_a\nabla^c H_{bc} - \nabla_b\nabla^c H_{ac} - m^2 H_{ab} = 0 .$$

Using the relation $\nabla_{[c}H_{ab]} = 0$, the Ricci identities and the decomposition of the curvature tensor [28] this equation can be converted into

$$\nabla^c\nabla_c H_{ab} - C_{ab}{}^{cd}H_{cd} + \left(\frac{R}{3} + m^2\right)H_{ab} = 0 . \tag{4.13}$$

The anti-self-dual part of this equation gives just the generalized Klein–Gordon equation (3.11).

4.2. Bosonic Fields with Higher Spin. In this section we consider bosonic fields whose spins are at least equal to two. As far as possible we will follow the considerations of the preceding section.

For simplicity we put $n + 1 = 2s$ and set

$$\varepsilon_{A_1 \ldots A_{2s}} := \varepsilon_{A_1 A_2} \varepsilon_{A_3 A_4} \ldots \varepsilon_{A_{2s-1} A_{2s}} . \tag{4.14}$$

With $\varphi \in \mathscr{S}_{2s,0}$, $\xi \in \mathscr{S}_{0,2s}$, $\chi \in \mathscr{S}_{2s-1,1}$ and $\vartheta \in \mathscr{S}_{1,2s-1}$ we define complex tensor fields H and U by

$$\begin{aligned} H_{a_1 \ldots a_{2s}} &\leftrightarrow \varphi_{A_1 \ldots A_{2s}} \varepsilon_{\dot{X}_1 \ldots \dot{X}_{2s}} + \xi_{\dot{X}_1 \ldots \dot{X}_{2s}} \varepsilon_{A_1 \ldots A_{2s}} , \\ U_{ba_3 \ldots a_{2s}} &\leftrightarrow \chi_{A_3 \ldots A_{2s} B \dot{Y}} \varepsilon_{\dot{X}_3 \ldots \dot{X}_{2s}} + \vartheta_{\dot{X}_3 \ldots \dot{X}_{2s} \dot{Y} B} \varepsilon_{A_3 \ldots A_{2s}} . \end{aligned} \tag{4.15}$$

The tensor fields H and U are called bivector of rank s and vector-bivector of rank s (see [22]). For example, the conformal curvature tensor in complexified space-times is a bivector of rank 2 (cf. [29]). We can give the following characterization of the tensor fields defined by (4.15) [22]:

Lemma 4.1. a) *A tensor H of rank $2s$ is a bivector of rank* s iff *it has the following symmetries*:

i) $H_{a_1 \ldots a_{2\nu-2}(a_{2\nu-1} a_{2\nu}) a_{2\nu+1} \ldots a_{2s}} = 0$,
ii) $H_{a_1 \ldots a_{2\nu-2} a_{2\rho-1} a_{2\rho} a_{2\nu+1} \ldots a_{2\rho-2} a_{2\nu-1} a_{2\nu} a_{2\rho+1} \ldots a_{2s}} = H_{a_1 \ldots a_{2s}}$

for all $\nu, \rho \in \{1, \ldots, s\}$,
iii) $^*H^*_{a_1 \ldots a_{2s}} \equiv \frac{1}{4} e_{a_1 a_2}{}^{b_1 b_2} e_{a_3 a_4}{}^{b_3 b_4} H_{b_1 b_2 b_3 b_4 a_5 \ldots a_{2s}} = -H_{a_1 \ldots a_{2s}}$,
iv) $g^{a_1 a_3} g^{a_2 a_4} H_{a_1 \ldots a_{2s}} = g^{a_1 a_3} g^{a_2 a_4} H^*_{a_1 \ldots a_{2s}} = 0$.

b) *A tensor U of rank $2s - 1$ is a vector-bivector of rank* s iff *it satisfies the relations* i), ..., iv) *with respect to the last $2s - 2$ indices and*

v) $g^{ba_3} U_{ba_3 \ldots a_{2s}} = g^{ba_3} U^*_{ba_3 \ldots a_{2s}} = 0$.

There was shown in [9, 24, 36] that the conditions iii) and iv) can be replaced by

iii)′ $g^{a_1 a_3} H_{a_1 \ldots a_{2s}} = 0$,
iv)′ $H_{[a_1 a_2 a_3] a_4 \ldots a_{2s}} = 0$.

Now we are going to generalize the differential operators (4.4) to fields of higher spin. We define d and δ by

$$\begin{aligned} (dU)_{a_1 \ldots a_{2s}} := \frac{1}{2s} \sum_{\nu=1}^{s} (&\nabla_{[a_{2\nu-1}} U_{a_{2\nu}] a_1 \ldots \hat{a}_{2\nu-1} \hat{a}_{2\nu} \ldots a_{2s} \\ &- \nabla_{[a^*_{2\nu-1}} U^*_{a_{2\nu}] a_1 \ldots \hat{a}_{2\nu-1} \hat{a}_{2\nu} \ldots a_{2s}) , \end{aligned} \tag{4.16}$$

$$(\delta H)_{ba_3 \ldots a_{2s}} := \nabla^c H_{bca_3 \ldots a_{2s}} , \tag{4.17}$$

where the hatted indices are to be omitted (cf. [22], note the modified factors). By Lemma 4.1, the operator d maps vector-bivector fields into bivector fields of rank s, whereas δ acts reversed. Moreover, one obtains the following spinor equivalents of the tensor fields dU and δH [22]:

$$(dU)_{a_1 \ldots a_{2s}} \leftrightarrow -\frac{1}{2} \{ N[\chi]_{A_1 \ldots A_{2s}} \varepsilon_{\dot{X}_1 \ldots \dot{X}_{2s}} + M[\vartheta]_{\dot{X}_1 \ldots \dot{X}_{2s}} \varepsilon_{A_1 \ldots A_{2s}} \} ,$$

$$\begin{aligned} (\delta H)_{ba_3 \ldots a_{2s}} \leftrightarrow -\{ &M[\varphi]_{A_3 \ldots A_{2s} B \dot{Y}} \varepsilon_{\dot{X}_3 \ldots \dot{X}_{2s}} \\ &+ N[\xi]_{\dot{X}_3 \ldots \dot{X}_{2s} \dot{Y} B} \varepsilon_{A_3 \ldots A_{2s}} \} . \end{aligned} \tag{4.18}$$

