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Bicovariant Quantum Algebras and Quantum Lie Algebras*

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Abstract. A bicovariant calculus of differential operators on a quantum group is constructed in a natural way, using invariant maps from $\mathrm{Fun}(\mathfrak{G}_q)$ to $U_q\mathfrak{g}$, given by elements of the pure braid group. These operators – the "reflection matrix" $Y \equiv L^+SL^-$ being a special case – generate algebras that linearly close under adjoint actions, i.e. they form generalized Lie algebras. We establish the connection between the Hopf algebra formulation of the calculus and a formulation in compact matrix form which is quite powerful for actual computations and as applications we find the quantum determinant and an orthogonality relation for Y in $SO_q(N)$.

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1. Introduction

In the classical theory of Lie algebras we start the construction of a bicovariant calculus by introducing a matrix $\Omega = A^{-1}dA \in \Gamma$ of one-forms that is invariant under left transformations,

$$A \to A'A\colon d \to d,\ \Omega \to \Omega\,, \tag{1}$$

and covariant under right transformations,

$$A \to AA'\colon d \to d,\ \Omega \to A'^{-1}\Omega A'\,. \tag{2}$$

The dual basis to the entries of this matrix Ω form a matrix X of vector fields with the same transformation properties as Ω:

$$\langle \Omega^i{}_j, X^k{}_l \rangle = \delta^i{}_l \delta^k{}_j \quad (classical)\,. \tag{3}$$

We find,

$$X = \left(A^T \frac{\partial}{\partial A} \right)^T \quad (classical)\,. \tag{4}$$

Woronowicz [1] was able to extend the definition of a bicovariant calculus to quantum groups. His approach via differential forms has the advantage that coactions (transformations) ${}_{\mathfrak{A}}\Delta\colon \Gamma \to \mathfrak{A} \otimes \Gamma$ and $\Delta_{\mathfrak{A}}\colon \Gamma \to \Gamma \otimes \mathfrak{A}$ can be introduced very easily through,

$${}_{\mathfrak{A}}\Delta(da) = (\mathrm{id} \otimes d)\Delta a\,, \tag{5}$$

$$\Delta_{\mathfrak{A}}(da) = (d \otimes \mathrm{id})\Delta a\,, \tag{6}$$

where $\mathfrak{A}$ is the Hopf algebra of "functions on the quantum groups," $a \in \mathfrak{A}$ and Δ is the coproduct in $\mathfrak{A}$. Equations (5, 6) rely on the existence of an invariant map $d\colon \mathfrak{A} \to \Gamma$ provided by the exterior derivative. A construction of the bicovariant calculus starting directly from the vector fields is much harder because simple formulae like (5, 6) do not seem to exist. We will show that in the case of a quasitriangular Hopf algebra $\mathfrak{U}$ invariant maps from $\mathfrak{A}$ to the quantized algebra of differential operators $\mathfrak{A} \rtimes \mathfrak{U}$ can arise from elements of the pure braid group on two strands. Using these maps we will then construct differential operators with simple transformation properties and in particular a bicovariant matrix of vector fields corresponding to (4).

Before proceeding we would like to recall some useful facts about quasitriangular Hopf algebras and quantum groups. A thorough introduction to these topics and additional references can be found in [2].

1.1. Quasitriangular Hopf Algebras. A Hopf algebra $\mathfrak{A}$ is an associative unital algebra $(\mathfrak{A}, \cdot, +, k)$ over a field k, equipped with a coproduct $\Delta: \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$, an antipode $S: \mathfrak{A} \to \mathfrak{A}$, and a counit $\varepsilon: \mathfrak{A} \to k$, satisfying

$$(\Delta \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \Delta)\Delta(a), \quad \text{(coassociativity)}, \tag{7}$$

$$\cdot(\varepsilon \otimes \mathrm{id})\Delta(a) = \cdot(\mathrm{id} \otimes \varepsilon)\Delta(a) = a, \quad \text{(counit)}, \tag{8}$$

$$\cdot(S \otimes \mathrm{id})\Delta(a) = \cdot(\mathrm{id} \otimes S)\Delta(a) = 1\varepsilon(a), \quad \text{(coinverse)}, \tag{9}$$

for all $a \in \mathfrak{A}$. These axioms are dual to the axioms of an algebra. There are also a number of consistency conditions between the algebra and the coalgebra structure,

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{10}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \tag{11}$$

$$S(ab) = S(b)S(a), \quad \text{(antihomomorphism)}, \tag{12}$$

$$\Delta(S(a)) = \tau(S \otimes S)\Delta(a), \quad \text{with } \tau(a \otimes b) \equiv b \otimes a, \tag{13}$$

$$\varepsilon(S(a)) = \varepsilon(a), \quad \text{and} \tag{14}$$

$$\Delta(1) = 1 \otimes 1, \quad S(1) = 1, \quad \varepsilon(1) = 1_k, \tag{15}$$

for all $a, b \in \mathfrak{A}$. We will often use Sweedler's [3] notation for the coproduct:

$$\Delta(a) \equiv a_{(1)} \otimes a_{(2)} \quad \textit{(summation is understood)}. \tag{16}$$

Note that a Hopf algebra is in general non-cocommutative, i.e. $\tau \circ \Delta \neq \Delta$.

A quasitriangular Hopf algebra $\mathfrak{U}$ [4] is a Hopf algebra with a *universal* $\mathcal{R} \in \mathfrak{U} \hat{\otimes} \mathfrak{U}$ that keeps the non-cocommutativity under control,

$$\tau(\Delta(a)) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \tag{17}$$

and satisfies,

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}, \quad \text{and} \tag{18}$$

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}, \tag{19}$$

where *upper* indices denote the position of the components of $\mathcal{R}$ in the tensor product *algebra* $\mathfrak{U} \hat{\otimes} \mathfrak{U} \hat{\otimes} \mathfrak{U}$: if $\mathcal{R} \equiv \alpha_i \otimes \beta_i$ (*summation is understood*), then e.g. $\mathcal{R}^{13} \equiv \alpha_i \otimes 1 \otimes \beta_i$. Equation (19) states that $\mathcal{R}$ generates an algebra map $\langle \mathcal{R}, \cdot \otimes \mathrm{id} \rangle: \mathfrak{U}^* \to \mathfrak{U}$ and an antialgebra map $\langle \mathcal{R}, \mathrm{id} \otimes \cdot \rangle: \mathfrak{U}^* \to \mathfrak{U}$.[1] The following equalities are consequences of the axioms:

$$\mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23} = \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12}, \quad \text{(quantum Yang–Baxter equation)}, \tag{20}$$

$$(S \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{-1}, \tag{21}$$

$$(\mathrm{id} \otimes S)\mathcal{R}^{-1} = \mathcal{R}, \quad \text{and} \tag{22}$$

$$(\varepsilon \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes \varepsilon)\mathcal{R} = 1. \tag{23}$$

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[1] Notation: "$\cdot$" denotes an argument to be inserted and "id" is the identity map, e.g. $\langle \mathcal{R}, \mathrm{id} \otimes f \rangle \equiv \alpha_i \langle \beta_i, f \rangle$; $\mathcal{R} \equiv \alpha_i \otimes \beta_i \in \mathfrak{U} \hat{\otimes} \mathfrak{U}$, where $f \in \mathfrak{U}^*$ has here replaced "$\cdot$"

An example of a quasitriangular Hopf algebra that is of particular interest here is the deformed universal enveloping algebra $U_q\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. Dual to $U_q\mathfrak{g}$ is the Hopf algebra of "functions on the quantum group" $\mathrm{Fun}(\mathfrak{G}_q)$; in fact, $U_q\mathfrak{g}$ and $\mathrm{Fun}(\mathfrak{G}_q)$ are *dually paired.* We call two Hopf algebras $\mathfrak{U}$ and $\mathfrak{A}$ dually paired if there exists a non-degenerate inner product $\langle\,,\rangle\colon \mathfrak{U}\otimes\mathfrak{A}\to k$, such that:

$$\langle xy, a\rangle = \langle x\otimes y, \Delta(a)\rangle \equiv \langle x, a_{(1)}\rangle\langle y, a_{(2)}\rangle\,, \tag{24}$$

$$\langle x, ab\rangle = \langle \Delta(x), a\otimes b\rangle \equiv \langle x_{(1)}, a\rangle\langle x_{(2)}, b\rangle\,, \tag{25}$$

$$\langle S(x), a\rangle = \langle x, S(a)\rangle\,, \tag{26}$$

$$\langle x, 1\rangle = \varepsilon(x), \quad \text{and} \quad \langle 1, a\rangle = \varepsilon(a)\,, \tag{27}$$

for all $x, y\in\mathfrak{U}$ and $a, b\in\mathfrak{A}$. In the following we will assume that $\mathfrak{U}$ (quasitriangular) and $\mathfrak{A}$ are dually paired Hopf algebras, always keeping $U_q\mathfrak{g}$ and $\mathrm{Fun}(\mathfrak{G}_q)$ as concrete realizations in mind.

In the next subsection we will sketch how to obtain $\mathrm{Fun}(\mathfrak{G}_q)$ as a matrix representation of $U_q\mathfrak{g}$.

