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Monopoles, Braid Groups, and the Dirac Operator

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Abstract. Using the relation between the space of rational functions on $\mathbb{C}$, the space of $SU(2)$-monopoles on $\mathbb{R}^3$, and the classifying space of the braid group, see [10], we show how the index bundle of the family of real Dirac operators coupled to $SU(2)$-monopoles can be described using permutation representations of Artin's braid groups. We also show how this implies the existence of a pair consisting of a gauge field A and a Higgs field Φ on $\mathbb{R}^3$ whose corresponding Dirac equation has an arbitrarily large dimensional space of solutions.

1. Introduction and Statement of Results

Let $\mathscr{M}_k$ denote the space of based, $SU(2)$ monopoles in $\mathbb{R}^3$ of charge k. Thus an element of $\mathscr{M}_k$ is represented by a configuration (A, Φ), where A, the gauge field, is a smooth connection on the trivial $SU(2)$ bundle P over $\mathbb{R}^3$ and Φ, the Higgs field, is a smooth section of the vector bundle associated to P via the adjoint representation. Since the bundle P is trivial A can be identified with a smooth 1-form on $\mathbb{R}^3$ with values in the Lie algebra $\mathfrak{su}(2)$ and Φ can be identified with a smooth function Φ: $\mathbb{R}^3 \to \mathfrak{su}(2)$. We equip $\mathbb{R}^3$ with its standard metric and orientation and $\mathfrak{su}(2)$ with its standard invariant inner product. The pair (A, Φ) is a monopole if it satisfies the following conditions:

(1) $1 - |\Phi| \in L^6(\mathbb{R}^3)$.
(2) The pair (A, Φ) has finite energy; that is the Yang–Mills–Higgs functional is finite

$$\mathscr{U}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|F_A|^2 + |D_A \Phi|^2)\, \mathrm{dvol} < \infty .$$

Here D_A is the covariant derivative operator defined by A and F_A is the curvature of A.

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(3) The pair (A, Φ) satisfies the Bogomolnyi equation

$$*F_A = D_A\Phi ,$$

where $*$ is the Hodge star operator on $\mathbb{R}^3$.

The charge of the monopole is defined as follows. Since Φ is smooth and $1 - |\Phi| \in L^6(\mathbb{R}^3)$ it follows that if $|x|$ is large enough $\Phi(x) \neq 0$. So for sufficiently large R, define

$$\Phi_R\colon S_R^2 \to S^2, \quad \Phi_R(x) = \Phi(x)/|\Phi(x)| ,$$

where S_R^2 is the sphere of radius R in $\mathbb{R}^3$ and the target is the unit sphere in $\mathfrak{su}(2)$. Then the degree of Φ_R is the charge of the monopole.

The based gauge group $\mathscr{G}$ is the space of bundle automorphisms $g\colon P \to P$ whose restriction to the fibre over the origin is the identity. This group acts on the pairs (A, Φ) in the usual way and the space $\mathscr{M}_k$ is the quotient of the space of monopoles with charge k by the action of $\mathscr{G}$. For more information concerning the geometry of the space $\mathscr{M}_k$ see [6] and [19].

Let us denote the space of pairs (A, Φ) which satisfy the asymptotic condition

$$1 - |\Phi| \in L^6(\mathbb{R}^3)$$

and has finite energy

$$\mathscr{U}(A, \Phi) = \frac{1}{2}\int_{\mathbb{R}^3} (|F_A|^2 + |D_A\Phi|^2)\mathrm{dvol} < \infty$$

by $\mathscr{A}$. The group $\mathscr{G}$ acts on $\mathscr{A}$ and we denote the quotient space $\mathscr{A}/\mathscr{G}$ by $\mathscr{B}$. The components of this space are determined by the charge of the configuration (A, Φ) which is defined as above. We use the notation $\mathscr{B}_k$ for the component with charge k.

Fundamental results concerning the topology of $\mathscr{M}_k$ have been proved by Taubes, Donaldson and Segal. In [25] Taubes proves by using Morse theoretic arguments that the inclusion $\mathscr{M}_k \to \mathscr{B}_k$ is a homotopy equivalence through dimension k. Taubes proves in [24] that $\mathscr{B}_k$ is homotopy equivalent to the space $\Omega_k^2 S^2$ of all base point preserving maps $S^2 \to S^2$ of degree k and so we see that $\mathscr{M}_k$ is homotopy equivalent to $\Omega_k^2 S^2$ through dimension k.

In [18] Donaldson proves that $\mathscr{M}_k$ is homeomorphic to the space Rat_k of based rational functions p/q of degree k. Here p and q are monic polynomials of degree k over $\mathbb{C}$ with no roots in common. The homotopy type of Rat_k was originally studied by Segal [23]. The space Rat_k is a subspace of $\Omega_k^2 S^2$ since it can be identified with the space of holomorphic maps $f\colon S^2 \to S^2$, where S^2 is the Riemann sphere $\mathbb{C} \cup \infty$, which have degree k and satisfy the basepoint condition $f(\infty) = 1$. Segal proved that the inclusion

$$i_k\colon \mathrm{Rat}_k \to \Omega_k^2 S^2$$

is a homotopy equivalence through dimension k. Thus combining the work of Segal and Donaldon gives a proof of the fact that $\mathscr{M}_k$ is homotopy equivalent to $\Omega_k^2 S^2$ through dimension k.

The homotopy type of Rat_k was studied further in [10, 11, and 17]. The basic method is to compare the filtration of $\Omega^2 S^2$ given by the degree of a holomorphic map with the combinatorial models of these loop spaces described in [20, 21, and 22]. To state the results we need to recall the relation between braid groups and

$\Omega^2 S^2$. Let β_q be Artin's braid group on q strings, and let $B\beta_q$ be its classifying space. There is a map α_q: $B\beta_q \to \Omega_q^2 S^2$, constructed explicitly in [22], which induces a monomorphism in homology and an isomorphism through dimension $[q/2]$. One of the main results of [10] relates the stable homotopy type of the space Rat_k with the braid groups. Recall that a stable homotopy equivalence between two finite cell complexes X and Y is a homotopy equivalence between the N-fold suspension spaces

$$\Sigma^N X \simeq \Sigma^N Y$$

for N sufficiently large; we denote this by $X \simeq_s Y$.

Theorem 1. *There is a stable homotopy equivalence*

$$\mathscr{M}_k \cong \mathrm{Rat}_k \simeq_s B\beta_{2k} .$$

This is proved in [10]. Thus Rat_k and therefore the monopole space $\mathscr{M}_k$ have the same homology, cohomology as the braid groups, and this is similarily true for any generalized homology or cohomology theory. Here we will study in detail the implications of the following corollary.

Corollary 2. *There are isomorphisms*

$$K(B\beta_{2k}) \cong K(\mathscr{M}_k), \quad KO(B\beta_{2k}) \cong KO(\mathscr{M}_k) ,$$

where K, KO are complex and real topological K-theory.

Here by K and KO we mean K^0 and KO^0. Of course the corollary is true for K^* and KO^* but here our principal interest is in the K^0-groups.

As is customary when working with the cohomology of groups we will denote $K(B\beta_q)$ by $K(\beta_q)$ and use similar notation for the other K-theories. Now representations of β_q give elements in the K-theory of β_q. On the other hand, since $\mathscr{M}_k$ is a moduli space of connections a natural method of constructing elements in the K-theory of $\mathscr{M}_k$ is to take the index bundles of families of Fredholm operators constructed using the connections. For each representation of $SU(2)$ there is a natural family of differential operators parameterised by $\mathscr{M}_k$, the Dirac operator coupled to $SU(2)$-monopoles using the given representation of $SU(2)$. The main goal of this paper is to use the above corollary to compute these index bundles in terms of representations of the braid groups. Before stating the results we will describe the twisted Dirac operator in more detail.

Let $\mathscr{S} = \mathscr{S}_3$ be the spin representation of Spin(3) and let

$$\partial\colon C^\infty(\mathbb{R}^3; \mathscr{S}) \to C^\infty(\mathbb{R}^3; \mathscr{S})$$

be the Dirac operator defined on smooth functions from $\mathbb{R}^3$ to $\mathscr{S}$. Furthermore let E be a representation of $SU(2)$. Now given a monopole $c = (A, \Phi) \in \mathscr{M}_k$ we may form the coupled Dirac operator $\partial_{c,E}$,

$$\partial_{c,E}\colon C^\infty(\mathbb{R}^3; \mathscr{S} \otimes E) \to C^\infty(\mathbb{R}^3; \mathscr{S} \otimes E) .$$

In coordinates this operator is given by the formula

$$\partial_{c,E}(\psi) = \sum_{i=1}^{3} (e_i \otimes 1) \cdot D_{A,i}(\psi) + (1 \otimes \Phi)\psi .$$

Here e_1, e_2, e_3 are the usual generators of the Clifford algebra of $\mathbb{R}^3$; the covariant derivative operator D_A on $C^\infty(\mathbb{R}^3; \mathscr{S} \otimes E)$ is defined using the spinor connection and the connection A; $D_{A,i}$ is the covariant derivative in the direction of the i^{th} coordinate in $\mathbb{R}^3$; e_i acts on $\mathscr{S}$ by the $SU(2)$ action on E.