Lemma 4.2. *The operator δ is the adjoint of the operator d and we have the following scalar products*:

$$
\begin{aligned}
(\bar{H}, dU) &= -2^{s-1}\{(\bar{\varphi}, M[\vartheta]) + (\bar{\xi}, N[\chi])\}\,,\\
(\delta\bar{H}, U) &= -2^{s-1}\{(N[\bar{\varphi}], \vartheta) + (M[\bar{\xi}], \chi)\}\,,\\
(\bar{H}, H) &= 2^{s}\{(\bar{\varphi}, \xi) + (\varphi, \bar{\xi})\}\,,\\
(\bar{U}, U) &= 2^{s-1}\{(\bar{\chi}, \vartheta) + (\bar{\vartheta}, \chi)\}\,.
\end{aligned}
\tag{4.19}
$$

Proof. From (4.15) and (4.18) one obtains immediately the scalar products (4.19). Then the operators d and δ are adjoined to each other by Proposition 3.2. ■

Now we are able to specify Theorem 1 for bosonic fields with higher spin:

Proposition 4.2. *Let $n = 2s - 1$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\mu = m^2$, $\nu = -1$ and the connection between the spinor and tensor fields be given by* (4.15). *Then the Lagrangian density* (3.21) *reads*

$$
\begin{aligned}
L^{(s)} = -2^{1-s}\Big(& a\{(\delta\bar{H}, U) + (\delta H, \bar{U})\} + b\{(\bar{H}, dU) + (H, d\bar{U})\}\\
& - (a+b)\left\{\frac{1}{2}(H, \bar{H}) + m^2(U, \bar{U})\right\}\Big)\,,
\end{aligned}
\tag{4.20}
$$

and the field equations (3.22), (3.23) *are*

$$
H = 2dU, \quad \delta H - m^2 U = 0\,. \tag{4.21}
$$

Proof. By Lemma 4.2, the Lagrangian density (4.20) is equivalent to (3.21) with $n = 2s - 1$ if the constants are specified in the given manner. (Note: The scalar product $(\cdot,\cdot)$ is symmetrical for bosonic fields.) By (4.18) and (4.15), the anti-self-dual part of the equations (4.21) gives just (3.22), whereas the self-dual part gives (3.23). Therefore, the proposition is already proved.

Corollary. *Using the field equations* (4.21) *the Lagrangian density* (4.20) *can be simplified to*

$$
L^{(s)} = 2^{1-s}(b-a)\left(-\frac{1}{2}(\bar{H}, H) + m^2(\bar{U}, U)\right). \tag{4.20'}
$$

Remark 4.1. Define the operator $\tilde{d}$ for vector-bivector fields by

$$
(\tilde{d}U)_{a_1 \ldots a_{2s}} := \nabla_{[a_1} U_{a_2]a_3 \ldots a_{2s}}\,. \tag{4.22}
$$

Then, by (4.15), we obtain

$$
(\bar{H}, \tilde{d}U) = -2^{s-1}\{(\bar{\varphi}, M[\vartheta]) + (\bar{\xi}, N[\chi])\} = (\bar{H}, dU)\,.
$$

Consequently, we can replace dU by $\tilde{d}U$ in the Lagrangian density; the scalar product "singles out" the correct symmetries. But from that we cannot derive the wrong field equations $2\tilde{d}U = H$. The differential operator d is the correct one because it maps U into a bivector of rank s.

Remark 4.2. If the field tensors are real, then $\xi = \bar{\varphi}$, $\vartheta = \bar{\chi}$ and the Lagrangian densities (4.20), (4.7) reduce to those considered in [19, 20]. In this case, the field equations (3.22) and (3.23) are identical (apart from complex conjugation).

Remark 4.3. The first equation of the pairs (4.8) and (4.21), namely $H = 2dU$, is known in connection with massless fields, too. Examples are the relations between the Maxwell field tensor and the electromagnetic potential ($s = 1$) or the conformal curvature tensor and the Lanczos potential ($s = 2$, cf. [22]). Therefore, one might comprehend U as potential for the field H. ■

The above statements show that the tensor form of the field equations is exceedingly difficult for $s \geqq 2$ in comparison with that of spin-1 fields. (There are at least two mathematical reasons for it: Firstly, the Poincaré lemma in the form

$$\nabla_{[a} H_{bc]} = 0 \;\Leftrightarrow\; H_{bc} = \nabla_{[b} U_{c]}$$

is generally not true if there are further free indices on U and H [on the contrary to flat space-times [9]]. Secondly, the divergence of U vanishes for $s > 1$ only if the space-time is conformally flat [see (3.6)]). Therefore, we present the field equations in an alternative manner without the two-sided dual (4.16) by use of the formula [32]

$$e_{abcd} e^{klmn} = -24 \delta_a^{[k} \delta_b^l \delta_c^m \delta_d^{n]} . \tag{4.23}$$

We give all the explicit formulas only for $s = 2$, because this case is of particular interest.

Lemma 4.3. *Let G be a tensor of rank 4 with the symmetries*

$$G_{abcd} = G_{[ab][cd]} = G_{cdab}, \quad g^{ac} g^{bd} G_{abcd} = 0 ,$$

and let $\tilde{G}$ be defined by

$$\tilde{G}_{ac} := g^{bd} G_{abcd} .$$

Then we have

$$*G^{*}_{ab}{}^{cd} = -G^{cd}{}_{ab} + 4 \delta^{[c}_{[a} \tilde{G}^{d]}_{b]} . \tag{4.24}$$

Proof. If the tensor G satisfies the assumptions of Lemma 4.3 then the tensor $\tilde{G}$ is symmetric and tracefree. Using (4.23) one obtains formula (4.24) after a straightforward calculation.