1.2. Dual Quantum Groups. We cannot speak about a quantum group $\mathfrak{G}_q$ directly, just as "phase space" loses its meaning in quantum mechanics, but in the spirit of geometry on noncommuting spaces the (deformed) functions on the quantum group $\mathrm{Fun}(\mathfrak{G}_q)$ still make sense. This can be made concrete, if we write $\mathrm{Fun}(\mathfrak{G}_q)$ as a pseudo-matrix group [5], generated by the elements of an $N\times N$ matrix $A\equiv(A^i{}_j)_{i,j=1\ldots N}\in M_N(\mathrm{Fun}(\mathfrak{G}_q))$.[2] We require that $\rho^i{}_j\equiv\langle\cdot, A^i{}_j\rangle$ be a matrix representation of $U_q\mathfrak{g}$, i.e.

$$\begin{aligned}&\rho^i{}_j\colon U_q\mathfrak{g}\to k\,,\\ &\rho^i{}_j(xy) = \sum_k \rho^i{}_k(x)\rho^k{}_j(y), \quad \text{for } \forall x, y\in U_q\mathfrak{g}\,,\end{aligned} \tag{28}$$

just like in the classical case.[3] The universal $\mathcal{R}\in U_q\mathfrak{g}\,\hat{\otimes}\, U_q\mathfrak{g}$ coincides in this representation with the numerical R-matrix:

$$\langle\mathcal{R}, A^i{}_k\otimes A^j{}_l\rangle = R^{ij}{}_{kl}\,. \tag{29}$$

It immediately follows from (24) and (28) that the coproduct of A is given by matrix multiplication [5, 6],

$$\Delta A = A\dot{\otimes} A, \quad \text{i.e.} \quad \Delta(A^i{}_j) = A^i{}_k\otimes A^k{}_j\,. \tag{30}$$

Equations (17), (25), and (28) imply [4, 6],

$$\begin{aligned}\langle x, A^j{}_s A^i{}_r\rangle &= \langle\Delta x, A^j{}_s\otimes A^i{}_r\rangle\\ &= \langle\tau\circ\Delta x, A^i{}_r\otimes A^j{}_s\rangle\\ &= \langle\mathcal{R}(\Delta x)\mathcal{R}^{-1}, A^i{}_r\otimes A^j{}_s\rangle\\ &= R^{ij}{}_{kl}\langle\Delta x, A^k{}_m\otimes A^l{}_n\rangle(R^{-1})^{mn}{}_{rs}\\ &= \langle x, R^{ij}{}_{kl}A^k{}_m A^l{}_n(R^{-1})^{mn}{}_{rs}\rangle\,,\end{aligned} \tag{31}$$

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[2] We are automatically dealing with $GL_q(N)$ unless there are explicit or implicit restrictions on the matrix elements of A

[3] The quintessence of this construction is that the coalgebra of $\mathrm{Fun}(\mathfrak{G}_q)$ is undeformed, i.e. we keep the familiar matrix group expressions of the classical theory

i.e. the matrix elements of A satisfy the following commutation relations,

$$R^{ij}{}_{kl}A^{k}{}_{m}A^{l}{}_{n} = A^{j}{}_{s}A^{i}{}_{r}R^{rs}{}_{mn} , \tag{32}$$

which can be written more compactly in tensor product notation as:

$$R_{12}A_1A_2 = A_2A_1R_{12} ; \tag{33}$$

$$R_{12} = (\rho_1 \otimes \rho_2)(\mathcal{R}) \equiv \langle \mathcal{R}, A_1 \otimes A_2 \rangle . \tag{34}$$

Lower numerical indices shall denote here the position of the respective matrices in the tensor product of *representation spaces* (*modules*). The contragredient representation [7] $\rho^{-1} = \langle \cdot, SA \rangle$ gives the antipode of $\mathrm{Fun}(\mathfrak{G}_q)$ in matrix form: $S(A^{i}{}_{j}) = (\mathrm{A}^{-1})^{i}{}_{j}$. The counit is: $\varepsilon(A^{i}{}_{j}) = \langle 1, A^{i}{}_{j} \rangle = \delta^{i}{}_{j}$.

Higher (tensor product) representations can be constructed from A: $A_1A_2, A_1A_2A_3, \ldots, A_1A_2 \ldots A_m$. We find numerical **R**-matrices [2] for any pair of such representations:

$$\begin{aligned} \mathbf{R}_{\underbrace{(1', 2', \ldots, n')}_{I}, \underbrace{(1, 2, \ldots, m)}_{II}} &\equiv \langle \mathcal{R}, A_{1'}A_{2'} \ldots A_{n'} \otimes A_1A_2 \ldots A_m \rangle \\ &= R_{1'm} \cdot R_{1'(m-1)} \cdot \ldots \cdot R_{1'1} \\ &\quad \cdot R_{2'm} \cdot R_{2'(m-1)} \cdot \ldots \cdot R_{2'1} \\ &\quad\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ &\quad \cdot R_{n'm} \cdot R_{n'(m-1)} \cdot \ldots \cdot R_{n'1} . \end{aligned} \tag{35}$$

Let $\mathbf{A}_I \equiv A_{1'}A_{2'} \ldots A_{n'}$ and $\mathbf{A}_{II} \equiv A_1A_2 \ldots A_m$, then:

$$\mathbf{R}_{I,II}\mathbf{A}_I\mathbf{A}_{II} = \mathbf{A}_{II}\mathbf{A}_I\mathbf{R}_{I,II} . \tag{36}$$

$\mathbf{R}_{I,II}$ is the "partition function" of exactly solvable models. We will need it in Sect. 3.

We can also write $U_q\mathfrak{g}$ in matrix form [6, 7] by taking representations ϱ – e.g. $\varrho = \langle \cdot, \mathbf{A} \rangle$ – of $\mathcal{R}$ in its first or second tensor product space,

$$L^{+}_{\varrho} \equiv (\mathrm{id} \otimes \varrho)(\mathcal{R}), \qquad L^{+} \equiv \langle \mathcal{R}^{21}, A \otimes \mathrm{id} \rangle , \tag{37}$$

$$SL^{-}_{\varrho} \equiv (\varrho \otimes \mathrm{id})(\mathcal{R}), \qquad SL^{-} \equiv \langle \mathcal{R}, A \otimes \mathrm{id} \rangle , \tag{38}$$

$$L^{-}_{\varrho} \equiv (\varrho \otimes \mathrm{id})(\mathcal{R}^{-1}), \quad L^{-} \equiv \langle \mathcal{R}, SA \otimes \mathrm{id} \rangle . \tag{39}$$

The commutation relations for all these matrices follow directly from the quantum Yang–Baxter equation, e.g.

$$\begin{aligned} 0 &= \langle \mathcal{R}^{23}\mathcal{R}^{13}\mathcal{R}^{12} - \mathcal{R}^{12}\mathcal{R}^{13}\mathcal{R}^{23}, \mathrm{id} \otimes A_1 \otimes A_2 \rangle \\ &= R_{12}L^{+}_{2}L^{+}_{1} - L^{+}_{1}L^{+}_{2}R_{12} , \end{aligned} \tag{40}$$

where upper "algebra" indices should not be confused with lower "matrix" indices. Similarly one finds:

$$R_{12}L^{-}_{2}L^{-}_{1} = L^{-}_{1}L^{-}_{2}R_{12} , \tag{41}$$

$$R_{12}L^{+}_{2}L^{-}_{1} = L^{-}_{1}L^{+}_{2}R_{12} . \tag{42}$$

2. Quantized Algebra of Differential Operators

Here we would like to establish the connection between the actions of differential operators [8], written as commutation relations of operator-valued matrices and the more abstract formulation of the calculus in the Hopf algebra language.

2.1. Actions and Coactions. A *left action* of an algebra A on a vector space V is a bilinear map, $\triangleright$: $A \otimes V \to V$: $x \otimes v \mapsto x \triangleright v$, such that: $(xy) \triangleright v = x \triangleright (y \triangleright v)$. V is called a left A-module. In the case of the left action of a Hopf algebra H on an algebra A' we can in addition ask that this action preserve the algebra structure of A', i.e. $x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b)$[4] and $x \triangleright 1 = 1\varepsilon(x)$, for all $x \in H$, $a, b \in A'$. A' is then called a left H-module algebra. Right actions and modules are defined in complete analogy. A left action of an algebra on a (finite dimensional) vector space induces a right action of the same algebra on the dual vector space and vice versa, via pullback. Of particular interest to us is the left action of $\mathfrak{U}$ on $\mathfrak{A}$ induced by the right multiplication in $\mathfrak{U}$:

$$\langle y, x \triangleright a \rangle := \langle yx, a \rangle = \langle y \otimes x, \Delta a \rangle = \langle y, a_{(1)} \langle x, a_{(2)} \rangle \rangle ,$$

$$\Rightarrow \quad x \triangleright a = a_{(1)} \langle x, a_{(2)} \rangle, \quad \text{for } \forall x, y \in \mathfrak{U}, a \in \mathfrak{A} , \tag{43}$$

where again $\Delta a \equiv a_{(1)} \otimes a_{(2)}$. This action of $\mathfrak{U}$ on $\mathfrak{A}$ respects the algebra structure of $\mathfrak{A}$, as can easily be checked. The action of $\mathfrak{U}$ on itself given by right or left multiplication does *not* respect the algebra structure of $\mathfrak{U}$; see however (62) as an example of an algebra-respecting "inner" action.

In the same sense as comultiplication is the dual operation to multiplication, *right* or *left coactions* are dual to left or right actions respectively. One therefore defines a right coaction of a coalgebra C on a vector space V to be a linear map, Δ_C: $V \to V \otimes C$: $v \mapsto \Delta_C(v) \equiv v^{(1)} \otimes v^{(2)'}$, such that, $(\Delta_C \otimes \mathrm{id})\Delta_C = (\mathrm{id} \otimes \Delta)\Delta_C$. Following [2] we have introduced here a notation for the coaction that resembles Sweedler's notation (16) of the coproduct. The prime on the second factor marks a right coaction. If we are dealing with the right coaction of a Hopf algebra H on an algebra A, we say that the coaction respects the algebra structure and A is a right H-comodule algebra, if $\Delta_H(a \cdot b) = \Delta_H(a) \cdot \Delta_H(b)$ and $\Delta_H(1) = 1 \otimes 1$, for all $a, b \in A$.