Taubes showed in [24] that for each representation E the operators $\partial_{c,E}$ form a continuous family of Fredholm operators ∂_E, defined on suitable Sobolev spaces, parameterised by $c = (A, \Phi) \in \mathscr{M}_k$. Associated to this family of Fredholm operators is the index bundle

$$\operatorname{ind}(\partial_E) \in K(\mathscr{M}_k) \cong K(B\beta_k)\,.$$

However, as we will show in Sect. 5, this K-theory class is zero. This fact contrasts sharply with the situation for the family of Dirac operators coupled to $SU(2)$-instantons on the four sphere S^4 via the fundamental representation. This was studied in [7] where it was shown that this Dirac family produces a non-trivial element in complex K-theory.

In order to get more information we will use a real analogue of this index class. More precisely, recall that the Dirac operator ∂ on $\mathbb{R}^3$ has a quaternionic structure. This will be described in more detail in Sect. 2. Thus if we couple the Dirac operator to a monopole c using a quaternionic representation E of $SU(2)$, for example the standard representation of $SU(2) = Sp(1)$ on $\mathbb{C}^2 = \mathbb{H}$, then the resulting operator has a real structure and so can be identified with the complexification of a real operator. We denote this real operator by $\delta_{c,E}$. The index bundle of the corresponding family of operators is real and so for each E in the quaternionic representation ring $RSp(SU(2))$ we obtain a bundle

$$\operatorname{ind}(\delta_E) \in KO(\mathscr{M}_k) \cong KO(\beta_k)\,.$$

Our main goal in this paper is to compute these real index bundles.

The most important single case is the case where E is the fundamental representation of $SU(2) = Sp(1)$ on $\mathbb{C}^2 = \mathbb{H}$. We will use the notation δ for the family of operators obtained by coupling the Dirac operator to monopoles using the fundamental representation. The index bundle now gives us the basic element

$$\operatorname{ind}(\delta) \in KO(\beta_k)\,.$$

Any real representation of β_q gives a real vector bundle over $B\beta_q$ and hence an element in $KO(\beta_q)$. This gives a homomorphism

$$RO(\beta_q) \to KO(\beta_q)$$

and we will not use any special notation to distinguish between elements of $RO(\beta_q)$ and the corresponding elements of $KO(\beta_q)$ unless it is absolutely necessary. The most basic representation of β_q is the permutation representation ρ_q given by composing the homomorphism $\beta_q \to \Sigma_q$, which maps a braid to the corresponding permutation of its end points, with the representation of Σ_q in $O(q)$ given by permuting the coordinates of $\mathbb{R}^q$. There is a close relation between the elements ρ_k and $\operatorname{ind}(\delta)$ in $KO(\beta_k)$ but to describe this precisely requires a little more background.

Given integers p, q with $p < q$ the inclusion u: $\beta_p \to \beta_q$ defines a map

$$Bu\colon B\beta_p \to B\beta_q\,.$$

In fact this map is stably split, that is, provided N is sufficiently large, there is a map

$$\tau\colon \Sigma^N B\beta_q \to \Sigma^N B\beta_p$$

such that the composite

$$\Sigma^N B\beta_p \xrightarrow{\Sigma^N Bu} \Sigma^N B\beta_q \xrightarrow{\tau} \Sigma^N B\beta_p$$

is homotopic to the identity. This map τ induces a map

$$\tau^*\colon KO(\beta_p) \to KO(\beta_q)$$

and it follows that the composite

$$KO(\beta_p) \xrightarrow{\tau^*} KO(\beta_q \xrightarrow{u^*} KO(\beta_p)$$

is the identity.

Theorem A. *Under the isomorphism $KO(\mathscr{M}_k) \cong KO(\beta_{2k})$ of Corollary* 2, *the element* $\mathrm{ind}(\delta) \in KO(\mathscr{M}_k)$ *corresponds to*

$$\rho_{2k} - \tau^*(\rho_k) \in KO(\beta_{2k}) .$$

This theorem shows that there is an intimate relation between the index bundle $\mathrm{ind}(\delta)$ defined by the fundamental representation of $SU(2)$ and the permutation representation of the braid groups. However twisting the Dirac operator using other representations does not produce any other interesting classes.

Let $\gamma\colon SU(2) \to Sp(n)$ be a symplectic representation. By identifying $SU(2) = Sp(1)$ with the sphere S^3, such a representation defines a homotopy class in $\pi_3(Sp(n))$. Now $\pi_3(Sp(n)) \cong \mathbb{Z}$ and we choose this isomorphism so that 1 corresponds to the homotopy class of the usual inclusion $Sp(1) \to Sp(n)$. Now define n_γ to be the integer corresponding to the homotopy class of γ. The definition of n_γ extends additively to virtual quaternionic representations, that is elements of $RSp(SU(2))$.

Theorem B. *Let $\gamma \in RSp(SU(2))$ be any virtual quaternionic representation of $SU(2)$. Then the index bundle of the family of real operators δ_γ is given by*

$$\mathrm{ind}(\delta_\gamma) = n_\gamma \mathrm{ind}(\delta) \in KO(\mathscr{M}_k) ,$$

which, by Theorem A, corresponds to

$$n_\gamma(\rho_{2k} - \tau^* \rho_k) \in KO(\beta_{2k})$$

under the isomorphism of Corollary 2.

We will show in Sect. 5 that $\mathrm{ind}(\delta)$ has order two in $KO(\mathscr{M}_k)$. Thus it follows that we do not produce any new elements of $KO(\mathscr{M}_k)$ by using representations other than the fundamental representation.

In fact the Dirac operator can be coupled to any pair (A, Φ) which satisfies only (1) and (2) in the definition of a monopole. This family of operators is extensively studied by Taubes in [24] and we frequently refer to this paper for details of the analysis associated to these, and other closely related operators. Taubes' motivation was to use computations with the Chern classes of the index bundle of this family of operators, as in [7], to force the existence of large spaces of solutions to the Dirac equation. This in turn was used to study the critical values of the Yang–Mills–Higgs energy functional. However as pointed out in [26] there is a difficulty with the calculation in [24] arising from the fact that the index bundle $\operatorname{ind}(\partial)$ of the complex Dirac operator coupled to pairs (A, Φ) using the fundamental representation of $SU(2)$ is trivial and so its Chern classes are all zero. This is the reason why we study the real operator δ in terms of real K-theory. Nonetheless the resulting theorem about critical values is true (Theorem A1.2 of [24]) and an alternative proof is given in [26]. In fact Taubes's original argument in [24], based on the topology of the family of Dirac operators can be completed as follows.

Recall that the space of pairs (A, Φ) which satisfy the asymptotic condition $1-|\Phi| \in L^6(\mathbb{R}^3)$ and have finite Yang–Mills–Higgs energy is denoted by $\mathscr{A}$ and its quotient by the gauge group $\mathscr{G}$ is denoted by $\mathscr{B}$. The components of $\mathscr{B}$ are determined by the charge of the configuration (A, Φ) which is defined as above. We can couple the Dirac operator to such a configuration $c=(A, \Phi)$ using the standard representation of $SU(2)$ and so we obtain an index bundle

$$\operatorname{ind}(\delta) \in KO(\mathscr{B}_k)\,.$$

By using computations of Stiefel–Whitney classes, rather than the computations with Chern classes in [7], we prove the following theorem which would complete the proof of Theorem A1.2 in [24] using the original argument given in Sect. C3 of that paper.

Theorem C. *Let N be any integer. Then for every k there exists a class $c=(A, \Phi) \in \mathscr{B}_k$ for which the space of solutions of the Dirac equation*

$$\delta_c \psi = 0$$

and the adjoint equation

$$\delta_c^* \psi = 0$$

both have dimension $\geqq N$.

This paper is set out as follows. In Sects. 2 and 3 we give the technical argument required to identify the index bundles homotopy theoretically. This argument is related to real Bott periodicity. The conclusion, for the case we are interested in, is given in Theorem 4.1; the statement is very natural but it requires a careful analysis to give a proof. In Sect. 4 we go on to complete the proof of Theorem A. In Sect. 5 we prove Theorem B and in Sect. 6 Theorem C. Some of the results in this paper are described very briefly in [15].

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2. The Coupled Dirac Operator

In this section we describe in detail the construction of the coupled Dirac operator and show how the process of forming the coupled Dirac operator and taking index bundles is related to the Bott equivalence $\Omega^2(Sp/U) \simeq \mathbb{Z} \times BO$. This equivalence is part of the real Bott peridicity theorem. Our argument is motivated by Atiyah's proof of Bott periodicity [5] using elliptic operators. However it is quite tricky in detail. The first thing which makes it so is that we are dealing with real K-theory. The second is that the process of coupling the Dirac operator is not the standard process of coupling an operator to a connection as the local formula given in Sect. 1 shows. Thus it requires a careful analysis to establish the relation between the family of coupled Dirac operators and the Bott equivalence.