Corollary. *The tensor $G_{abcd} - *G^{*}{}_{abcd}$ is trace-free in all pairs of indices.*

Example. We can give a simple application of Lemma 4.3. The curvature tensor satisfies the assumptions of this lemma if the scalar curvature vanishes. Then one obtains by (4.24),

$$C_{abcd} = \frac{1}{2}(R_{abcd} - *R^{*}_{abcd}) .$$ ■

After these preliminaries we are able to deal with the differential operator d defined by (4.16). We put

$$G_{abcd} = \frac{1}{4}(\nabla_{[a} U_{b]cd} + \nabla_{[c} U_{d]ab}) ,$$

$$\tilde{G}_{ac} = -\frac{1}{4} \nabla^{e} U_{(ac)e} .$$

By Lemma 4.1, the assumptions of Lemma 4.3 are satisfied. Consequently, by (4.16) and (4.24), we obtain the field equations (4.21) for $s = 2$ in the equivalent form

$$\begin{aligned} H_{abcd} = \nabla_{[a} U_{b]cd} + \nabla_{[c} U_{d]ab} \\ + \frac{1}{2}(g_{ac} \nabla^e U_{(bd)e} + g_{bd} \nabla^e U_{(ac)e} \\ - g_{ad} \nabla^e U_{(bc)e} - g_{bc} \nabla^e U_{(ad)e}) , \end{aligned} \tag{4.25}$$

$$\nabla^e H_{aebc} - m^2 U_{abc} = 0 . \tag{4.26}$$

As for spin-1 fields, there are different possibilities to obtain the second-order equations. We will give the explicit form of

$$2d\delta H - m^2 H = 0$$

only because it is of normal hyperbolic type as has been shown in Sect. 3.2. By (3.11) and (3.12) we can obtain the desired equation by "translation" of

$$\nabla^a \nabla_a \varphi_{ABCD} - 6\Psi_{(AB}{}^{EF} \varphi_{CD)EF} + (12\Lambda + m^2) \varphi_{ABCD} = 0 \tag{4.27}$$

into tensor form. Before we can do this we must transform the term containing the Weyl spinor. We obtain

$$\begin{aligned} \Psi_{(AB}{}^{EF} \varphi_{CD)EF} = \frac{1}{2}(\Psi_{AB}{}^{EF} \varphi_{CDEF} + \Psi_{CD}{}^{EF} \varphi_{ABEF}) \\ - \frac{1}{6}(\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BC}) \Psi^{EFGK} \varphi_{EFGK} . \end{aligned}$$

Now we define the anti-self-dual part of H by

$$\mathscr{H}_{abcd} := \frac{1}{2}(H_{abcd} + iH^*_{abcd}) .$$

Consequently, by (4.15),

$$\mathscr{H}_{abcd} \leftrightarrow \varphi_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} .$$

Using

$$C_{ab}{}^{ef} \mathscr{H}_{cdef} \leftrightarrow 2\Psi_{AB}{}^{EF} \varphi_{CDEF} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} ,$$

$$C^{efgk} \mathscr{H}_{efgk} \leftrightarrow 4\, \Psi^{EFGK} \varphi_{EFGK} ,$$

$$2g_{a[c} g_{d]b} + ie_{abcd} \leftrightarrow (\varepsilon_{AC} \varepsilon_{BD} + \varepsilon_{AD} \varepsilon_{BC}) \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{C}\dot{D}} ,$$

we obtain the following tensor equivalent of (4.27):

$$\begin{aligned} \nabla^e \nabla_e \mathscr{H}_{abcd} - \frac{3}{2}(C_{ab}{}^{ef} \mathscr{H}_{cdef} + C_{cd}{}^{ef} \mathscr{H}_{abef}) \\ + \frac{1}{4}(2g_{a[c} g_{d]b} + ie_{abcd}) C^{efgk} \mathscr{H}_{efgk} + \left(\frac{R}{2} + m^2\right) \mathscr{H}_{abcd} = 0 . \end{aligned} \tag{4.28}$$

One can deduce an analogous equation for the self-dual part of H.

4.3. Dirac Fields. In the Dirac theory [10], the "field functions" are 4-component spinor fields Ψ ("bispinors", indices suppressed). If we denote the Dirac matrices by γ^a, the differential operator $\nabla\!\!\!\!/$ by

$$\nabla\!\!\!\!/\, \Psi = i\gamma^a \nabla_a \Psi \tag{4.29}$$

and the conjugated bispinor by Ψ^+, then the Lagrangian density for massive spin-$\frac{1}{2}$ fields reads

$$L^{(\frac{1}{2})} = \frac{1}{2}(\Psi^+ \vec{\nabla\!\!\!\!/}\Psi - \Psi^+ \overleftarrow{\nabla\!\!\!\!/}\Psi) + m\Psi^+\Psi\,, \tag{4.30}$$

and the field equations for Ψ and Ψ^+ are

$$\vec{\nabla\!\!\!\!/}\Psi + m\Psi = 0\,, \tag{4.31}$$

$$\Psi^+\overleftarrow{\nabla\!\!\!\!/} - m\Psi^+ = 0\,, \tag{4.32}$$

respectively (cf. [2, 4, 26, 32, 35]). To compare (4.30) ... (4.32) with (3.21) ... (3.23) we have to "translate" the given equations into the 2-component spinor calculus using the Weyl representation (see [12, 13, 32], cf. also Remark 2.3.b).