If the coalgebra C is dual to an algebra A in the sense of (24–27), then a *right* coaction of C on V will induce a *left* action of A on V and vice versa, via

$$x \triangleright v = v^{(1)} \langle x, v^{(2)'} \rangle, \quad (general) , \tag{44}$$

for all $x \in A$, $v \in V$. Applying this general formula to the specific case of our dually paired Hopf algebras $\mathfrak{U}$ and $\mathfrak{A}$, we see that the right coaction $\Delta_{\mathfrak{A}}$ of $\mathfrak{A}$ on itself, corresponding to the left action of $\mathfrak{U}$ on $\mathfrak{A}$, as given by (43), is just the coproduct Δ in $\mathfrak{A}$, i.e. we pick:

$$\Delta_{\mathfrak{A}}(a) \equiv a^{(1)} \otimes a^{(2)'} = a_{(1)} \otimes a_{(2)}, \quad \text{for } \forall a \in \mathfrak{A} . \tag{45}$$

To get an intuitive picture we may think of the left action (43) as being a generalized *specific left translation* generated by a left invariant "tangent vector" $x \in \mathfrak{U}$ of the quantum group. The coaction $\Delta_{\mathfrak{A}}$ is then the generalization of an *unspecified translation.* If we supply for instance a vector $x \in \mathfrak{U}$ as transformation

[4] $x \triangleright$ is called a *generalized derivation*

parameter, we recover the generalized specific transformation (43); if we use $1 \in \mathfrak{U}$, i.e. evaluate at the "identity of the quantum group," we get the identity transformation; but the quantum analog of a classical finite translation through left or right multiplication by a *specific* group element does not exist.

The dual quantum group in matrix form stays very close to the classical formulation and we want to use it to illustrate some of the above equations. For the matrix $A \in M_N(\mathrm{Fun}(\mathfrak{G}_q))$ and $x \in U_q\mathfrak{g}$ we find,

$$\mathrm{Fun}(\mathfrak{G}_q) \to \mathrm{Fun}(\mathfrak{G}_q) \otimes \mathrm{Fun}(\mathfrak{G}_q):$$

$$\Delta_{\mathfrak{A}} A = AA', \quad \text{(right coaction)}, \tag{46}$$

$$\mathrm{Fun}(\mathfrak{G}_q) \to \mathrm{Fun}(\mathfrak{G}_q) \otimes \mathrm{Fun}(\mathfrak{G}_q):$$

$$_{\mathfrak{A}}\Delta A = A'A, \quad \text{(left coaction)}, \tag{47}$$

$$U_q\mathfrak{g} \otimes \mathrm{Fun}(\mathfrak{G}_q) \to \mathrm{Fun}(\mathfrak{G}_q):$$

$$x \triangleright A = A\langle x, A\rangle, \quad \text{(left action)}, \tag{48}$$

where matrix multiplication is implied. Following common custom we have used a prime to distinguish copies of the matrix A in different tensor product spaces. We see that in complete analogy to the classical theory of Lie algebras, we first evaluate $x \in U_q\mathfrak{g}$, interpreted as a left invariant vector field, on $A \in M_n(\mathrm{Fun}(\mathfrak{G}_q))$ at the "identity of $\mathfrak{G}_q$," giving a numerical matrix $\langle x, A\rangle \in M_n(k)$, and then shift the result by left matrix multiplication with A to an unspecified "point" on the quantum group. Unlike a Lie group, a quantum group is not a manifold in the classical sense and hence we cannot talk about its elements, except for the identity (which is also the counit of $\mathrm{Fun}(\mathfrak{G}_q)$). For $L^+ \in M_N(U_q\mathfrak{g})$ Eq. (48) becomes,

$$L_2^+ \triangleright A_1 = A_1\langle L_2^+, A_1\rangle = A_1 R_{12}, \tag{49}$$

and similarly for $L^- \in M_N(U_q\mathfrak{g})$:

$$L_2^- \triangleright A_1 = A_1\langle L_2^-, A_1\rangle = A_1 R_{21}^{-1}. \tag{50}$$

2.2. Commutation Relations. The left action of $x \in \mathfrak{U}$ on products in $\mathfrak{A}$, say bf, is given via the coproduct in $\mathfrak{U}$,

$$\begin{aligned} x \triangleright bf &= (bf)_{(1)}\langle x, (bf)_{(2)}\rangle \\ &= b_{(1)}f_{(1)}\langle \Delta(x), b_{(2)} \otimes f_{(2)}\rangle \\ &= \cdot\, \Delta x \triangleright (b \otimes f) = b_{(1)}\langle x_{(1)}, b_{(2)}\rangle x_{(2)} \triangleright f. \end{aligned} \tag{51}$$

Dropping the "$\triangleright$" we can write this for arbitrary functions f in the form of commutation relations,

$$xb = \Delta x \triangleright (b \otimes \mathrm{id}) = b_{(1)}\langle x_{(1)}, b_{(2)}\rangle x_{(2)}. \tag{52}$$

This commutation relation provides $\mathfrak{A} \otimes \mathfrak{U}$ with an algebra structure via the *cross product*,

$$\cdot : (\mathfrak{A} \otimes \mathfrak{U}) \otimes (\mathfrak{A} \otimes \mathfrak{U}) \to \mathfrak{A} \otimes \mathfrak{U}:$$

$$ax \otimes by \mapsto ax \cdot by = ab_{(1)}\langle x_{(1)}, b_{(2)}\rangle x_{(2)}y. \tag{53}$$

That $\mathfrak{A} \otimes \mathfrak{U}$ is indeed an associative algebra with this multiplication follows from the Hopf algebra axioms; it is denoted $\mathfrak{A} \rtimes \mathfrak{U}$ and we call it the *quantized algebra of* *differential operators*. The commutation relation (52) should be interpreted as a product in $\mathfrak{A} \rtimes \mathfrak{U}$. (Note that we omit $\otimes$-signs wherever they are obvious, but we sometimes insert a product sign "$\cdot$" for clarification of the formulas.) Right actions and the corresponding commutation relations are also possible: $b \triangleleft \mathring{x} = \langle \mathring{x}, b_{(1)} \rangle b_{(2)}$ and $b\mathring{x} = \mathring{x}_{(1)} \langle \mathring{x}_{(2)}, b_{(1)} \rangle b_{(2)}$.

Equation (52) can be used to calculate arbitrary inner products of $\mathfrak{U}$ with $\mathfrak{A}$, if we define a *right vacuum* "$\rangle$" to act like the counit in $\mathfrak{U}$ and a *left vacuum* "$\langle$" to act like the counit in $\mathfrak{A}$,

$$\begin{aligned} \langle xb \rangle &= \langle b_{(1)} \langle x_{(1)}, b_{(2)} \rangle x_{(2)} \rangle \\ &= \varepsilon(b_{(1)}) \langle x_{(1)}, b_{(2)} \rangle \varepsilon(x_{(2)}) \\ &= \langle \cdot (\mathrm{id} \otimes \varepsilon) \Delta(x), \cdot (\varepsilon \otimes \mathrm{id}) \Delta(b) \rangle \\ &= \langle x, b \rangle, \quad \text{for } \forall x \in \mathfrak{U}, b \in \mathfrak{A} . \end{aligned} \tag{54}$$

Using only the right vacuum we recover formula (43) for left actions,

$$\begin{aligned} xb \rangle &= b_{(1)} \langle x_{(1)}, b_{(2)} \rangle x_{(2)} \rangle \\ &= b_{(1)} \langle x_{(1)}, b_{(2)} \rangle \varepsilon(x_{(2)}) \\ &= b_{(1)} \langle x, b_{(2)} \rangle \\ &= x \triangleright b, \quad \text{for } \forall x \in \mathfrak{U}, b \in \mathfrak{A} . \end{aligned} \tag{55}$$

As an example we will write the preceding equations for A, L^+, and L^-:

$$L_2^+ A_1 = A_1 R_{12} L_2^+ , \quad \text{(commutation relation for } L^+ \text{ with } A) , \tag{56}$$

$$L_2^- A_1 = A_1 R_{21}^{-1} L_2^- , \quad \text{(commutation relation for } L^- \text{ with } A) , \tag{57}$$

$$\langle A = I \langle , \quad \text{(left vacuum for } A) , \tag{58}$$

$$L^+ \rangle = L^- \rangle = \rangle I, \quad \text{(right vacua for } L^+ \text{ and } L^-) . \tag{59}$$

Equation (55) is not the only way to define left actions of $\mathfrak{U}$ on $\mathfrak{A}$ in terms of the product in $\mathfrak{A} \rtimes \mathfrak{U}$. An alternate definition utilizing the coproduct and antipode in $\mathfrak{U}$,

$$\begin{aligned} x_{(1)} b S(x_{(2)}) &= b_{(1)} \langle x_{(1)}, b_{(2)} \rangle x_{(2)} S(x_{(3)})^5 \\ &= b_{(1)} \langle x_{(1)}, b_{(2)} \rangle \varepsilon(x_{(2)}) \\ &= b_{(1)} \langle x, b_{(2)} \rangle \\ &= x \triangleright v, \quad \text{for } \forall x \in \mathfrak{U}, b \in \mathfrak{A} , \end{aligned} \tag{60}$$

is in a sense more satisfactory because it readily generalizes to left actions of $\mathfrak{U}$ on $\mathfrak{A} \rtimes \mathfrak{U}$,

$$\begin{aligned} x \triangleright by &:= x_{(1)} by S(x_{(2)}) \\ &= x_{(1)} bS(x_{(2)}) x_{(3)} y S(x_{(4)})^5 \\ &= (x_{(1)} \triangleright b)(x_{(2)} \overset{\mathrm{ad}}{\triangleright} y), \quad \text{for } \forall x, y \in \mathfrak{U}, b \in \mathfrak{A} , \end{aligned} \tag{61}$$

[5] Notation: $(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)} = \Delta^2(x)$, $x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \otimes x_{(4)} = \Delta^3(x)$, etc., see [2]

where we have introduced the left adjoint (inner) action in $\mathfrak{U}$:

$$x \overset{\mathrm{ad}}{\triangleright} y = x_{(1)} y S(x_{(2)}), \quad \text{for } \forall x, y \in \mathfrak{U} . \tag{62}$$

Having extended the left $\mathfrak{U}$-module $\mathfrak{A}$ to $\mathfrak{A} \rtimes \mathfrak{U}$, we would now like to also extend the definition of the coaction of $\mathfrak{A}$ to $\mathfrak{A} \rtimes \mathfrak{U}$, making the quantized algebra of differential operators an $\mathfrak{A}$-bicomodule.