The first step is to describe in detail the coupling process at the following level of generality. Let (A, Φ) be a pair consisting of a smooth connection A on the trivial $Sp(n)$-bundle over $\mathbb{R}^3$ and a smooth map $\Phi\colon \mathbb{R}^3 \to \mathfrak{sp}(n)$. This pair is required to statisfy the following conditions. Firstly (A, Φ) is required to have finite energy;

$$\mathscr{U}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|F_A|^2 + |D_A\Phi|^2)\,\mathrm{dvol} < \infty ,$$

where D_A is the covariant derivative operator defined by A and F_A is the curvature of A. Secondly we must impose asymptotic conditions on Φ. Inside the Lie algebra $\mathfrak{sp}(n)$ we can look at the orbit of the quaternionic matrix

$$\begin{pmatrix} i & 0 & \cdots & 0 \\ 0 & i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & i \end{pmatrix},$$

where $i = \sqrt{-1} \in \mathbb{C} \subset \mathbb{H}$, under the adjoint action of $Sp(n)$. This orbit is the homogeneous space $Sp(n)/U(n)$. Consider the function

$$d_\Phi(x) = d(\Phi(x), Sp(n)/U(n)) = \inf_{y \in Sp(n)/U(n)} |\Phi(x) - y| .$$

This function is continuous and we assume that

$$d_\Phi(x) \in L^6(\mathbb{R}^3) .$$

Notice that in the case $n = 1$, $Sp(1)/U(1) = S^2$ and the function d_Φ is exactly $|\Phi| - 1$. Since $\pi_2(Sp(n)/U(n)) = \mathbb{Z}$ the charge of the pair (A, Φ) is defined in exactly the same way as in the case $n = 1$.

Now define $\mathscr{A}(n)$ to be the space of pairs (A, Φ) which have finite energy and satisfy the above asymptotic condition. Once more the gauge group $\mathscr{G}(n)$ consisting of automorphisms of the trivial bundle which are the identity at the origin acts and the quotient space $\mathscr{A}(n)/\mathscr{G}(n)$ will be denoted by $\mathscr{B}(n)$. Now define $\mathscr{B}_k(n)$ to be the space of those pairs with charge k. There are stabilisation maps

$$\mathscr{B}(n) \to \mathscr{B}(n+1)$$

given by forming the direct sum with $c_0 \in \mathscr{B}_0(1)$ where c_0 is the pair with $A = 0$ and $\Phi = i$ is constant. We denote the limit by

$$\mathscr{B}(\infty) = \lim_{n \to \infty} \mathscr{B}(n) .$$

Theorem 2.1. *There are natural homotopy equivalences*

$$I_n\colon \Omega^2(Sp(n)/U(n)) \to \mathscr{B}(n)$$

which make the following diagrams commute

$$\begin{array}{ccc} \Omega^2(Sp(n)/U(n)) & \to & \mathscr{B}(n) \\ \downarrow & & \downarrow \\ \Omega^2(Sp(n+1)/U(n+1)) & \to & \mathscr{B}(n+1) \end{array}$$

where the vertical arrows are the natural stabilisation maps.

The proof of this theorem is given by repeating the argument used by Taubes [24, Sect. B1] in the case where $n = 1$ in the general case. From the real Bott periodicity theorem there is a homotopy equivalence

$$\Omega^2(Sp/U) \simeq \mathbb{Z} \times BO$$

and so we immediately obtain the following corollary.

Corollary 2.2. *The stabilisation maps $\mathscr{B}(n) \to \mathscr{B}(n+1)$ are homotopy equivalences through dimension $2n$ and there is a homotopy equivalence*

$$\mathscr{B}(\infty) \to \mathbb{Z} \times BO\ .$$

Now we describe how to couple the Dirac operator on $\mathbb{R}^3$ to the pair $c = (A, \Phi) \in \mathscr{B}(n)$ using a quaternionic representation E of $Sp(n)$. Form the $Sp(n)$ connection

$$\alpha = A + \Phi dt$$

on the trivial bundle over $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. This connection is independent of t and a standard computation shows that (A, Φ) satisfies the Bogomolnyi equation if and only if α is self dual. We can now form the Dirac operator on $\mathbb{R}^4$ coupled to the connection α using the representation E

$$\partial_{\alpha, E}\colon C^\infty(\mathbb{R}^4; \mathscr{S}_4^+ \otimes E) \to C^\infty(\mathbb{R}^4; \mathscr{S}_4^- \otimes E)\ ,$$

where $\mathscr{S}_4^\pm$ are the positive and negative spin representations of Spin(4). More explicitly the operator ∂_α is defined to be the composite

$$C^\infty(\mathbb{R}^4; \mathscr{S}_4^+ \otimes E) \xrightarrow{D_\alpha} \Omega^1(\mathbb{R}^4; \mathscr{S}_4^+ \otimes E) \xrightarrow{\mu} C^\infty(\mathbb{R}^4; \mathscr{S}_4^- \otimes E)\ .$$

Here $\Omega^1(\mathbb{R}^4; \mathscr{S}_4^+ \otimes E)$ is the space of one-forms on $\mathbb{R}^4$ with values in $\mathscr{S}_4^+ \otimes E$; D_α is the covariant derivative operator defined by the spinor connection and the connesction α; and μ is Clifford multiplication.

We now restrict this operator to the subspace of functions on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ which are independent of t. Since α is independent of t we get an operator

$$\partial_\alpha\colon C^\infty(\mathbb{R}^3; \mathscr{S}_4^+ \otimes E) \to C^\infty(\mathbb{R}^3; \mathscr{S}_4^- \otimes E)\ .$$

Now let $e_1, \ldots, e_n$ be the usual generators for the Clifford algebra C_n of $\mathbb{R}^n$. Then, in the case $n = 4$, we use e_4 to identify $\mathscr{S}_4^-$ with $\mathscr{S}_4^+$ and so form the operator

$$e_4 \cdot \partial_\alpha\colon C^\infty(\mathbb{R}^3; \mathscr{S}_4^+ \otimes E) \to C^\infty(\mathbb{R}^3; \mathscr{S}_4^+ \otimes E)\ .$$

Finally using the identification of the C_3-module $\mathscr{S}_3$ with $\mathscr{S}_4^+$ where $e_i \in C_3$ acts as $e_4 e_i$ we end up with the operator

$$\partial_{c,E}\colon C^\infty(\mathbb{R}^3; \mathscr{S}_3 \otimes E) \to C^\infty(\mathbb{R}^3; \mathscr{S}_3 \otimes E)\,,$$

where c refers to the pair (A, Φ) used to define α. In coordinates this operator is given by

$$\partial_{c,E}(\psi) = \sum_{i=1}^{3} (e_i \otimes 1)\cdot D_{A,i}(\psi) + (1 \otimes \Phi)\psi$$

compare Sect. 1.

Now $\mathrm{Spin}(3) \cong Sp(1)$ and under this isomorphism the spin representation $\mathscr{S}_3$ becomes the canonical representation of $Sp(1)$ on the quaternions. In particular the spin representation has a quaternionic structure. The representation E has a quaternionic structure and so $\mathscr{S}_3 \otimes E$ has a real structure. Furthermore the Dirac operator preserves this real structure. Therefore there is a real representation $E_{\mathbb{R}}$ of $Sp(1) \times Sp(n)$ whose complexification is $\mathscr{S}_3 \otimes E$ and a real operator

$$\delta_{c,E}\colon C^\infty(\mathbb{R}^3; E_{\mathbb{R}}) \to C^\infty(\mathbb{R}^3; E_{\mathbb{R}})$$

whose complexification is $\partial_{c,E}$. We refer to $\delta_{c,E}$ as the real Dirac operator coupled to c using the representation E.

Following [24], define a norm on the space $C^\infty(\mathbb{R}^3; E_{\mathbb{R}})$ by the formula

$$\|\psi\|_c^2 = \int_{\mathbb{R}^3} (|D_A\psi|^2 + |(1 \otimes \Phi)\psi|^2)\mathrm{dvol}\,.$$

Taubes shows in [24, Sect. C] that $\delta_{c,E}$ extends to a real Fredholm operator

$$\delta_{c,E}\colon \mathscr{H}_c(E_{\mathbb{R}}) \to L^2(E_{\mathbb{R}})\,,$$

where $\mathscr{H}_c(E_{\mathbb{R}})$ is the completion of the space of compactly supported smooth functions $C_c^\infty(\mathbb{R}^3; E_{\mathbb{R}})$ with respect to the norm $\|\psi\|_c$. Taubes also shows that the family of real operators $\delta_{c,E}$ parametrised by $c = (A, \Phi) \in \mathscr{B}_k(n)$ is continuous in c.

There is technical point to take care of in this construction. The Hilbert spaces $\mathscr{H}_c$ form a bundle of Hilbert spaces over $\mathscr{A}(n)$ and $\mathscr{G}(n)$ acts on this bundle. The quotient by this action of $\mathscr{G}(n)$ gives a bundle of Hilbert spaces over $\mathscr{B}(n)$ and the Dirac family gives a Fredholm operator $\delta_{c,E}$ on the fibre of this bundle over $c \in \mathscr{B}(n)$. Now using Kuiper's theorem that the unitary group of Hilbert space is contractible this bundle can be trivialised. Thus we get a map

$$\delta_E\colon \mathscr{B}(n) \to \mathscr{F}_{\mathbb{R}}\,,$$

where $\mathscr{F}_{\mathbb{R}}$ is the space of Fredholm operators on a real Hilbert space. The homotopy class of the resulting map does not depend on the choice of the trivialisation; compare [4].