In the Weyl representation, the Dirac matrices have the explicit form

$$\gamma_a = \sqrt{2}\begin{pmatrix} 0 & \sigma_a{}^{A\dot{X}} \\ \sigma_{a\dot{Y}B} & 0 \end{pmatrix} \tag{4.33}$$

and the matrix η defined by

$$\eta := -\frac{i}{24}e^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d \tag{4.34}$$

(One often finds the notation γ_5 for η in the literature) reads

$$\eta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{4.35}$$

The chiral parts of the Dirac spinor are

$$\Psi^{(\pm)} := \tfrac{1}{2}(\mathbb{1} \pm \eta)\Psi \tag{4.36}$$

and may be regarded as 2-component spinors of first and second kind, respectively. Thus, by (4.35), we have

$$\Psi \leftrightarrow \begin{pmatrix} \Psi^{(+)} \\ \Psi^{(-)} \end{pmatrix} \leftrightarrow \begin{pmatrix} \varphi^B \\ \chi_{\dot{X}} \end{pmatrix}, \tag{4.37}$$

where $\varphi \in \mathscr{S}_{1,0}$ and $\chi \in \mathscr{S}_{0,1}$ and

$$\Psi^+ = \overline{\Psi^T}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\bar{\chi}_A \bar{\varphi}^{\dot{Y}})\,. \tag{4.38}$$

Using (4.29), (4.33), (4.37), (4.38) and Definition 3.1 we eventually obtain

$$\vec{\nabla\!\!\!\!/}\Psi \leftrightarrow i\sqrt{2}\begin{pmatrix} 0 & \nabla^{A\dot{X}} \\ \nabla_{\dot{Y}B} & 0 \end{pmatrix}\begin{pmatrix} \varphi^B \\ \chi_{\dot{X}} \end{pmatrix} = i\sqrt{2}\begin{pmatrix} \nabla^{A\dot{X}}\chi_{\dot{X}} \\ \nabla_{B\dot{Y}}\varphi^B \end{pmatrix} = i\sqrt{2}\begin{pmatrix} N[\chi]^A \\ -M[\varphi]_{\dot{Y}} \end{pmatrix}, \tag{4.39}$$

$$\Psi^+\overleftarrow{\nabla\!\!\!\!/} \leftrightarrow i\sqrt{2}(-N[\bar{\varphi}]_B \quad M[\bar{\chi}]^{\dot{X}})\,. \tag{4.40}$$

By (4.38) and Remark 3.1, the bispinor $-\Psi^+\overleftarrow{\nabla\!\!\!\!/}$ is just the conjugated of $\vec{\nabla\!\!\!\!/}\Psi$.

After these considerations we notice that only two independent spinor fields φ and χ are needed for $s = \frac{1}{2}$ instead of four in (3.21). However, for $n = 0$, we have $\mathscr{S}_{n+1,0} \equiv \mathscr{S}_{1,n}$ and $\mathscr{S}_{0,n+1} = \mathscr{S}_{n,1}$ therefore we can identify the fields φ and ϑ as well as χ and ξ.

Proposition 4.3. *Let* $n = 0, \varphi \equiv \vartheta, \chi = \xi, a = b = -i\sqrt{2}/4, \mu = \nu = im/\sqrt{2}$ *and the connection between the bispinor* Ψ *and the spinors* $\varphi \in \mathscr{S}_{1,0}$ *and* $\chi \in \mathscr{S}_{0,1}$ *be given by* (4.37). *Then the Lagrangian density* $L^{(\frac{1}{2})}$ *of Theorem* 1 *is equivalent to* (4.30) *and the field equations* (4.31), (4.32) *are identically with* (3.22) *and* (3.23).

Proof. By use of (4.37) . . . (4.40) the Lagrangian density (4.30) reads

$$L^{(\frac{1}{2})} = \frac{i\sqrt{2}}{2}((\bar{\chi}, N[\chi]) + (\bar{\varphi}, M[\varphi]) + (N[\bar{\varphi}], \varphi) + (M[\bar{\chi}], \chi))$$
$$+ m((\bar{\chi}, \varphi) - (\bar{\varphi}, \chi)),$$

whereas the field equations (4.31), (4.32) are

$$i\sqrt{2}N[\chi] + m\varphi = 0, \quad -i\sqrt{2}M[\varphi] + m\chi = 0,$$
$$-i\sqrt{2}N[\bar{\varphi}] - m\bar{\chi} = 0, \quad i\sqrt{2}M[\bar{\chi}] - m\bar{\varphi} = 0.$$

These formulas are just those of Theorem 1 for $n = 0$ if the constants are specified in the given manner. (Note that the scalar product $(\cdot,\cdot)$ is antisymmetric for fermionic fields.) Thus the proposition is already proved and we see by Remark 3.1, that the second pair of the field equations is the complex conjugate of the first for $s = \frac{1}{2}$.

4.4. Fermionic Fields with Higher Spin. If one compares the theories of Fierz and Rarita–Schwinger (see Remark 2.3), then one obtains – roughly speaking – the field function of the fermionic field with spin $s = t + \frac{1}{2}$ ($t = 1, 2, \ldots$) as the tensor product of a bosonic field of spin t and a bispinor, restricted by the second equation of (2.4). We will do here the same: The "field functions" $\Psi_{a_1 \ldots a_{2t}}$ are the tensor product of a bivector field of rank t (the bosonic field tensor in the sense of Remark 4.3) and a bispinor satisfying the condition

$$\gamma^{a_1}\gamma^{a_2}\Psi_{a_1 \ldots a_{2t}} = 0 \tag{4.41}$$

(cf. also [12, 34] for flat space-times).