2.3. Bicovariant Calculus. In this subsection we would like to study the transformation properties of the differential operators in $\mathfrak{A} \rtimes \mathfrak{U}$ under left and right translations, i.e. the coactions ${}_{\mathfrak{A}}\Delta$ and $\Delta_{\mathfrak{A}}$ respectively. We will require,

$${}_{\mathfrak{A}}\Delta(by) = {}_{\mathfrak{A}}\Delta(b)\,{}_{\mathfrak{A}}\Delta(y) = \Delta(b)\,{}_{\mathfrak{A}}\Delta(y) \in \mathfrak{A} \otimes \mathfrak{A} \rtimes \mathfrak{U} , \tag{63}$$

$$\Delta_{\mathfrak{A}}(by) = \Delta_{\mathfrak{A}}(b)\Delta_{\mathfrak{A}}(y) = \Delta(b)\Delta_{\mathfrak{A}}(y) \in \mathfrak{A} \rtimes \mathfrak{U} \otimes \mathfrak{A} , \tag{64}$$

for all $b \in \mathfrak{A}$, $y \in \mathfrak{U}$, so that we are left only to define ${}_{\mathfrak{A}}\Delta$ and $\Delta_{\mathfrak{A}}$ on elements of $\mathfrak{U}$. We already mentioned that we would like to interpret $\mathfrak{U}$ as the algebra of *left invariant* vector fields; consequently we will try

$${}_{\mathfrak{A}}\Delta(y) = 1 \otimes y \in \mathfrak{A} \otimes \mathfrak{U} , \tag{65}$$

as a left coaction. It is easy to see that this coaction respects not only the left action (43) of $\mathfrak{U}$ on $\mathfrak{A}$,

$$\begin{aligned} {}_{\mathfrak{A}}\Delta(x \triangleright b) &= {}_{\mathfrak{A}}\Delta(b_{(1)})\langle x, b_{(2)} \rangle \\ &= 1 b_{(1)} \otimes b_{(2)} \langle x, b_{(3)} \rangle \\ &= x^{(1)'} b_{(1)} \otimes (x^{(2)} \triangleright b_{(2)}) \\ &=: {}_{\mathfrak{A}}\Delta(x) \triangleright {}_{\mathfrak{A}}\Delta(b) , \end{aligned} \tag{66}$$

but also the algebra structure (52) of $\mathfrak{A} \rtimes \mathfrak{U}$,

$$\begin{aligned} {}_{\mathfrak{A}}\Delta(x \cdot b) &= {}_{\mathfrak{A}}\Delta(b_{(1)})\langle x_{(1)}, b_{(2)} \rangle\, {}_{\mathfrak{A}}\Delta(x_{(2)}) \\ &= b_{(1)} 1 \otimes b_{(2)} \langle x_{(1)}, b_{(3)} \rangle x_{(2)} \\ &= 1 b_{(1)} \otimes b_{(2)} \langle x_{(1)}, b_{(3)} \rangle x_{(2)} \\ &= x^{(1)'} b_{(1)} \otimes (x^{(2)} \cdot b_{(2)}) \\ &=: {}_{\mathfrak{A}}\Delta(x) \cdot {}_{\mathfrak{A}}\Delta(b) . \end{aligned} \tag{67}$$

The right coaction, $\Delta_{\mathfrak{A}} : \mathfrak{U} \to \mathfrak{U} \otimes \mathfrak{A}$, is considerably harder to find. We will approach this problem by extending the commutation relation (52) for elements of $\mathfrak{U}$ with elements of $\mathfrak{A}$ to a generalized commutation relation for elements of $\mathfrak{U}$ with elements of $\mathfrak{A} \rtimes \mathfrak{U}$,

$$x \cdot by =: (by)^{(1)} \langle x_{(1)}, (by)^{(2)'} \rangle x_{(2)} , \tag{68}$$

for all $x, y \in \mathfrak{U}$, $b \in \mathfrak{A}$. In the special case $b = 1$ this states,

$$x \cdot y = y^{(1)} \langle x_{(1)}, y^{(2)'} \rangle x_{(2)}, \quad x, y \in \mathfrak{U} , \tag{69}$$

and gives an implicit definition of the right coaction $\Delta_{\mathfrak{A}}(y) \equiv y^{(1)} \otimes y^{(2)'}$ of $\mathfrak{A}$ on $\mathfrak{U}$. Let us check whether $\Delta_{\mathfrak{A}}$ defined in this way respects the left action (43) of $\mathfrak{U}$ on $\mathfrak{A}$:

$$\begin{aligned}
\langle z \otimes y, \Delta_{\mathfrak{A}}(x \triangleright b)\rangle &= \langle zy, x \triangleright b\rangle \\
&= \langle zy, b_{(1)}\rangle \langle x, b_{(2)}\rangle \\
&= \langle zyx, b\rangle \\
&= \langle z(x^{(1)}\langle y_{(1)}, x^{(2)'}\rangle y_{(2)}), b\rangle \\
&= \langle zx^{(1)} \otimes y_{(1)} \otimes y_{(2)}, b_{(1)} \otimes x^{(2)'} \otimes b_{(2)}\rangle \\
&= \langle zx^{(1)} \otimes y, b_{(1)} \otimes x^{(2)'} b_{(2)}\rangle \\
&= \langle z \otimes y, (x^{(1)} \triangleright b_{(1)}) \otimes x^{(2)'} b_{(2)}\rangle \\
&=: \langle z \otimes y, \Delta_{\mathfrak{A}}(x) \triangleright \Delta_{\mathfrak{A}}(b)\rangle \, , \qquad (70)
\end{aligned}$$

for all $x, y, z \in \mathfrak{U}$, $b \in \mathfrak{A}$. q.e.d. It is straightforward also to check that $\Delta_{\mathfrak{A}}$ respects the algebra structure of $\mathfrak{A} \bowtie \mathfrak{U}$ as well.

Remark. If we know a linear basis $\{e_i\}$ of $\mathfrak{U}$ and the dual basis $\{f^j\}$ of $\mathfrak{A} = \mathfrak{U}^*$, $\langle e_i, f^j\rangle = \delta_i^j$, then we can derive an explicit expression for $\Delta_{\mathfrak{A}}$ from (69):

$$\Delta_{\mathfrak{A}}(e_i) = (e_j \overset{\mathrm{ad}}{\triangleright} e_i) \otimes f^j \, , \qquad (71)$$

or equivalently, by linearity of $\Delta_{\mathfrak{A}}$:

$$\Delta_{\mathfrak{A}}(y) = (e_j \overset{\mathrm{ad}}{\triangleright} y) \otimes f^j, \quad y \in \mathfrak{U} \, . \qquad (72)$$

It is then easy to show that,

$$(\Delta_{\mathfrak{A}} \otimes \mathrm{id})\Delta_{\mathfrak{A}}(e_i) = (\mathrm{id} \otimes \Delta)\Delta_{\mathfrak{A}}(e_i) \, , \qquad (73)$$

$$(\mathrm{id} \otimes \varepsilon)\Delta_{\mathfrak{A}}(e_i) = e_i \, , \qquad (74)$$

proving that $\Delta_{\mathfrak{A}}$ satisfies the requirements of a coaction on $\mathfrak{U}$, and,

$$\Delta_{\mathfrak{A}}(e_i e_k) = \Delta_{\mathfrak{A}}(e_i)\Delta_{\mathfrak{A}}(e_k) \, , \qquad (75)$$

showing that $\Delta_{\mathfrak{A}}$ is an $\mathfrak{U}$-algebra homomorphism. Note however that $\Delta_{\mathfrak{A}}$ is in general not a $\mathfrak{U}$-Hopf algebra homomorphism.

In the next subsection we will describe a map, $\Phi\colon \mathfrak{A} \to \mathfrak{U}$, that is invariant under (right) coactions and can hence be used to find $\Delta_{\mathfrak{A}}$ on specific elements $\Phi(b) \in \mathfrak{U}$ in terms of $\Delta_{\mathfrak{A}}$ on $b \in \mathfrak{A}$: $\Delta_{\mathfrak{A}}(\Phi(b)) = (\Phi \otimes \mathrm{id})\Delta_{\mathfrak{A}}(b)$.

2.4. Invariant Maps and the Pure Braid Group. A basis of generators for the pure braid group B_n on n strands can be realized in $\mathfrak{U}$, or for that matter $U_q\mathfrak{g}$, as follows in terms of the universal $\mathcal{R}$:

$$\mathcal{R}^{21}\mathcal{R}^{12}, \; \mathcal{R}^{21}\mathcal{R}^{31}\mathcal{R}^{13}\mathcal{R}^{12} \equiv (\mathrm{id} \otimes \Delta)\mathcal{R}^{21}\mathcal{R}^{12}, \ldots ,$$

$$\mathcal{R}^{21} \ldots \mathcal{R}^{n1}\mathcal{R}^{1n} \ldots \mathcal{R}^{12} \equiv (\mathrm{id}^{(n-2)} \otimes \Delta)(\mathrm{id}^{(n-3)} \otimes \Delta) \ldots (\mathrm{id} \otimes \Delta)\mathcal{R}^{21}\mathcal{R}^{12},$$

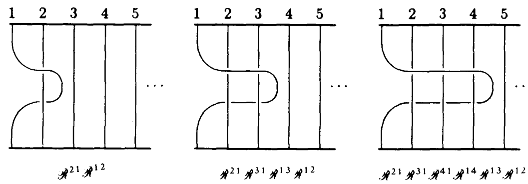

Fig. 1. Generators of the pure braid group

and their inverses; see Fig. 1 and ref. [7]. All polynomials in these generators are central in $\Delta^{(n-1)}\mathfrak{U} \equiv \{\Delta^{(n-1)}(x) | x \in \mathfrak{U}\}$; in fact we can take,

$$\operatorname{span}\{B_n\} := \{\mathscr{Z}_n \in \mathfrak{U}^{\hat{\otimes} n} | \mathscr{Z}_n \Delta^{(n-1)}(x) = \Delta^{(n-1)}(x)\mathscr{Z}_n, \text{ for } \forall x \in \mathfrak{U}\} , \tag{76}$$

as a definition.