Now we specialise to the case where $E = \mathbb{H}^n$ is the fundamental representation of $Sp(n)$. We use the notation

$$\delta(n)\colon \mathscr{B}(n) \to \mathscr{F}_{\mathbb{R}}$$

for the corresponding map. Now let $i\colon \mathscr{B}(n) \to \mathscr{B}(n+1)$ be the inclusion. Then for $c \in \mathscr{B}(n)$ note that

$$\delta(n+1)_{i(c)} = \delta(n)_c \oplus \delta_{c_0}\,,$$

where δ_{c_0} is the real Fredholm operator given by coupling the Dirac operator to the pair c_0 defining the stabilisation map $\mathscr{B}(n) \to \mathscr{B}(n+1)$. The operator δ_{c_0} has index zero since Taubes proves in [24, Lemma C3.3] that the index of δ_c is given by the charge of c and c_0 has charge 0. Now since δ_{c_0} has index zero, the map $\mathscr{F}_{\mathbb{R}} \to \mathscr{F}_{\mathbb{R}}$ defined by taking direct sum with δ_{c_0} is homotopic to the identity and so we see that the diagram

$$\begin{array}{ccc} \mathscr{B}(n) & \xrightarrow{i} & \mathscr{B}(n+1) \\ {\scriptstyle \delta(n)} \downarrow & & \downarrow {\scriptstyle \delta(n+1)} \\ \mathscr{F}_{\mathbb{R}} & \xrightarrow[=]{} & \mathscr{F}_{\mathbb{R}} \end{array}$$

commutes up to homotopy. It follows that there is a map

$$\delta = \delta(\infty)\colon \mathscr{B}(\infty) \to \mathscr{F}_{\mathbb{R}}$$

whose restriction to $\mathscr{B}(n)$ is homotopic to $\delta(n)$. This map δ is not necessarily uniquely determined, but any two choices will be homotopic on any finite skeleton of $\mathscr{B}(\infty)$ and this is sufficient for our purposes.

Loop sum gives a composition law on the space $\Omega^2(Sp(n)/U(n))$ and composition of Fredholm operators gives a composition law on $\mathscr{F}_{\mathbb{R}}$. The argument given in [24, Sect. C.5] shows that the composite

$$\Omega^2(Sp(n)/U(n)) \xrightarrow{I_n} \mathscr{B}(n) \xrightarrow{\delta(n)} \mathscr{F}_{\mathbb{R}}$$

is an H-map; that is it preserves composition up to homotopy. Indeed Taubes's argument shows that the composite

$$\Omega^2(Sp(n)/U(n)) \xrightarrow{I_n} \mathscr{B}(n) \xrightarrow{\delta_E} \mathscr{F}_{\mathbb{R}}$$

is an H-map for any quaternionic representation E of $Sp(n)$.

Now we prove the following theorem.

Theorem 2.3. *The map $\delta\colon \mathscr{B}(\infty) \to \mathscr{F}_{\mathbb{R}}$ is a homotopy equivalence.*

Proof. From [4] we know that $\mathscr{F}_{\mathbb{R}}$ is a classifying space for the functor $KO(X)$; that is for any compact space X

$$[X, \mathscr{F}_{\mathbb{R}}] \cong KO(X)\,;$$

or equivalently

$$\mathscr{F}_{\mathbb{R}} \cong \mathbb{Z} \times BO\,.$$

Now note that taking the index of Fredholm operators gives a bijection

$$\pi_0(\mathscr{F}_{\mathbb{R}}) \to \mathbb{Z}$$

and Taubes shows in [24, Lemma C3.3] that if $c \in \mathscr{B}_k(n)$ then

$$\operatorname{index} \delta_c = k\,.$$

Thus δ induces an isomorphism on π_0.

Now we construct a map

$$\nu\colon \mathscr{B}_k(n) \times G_q(\mathbb{R}^N) \to \mathscr{B}_k(qn)\,,$$

where $G_q(\mathbb{R}^N)$ is the Grassmannian of q-planes in $\mathbb{R}^N$. Given $c = (A, \Phi) \in \mathscr{B}_k(n)$ we first form the t-invariant connection α on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ and then the covariant derivative operator

$$D_\alpha : C^\infty(\mathbb{R}^4; E_n) \to C^\infty(\mathbb{R}^4; \mathbb{R}^4 \otimes E_n),$$

where $E_n = \mathbb{H}^n$ is the fundamental representation of $Sp(n)$. Now let $V \subset \mathbb{R}^N$ be a q-dimensional subspace of $\mathbb{R}^N$ and form the operator

$$D_\alpha \otimes 1 : C^\infty(\mathbb{R}^4; E_n) \otimes V \to \Omega^1(\mathbb{R}^4; E_n) \otimes V.$$

Now $E_n \otimes V$ is naturally isomorphic to E_{nq} and this operator is the covariant derivative defined by a unique t-invariant, $Sp(nq)$ connection β which in turn corresponds to a unique element $c' \in \mathscr{B}_k(qn)$. The map v is given by

$$(c, V) \mapsto c'.$$

Suppose Y is a compact, connected space and V is a vector bundle over Y classified by a map

$$Y \to G_q(\mathbb{R}^N), \quad y \mapsto V_y.$$

Then for fixed $c \in \mathscr{B}(n)$ we get a map

$$f_{c,V} : Y \to \mathscr{B}(qn), \quad y \mapsto v(c, V_y).$$

It is routine to check that the index bundle of the family $\delta \circ f_{c,V}$ of coupled Dirac operators is given by

$$\operatorname{ind}(\delta(qn) \circ f_{c,V}) = \operatorname{ind}(\delta_c) V,$$

where $\operatorname{ind}(\delta_c) \in \mathbb{Z}$ is the index of the Fredholm operator δ_c. Therefore if $c \in \mathscr{B}_1(n)$ it follows that

$$\operatorname{ind}(\delta \circ f_{c,V}) = V.$$

Now if Y is a finite CW complex we can choose q and N large enough so that

$$[Y; BO] = [Y; G_q(\mathbb{R}^N)]$$

and the above argument shows that

$$\delta_* : [Y; \mathscr{B}_1(qn)] \to [Y; BO]$$

is surjective.

Now take Y to be a sphere and pass to the limit. We deduce that

$$\delta_* : \pi_j(\mathscr{B}_1(\infty)) \to \pi_j(BO)$$

is surjective. However, in view of Corollary 2.2, the homotopy groups of both spaces are abstractly isomorphic cyclic groups and therefore δ_* must be an isomorphism.

This shows that $\delta : \mathscr{B}_1(\infty) \to BO$ is a homotopy equivalence. However $\delta : \mathscr{B}(\infty) \to \mathbb{Z} \times BO$ is an H-map; its restriction to $\mathscr{B}_1(\infty)$ is a homotopy equivalence $\mathscr{B}_1(\infty) \to 1 \times BO$ and it induces an isomorphism on π_0. Thus it is a homotopy equivalence. □

Note how Theorem 2.3 and Theorem 2.1 show that the composite

$$\Omega^2(Sp/U) \xrightarrow{I_\infty} \mathscr{B}(\infty) \xrightarrow{\delta(\infty)} \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO$$

is a homotopy equivalence. The proof given here actually uses Bott's theorem that there is a homotopy equivalence between $\Omega^2(Sp/U)$ and $\mathbb{Z} \times BO$. However a more careful analysis, based on [5], shows that the argument can be made into a proof of Bott's theorem. We will not go into the details here.

We now have two equivalences of $\Omega^2(Sp/U)$ with $\mathbb{Z} \times BO$. The first is the Bott equivalence

$$\beta \colon \Omega^2(Sp/U) \to \mathbb{Z} \times BO$$

and the second is the composite

$$\Omega^2(Sp/U) \xrightarrow{I_\infty} \mathscr{B}(\infty) \xrightarrow{\delta(\infty)} \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO \ .$$

We need to know that these two equivalences are homotopic. Combining them we get a self equivalence

$$\lambda \colon \mathbb{Z} \times BO \to \mathbb{Z} \times BO \ .$$

This map is an H-map and so it defines a natural isomomorphism

$$\lambda \colon KO(X) \to KO(X)$$

for any space X. We show in the next section that any such natural isomorphism must be ± 1 and this will allow us to conclude that these two equivalences are homotopic.

3. *K*-Theory

We now establish the general facts about K-theory we need. Let

$$\lambda \colon KO(X) \to KO(X)$$

be a natural homomorphism of groups. We refer to λ as an additive operation and use the notation Λ_O for the group of additive operations $KO(X) \to KO(X)$. The purpose of this section is to prove the following result.

Theorem 3.1. (1) *Let λ be an element of Λ_O. Then λ is zero if and only if λ is zero on $X = S^{4n}$ for all $n \geqq 0$.*

(2) *Let λ be an element of Λ_O which is an isomorphism for any finite* CW*-complex X; then $\lambda = \pm 1$.*

The method of proof for this theorem is quite standard in the study of operations in K-theory, compare [1, 2]. We should emphasize here that we only study natural transformations of the single functor $KO = KO^0$ not natural transformations of the generalized cohomology theory KO^*.