Splitting Ψ into 2-component spinors $\Psi^{(\pm)}$ we obtain by (4.15) and (4.37),

$$\Psi_{a_1 \ldots a_{2t}} \leftrightarrow \begin{pmatrix} \Psi^{(+)}_{a_1 \ldots a_{2t}} \\ \Psi^{(-)}_{a_1 \ldots a_{2t}} \end{pmatrix}$$
$$\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_{A_1 \ldots A_{2t}}{}^{B}\varepsilon_{\dot{X}_1 \ldots \dot{X}_{2t}} + \vartheta_{\dot{X}_1 \ldots \dot{X}_{2t}}{}^{B}\varepsilon_{A_1 \ldots A_{2t}} \\ \chi_{A_1 \ldots A_{2t}\dot{X}}\varepsilon_{\dot{X}_1 \ldots \dot{X}_{2t}} + \xi_{\dot{X}_1 \ldots \dot{X}_{2t}\dot{X}}\varepsilon_{A_1 \ldots A_{2t}} \end{pmatrix}, \tag{4.42}$$

where the spinors φ and χ are symmetrical with respect to the indices $A_1 \ldots A_{2t}$, whereas ϑ and ξ are symmetrical with respect to $\dot{X}_1 \ldots \dot{X}_{2t}$. By (4.33), the condition (4.41) yields

$$\varphi_{BA_2 \ldots A_{2t}}{}^{B} = 0, \quad \xi^{\dot{X}}{}_{\dot{X}_2 \ldots \dot{X}_{2t}\dot{X}} = 0,$$

therefore we obtain $\varphi \in \mathscr{S}_{2t+1,0}$, $\vartheta \in \mathscr{S}_{1,2t}$, $\chi \in \mathscr{S}_{2t,1}$, $\xi \in \mathscr{S}_{0,2t+1}$. The conjugated field Ψ^+ reads by (4.38),

$$\Psi^+_{a_1 \dots a_{2t}} \leftrightarrow$$

$$\frac{1}{\sqrt{2}}(\bar{\xi}_{A_1 \dots A_{2t}A}\varepsilon_{\dot{X}_1 \dots \dot{X}_{2t}} + \bar{\chi}_{\dot{X}_1 \dots \dot{X}_{2t}A}\varepsilon_{A_1 \dots A_{2t}}$$

$$\bar{\vartheta}_{A_1 \dots A_{2t}}{}^{\dot{Y}}\varepsilon_{\dot{X}_1 \dots \dot{X}_{2t}} + \bar{\varphi}_{\dot{X}_1 \dots \dot{X}_{2t}}{}^{\dot{Y}}\varepsilon_{A_1 \dots A_{2t}}) . \tag{4.43}$$

The differential operator $\nabla\!\!\!\!/$ for bivector-bispinor fields is defined by generalization of (4.39):

$$\vec{\nabla\!\!\!\!/}\Psi_{a_1 \dots a_{2t}} \leftrightarrow i\begin{pmatrix} N[\chi]_{A_1 \dots A_{2t}}{}^{A}\varepsilon_{\dot{X}_1 \dots \dot{X}_{2t}} + N[\xi]_{\dot{X}_1 \dots \dot{X}_{2t}}{}^{A}\varepsilon_{A_1 \dots A_{2t}} \\ -M[\varphi]_{A_1 \dots A_{2t}\dot{Y}}\varepsilon_{\dot{X}_1 \dots \dot{X}_{2t}} - M[\vartheta]_{\dot{X}_1 \dots \dot{X}_{2t}\dot{Y}}\varepsilon_{A_1 \dots A_{2t}} \end{pmatrix}, \tag{4.44}$$

whereas $\Psi^+\overleftarrow{\nabla\!\!\!\!/}$ is the conjugated of $-\vec{\nabla\!\!\!\!/}\Psi$ accordant with (4.40).

Lemma 4.4. *We have the following scalar products*:

$$\Psi^+_{a_1 \dots a_{2t}}\vec{\nabla\!\!\!\!/}\Psi^{a_1 \dots a_{2t}} = i\sqrt{2}2^{t-1}((\bar{\xi}, N[\chi]) + (\bar{\chi}, N[\xi]) + (\bar{\vartheta}, M[\varphi]) + (\bar{\varphi}, M[\vartheta])) , \tag{4.45}$$

$$\Psi^+_{a_1 \dots a_{2t}}\overleftarrow{\nabla\!\!\!\!/}\Psi^{a_1 \dots a_{2t}} = -i\sqrt{2}2^{t-1}((N[\bar{\varphi}], \vartheta) + (N[\bar{\vartheta}], \varphi) + (M[\bar{\chi}], \xi) + (M[\bar{\xi}], \chi)) , \tag{4.46}$$

$$\Psi^+_{a_1 \dots a_{2t}}\Psi^{a_1 \dots a_{2t}} = 2^{t-1}((\bar{\xi}, \varphi) + (\bar{\chi}, \vartheta) - (\bar{\vartheta}, \chi) - (\bar{\varphi}, \xi)) . \tag{4.47}$$

Proof. The relations (4.45) ... (4.47) follow immediately from (4.42) ... (4.44) and Definition 3.2.

Corollary. *Because of the antisymmetry of the scalar product* $(\cdot, \cdot)$ *for fermionic fields and Remark* 3.1 *we have*

$$\Psi^+\vec{\nabla\!\!\!\!/}\Psi = -\overline{\Psi^+\vec{\nabla\!\!\!\!/}\Psi}, \quad \Psi^+\Psi = \overline{\Psi^+\Psi} . \tag{4.48}$$

Proposition 4.4. *Let* $n = 2t$, $a = b = -i\sqrt{2}/4$, $\mu = \nu = im/\sqrt{2}$ *and the connection between the bivector-bispinor field and the (2-component) spinor fields given by* (4.42). *Then the Lagrangian density* $L^{(t+\frac{1}{2})}$ *of Theorem* 1 *is equivalent to*

$$L^{(t+\frac{1}{2})} = 2^{-t}\{\tfrac{1}{2}(\Psi^+_{a_1 \dots a_{2t}}\vec{\nabla\!\!\!\!/}\Psi^{a_1 \dots a_{2t}} - \Psi^+_{a_1 \dots a_{2t}} \times \overleftarrow{\nabla\!\!\!\!/}\Psi^{a_1 \dots a_{2t}}) + m\Psi^+_{a_1 \dots a_{2t}}\Psi^{a_1 \dots a_{2t}}\} , \tag{4.49}$$

and the field equations (3.22), (3.23) *are*

$$\vec{\nabla\!\!\!\!/}\Psi_{a_1 \dots a_{2t}} + m\Psi_{a_1 \dots a_{2t}} = 0 . \tag{4.50}$$

Proof. The proof follows immediately from Lemma 4.4 and (4.44).