Remark. Elements of span$\{B_n\}$ do not have to be written in terms of the universal $\mathscr{R}$, they also arise from central elements and coproducts of central elements. This is particularly important in cases where $\mathfrak{U}$ is not a quasitriangular Hopf algebra.

There is a map, $\Phi_n: \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{U}^{\otimes(n-1)} \subsetneq (\mathfrak{A} \rtimes \mathfrak{U})^{\otimes(n-1)}$, associated to each element of span$\{B_n\}$:

$$\Phi_n(a) := \mathscr{Z}_n \triangleright (a \otimes \mathrm{id}^{(n-1)}), \quad \text{with} \quad \mathscr{Z}_n \in \operatorname{span}\{B_n\}, \quad a \in \mathfrak{A} . \tag{77}$$

We will first consider the case $n = 2$. Let $\mathscr{Y} \equiv \mathscr{Y}_{1_i} \otimes \mathscr{Y}_{2_i}$ be an element of span$\{B_2\}$ and $\Phi(b) = \mathscr{Y} \triangleright (b \otimes \mathrm{id}) = b_{(1)} \langle \mathscr{Y}_{1_i}, b_{(2)} \rangle \mathscr{Y}_{2_i}$, for $b \in \mathfrak{A}$. We compute,

$$\begin{aligned} x \cdot \Phi(b) &= \Delta(x) \triangleright \Phi(b) \\ &= \Delta(x)\mathscr{Y} \triangleright (b \otimes \mathrm{id}) \\ &= \mathscr{Y}\Delta(x) \triangleright (b \otimes \mathrm{id}) \\ &= \mathscr{Y} \triangleright (x \cdot b) \\ &= \Phi(b_{(1)}) \langle x_{(1)}, b_{(2)} \rangle x_{(2)} , \end{aligned} \tag{78}$$

which, when compared with the *generalized* commutation relation (68), i.e.

$$x \cdot \Phi(b) = [\Phi(b)]^{(1)} \langle x_{(1)}, [\Phi(b)]^{(2)'} \rangle x_{(2)} , \tag{79}$$

gives,

$$\begin{aligned} \Delta_{\mathfrak{A}}(\Phi(b)) \equiv [\Phi(b)]^{(1)} \otimes [\Phi(b)]^{(2)'} &= \Phi(b_{(1)}) \otimes b_{(2)} \\ \Rightarrow \Delta_{\mathfrak{A}}(\Phi(b)) &= (\Phi \otimes \mathrm{id})\Delta_{\mathfrak{A}}(b) , \end{aligned} \tag{80}$$

as promised. However we are especially interested in the transformation properties of elements of $\mathfrak{U}$, so let us define,

$$Y_b := \langle \mathscr{Y}, b \otimes \mathrm{id} \rangle = \langle \mathscr{Y}_{1_i}, b \rangle \mathscr{Y}_{1_i} , \tag{81}$$

for $\mathscr{Y} \in \operatorname{span}(B_2)$, $b \in \mathfrak{A}$. From (64, 80) we find:

$$\Delta_{\mathfrak{A}}(Y_b) = Y_{b_{(2)}} \otimes S(b_{(1)}) b_{(3)} . \tag{82}$$

Here are a few important examples: For the simplest non-trivial example, $\mathscr{Y} \equiv \mathscr{R}^{21}\mathscr{R}^{12}$ and $b \equiv A^i{}_j$, we obtain the "reflection-matrix" $Y \in M_n(\mathfrak{U})$, which has been introduced before by other authors [9, 10, 11, 12] in connection with integrable models and the differential calculus on quantum groups,

$$\begin{aligned} Y^i{}_j &:= Y_{A^i{}_j} \\ &= \langle \mathcal{R}^{21}\mathcal{R}^{12}, A^i{}_j \otimes \mathrm{id}\rangle \\ &= (\langle \mathcal{R}^{31}\mathcal{R}^{23}, A \dot{\otimes} A \otimes \mathrm{id}\rangle)^i{}_j \\ &= (\langle \mathcal{R}^{21}, A \otimes \mathrm{id}\rangle \langle \mathcal{R}^{12}, A \otimes \mathrm{id}\rangle)^i{}_j \\ &= (L^+ S L^-)^i{}_j \,, \end{aligned} \tag{83}$$

with transformation properties,

$$A \to AA': \quad Y^i{}_j \to \Delta_{\mathfrak{A}}(Y^i{}_j) = Y^k{}_l \otimes S(A^i{}_k)A^l{}_j \equiv ((A')^{-1} Y A')^i{}_j \,, \tag{84}$$

$$A \to A'A: \quad Y^i{}_j \to {}_{\mathfrak{A}}\Delta(Y^i{}_j) = 1 \otimes Y^i{}_j \,. \tag{85}$$

The commutation relation (52) becomes in this case,

$$\begin{aligned} Y_2 A_1 &= L_2^+ S L_2^- A_1 \\ &= L_2^+ A_1 S L_2^- R_{21} \\ &= A_1 R_{12} L_2^+ S L_2^- R_{21} \\ &= A_1 R_{12} Y_2 R_{21} \,, \end{aligned} \tag{86}$$

where we have used (56), (57), and the associativity of the cross product (53); note that we did not have to use any explicit expression for the coproduct of Y. The matrix $\Phi(A^i{}_j) = A^i{}_k Y^k{}_j$ transforms exactly like A, as expected, and interestingly even satisfies the same commutation relation as A,

$$R_{12}(AY)_1(AY)_2 = (AY)_2(AY)_1 R_{12} \,, \tag{87}$$

as can be checked by direct computation.

The choice, $\mathcal{Y} \equiv (1 - \mathcal{R}^{21}\mathcal{R}^{12})/\lambda$, where $\lambda \equiv q - q^{-1}$, and again $b \equiv A^i{}_j$ gives us a matrix $X \in M_n(\mathfrak{U})$,

$$X^i{}_j := \langle (1 - \mathcal{R}^{21}\mathcal{R}^{12})/\lambda, A^i{}_j \otimes \mathrm{id}\rangle = ((I - Y)/\lambda)^i{}_j \,, \tag{88}$$

that we will encounter again in Sect. 4. X has the same transformation properties as Y and is the quantum analog of the classical matrix (4) of vector fields.

Finally, the particular choice $b \equiv \det_q A$ in conjunction with $\mathcal{Y} \equiv \mathcal{R}^{21}\mathcal{R}^{12}$ can serve as the definition of the quantum determinant of Y,

$$\mathrm{Det}\, Y := Y_{\det_q A} = \langle \mathcal{R}^{21}\mathcal{R}^{12}, \det_q A \otimes \mathrm{id}\rangle \,; \tag{89}$$

we will come back to this in the next section, but let us just mention that this definition of Det Y agrees with [13],

$$\begin{aligned} \det_q(AY) &= \det_q(A\langle \mathcal{R}^{21}\mathcal{R}^{12}, A \otimes \mathrm{id}\rangle) \\ &= \det_q A \langle \mathcal{R}^{21}\mathcal{R}^{12}, \det_q A \otimes \mathrm{id}\rangle \\ &= \det_q A \,\mathrm{Det}\, Y \,. \end{aligned} \tag{90}$$

Before we can consider maps Φ_n for $n > 2$ we need to extend the algebra and coalgebra structure of $\mathfrak{A} \rtimes\!\!\!\lhd \mathfrak{U}$ to $(\mathfrak{A} \rtimes\!\!\!\lhd \mathfrak{U})^{\otimes(n-1)}$. It is sufficient to consider $(\mathfrak{A} \rtimes\!\!\!\lhd \mathfrak{U})^{\otimes 2}$; all other cases follow by analogy. If we let

$$(a \otimes b)(x \otimes y) = ax \otimes by, \quad \text{for } \forall a, b \in \mathfrak{A},\, x, y \in \mathfrak{U}\,, \tag{91}$$

then it follows that

$$\begin{aligned} x \cdot a \otimes y \cdot b &= a_{(1)} \langle x_{(1)}, a_{(2)} \rangle x_{(2)} \otimes b_{(1)} \langle y_{(1)}, b_{(2)} \rangle y_{(2)} \\ &= (a \otimes b)_{(1)} \langle (x \otimes y)_{(1)}, (a \otimes b)_{(2)} \rangle (x \otimes y)_{(2)} \\ &= (x \otimes y) \cdot (a \otimes b), \quad \text{for } \forall a, b \in \mathfrak{A},\, x, y \in \mathfrak{U}\,, \end{aligned} \tag{92}$$

as expected from a tensor product algebra. If we coact with $\mathfrak{A}$ on $(\mathfrak{A} \rtimes\!\!\!\lhd \mathfrak{U})^{\otimes 2}$, or higher powers, we simply collect all the contributions of $\Delta_{\mathfrak{A}}$ from each tensor product space in one space on the right:

$$\begin{aligned} \Delta_{\mathfrak{A}}(ax \otimes by) &= (ax)^{(1)} \otimes (by)^{(1)} \otimes (ax)^{(2)'} (by)^{(2)'}\,, \\ &\text{for} \quad \forall a, b \in \mathfrak{A},\, x, y \in \mathfrak{U}\,. \end{aligned} \tag{93}$$