The first step in the proof is to establish the following lemma. Let ζ be the universal oriented 2-plane bundle over $\mathbb{CP}^\infty = BSO(2)$ and let

$$u = \zeta - 2 \in KO(\mathbb{CP}^\infty) \ .$$

According to [3]

$$KO(\mathbb{CP}^\infty) = \mathbb{Z}[\![u]\!]$$

is the ring of formal power series in u. Now define a homomorphism

$$\Lambda_O \to KO(\mathbb{CP}^\infty), \quad \lambda \mapsto \lambda(\zeta) \ .$$

Lemma 3.2. *This homomorphism*

$$\Lambda_O \to KO(\mathbb{CP}^\infty) = \mathbb{Z}[\![u]\!]$$

is injective with image the subgroup of power series with even constant term.

Proof. The splitting principle for real bundles tells us that an additive operation λ is uniquely determined by

$$\lambda(1) \in KO(\mathrm{pt}), \quad \lambda(\eta) \in KO(\mathbb{RP}^\infty), \quad \lambda(\zeta) \in KO(\mathbb{CP}^\infty)\,,$$

where 1 is the unit in $KO(\mathrm{pt})$, η is the universal real line bundle over $BO(1) = \mathbb{RP}^\infty$, and as above, ζ is the universal 2-plane bundle over $BSO(2) = \mathbb{CP}^\infty$. We first show that $\lambda(\zeta)$ uniquely determines the others. The naturality of λ with respect to the map $\mathbb{CP}^\infty \to \mathrm{pt}$ shows that $\lambda(\zeta)$ determines $\lambda(1)$.

Now let f: $\mathbb{RP}^\infty \to \mathbb{CP}^\infty$ be the standard inclusion. Then $f^*(\zeta) = 2\eta$ and so by the naturality of λ

$$f^*(\lambda(\zeta)) = 2\lambda(\eta) \in KO(\mathbb{RP}^\infty)\,.$$

However we know that

$$KO^0(\mathbb{RP}^\infty) = \mathbb{Z} \otimes \hat{\mathbb{Z}}_2\,,$$

where $\hat{\mathbb{Z}}$ is the 2-adic integers. Therefore multiplication by 2 is injective on $KO^0(\mathbb{RP}^\infty)$. Thus $\lambda(\eta)$ is determined by $\lambda(\zeta)$ and this shows that the homomorphism $\Lambda_O \to KO(\mathbb{CP}^\infty)$ is injective.

To determine its image we construct operations $\rho^k \in \Lambda_O$, where $\rho^0(x) = \mathrm{rank}(x)$, such that

$$\rho^k(\zeta) = u^k$$

for $k > 0$. These will be constructed using the (real) Adams operations ψ^k. According to [3] there is a polynomial $T_k(x)$ of degree k with leading term x^k such that

$$\psi^k(u) = T_k(u)\,.$$

In fact T_k is uniquely determined by the equation

$$T_k(z - 2 + z^{-1}) = z^k - 2 + z^{-k}\,.$$

Now $\psi^k(\zeta) = \psi^k(u + 2) = \psi^k(u) + 2$ and it is straightforward to construct ρ^k as a linear combination of the operations ψ^k. By taking formally infinite sums,

$$\lambda = \sum \lambda_i \rho^i\,,$$

where the λ_i are integers, it then follows that the above homomorphism has as its image those power series with even constant term. Note that on a finite CW-complex only a finite number of the ρ^i in this formally infinite sum will be non-zero so the above expression makes sense. $\square$

Now let $\rho^k \in \Lambda_O$ be the unique additive operation such that $\rho^k(\zeta) = u^k$ in $KO(\mathbb{CP}^\infty)$. We now compute the effect of ρ^k on the reduced K-groups $\widetilde{KO}(S^{4n})$.

Lemma 3.3. *The homomorphism*

$$\rho^k\colon \widetilde{KO}(S^{4n}) \to \widetilde{KO}(S^{4n})$$

is zero if $n < k$ and

$$\rho^k\colon \widetilde{KO}(S^{4k}) \cong \mathbb{Z} \to \widetilde{KO}(S^{4k}) \cong \mathbb{Z}$$

is multiplication by $(2k)!/2^k$.

Proof. Let ph: $KO(X) \to H^*(X; \mathbb{Q})$ be the Pontryagin character and let ph_n be the component of ph in $H^{4n}(X; \mathbb{Q})$. Then, computing in $KO(\mathbb{CP}^\infty)$ we see that

$$\mathrm{ph}(u) = \sum_{i \geqq 1} \frac{x^{2i}}{(2i)!},$$

$$\mathrm{ph}(\rho^k(\zeta)) = \mathrm{ph}(u^k) = \left(\sum_{i \geqq 1} \frac{x^{2i}}{(2i)!}\right)^k,$$

where $x \in H^2(\mathbb{CP}^\infty)$ is the usual generator. Therefore we obtain the following formulas:

$$\mathrm{ph}_n(\rho^k(\zeta)) = 0, \quad \text{if } n < k,$$

$$\mathrm{ph}_k(\rho^k(\zeta)) = \frac{(2k)!}{2^k}\,\mathrm{ph}_k(u).$$

Since the Pontryagin character is additive, the splitting principle shows that

$$\mathrm{ph}_n(\rho^k(\xi)) = 0, \quad \text{if } n < k,$$

$$\mathrm{ph}_k(\rho(\xi)) = \frac{(2k)!}{2^k}\,\mathrm{ph}_k(\xi).$$

for any element $\xi \in \widetilde{KO}(X)$. In particular this is true when X is a sphere and the lemma follows since the Pontryagin character is injective when X is a sphere. □

Proof of Theorem 3.1. Part (1) is a straightforward deduction from the previous lemma. To prove part (2) let

$$\lambda = \sum \lambda_i \rho^i$$

be an additive operation which is an isomorphism on $KO(S^{4k})$ for all k. Then since ρ^1 is the identity and λ is an isomorphism on S^4 it follows from Lemma 3.3 that $\lambda_1 = \pm 1$. Also from this lemma it follows that λ acts on $\widetilde{KO}(S^8) \cong \mathbb{Z}$ by multiplication by

$$6\lambda_2 \pm 1.$$

Now since λ is an isomorphism it follows that

$$\lambda_2 = 0.$$

Continuing in this way we see that

$$\lambda_i = 0, \quad \text{for } i \geqq 2$$

and this proves part (2) of Theorem 3.1. □

4. The Fundamental Representation

Now we use the results of the previous section to prove Theorem A in the introduction. First we summarise the conclusions of the previous two sections about the case we are interested in, which is the case $\mathscr{B} = \mathscr{B}(1)$. Recall

$$SU(2)/U(1) = Sp(1)/U(1) = S^2 ,$$

and so we get a map

$$\eta\colon \Omega^2 S^2 = \Omega^2(Sp(1)/U(1)) \to \Omega^2(Sp/U) \simeq \mathbb{Z} \times BO ,$$

where we are of course using the Bott equivalence $\Omega^2(Sp/U) \simeq \mathbb{Z} \times BO$. Further using the coupled Dirac operator we get a map

$$\delta = \delta(1)\colon \mathscr{B} \to \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO .$$

From [24], compare Theorem 2.1, we know that there is a homotopy equivalence

$$I\colon \Omega^2 S^2 \to \mathscr{B} .$$

Theorem 4.1.

$$\delta \circ I \simeq \pm\eta\colon \Omega^2 S^2 \to \mathbb{Z} \times BO .$$

Proof. Consider the maps

$$\mathbb{Z} \times BO \xleftarrow{\delta(\infty)\circ I_\infty} \Omega^2(Sp/U) \xrightarrow{\beta} \mathbb{Z} \times BO ,$$

where β is the Bott equivalence. Then both are H-maps which are homotopy equivalences. Thus they define a natural automorphism of $KO(X)$ and by Theorem 3.1 this must be ± 1. Therefore

$$\delta(\infty)\circ I_\infty \simeq \pm\beta\colon \Omega^2(Sp/U) \to \mathbb{Z} \times BO ,$$

and the lemma follows by composing with the above map

$$\Omega^2 S^2 \to \Omega^2(Sp/U) . \quad \square$$

It follows that replacing the Bott equivalence β by $-\beta$ if necessary we can assume that

$$\delta \circ I \simeq \eta\colon \Omega^2 S^2 \to \mathbb{Z} \times BO ,$$

and so in K-theory

$$\operatorname{ind}(\delta) = \eta \in KO(\Omega^2 S^2) \cong KO(\mathscr{B}) ,$$

where $\operatorname{ind}(\delta)$ is the index bundle for the family of Dirac operators coupled to pairs $c = (A, \Phi)$ using the fundamental representations of $SU(2)$ on $\mathbb{H}$.

Now we compute the element in $KO(\mathscr{M}_k)$ given by restricting η to $\mathscr{M}_k$. In view of Lemma 4.1 this is the index bundle for the family of Dirac operators coupled to monopoles using the fundamental representation of $SU(2)$. To study this we must recall some of the homotopy theory of the spaces $\mathscr{M}_k$ and how they are related to the braid groups.