Remark 4.4. In the Lagrangian density (4.49) one can replace the differential operator $\nabla\!\!\!\!/$ by $i\gamma^a\nabla_a$ because the scalar product singles out the correct symmetries (cf. Remark 4.1). But the field equations $i\gamma^a\nabla_a\Psi + m\Psi = 0$ are not correct for $t > 0$ because the field $\gamma^a\nabla_a\Psi$ does not satisfy (4.41).

We can define the differential operator $\nabla\!\!\!\!/$ and the field equations (4.50) without any use of 2-component spinor fields. But we give all explicit formulas only for $t = 1$ (i.e. for $s = 3/2$) because this case is of particular interest.

At the beginning we quote three formulas for Dirac matrices (cf. [12, 32]):

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2g^{ab}\mathbb{1} \,, \tag{4.51}$$

$$\gamma_{[a}\gamma_{b]}{}^* = -i\eta\gamma_{[a}\gamma_{b]} \,, \tag{4.52}$$

$$\gamma_e\gamma_{[a}\gamma_{b]} + 2\gamma_{[a}g_{b]e} = -i\eta\, e_{abe}{}^f\gamma_f \,. \tag{4.53}$$

Lemma 4.5. *The condition* (4.41) *is for* $t = 1$ *equivalent to*

$$\gamma^a(\Psi_{ab} + i\eta\Psi^*_{ab}) = 0 \,. \tag{4.54}$$

Proof. Contracting (4.53) by some tensor-bispinor Ψ^{ab} we obtain the identity

$$\gamma_e\gamma^a\gamma^b\Psi_{[ab]} = -2(\gamma^a\Psi_{[ae]} - i\eta\gamma^f\Psi^*_{[fe]}) \,. \tag{4.55}$$

The assertion follows immediately from this relation. (Note: The matrices η and γ^a anticommute.)

Corollary. *If the bivector-bispinor* Ψ *satisfies the condition* (4.41), *then its conjugate* Ψ^+ *satisfies*

$$(\Psi^+_{ab} + i(\Psi^+_{ab})^*\eta)\gamma^a = 0 \,. \tag{4.56}$$

The formulas (4.54) and (4.56) contain the essential information how the differential operator $\nabla\!\!\!\!/$ has to be defined.

Proposition 4.5. *Let* Ψ_{ab} *be a bivector-bispinor satisfying* (4.41). *Then we have*

$$\nabla\!\!\!\!/\,\Psi_{ab} = i\gamma^c\nabla_c\Psi_{ab} - \frac{2}{3}(i\gamma_{[a}\nabla^d\Psi_{|d|b]} - \eta\gamma_{[a}\nabla^d\Psi_{|d|b]}{}^*) \,. \tag{4.57}$$

Proof. We give two independent possibilities to prove this proposition. The first uses the splitting (4.42) into 2-component spinors, whereas the second does not need such a splitting.

1. By (4.33) and (4.42) we obtain

$$\gamma^c\nabla_c\Psi_{kl} \leftrightarrow \begin{pmatrix} \nabla^{A\dot X}\chi_{KL\dot X}\varepsilon_{\dot K\dot L} + \nabla^{A\dot X}\xi_{\dot K\dot L\dot X}\varepsilon_{KL} \\ \nabla_{B\dot Y}\varphi_{KL}{}^B\varepsilon_{\dot K\dot L} + \nabla_{B\dot Y}\vartheta_{\dot K\dot L}{}^B\varepsilon_{KL} \end{pmatrix},$$

and further, by Definition 3.1 and (3.7) ,

$$\gamma^c\nabla_c\Psi_{kl} \leftrightarrow \begin{pmatrix} N[\chi]^A{}_{KL}\varepsilon_{\dot K\dot L} + N[\xi]^A{}_{\dot K\dot L}\varepsilon_{KL} + \frac{2}{3}\delta^A{}_{(K}\nabla^{C\dot Z}\chi_{L)C\dot Z}\varepsilon_{\dot K\dot L} \\ -M[\varphi]_{KL\dot Y}\varepsilon_{\dot K\dot L} - M[\vartheta]_{\dot K\dot L\dot Y}\varepsilon_{KL} + \frac{2}{3}\varepsilon_{\dot Y(\dot K}\nabla^{C\dot Z}\vartheta_{\dot L)\dot ZC}\varepsilon_{KL} \end{pmatrix}. \tag{4.58}$$

From (4.33) and (4.42) it follows after some calculations

$$\gamma_{[k}\nabla^d\Psi_{|d|l]} \leftrightarrow \frac{1}{2}\begin{pmatrix} (\nabla^{\dot D}_{(\dot K}\chi_D{}^A{}_{\dot L)} + \nabla^{A\dot D}\xi_{\dot D\dot K\dot L})\varepsilon_{KL} + \delta^A_{(K}\nabla^{C\dot Z}\chi_{L)C\dot Z}\varepsilon_{\dot K\dot L} \\ -(\nabla^D_{(K}\vartheta_{L)\dot D\dot Y} + \nabla^D_{\dot Y}\varphi_{DKL})\varepsilon_{\dot K\dot L} + \varepsilon_{\dot Y(\dot K}\nabla^{C\dot Z}\vartheta_{\dot L)\dot ZC}\varepsilon_{KL} \end{pmatrix},$$

and by use of (4.35) ,

$$i\gamma_{[k}\nabla^{d}\Psi_{|d|l]} - \eta\gamma_{[k}\nabla^{d}\Psi_{|d|l]}^{*} \leftrightarrow i\begin{pmatrix}\delta^{A}_{(K}\nabla^{C\dot{Z}}\chi_{L)C\dot{Z}}\varepsilon_{\dot{K}\dot{L}} \\ \varepsilon_{\dot{Y}(\dot{K}}\nabla^{C\dot{Z}}\vartheta_{\dot{L})\dot{Z}C}\varepsilon_{KL}\end{pmatrix}. \tag{4.59}$$

From (4.58) and (4.59) we obtain (4.57).
2. By virtue of (4.56) we have

$$(\Psi^{+})^{ab}(\gamma_{[a}\nabla^{d}\Psi_{|d|b]} + i\eta\gamma_{[a}\nabla^{d}\Psi_{|d|b]}^{*}) = 0 .$$

Therefore, by Remark 4.4, the proposition is proved if the right-hand side of (4.57) satisfies the condition (4.41).