Let Φ_n be defined as in (77) and compute in analogy with (78):

$$\begin{aligned} \Delta^{(n-2)}(x) \cdot \Phi_n(b) &= \Delta^{(n-1)}(x) \triangleright \Phi_n(b) \\ &= \Delta^{(n-1)}(x) \mathscr{Z}_n \triangleright (b \otimes \mathrm{id}^{(n-1)}) \\ &= \mathscr{Z}_n \Delta^{(n-1)}(x) \triangleright (b \otimes \mathrm{id}^{(n-1)}) \\ &= \mathscr{Z}_n \triangleright (\Delta^{(n-2)}(x) \cdot b) \\ &= \Phi_n(b_{(1)}) \langle x_{(1)}, b_{(2)} \rangle x_{(2)} \otimes \cdots \otimes x_{(n)}\,. \end{aligned} \tag{94}$$

Compare this to the *generalized* commutation relation,

$$\Delta^{(n-2)}(x) \cdot \Phi_n(b) = [\Phi_n(b)]^{(1)} \langle x_1, [\Phi_n(b)]^{(2)'} \rangle x_{(2)} \otimes \cdots \otimes x_{(n)}\,, \tag{95}$$

to find:

$$\begin{aligned} &\Delta_{\mathfrak{A}}(\Phi_n(b)) \equiv [\Phi_n(b)]^{(1)} \otimes [\Phi_n(b)]^{(2)'} = \Phi_n(b_{(1)}) \otimes b_{(2)} \\ \Rightarrow\ &\Delta_{\mathfrak{A}}(\Phi_n(b)) = (\Phi_n \otimes \mathrm{id}) \Delta_{\mathfrak{A}}(b) \in (\mathfrak{A} \rtimes\!\!\!\lhd \mathfrak{U})^{\otimes(n-1)} \otimes \mathfrak{A}\,. \end{aligned} \tag{96}$$

Following the $n = 2$ case we also define $Z_{n,b} := \langle \mathscr{Z}_n, b \otimes \mathrm{id}^{(n-1)} \rangle$ and get:

$$\Delta_{\mathfrak{A}}(Z_{n,b}) = Z_{n,b_{(2)}} \otimes S(b_{(1)}) b_{(3)}\,. \tag{97}$$

As an example for $n = 3$ consider $\mathscr{Z}_3 \equiv \mathscr{R}^{21}\mathscr{R}^{31}\mathscr{R}^{13}\mathscr{R}^{12}$ and $b = A^i{}_j$, then

$$\begin{aligned} Z_{3,A^i{}_j} &= \langle \mathscr{R}^{21}\mathscr{R}^{31}\mathscr{R}^{13}\mathscr{R}^{12}, A^i{}_j \otimes \mathrm{id}^2 \rangle \\ &= \langle (\mathrm{id} \otimes \Delta)\mathscr{R}^{21}\mathscr{R}^{12}, A^i{}_j \otimes \mathrm{id}^2 \rangle \\ &= \Delta(Y^i{}_j)\,, \end{aligned} \tag{98}$$

is nothing but the coproduct of Y which, as we can see from Eq. (97), transforms exactly like Y itself. We see that $\Delta_{\mathfrak{A}}$ is actually a $\mathfrak{U}$-*co*algebra homomorphism on the subset $\{Y_b | b \in \mathfrak{A}\}$.

3. $\mathscr{R}$-Gymnastics

In this section we would like to study for the example of $Y \in M_N(\mathfrak{U})$ the matrix form of $\mathfrak{U}$ as introduced at the end of Sect. 1.2. Let us first derive commutation relations for Y from the quantum Yang–Baxter equation (QYBE): Combine the following two copies of the QYBE,

$$\mathscr{R}^{12}\mathscr{R}^{13}\mathscr{R}^{23} = \mathscr{R}^{23}\mathscr{R}^{13}\mathscr{R}^{12}, \quad \text{and} \quad \mathscr{R}^{21}\mathscr{R}^{31}\mathscr{R}^{32} = \mathscr{R}^{32}\mathscr{R}^{31}\mathscr{R}^{21} ,$$

resulting in,

$$\mathscr{R}^{21}\mathscr{R}^{31}\underline{\mathscr{R}^{32}\mathscr{R}^{12}\mathscr{R}^{13}}\mathscr{R}^{23} = \mathscr{R}^{32}\underline{\mathscr{R}^{31}\mathscr{R}^{21}\mathscr{R}^{23}}\mathscr{R}^{13}\mathscr{R}^{12} ,$$

and apply the QYBE to the underlined part to find,

$$\mathscr{R}^{21}(\mathscr{R}^{31}\mathscr{R}^{13})\mathscr{R}^{12}(\mathscr{R}^{32}\mathscr{R}^{23}) = (\mathscr{R}^{32}\mathscr{R}^{23})\mathscr{R}^{21}(\mathscr{R}^{31}\mathscr{R}^{13})\mathscr{R}^{12} ,$$

which, when evaluated on $\langle \cdot, A_1 \otimes A_2 \otimes \mathrm{id} \rangle$, gives:

$$R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12} . \tag{99}$$

3.1. Higher Representations and the •-Product. As was pointed out in Sect. 1.2, tensor product representations of $\mathfrak{U}$ can be constructed by combining A-matrices. This product of A-matrices defines a new product for $\mathfrak{U}$ which we will denote "•." The idea is to combine Y-matrices (or L^+, L^- matrices) in the same way as A-matrices to get higher dimensional matrix representations,

$$Y_1 \bullet Y_2 := \langle \mathscr{R}^{21}\mathscr{R}^{12}, A_1 A_2 \otimes \mathrm{id} \rangle , \tag{100}$$

$$L_1^+ \bullet L_2^+ := \langle \mathscr{R}^{21}, A_1 A_2 \otimes \mathrm{id} \rangle , \tag{101}$$

$$SL_1^- \bullet SL_2^- := \langle \mathscr{R}^{12}, A_1 A_2 \otimes \mathrm{id} \rangle . \tag{102}$$

Let us evaluate (100) in terms of the ordinary product in $\mathfrak{U}$,

$$\begin{aligned} Y_1 \bullet Y_2 &= \langle (\Delta \otimes \mathrm{id})\mathscr{R}^{21}\mathscr{R}^{12}, A_1 \otimes A_2 \otimes \mathrm{id} \rangle \\ &= \langle \mathscr{R}^{32}\mathscr{R}^{31}\mathscr{R}^{13}\mathscr{R}^{23}, A_1 \otimes A_2 \otimes \mathrm{id} \rangle \\ &= \langle (\mathscr{R}^{-1})^{12}\mathscr{R}^{31}\mathscr{R}^{13}\mathscr{R}^{12}\mathscr{R}^{32}\mathscr{R}^{23}, A_1 \otimes A_2 \otimes \mathrm{id} \rangle \\ &= R_{12}^{-1} Y_1 R_{12} Y_2 , \end{aligned} \tag{103}$$

where we have used,

$$\begin{aligned} \mathscr{R}^{32}\mathscr{R}^{31}\mathscr{R}^{13}\mathscr{R}^{23} &= ((\mathscr{R}^{-1})^{12}\underline{\mathscr{R}^{12})\mathscr{R}^{32}\mathscr{R}^{31}}\mathscr{R}^{13}\mathscr{R}^{23} \\ &= (\mathscr{R}^{-1})^{12}\mathscr{R}^{31}\underline{\mathscr{R}^{32}\mathscr{R}^{12}\mathscr{R}^{13}}\mathscr{R}^{23} \\ &= (\mathscr{R}^{-1})^{12}\mathscr{R}^{31}\mathscr{R}^{13}\mathscr{R}^{12}\mathscr{R}^{32}\mathscr{R}^{23} . \end{aligned}$$

Similar expressions for L^+ and SL^- are:

$$L_1^+ \bullet L_2^+ = L_2^+ L_1^+ , \tag{104}$$

$$SL_1^- \bullet SL_2^- = SL_1^- SL_2^- . \tag{105}$$

All matrices in $M_N(\mathfrak{U})$ satisfy by definition the same commutation relations (33) as A, when written in terms of the $\bullet$-product,

$$R_{12}L_1^+ \bullet L_2^+ = L_2^+ \bullet L_1^+ R_{12} \Leftrightarrow R_{12}L_2^+ L_1^+ = L_1^+ L_2^+ R_{12}\,, \tag{106}$$

$$R_{12}SL_1^+ \bullet SL_2^+ = SL_2^+ \bullet SL_1^+ R_{12} \Leftrightarrow R_{12}SL_1^+ SL_2^+ = SL_2^+ SL_1^+ R_{12}\,, \tag{107}$$

$$\begin{aligned} R_{12}Y_1 \bullet Y_2 = Y_2 \bullet Y_1 R_{12} &\Leftrightarrow R_{12}(R_{12}^{-1} Y_1 R_{12} Y_2) \\ &\quad = (R_{21}^{-1} Y_2 R_{21} Y_1) R_{12} \\ &\Leftrightarrow R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12}\,. \end{aligned} \tag{108}$$

Remark. Equations incorporating the $\bullet$-product are mathematically very similar to the expressions introduced in ref. [14] for braided linear algebras – our analysis was in fact motivated by that work – but on a conceptional level things are quite different: we are not dealing with a braided algebra with a braided multiplication but rather with a rule for combining matrix representations that turns out to be very useful, as we will see, to find conditions on the matrices in $M_N(\mathfrak{U})$ from algebraic relations for matrices in $M_N(\mathfrak{A})$.