Let $\tilde{C}_k = \tilde{C}_k(\mathbb{C})$ be the configuration space of k distinct ordered points in $\mathbb{C}$. The symmetric group Σ_k acts freely on $\tilde{C}_k$ and the quotient space C_k is the configuration space of k distinct unordered points in $\mathbb{C}$. The space C_k is an Eilenberg–MacLane space with $\pi_1(C_k) = \beta_k$ and so

$$C_k = B\beta_k \,.$$

Then as mentioned in Sect. 1 there is a natural map

$$\alpha_k\colon C_k \to \Omega_k^2 S^2 \,,$$

and this map is an isomorphism in homology through dimension $[k/2]$. Now let

$$i_k\colon \mathcal{M}_k \to \Omega_k^2 S^2$$

be the map corresponding to the inclusion of the space of monopoles in the space $\mathcal{B}_k$ under Taubes's equivalence of $\Omega_k^2 S^2$ with $\mathcal{B}_k$. Our aim is to relate the maps

$$i_k\colon \mathcal{M}_k \to \Omega_k^2 S^2, \quad \alpha_k\colon C_k \to \Omega_k^2 S^2$$

and to do this we use the gluing construction for monopoles described in [24].

Recall from [6] that a monopole of charge 1 is determined by its centre in $\mathbb{R}^3$ and a phase angle in S^1 and so

$$\mathcal{M}_1 \cong \mathbb{R}^3 \times S^1 \simeq S^1 \,.$$

Taubes shows that k one-monopoles whose centres are sufficiently far apart can be patched together to give a single k-monopole. This construction gives a continuous map

$$\lambda_k\colon \tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k \to \mathcal{M}_k \,.$$

The importance of this map is given by the following theorem proved in [11].

Theorem 4.2. *For a sufficiently large N there is a map*

$$\mu_k\colon \Sigma^N \mathcal{M}_k \to \Sigma^N \tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k$$

such that

$$\Sigma^N \lambda_k \circ \mu_k \simeq 1 \,.$$

In [11] splittings of the spaces Rat_k of rational functions were considered. Theorem 4.2 follows from this work by applying Donaldson's homeomorphism $\mathcal{M}_k \cong \mathrm{Rat}_k$ [18]. Therefore we have the following corollary.

Corollary 4.3. *The induced homomorphism*

$$\lambda_k^*\colon KO(\mathcal{M}_k) \to KO(\tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k)$$

is injective.

Now the space

$$\tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k$$

is the total space of a bundle over C_k with fibre

$$(\mathcal{M}_1)^k \simeq (S^1)^k \,,$$

and so it is an Eilenberg–MacLane space. The fundamental group is given by the semi-direct product

$$\beta_{2,k} = \beta_k \rtimes (\mathbb{Z})^k \,,$$

where the braid group β_k acts on $(\mathbb{Z})^k$ by permuting factors. Further there is a natural homomorphism

$$j_k\colon \beta_{2,k} \to \beta_{2k}$$

defined by "cabling" as follows. Start with k pairs of pieces of string and twist the i^{th} pair n_i times, where $n_i \in \mathbb{Z}$. Now braid the k pairs according to the braid $b \in \beta_k$. This gives a braid on $2k$ strings and the homomorphism j_k maps $(b; n_1, \ldots, n_k)$ to this braid. This gives a map

$$Bj_k\colon B\beta_{2,k} = \tilde{C}_k \times_{\Sigma_k} (S^1)^k \to B\beta_{2k} = C_{2k}\,.$$

This map also has a stable splitting, indeed we have the following result proved in [9].

Theorem 4.4. *For sufficiently large N there is a map*

$$\tau_{2k}\colon \Sigma^N C_{2k} \to \Sigma^N \tilde{C}_k \times_{\Sigma_k} (S^1)^k$$

such that

$$\Sigma^N Bj_k \circ \tau_{2k} \simeq 1\,.$$

Corollary 4.5. *The induced homomorphism*

$$Bj_k^*\colon KO(C_k) \to KO(\tilde{C}_k \times_{\Sigma_k} (S^1)^k)$$

is injective.

For our purposes it is easier to consider the stable splitting of the monopole space before the stable splitting of the configuration space C_k but in fact the stable splitting of the monopole space is essentially a consequence of the stable splitting of the configuration space, as was shown in [10, 11]. See [11] for a detailed analysis of these splittings and the relation between them.

We can now choose N large enough and consider the composite

$$\Sigma^N C_{2k} \xrightarrow{\tau_{2k}} \Sigma^N \tilde{C}_k \times_{\Sigma_k} (S^1)^k \xrightarrow{\Sigma^N \lambda_k} \Sigma^N \mathscr{M}_k\,.$$

One of the main results of [11] is the following theorem.

Theorem 4.6. *The composite*

$$\Sigma^N \lambda_k \circ \tau_{2k}\colon \Sigma^N C_{2k} \to \Sigma^N \mathscr{M}_k$$

is a homotopy equivalence.

Next we must be careful in the choice of the stable splittings of the map

$$\tilde{C}_k \times_{\Sigma_k} (S^1)^k \to C_{2k}\,.$$

Let

$$s_k\colon C_k \to \tilde{C}_k \times_{\Sigma_k} (S^1)^k$$

be the natural section. Then the following lemma follows directly from [13, 14].

Lemma 4.7. *The stable splitting*

$$\tau_{2k}\colon \Sigma^N C_{2k} \to \Sigma^N \tilde{C}_k \times_{\Sigma_k} (S^1)^k$$

of Theorem 4.4 *can be chosen so that*

$$\Sigma^N C_k \to \Sigma^N C_{2k} \to \Sigma^N \tilde{C}_k \times_{\Sigma_k} (S^1)^k$$

is homotopic $\Sigma^N s_k$.

We will always assume that the stable splitting τ_{2k} of Theorem 4.4 is chosen so that it satisfies Lemma 4.7. Now to prove Theorem A we must analyse the composite

$$\begin{aligned} KO(\mathcal{M}_k) &\xrightarrow{\lambda_k^*} KO(\tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k) \\ &\cong KO(\tilde{C}_k \times_{\Sigma_k} (S^1)^k) \\ &\xrightarrow{\tau_k^*} KO(C_{2k}) \,. \end{aligned}$$

Here we have written τ_{2k}^* for the map of K-theory induced by the stable map $\Sigma^N \tau_{2k}$.

Let us use the notation

$$\xi_k = i_k^*(\eta) \in KO(\mathcal{M}_k) ;$$

so ξ_k is the index bundle defined by the family of Dirac operators coupled to $SU(2)$ monopoles using the fundamental representation of $SU(2)$. First we compute the element

$$\lambda_k^*(\xi_k) = \lambda_k^* i_k^*(\eta) \in KO(\tilde{C}_k \times_{\Sigma_k} (S^1)^k)$$

in terms of representations of

$$\pi_1(\tilde{C}_k \times_{\Sigma_k} (S^1)^k) = \beta_{2,k} \,.$$

Let π_k be the representation of $\beta_{2,k}$ defined as follows:

$$\beta_{2,k} = \beta_k \rtimes (\mathbb{Z})^k \to \Sigma_k \rtimes (\mathbb{Z}/2)^k \subset O(k) \,.$$

Here the first homomorphism is the obvious quotient map and $\Sigma_k \rtimes (\mathbb{Z}/2)^k$ is identified with the subgroup of $O(k)$ generated by the permutation matrices and the matrices with $\pm$ 1's on the diagonal. Thus in the representation π_k the generator of the i^{th} copy of $\mathbb{Z}$ acts by changing the sign of the i^{th} basis vector and a braid acts by permuting the basis.

Now we can do the first computation required to prove Theorem A.

Lemma 4.8.

$$\lambda_k^*(\xi_k) = \pi_k \in KO(\beta_{2,k}) \,.$$

Proof. The composite map

$$\tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k \xrightarrow{\lambda_k} \mathcal{M}_k \xrightarrow{i_k} \Omega_k^2 S^2$$

can be described in purely homotopy theoretic terms as follows. Recall that, since $\Omega^2 S^2$ is a double loop space, there are extended power maps

$$\theta_k \colon \tilde{C}_k \times_{\Sigma_k} (\Omega_l^2 S^2)^k \to \Omega_{kl}^2 S^2 \,.$$

The construction and properties of these maps can be found in [20]. It is proved in [8] that the following diagram:

$$\begin{array}{ccc} \tilde{C}_k \times_{\Sigma_k} (\mathcal{M}_1)^k & \xrightarrow{\lambda_k} & \mathcal{M}_k \\ {\scriptstyle 1\times_{\Sigma_k}(i_1)^k}\downarrow & & \downarrow{\scriptstyle i_k} \\ \tilde{C}_k \times_{\Sigma_k} (\Omega_1^2 S^2)^k & \xrightarrow[\theta_k]{} & \Omega_k^2 S^2 \end{array}$$

commutes up to homotogy.