By (4.52) we obtain

$$\gamma^{a}\gamma^{b}\eta\gamma_{[a}\nabla^{d}\Psi_{|d|b]}^{*} = -i\gamma^{a}\gamma^{b}\gamma_{[a}\nabla^{d}\Psi_{|d|b]} ,$$

and by use of (4.51) ,

$$\gamma^{a}\gamma^{b}(i\gamma_{[a}\nabla^{d}\Psi_{|d|b]} - \eta\gamma_{[a}\nabla^{d}\Psi_{|d|b]}^{*}) = 2i\gamma^{a}\gamma^{b}\gamma_{[a}\nabla^{d}\Psi_{|d|b]} = -6i\gamma^{b}\nabla^{d}\Psi_{db} . \tag{4.60}$$

Further we obtain by (4.51)

$$\gamma^{a}\gamma^{b}\gamma^{c}\nabla_{c}\Psi_{ab} = -4\gamma^{b}\nabla^{a}\Psi_{ab} , \tag{4.61}$$

since Ψ_{ab} satisfies the condition (4.41). From (4.61) and (4.60) it follows that the right-hand side of (4.57) satisfies the condition (4.41). Thus the proposition is proved.

5. The Electromagnetic Field as Gauge Field

The complex character of the fields considered in the preceding chapter allows the action of the gauge group $U(1)$ on the "field functions." By (4.15) and (4.42), a gauge transformation

$$H \to H' = \exp(ie\tau)H, \quad U \to U' = \exp(ie\tau)U$$

for bosonic fields and

$$\Psi \to \Psi' = \exp(ie\tau)\Psi$$

for fermionic fields (e: electric charge) gives rise to a gauge transformation of the related spinor fields according to

$$\varphi \to \varphi' = \exp(ie\tau)\varphi, \quad \vartheta \to \vartheta' = \exp(ie\tau)\vartheta ,$$
$$\chi \to \chi' = \exp(ie\tau)\chi, \quad \xi \to \xi' = \exp(ie\tau)\xi . \tag{5.1}$$

Therefore, using the spinor description of Sect. 3.3, we can deal with all fields of spin $s \geqq \frac{1}{2}$ in an uniform manner.

Obviously, the Lagrangian density (3.21) is invariant under a transformation (5.1) if $\tau \in \mathbb{R}$, $\tau = \text{const.}$ (global symmetry). If we have a local gauge transformation (5.1) with a real function $\tau = \tau(x)$ we must replace

$$\nabla_{A\dot{X}} \to \nabla_{A\dot{X}} - ieA_{A\dot{X}} , \tag{5.2}$$

where the gauge field A_a may be considered as electromagnetic potential [14]. We can state the following:

Theorem 2. *Let* $B_{A\dot{X}} = ie\sigma^a{}_{A\dot{X}}A_a$ *with* $\bar{A}_a = A_a$, $F_{ab} = 2\nabla_{[a}A_{b]}$, *and the spinor fields* φ, χ, ξ, ϑ *be as in Theorem* 1. *The Lagrangian density*

$$\begin{aligned} L_{\mathrm{elm}}^{(\frac{n+1}{2})} &= a\{(M^{(-)}[\varphi], \bar{\vartheta}) + (\chi, M^{(+)}[\bar{\xi}])\} + b\{(\varphi, N^{(+)}[\bar{\vartheta}]) + (N^{(-)}[\chi], \bar{\xi})\} \\ &+ \bar{a}\{(N^{(+)}[\bar{\varphi}], \vartheta) + (\bar{\chi}, N^{(-)}[\xi])\} + \bar{b}\{(\bar{\varphi}, M^{(-)}[\vartheta]) + (M^{(+)}[\bar{\chi}], \xi)\} \\ &+ (a+b)\{\mu(\chi, \bar{\vartheta}) - \nu(\varphi, \bar{\xi})\} \\ &+ (\bar{a}+\bar{b})\{\bar{\mu}(\bar{\chi}, \vartheta) - \bar{\nu}(\bar{\varphi}, \xi)\} - \frac{1}{4}F_{ab}F^{ab} \end{aligned} \tag{5.3}$$

is invariant under a local gauge transformation (5.1) *if the gauge field* A_a *is transformed according to*

$$A_a \to A'_a = A_a - \nabla_a \tau\,. \tag{5.4}$$

The Euler–Lagrange equations of the related variational problems are

$$M^{(-)}[\varphi] + \mu\chi = 0, \quad N^{(-)}[\chi] - \nu\varphi = 0\,, \tag{5.5}$$

$$N^{(-)}[\xi] + \bar{\mu}\vartheta = 0, \quad M^{(-)}[\vartheta] - \bar{\nu}\xi = 0\,, \tag{5.6}$$

$$\nabla^b(\nabla_b A_a - \nabla_a A_b) = j_a\,, \tag{5.7}$$

where the current vector j is given by

$$\begin{aligned} j_{A\dot{X}} = ie\{&(a+b)(\varphi_{AA_1\ldots A_n}\bar{\vartheta}^{A_1\ldots A_n}{}_{\dot{X}} + \chi_{A_1\ldots A_n\dot{X}}\bar{\xi}_A{}^{A_1\ldots A_n}) \\ &- (\bar{a}+\bar{b})(\bar{\varphi}_{\dot{X}\dot{X}_1\ldots\dot{X}_n}\vartheta_A{}^{\dot{X}_1\ldots\dot{X}_n} + \bar{\chi}_{A\dot{X}_1\ldots\dot{X}_n}\xi_{\dot{X}}{}^{\dot{X}_1\ldots\dot{X}_n})\}\,. \end{aligned} \tag{5.8}$$