3.2. Multiple $\bullet$-Products. We can define multiple (associative) $\bullet$-products by,

$$Y_1 \bullet Y_2 \bullet \ldots \bullet Y_k := \langle \mathscr{R}^{21}\mathscr{R}^{12}, A_1 A_2 \ldots A_k \otimes \mathrm{id} \rangle\,, \tag{109}$$

but this equation is not very useful to evaluate these multiple $\bullet$-products in practice. However, the "big" $\mathbf{R}$-matrix of Eq. (35) can be used to calculate multiple $\bullet$-products recursively: Let $\mathbf{Y}_I \equiv Y_{1'} \bullet Y_{2'} \bullet \ldots \bullet Y_{n'}$ and $\mathbf{Y}_{II} \equiv Y_1 \bullet Y_2 \bullet \ldots \bullet Y_m$, then:

$$\mathbf{Y}_I \bullet \mathbf{Y}_{II} = \mathbf{R}_{I,II}{}^{-1} \mathbf{Y}_I \mathbf{R}_{I,II} \mathbf{Y}_{II}\,; \tag{110}$$

compare to (36) and (103). The analog of Eq. (108) is also true:

$$\mathbf{R}_{I,II} \mathbf{Y}_I \bullet \mathbf{Y}_{II} = \mathbf{Y}_{II} \bullet \mathbf{Y}_I \mathbf{R}_{I,II} \tag{111}$$

$$\Leftrightarrow\ \mathbf{R}_{II,I} \mathbf{Y}_I \mathbf{R}_{I,II} \mathbf{Y}_{II} = \mathbf{Y}_{II} \mathbf{R}_{II,I} \mathbf{Y}_I \mathbf{R}_{I,II}\,. \tag{112}$$

The $\bullet$-product of three Y-matrices, for example, reads in terms of the ordinary multiplication in $\mathfrak{U}$ as,

$$\begin{aligned} Y_1 \bullet (Y_2 \bullet Y_3) &= \mathbf{R}_{1,(23)}^{-1} Y_1 \mathbf{R}_{1,(23)} (Y_2 \bullet Y_3) \\ &= (R_{12}^{-1} R_{13}^{-1} Y_1 R_{13} R_{12})(R_{23}^{-1} Y_2 R_{23}) Y_3\,. \end{aligned} \tag{113}$$

This formula generalizes to higher $\bullet$-products,

$$\mathbf{Y}_{(1 \cdots k)} \equiv \prod_{i=1}^{k} \bullet\, Y_i = \prod_{i=1}^{k} Y^{(i)}_{1 \cdots k},^{6} \text{ where :}$$

$$Y^{(i)}_{1 \cdots k} = \begin{cases} R_{i(i+1)}^{-1} R_{i(i+2)}^{-1} \cdot \ldots R_{ik}^{-1} Y_i R_{ik} \ldots R_{i(i+1)}, & 1 \leqq i < k, \\ Y_k,\ i = k. \end{cases} \tag{114}$$

---

[6] All products are ordered according to increasing multiplication parameter, e.g.

$$\prod_{i=1}^{k} \bullet\, Y_i \equiv Y_1 \bullet Y_2 \bullet \ldots \bullet Y_k$$

3.3. Quantum Determinants. Assuming that we have defined the quantum determinant $\det_q A$ of A in a suitable way – e.g. through use of the quantum ε_q-tensor, which in turn can be derived from the quantum exterior plane – we can then use the invariant maps Φ_n for $n = 2$ to find the corresponding expressions in $\mathfrak{U}$; see (89). Let us consider a couple of examples:

$$\mathrm{Det}\, Y := \langle \mathcal{R}^{21}\mathcal{R}^{12}, \det_q A \otimes \mathrm{id} \rangle \ , \tag{115}$$

$$\mathrm{Det}\, L^+ := \langle \mathcal{R}^{21}, \det_q A \otimes \mathrm{id} \rangle \ , \tag{116}$$

$$\mathrm{Det}\, SL^- := \langle \mathcal{R}^{12}, \det_q A \otimes \mathrm{id} \rangle \ . \tag{17}$$

Because of Eqs. (104) and (105) we can identify,

$$\mathrm{Det}\, L^+ \equiv \det_{q^{-1}} L^+, \quad \mathrm{Det}\, SL^- \equiv \det_q SL^- \ . \tag{118}$$

Properties of $\det_q A$, namely:

$$A \det_q A = \det_q A A \quad (\textit{central}) \ , \tag{119}$$

$$\Delta(\det_q A) = \det_q A \otimes \det_q A \quad (\textit{group-like}) \ , \tag{120}$$

translate into corresponding properties of "Det." For example, here is a short proof of the centrality of $\mathrm{Det}\, Y \equiv Y_{\det_q A}$ based on Eqs. (69) and (82):[7]

$$\begin{aligned} x Y_b &= Y_{b_{(2)}} \langle x_{(1)}, S(b_{(1)}) b_{(3)} \rangle x_{(2)}, \quad \forall x \in \mathfrak{U} \ ; \\ \Rightarrow x Y_{\det_q A} &= Y_{\det_q A} \langle x_{(1)}, S(\det_q A) \det_q A \rangle x_{(2)} \\ &= Y_{\det_q A} \langle x_{(1)}, 1 \rangle x_{(2)} \\ &= Y_{\det_q A} x, \quad \forall x \in \mathfrak{U}. \end{aligned} \tag{121}$$

The determinant of Y is central in the algebra, so its matrix representation must be proportional to the identity matrix,

$$\langle \mathrm{Det}\, Y, A \rangle = \kappa I \ , \tag{122}$$

with some proportionality constant κ that is equal to one in the case of special quantum groups; note that (122) is equivalent to:

$$\det_1 (R_{21} R_{12}) = \kappa I_{12} \ , \tag{123}$$

where $\det_1$ is the ordinary determinant taken in the first pair of matrix indices. We can now compute the commutation relation of $\mathrm{Det}\, Y$ with A [7],

$$\begin{aligned} (\mathrm{Det}\, Y) A &= A \langle \mathrm{Det}\, Y, A \rangle (\mathrm{Det}\, Y) \\ &= \kappa A (\mathrm{Det}\, Y) \ , \end{aligned} \tag{124}$$

showing that in the case of special quantum groups the determinant of Y is actually central in $\mathfrak{A} \rtimes \mathfrak{U}$.[8]

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[7] This proof easily generalizes to show the centrality of *any* (right) invariant $c \in \mathfrak{U}$, $\Delta_{\mathfrak{A}}(c) = c \otimes 1$, an example being the invaraint traces $\mathrm{tr}(D^{-1} Y^k)$ [6]

[8] The invariant traces are central only in $\mathfrak{U}$ because they are not group-like

Using (120) in the definition of Det Y,

$$\begin{aligned}\operatorname{Det} Y &= \langle \mathscr{R}^{21}\mathscr{R}^{12}, \det_q A \otimes \mathrm{id}\rangle \\ &= \langle \mathscr{R}^{31}\mathscr{R}^{23}, \Delta(\det_q A) \otimes \mathrm{id}\rangle \\ &= \langle \mathscr{R}^{31}\mathscr{R}^{23}, \det_q A \otimes \det_q A \otimes \mathrm{id}\rangle \\ &= \det_{q^{-1}} L^+ \cdot \det_q SL^- \,, \end{aligned} \tag{125}$$

we see that "Det Y" coincides with the definition of the determinant of Y given in [12].

A practical calculation of Det Y in terms of the matrix elements of Y starts from,

$$\det_q A\, \varepsilon_q^{i_1 \dots i_N} = \left(\prod_{k=1}^{N} A_k\right)^{i_1 \dots i_N}_{\qquad j_1 \dots j_N} \varepsilon_q^{j_1 \dots j_N} \,, \tag{126}$$

and uses Det $Y = \det_q{\bullet}\, Y$, i.e. the q-determinant with the $\bullet$-multiplication:

$$\operatorname{Det} Y\, \varepsilon_q^{i_1 \dots i_N} = \left(\prod_{k=1}^{N} \bullet\, Y_k\right)^{i_1 \dots i_N}_{\qquad j_1 \dots j_N} \varepsilon_q^{j_1 \dots j_N} \,. \tag{127}$$

Now we use Eq. (114) and get:

$$\operatorname{Det} Y\, \varepsilon_q^{i_1 \dots i_N} = \left(\prod_{k=1}^{N} Y^{(k)}_{1 \dots N}\right)^{i_1 \dots i_N}_{\qquad j_1 \dots j_N} \varepsilon_q^{j_1 \dots j_N}, \quad \text{where:}$$

$$Y^{(i)}_{1 \cdots k} = \begin{cases} R^{-1}_{i(i+1)} R^{-1}_{i(i+2)} \dots R^{-1}_{ik} Y_i R_{ik} \dots R_{i(i+1)}, & 1 \leqq i < k \,, \\ Y_k, \quad i = k \,. \end{cases} \tag{128}$$

It is interesting to see what happens if we use a matrix $T \in M_N(\mathfrak{A})$ with determinant $\det_q T = 1$, e.g. $T := A/(\det_q A)^{1/N}$, to define a matrix $Z \in M_N(\mathfrak{U})$ [7] in analogy to Eq. (83),

$$Z := \langle \mathscr{R}^{21}\mathscr{R}^{12}, T \otimes \mathrm{id}\rangle; \tag{129}$$

we find that Z is automatically of unit determinant:

$$\begin{aligned}\operatorname{Det} Z &:= \langle \mathscr{R}^{21}\mathscr{R}^{12}, \det_q T \otimes \mathrm{id}\rangle \\ &= \langle \mathscr{R}^{21}\mathscr{R}^{12}, 1 \otimes \mathrm{id}\rangle \\ &= (\varepsilon \otimes \mathrm{id})(\mathscr{R}^{21}\mathscr{R}^{12}) = 1 \,. \end{aligned} \tag{130}$$