Now the inclusion of $i_1\colon \mathcal{M}_1 \simeq S^1 \to \Omega_1^2 S^2$ corresponds to the generator of $\pi_1(\Omega_1^2 S^2) \cong \mathbb{Z}$ and from Lemma 4.1 it follows that $\xi_1 = i_1^*(\eta)$ is the real Hopf line bundle over $S^1 \simeq \mathcal{M}_1$. By construction the map $\eta\colon \Omega^2 S^2 \to \mathbb{Z} \times BO$ is a double loop map and so it commutes with extended power maps. From this it follows that

$$\theta_k^*(\eta) = \tilde{C}_k \times_{\Sigma_k} (\eta)^k ,$$

$$\lambda_k^*(\xi_k) = \tilde{C}_k \times_{\Sigma_k} (\xi_1)^k$$

and it is straightforward to check that this bundle corresponds to the representation π_k. $\square$

This shows that ξ_k is completely determined by a representation of the group $\beta_{2,k}$ but now we must relate the element $\pi_k \in KO(\beta_{2,k})$ to representations of the braid group β_{2k}, that is we must compute

$$\tau_{2k}^*(\pi_k) \in KO(C_{2k}) .$$

The first step is to establish the following lemma. Recall that ρ_k is the permutation representation of β_k in $O(k)$. We will use the notation

$$q_k\colon \beta_{2,k} \to \beta_k$$

for the natural quotient map and

$$j_k\colon \beta_{2,k} \to \beta_{2k}$$

For the inclusion given by cabling.

Lemma 4.9. *The representation $j_k^*(\rho_{2k})$ of β_{2k} is given by*

$$j_k^*(\rho_{2k}) = \pi_k \oplus q_k^*(\rho_k) .$$

Proof. Let $e_1, \ldots, e_{2k}$ be the standard basis for $\mathbb{R}^{2k}$. Then in the representation $j_k^*(\rho_{2k})$ the generator of the i^{th} factor of $\mathbb{Z}$ in $\beta_{2,k}$ acts by interchanging e_{2i-1}, e_{2i} and leaving the other basis vectors fixed while a braid acts by permuting the k pairs $(e_1, e_2), \ldots, (e_{2k-1}, e_{2k})$. Therefore in the basis

$$f_1 = e_1 - e_2, \ldots, \quad f_k = e_{2k-1} - e_{2k} ,$$

$$f_{k+1} = e + e_2, \ldots, \quad f_{2k} = e_{2k-1} + e_{2k} ,$$

the representation $j_k^*(\rho_{2k})$ is precisely $\pi_k \oplus q_k^*(\rho_k)$. $\square$

Now we complete the proof of Theorem A which, by Lemma 4.8, amounts to computing $\tau_{2k}^*(\pi_k)$. From Lemma 4.9 we know that

$$\pi_k = j_k^*(\rho_{2k}) - q_k^*(\rho_k) \in KO(\tilde{C}_k \times_{\Sigma_k} (S^1)^k) ,$$

and so, using Theorem 4.4, it follows that

$$\tau_{2k}^*(\pi_k) = \rho_{2k} - \tau_{2k}^* q_k^*(\rho_k) .$$

The splitting τ is the composite

$$\Sigma^N C_{2k} \xrightarrow{\tau_{2k}} \Sigma^N \tilde{C}_k \times_{\Sigma_k} (S^1)^k = \Sigma^N (B\beta_{2,k}) \xrightarrow{\Sigma^N q_k} \Sigma^N C_k .$$

Note that this stable map is a stable splitting of the map

$$B\beta_k \to B\beta_{2k}$$

induced by the inclusion of braid groups because we have chosen τ_{2k} so that it satisfies Lemma 4.7. Theorem A now follows easily.

5. Other Representations

Given any quaternionic representation γ of $SU(2)$ we have described how to construct the family of real Dirac operators $\delta_{c,\gamma}$, parametrised by monopoles $c = (A, \varphi) \in \mathcal{M}_k$. The index bundle of this family of operators gives us a class

$$\operatorname{ind}(\delta_\gamma) \in KO(\mathcal{M}_k) .$$

We have also computed this element in the case where γ is the fundamental representation. The goal of this section is to give a general formula for the K-theory classes $\operatorname{ind}(\delta_\gamma)$, that is prove Theorem B in Sect. 1. We also prove that $\operatorname{ind}(\partial) \in K(\mathcal{M}_k)$ is trivial, where ∂ is the complex Dirac family, and that $\operatorname{ind}(\delta) \in KO(\mathcal{M}_k)$ has order two.

First we give the proof of Theorem B. Let γ: $SU(2) = Sp(1) \to Sp(n)$ be a symplectic representation. By identifying $Sp(1)$ with the sphere S^3, such a representation defines an integer $n_\gamma \in \pi_3(Sp(n)) \cong \mathbb{Z}$. Then we must prove that

$$\operatorname{ind}(\delta_\gamma) = n_\gamma \operatorname{ind}(\delta) \in KO(\mathcal{M}_k) .$$

Now γ defines a map

$$f_\gamma\colon Sp(1)/U(1) = S^2 \to Sp(n)/U(n) .$$

The homotopy class in $\pi_2(Sp(n)/U(n)) \cong \mathbb{Z}$ of f_γ is also given by $n_\gamma \in \mathbb{Z}$. Furthermore γ and f_γ induce a map

$$F_\gamma\colon \mathcal{B}(1) \to \mathcal{B}(n)$$

in the natural way. The obvious compatibility of these constructions gives the following lemma.

Lemma 5.1. *The diagram*

$$\begin{array}{ccccc} \Omega^2 S^2 & \xrightarrow{I_1} & \mathcal{B}(1) & \xrightarrow{\delta_\gamma} & \mathcal{F}_{\mathbb{R}} \\ {\scriptstyle \Omega^2 f_\gamma}\downarrow & & {\scriptstyle F_\gamma}\downarrow & & \downarrow{\scriptstyle =} \\ \Omega^2(Sp(n)/U(n)) & \xrightarrow{I_n} & \mathcal{B}(n) & \xrightarrow{\delta(n)} & \mathcal{F}_{\mathbb{R}} \end{array}$$

commutes up to homotopy.

In the statement of Lemma 5.1 the map

$$\delta(n)\colon \mathscr{B}(n) \to \mathscr{F}_{\mathbb{R}}$$

is, as in Sect. 2, the map defined by the real Dirac family coupled to pairs $c = (A, \Phi)$ using the fundamental representation of $Sp(n)$. Now by construction, the composite

$$\mathscr{M}_k \to \mathscr{B}(1) \xrightarrow{\delta_\gamma} \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO$$

represents the element $\operatorname{ind}(\delta_\gamma) \in KO(\mathscr{M}_k)$.

Lemma 5.2. *The map*

$$\mathscr{B}(1) \xrightarrow{\delta_\gamma} \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO$$

represents the element $n_\gamma \cdot \delta(1) \in KO(\mathscr{B}(1))$.

Proof. Since $I_1\colon \Omega^2 S^2 \to \mathscr{B}(1)$ is a homotopy equivalence, it suffices to prove that

$$\delta_\gamma \circ I_1\colon \Omega^2 S^2 \to \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO$$

represents the element in $KO(\Omega^2 S^2)$ given by n_γ times the class represented by

$$\delta(1) \circ I_1\colon \Omega^2 S^2 \to \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO\ .$$

Now Lemma 5.1 shows that

$$\delta_\gamma \circ I_1 \simeq \delta(n) \circ I_n \circ \Omega^2 f_\gamma\ .$$

But by repeating the argument used to prove Theorem 4.1,

$$\delta(n) \circ I_n\colon \Omega^2(Sp(n)/U(n)) \to \mathscr{F}_{\mathbb{R}} \simeq \mathbb{Z} \times BO$$

is homotopic to the composition

$$\Omega^2(Sp(n)/U(n)) \to \Omega^2(Sp/U) \xrightarrow{\beta} \mathbb{Z} \times BO\ ,$$

where β is the Bott equivalence. Hence $\delta_\gamma \circ I_1$ is homotopic to the composition

$$\Omega^2 S^2 \xrightarrow{\Omega^2 f_\gamma} \Omega^2(Sp(n)/U(n)) \to \Omega^2(Sp/U) \xrightarrow{\beta} \mathbb{Z} \times BO\ .$$

Since the Bott equivalence β is an equivalence of 2-fold loop spaces (in fact infinite loop spaces) we see that $\delta_\gamma \circ I_1$ is homotopic to a 2-fold loop map

$$\Omega^2 S^2 \to \mathbb{Z} \times BO \simeq \Omega^2(Sp/U)\ .$$

But any 2-fold loop map $\Omega^2 g\colon \Omega^2 S^2 \to \Omega^2 Y$ is determined up to homotopy by the homotopy class $[g] \in \pi_2(Y)$. In our case this homotopy class is $n_\gamma \in \pi_2(Sp/U) \cong \mathbb{Z}$. Using the fact that $\pi_0(\Omega^2 Y) = \pi_2(Y)$, this discussion proves that the map

$$\delta_\gamma \circ I_1\colon \Omega^2 S^2 \to \mathbb{Z} \times BO$$

is characterized, up to homotopy, by the properties that

(1) it is a 2-fold loop map, and
(2) on π_0 it is the homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by n_γ.