Proof. Consider the first term of the Lagrangian density (5.3). If the spinor fields φ and ϑ undergo a gauge transformation (5.1) we obtain by Definition 3.1,

$$\begin{aligned} &(\nabla^A_{\dot{X}} - ieA^A_{\dot{X}})\varphi'_{AA_1\ldots A_n}\bar{\vartheta}'^{A_1\ldots A_n\dot{X}} \\ &= (\nabla^A_{\dot{X}} - ieA^A_{\dot{X}})\{\exp(ie\tau)\varphi_{AA_1\ldots A_n}\}\exp(-ie\tau)\bar{\vartheta}^{A_1\ldots A_n\dot{X}} \\ &= (\nabla^A_{\dot{X}} - ie[A^A_{\dot{X}} - \nabla^A_{\dot{X}}\tau])\varphi_{AA_1\ldots A_n}\bar{\vartheta}^{A_1\ldots A_n\dot{X}}. \end{aligned}$$

For the second term we obtain

$$\begin{aligned} &\chi'_{A_1\ldots A_n\dot{X}}(\nabla^{A\dot{X}} + ieA^{A\dot{X}})\bar{\xi}'_A{}^{A_1\ldots A_n} \\ &= \exp(ie\tau)\chi_{A_1\ldots A_n\dot{X}}(\nabla^{A\dot{X}} + ieA^{A\dot{X}})\{\exp(-ie\tau)\xi_A{}^{A_1\ldots A_n}\} \\ &= \chi_{A_1\ldots A_n\dot{X}}(\nabla^{A\dot{X}} + ie[A^{A\dot{X}} - \nabla^{A\dot{X}}\tau])\xi_A{}^{A_1\ldots A_n}. \end{aligned}$$

In the same manner one calculates all summands of (5.3). Therefore the gauge invariance of the Lagrangian density (5.3) is already proved. We remark that $M^{(+)}[\bar{\xi}]$ is just the complex conjugate of $N^{(-)}[\xi]$ (cf. Remark 3.1).

The derivation of the field equations (5.5) and (5.6) is carried out as in Sect. 3.3 by variation of the fields $\bar{\vartheta}$, $\bar{\xi}$, $\bar{\varphi}$ and $\bar{\chi}$, respectively.

The "interaction part" of the Lagrangian density (5.3) reads by Definition 3.1

$$
\begin{aligned}
L_{\mathrm{int}}^{(\frac{n+1}{2})} = {} & a\big(-B^{A}_{\dot{X}}\varphi_{AA_1\ldots A_n}\bar{\vartheta}^{A_1\ldots A_n\dot{X}} + \chi_{A_1\ldots A_n\dot{X}}B^{A\dot{X}}\bar{\xi}_{A}{}^{A_1\ldots A_n}\big) \\
& + b\big(\varphi_{AA_1\ldots A_n}B^{A\dot{X}}\bar{\vartheta}^{A_1\ldots A_n}{}_{\dot{X}} - B^{\dot{X}}_{A}\chi_{A_1\ldots A_n\dot{X}}\bar{\xi}^{AA_1\ldots A_n}\big) \\
& + \bar{a}\big(B^{\dot{X}}_{A}\bar{\varphi}_{\dot{X}\dot{X}_1\ldots\dot{X}_n}\vartheta^{A\dot{X}_1\ldots\dot{X}_n} - \bar{\chi}_{A\dot{X}_1\ldots\dot{X}_n}B^{A\dot{X}}\xi_{\dot{X}}{}^{\dot{X}_1\ldots\dot{X}_n}\big) \\
& + \bar{b}\big(-\bar{\varphi}_{\dot{X}\dot{X}_1\ldots\dot{X}_n}B^{A\dot{X}}\vartheta_{A}{}^{\dot{X}_1\ldots\dot{X}_n} + B^{A}_{\dot{X}}\bar{\chi}_{A\dot{X}_1\ldots\dot{X}_n}\xi^{\dot{X}\dot{X}_1\ldots\dot{X}_n}\big)\,. \qquad (5.9)
\end{aligned}
$$

From (5.9) we obtain

$$\frac{\partial L}{\partial A^a} = j_a$$

with j_a given by (5.8). Consequently, Eq. (5.7) is the Euler–Lagrange equation related to the vector field A_a and the theorem is proved.

Remark 5.1. Equations (5.7) are of normal hyperbolic type if A_a satisfies the Lorentz gauge condition.

In flat space-times, the current vector j of spin-s fields is known since the paper by Fierz [16]. Using the results of Chapter 4 we can give it in an alternative manner by "translation" of (5.8) into tensor and bispinor form, respectively. Because the calculation is straightforward we give only the results: The current vector for bosonic fields with spin $s \geqq 1$ reads

$$j_a = ie(a+b)2^{1-s}(\bar{H}_{aba_3\ldots a_{2s}}U^{ba_3\ldots a_{2s}} - H_{aba_3\ldots a_{2s}}\bar{U}^{ba_3\ldots a_{2s}}) \qquad (5.10)$$

(a and b are real constants, see Propositions 4.1 and 4.2) and that for fermionic fields with spin $t + \frac{1}{2}$ $(t \geqq 0)$ is

$$j_a = e2^{-t}\Psi^{+}_{a_1\ldots a_{2t}}\gamma_a\Psi^{a_1\ldots a_{2t}}\,. \qquad (5.11)$$

One obtains the same result, if one generalizes the Lagrangian densities (4.20) and (4.49) as in Theorem 2. The current vector j satisfies the continuity equation for all values of s if the (tensor or spinor) fields satisfy the field equations.

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