Now by Lemma 4.1 the map

$$\delta(1)\circ I_1\colon \Omega^2 S^2 \to \mathbb{Z}\times BO$$

is also a 2-fold loop map, and it induces the identity on π_0. Hence the class $n_\gamma\cdot(\delta(1)\circ I_1)\in KO(\Omega^2 S^2)$ is represented by the unique (up to homotopy) 2-fold loop map which induces multiplication by n_γ on path components. That is, $n_\gamma\cdot(\delta(1)\circ I_1)\in KO(\Omega^2 S^2)$ is represented by $\delta_\gamma\circ I_1$. □

We now complete the proof of Theorem B in the introduction. Given a representation γ: $SU(2)\to Sp(n)$ the index class $\operatorname{ind}(\delta_\gamma)\in KO(\mathcal{M}_k)$ is represented by the composition

$$\mathcal{M}_k \to \mathcal{B}(1) \xrightarrow{\delta_\gamma} \mathcal{F}_{\mathbb{R}} \simeq \mathbb{Z}\times BO\ .$$

In particular, the index class $\operatorname{ind}(\delta)$ corresponding to the fundamental representation is given by the composition

$$\mathcal{M}_k \to \mathcal{B}(1) \xrightarrow{\delta(1)} \mathcal{F}_{\mathbb{R}} \simeq \mathbb{Z}\times BO\ .$$

Thus by Lemma 5.2 we have

$$\operatorname{ind}(\delta_\gamma) = n_\gamma\cdot\operatorname{ind}(\delta)\in KO(\mathcal{M}_k)\ ,$$

and this completes the proof of Theorem B.

To complete the study of these index bundles we now prove the following lemma.

Lemma 5.3. (1) *The index bundle* $\operatorname{ind}(\partial)\in K(\mathcal{M}_k) = K(\beta_{2k})$ *of the complex family of Dirac operators coupled to monopoles using the fundamental representation of* $SU(2)$ *is trivial.*

(2) *The index bundle* $\operatorname{ind}(\delta)\in KO(\mathcal{M}_k) = K(\beta_{2k})$ *of the real family of Dirac operators coupled to monopoles using the fundamental representation of* $SU(2)$ *has order* 2.

Proof. First note that by definition $\operatorname{ind}(\partial)$ is the complexification of $\operatorname{ind}(\delta)$. Let

$$r\colon K(\mathcal{M}_k)\to KO(\mathcal{M}_k)$$

be the map given by forgetting the complex structure. It now follows that

$$r(\operatorname{ind}(\partial)) = 2\operatorname{ind}(\delta)$$

and so the second part of the lemma follows from the first part.

To prove the first part we start by using Theorem A to tell us that under the stable homotopy equivalence $B\beta_{2k}\simeq_s \mathcal{M}_k$ the index bundle $\operatorname{ind}(\delta)$ corresponds to $\rho_{2k}-\tau^*\rho_k\in KO(\beta_{2k})$. Hence it is sufficient to show that for every j, the complexification of the representation ρ_j of β_j is trivial in complex K-theory; that is

$$\rho_j^{\mathbb{C}} = 0\in K(\beta_j)\ .$$

This statement was essentially proved in [12] in a somewhat different context. For the sake of completeness we give a simple direct argument due to E.H. Brown.

By definition ρ_j is the representation given by the composition

$$\beta_j \to \Sigma_j \to O(j)\,,$$

where the first of these maps sends a braid to the permutation of its endpoints, and the second is the permutation representation. This representation gives the following j-dimensional bundle

$$\tilde{C}_j \times_{\Sigma_j} \mathbb{R}^j \to \tilde{C}_j/\Sigma_j = C_j$$

over $B\beta_j = C_j$. Here, as before, $\tilde{C}_j$ and C_j denote the configuration spaces of j distinct ordered and unordered points in $\mathbb{C}$ respectively. In this bundle the symmetric group Σ_j acts on $\mathbb{R}^j$ by permuting coordinates. Thus the complexification of the representation ρ_j gives the bundle

$$\tilde{C}_j \times_{\Sigma_j} \mathbb{C}^j \to \tilde{C}_j/\Sigma_j = C_j\,.$$

A trivialization

$$\Psi\colon \tilde{C}_j \times_{\Sigma_j} \mathbb{C}^j \to \mathbb{C}^j$$

of this complex bundle is defined as follows. Given

$$[z_1, \ldots, z_j; u_1, \ldots, u_j] \in \tilde{C}_j \times_{\Sigma_j} \mathbb{C}^j$$

define

$$\psi([z_1, \ldots, z_j; u_1, \ldots, u_j]) = \begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_j \\ z_1^2 & \cdots & z_j^2 \\ \vdots & \ddots & \vdots \\ z_1^{j-1} & \cdots & z_j^{j-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_j \end{pmatrix}.$$

Then Ψ is well-defined. The determinant of the above Vandermonde matrix is

$$\prod_{r<s} (z_r - z_s)\,.$$

This determinant is nonzero since the z_i are assumed to be distinct. Thus Ψ is a linear isomorphism on each fibre and this gives the required trivialisation. □

6. Solutions of the Dirac Equation

In this section we prove Theorem C of Sect. 1; we show that given any integer N then in each space $\mathscr{B}_k = \mathscr{B}_k(1)$ there is a pair $c = (A, \Phi)$ such that

$$\dim \ker \delta_c \geqq N, \quad \dim \ker \delta_c^* \geqq N\,.$$

In fact we prove the following theorem which clearly implies Theorem C.

Theorem 6.1. *The element*

$$\operatorname{ind} \delta \in KO^0(\mathscr{B}_k)$$

cannot be expressed as a difference of finite dimensional bundles.

Proof. Each of the spaces $\mathscr{B}_k$ are homotopy equivalent and these equivalences can be chosen so that in K-theory they preserve the index bundles. Thus it is sufficient to prove this result with $k = 0$. Now we can use the following identifications

$$\mathscr{B}_0 \simeq \Omega_0^2 S^2 \simeq \Omega^2 S^3 ,$$

where the second equivalence is induced by the Hopf map $S^3 \to S^2$. Under these equivalences, and in view of Lemma 4.1, the index bundle $\operatorname{ind}(\delta)$ corresponds to the following bundle. The composite

$$S^3 \to S^2 = Sp(1)/U(1) \to Sp/U ,$$

where the first map is the Hopf map, represents the generator of $\pi_3(Sp/U)$. Now applying Ω^2 to this composite and using the equivalence $\Omega^2(Sp/U) \simeq \mathbb{Z} \times BO$ gives a map

$$\zeta\colon \Omega^2 S^3 \to \mathbb{Z} \times BO .$$

The corresponding element in K-theory is the index bundle.

The mod 2 homology of $\Omega^2 S^3$ is well-known. For the remainder of this section H_* will denote homology with coefficients in $\mathbb{Z}/2$. Using the loop structure to define a product $\Omega^2 S^3 \times \Omega^2 S^3 \to \Omega^2 S^3$, and therefore a Pontryagin product in homology

$$H_*(\Omega^2 S^3) = \mathbb{Z}/2[x_{2^i-1}\colon i \geqq 1], \quad x_{2^i-1} \in H_{2^i-1}(\Omega^2 S^3)$$

is a polynomial ring. The coproduct in homology

$$\Delta_*\colon H_*(\Omega^2 S^3) \to H_*(\Omega^2 S^3) \otimes H_*(\Omega^2 S^3) ,$$

induced by the diagonal map $\Delta\colon \Omega^2 S^3 \to \Omega^2 S^3 \times \Omega^2 S^3$ is determined by the fact that it is a ring homomorphism and the formula

$$\Delta_*(x_{2^i-1}) = x_{2^i-1} \otimes 1 + 1 \otimes x_{2^i-1} .$$

Now let $v_n = v_n(\zeta) \in H^n(\Omega^2 S^3)$ be the n^{th} Wu classes of ζ. Then, see for example [16], it is known that for all i,

$$\langle v_{2^i-1}, x_{2^i-1} \rangle = 1 ,$$

where $\langle , \rangle$ is the pairing between cohomology and homology. Now suppose that $\zeta = E - F \in KO(\Omega^2 S^3)$, where E and F are finite dimensional bundles. Then

$$V(\zeta) = V(E)\, V(F)^{-1} ,$$

where $V = 1 + v_1 + \cdots$ is the total Wu-class and $V(F)^{-1}$ is the power series in the Wu classes $v_i(F)$ given by inverting the formal power series $V(F)$. Since E and F are finite dimensional bundles it follows that $v_k(E)$ and $v_k(F)$ are decomposable elements in the cohomology ring $H^*(\Omega^2 S^3)$ for $k > \max(\dim E, \dim F)$.

It therefore follows that $v_i(\zeta)$ is decomposable for $i > \max(\dim E, \dim F)$. However if $v = \alpha \cdot \beta \in H^*(\Omega^2 S^3)$, where $\alpha, \beta \neq 1$, is a decomposable element then

$$\begin{aligned} \langle v, x_{2^k-1} \rangle &= \langle \Delta^*(\alpha \otimes \beta), x_{2^k-1} \rangle \\ &= \langle \alpha \otimes \beta, \Delta_*(x_{2^k-1}) \rangle \\ &= \langle \alpha \otimes \beta, x_{2^k-1} \otimes 1 + 1 \otimes x_{2^k-1} \rangle \\ &= 0 . \end{aligned}$$

Thus

$$\langle v_{2^k-1}, x_{2^k-1} \rangle = 0, \quad 2^k - 1 > \max(\dim E, \dim F) ,$$

and this contradicts the fact that $\langle v_{2^k-1}, x_{2^i-1} \rangle = 1$ for all i. $\square$

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