3.4. An Orthogonality Relation for Y. If we want to consider only such transformations

$$x \mapsto {}_{\mathfrak{A}}\Delta(x) = A \dot{\otimes} x, \quad x \in \mathbb{C}_q^N,\ A \in M_N(\mathfrak{A}) \,, \tag{131}$$

of the quantum plane that leave lengths invariant, we need to impose an orthogonality condition on A; see [6]. Let $C \in M_N(k)$ be the appropriate metric and $x^T C x$ the length squared of x, then we find,

$$A^T C A = C \quad (\textit{orthogonality}) \,, \tag{132}$$

as the condition for an invariant length,

$$x^T C x \mapsto {}_{\mathfrak{A}}\Delta(x^T C x) = 1 \otimes x^T C x \,. \tag{133}$$

If we restrict A – and thereby $\mathfrak{A}$ – in this way we should also impose a corresponding orthogonality condition in $\mathfrak{U}$. Use of the $\bullet$-product makes this, as in the case of the quantum determinants, an easy task: we can simply copy the orthogonality condition for A and propose,

$$(L^+)^T \bullet CL^+ = C \;\Rightarrow\; L^+C^T(L^+)^T = C^T, \tag{134}$$

$$(SL^-)^T \bullet CSL^- = C \;\Rightarrow\; (SL^-)^T CSL^- = C, \tag{135}$$

$$Y^T \bullet CY = C, \quad \text{(matrix multiplication understood)}, \tag{136}$$

as orthogonality conditions in $\mathfrak{U}$. The first two equations were derived before in [6] in a different way. Let us calculate the condition on Y in terms of the ordinary multiplication in $\mathfrak{U}$,

$$\begin{aligned} C_{ij} &= Y^k{}_i \bullet C_{kl} Y^l{}_j \\ &= C_{kl}(Y_1 \bullet Y_2)^{kl}{}_{ij} \\ &= C_{kl}(R_{12}^{-1} Y_1 R_{12} Y_2)^{kl}{}_{ij} , \end{aligned} \tag{137}$$

or, using $C_{ij} = q^{(N-1)} R^{lk}{}_{ij} C_{kl}$:

$$C_{ij} = q^{(N-1)} C_{mn}(Y_1 R_{12} Y_2)^{nm}{}_{ij}. \tag{138}$$

Remark. Algebraic relations on the matrix elements of Y like the ones given in the previous two sections also give implicit conditions on $\mathcal{R}$; however we purposely did not specify $\mathcal{R}$, but rather formally assume its existence and focus on the numerical R-matrices that appear in all final expressions. Numerical R-matrices are known for most deformed Lie algebras of interest [6] and many other quantum groups. One could presumably use some of the techniques outlined in this article to actually derive relations for numerical R-matrices or even for the universal $\mathcal{R}$.

3.5. About the Coproduct of Y. It would be nice if we could express the coproduct of Y,

$$\Delta(Y) = \langle (\mathrm{id} \otimes \Delta)\mathcal{R}^{21}\mathcal{R}^{12}, A \otimes \mathrm{id} \rangle , \tag{139}$$

in terms of the matrix elements of the matrix Y itself, as it is possible for the coproducts of the matrices L^+ and L^-. Unfortunately, simple expressions have only been found in some special cases; see e.g. [15, 16, 17]. A short calculation gives,

$$\Delta(Y^i{}_j) = (\mathcal{R}^{-1})^{12}\,(1 \otimes Y^i{}_k)\mathcal{R}^{12}(Y^k{}_j \otimes 1) ; \tag{140}$$

this could be interpreted as some kind of braided tensor product [14, 18],

$$\Delta(Y^i{}_j) =: Y^i{}_k \tilde{\otimes}\, Y^k{}_j , \tag{141}$$

but for practical purposes one usually introduces a new matrix,

$$O_{(ij)}{}^{(kl)} := (L^+)^i{}_k\, S(L^-)^l{}_j \in M_{N\times N}(\mathfrak{U}) , \tag{142}$$

such that,

$$\Delta(Y_A) = O_A{}^B \otimes Y_B , \tag{143}$$

where capital letters stand for pairs of indices. The coproduct of $X^i{}_j = (I - Y)^i{}_j/\lambda$ is in this notation:

$$\Delta(X_A) = X_A \otimes 1 + O_A{}^B \otimes X_B \,. \tag{144}$$

We will only use $O_A{}^B$ in formal expressions involving the coproduct of Y. It will usually not show up in any practical calculation, because commutation relation (86) already implicitly contains $\Delta(Y)$ and all inner products of Y with strings of A-matrices following from it.

4. Quantum Lie Algebras

Classically the (left) adjoint actions of the generators χ_i of a Lie algebra $\mathfrak{g}$ on each other are given by the commutators,

$$\chi_i \overset{\mathrm{ad}}{\triangleright} \chi_j = [\chi_i, \chi_j] = \chi_k f_i{}^k{}_j \,, \tag{145}$$

expressible in terms of the structure constants $f_i{}^k{}_j$, whereas the (left) adjoint action of elements of the corresponding Lie group $\mathfrak{G}$ is given by conjugation,

$$h \overset{\mathrm{ad}}{\triangleright} g = hgh^{-1}, \quad h, g \in \mathfrak{G} \,. \tag{146}$$

Both formulas generalize in Hopf algebra language to the same expression,

$$\chi_i \overset{\mathrm{ad}}{\triangleright} \chi_j = \chi_{i_{(1)}} \chi_j S(\chi_{i_{(2)}}), \quad \text{with:} \quad S(\chi) = -\chi \,,$$

$$\Delta(\chi) \equiv \chi_{(1)} \otimes \chi_{(2)} = \chi \otimes 1 + 1 \otimes \chi, \quad \text{for } \forall \chi \in \mathfrak{g} \,, \tag{147}$$

$$h \overset{\mathrm{ad}}{\triangleright} g = h_{(1)} g S(h_{(2)}), \quad \text{with:} \quad S(h) = h^{-1} \,,$$

$$\Delta(h) \equiv h_{(1)} \otimes h_{(2)} = h \otimes h, \quad \text{for } \forall h \in \mathfrak{G} \,, \tag{148}$$

and agree with our formula (62) for the (left) adjoint action in $\mathfrak{U}$. We can derive two *generalized Jacobi identities* for double adjoint actions,

$$\begin{aligned} x \overset{\mathrm{ad}}{\triangleright} (y \overset{\mathrm{ad}}{\triangleright} z) &= (xy) \overset{\mathrm{ad}}{\triangleright} z \\ &= ((x_{(1)} \overset{\mathrm{ad}}{\triangleright} y)\, x_{(2)}) \overset{\mathrm{ad}}{\triangleright} z \\ &= (x_{(1)} \overset{\mathrm{ad}}{\triangleright} y) \overset{\mathrm{ad}}{\triangleright} (x_{(2)} \overset{\mathrm{ad}}{\triangleright} z) \,, \end{aligned} \tag{149}$$

and,

$$\begin{aligned} (x \overset{\mathrm{ad}}{\triangleright} y) \overset{\mathrm{ad}}{\triangleright} z &= (x_{(1)} y S(x_{(2)})) \overset{\mathrm{ad}}{\triangleright} z \\ &= x_{(1)} \overset{\mathrm{ad}}{\triangleright} (y \overset{\mathrm{ad}}{\triangleright} (S(x_{(2)}) \overset{\mathrm{ad}}{\triangleright} z)) \,. \end{aligned} \tag{150}$$

Both expressions become the ordinary Jacobi identity in the classical limt and they are not independent: Using the fact that $\stackrel{\text{ad}}{\triangleright}$ is an action they imply each other.

In the following we would like to derive the quantum version of (145) with "quantum commutator" and 'quantum structure constants." The idea is to utilize the (passive) transformations that we have studied in great detail in Sects. 2.3 and 2.4 to find an expression for the corresponding active transformations or actions. The effects of passive transformations are the inverse of active transformations, so here is the inverse or right adjoint action for a group:

$$h^{-1} \stackrel{\text{ad}}{\triangleright} g = g \stackrel{\text{ad}}{\triangleleft} h = S(h_{(1)})\, g h_{(2)} \,. \tag{151}$$

This gives rise to a (right) adjoint coaction in $\mathrm{Fun}(\mathfrak{G})$:

$$A \mapsto S(A')\,AA', \quad \text{i.e.}$$

$$\mathrm{Fun}(\mathfrak{G}_q) \ni A^i{}_j \mapsto A^k{}_l \otimes S(A^i{}_k)\,A^l{}_j \in \mathrm{Fun}(\mathfrak{G}_q) \otimes \mathrm{Fun}(\mathfrak{G}_q)\,; \tag{152}$$

here we have written "$\mathrm{Fun}(\mathfrak{G}_q)$" instead of "$\mathrm{Fun}(\mathfrak{G})$" because the coalgebra of $\mathrm{Fun}(\mathfrak{G}_q)$ is in fact the same undeformed coalgebra as the one of $\mathrm{Fun}(\mathfrak{G})$. In Sect. 2.4 we saw that the Y-matrix has particularly nice transformation properties:

$$A \mapsto S(A')\,A: \quad Y \mapsto 1 \otimes Y\,,$$

$$A \mapsto AA': \quad Y \mapsto S(A')\,YA'\,.$$

It follows that:

$$A \mapsto S(A')\,AA': \quad Y^i{}_j \mapsto Y^k{}_l \otimes S(A^i{}_k)\,A^l{}_j\,. \tag{153}$$

This is the "unspecified" adjoint right *coaction* for Y; we recover the "specific" left adjoint *action*,

$$x \stackrel{\text{ad}}{\triangleright} Y^i{}_j = x_{(1)}\, Y^i{}_j S(x_{(2)})\,,$$

of an arbitrary $x \in U_q\mathfrak{g}$ by evaluating the second factor of the adjoint coaction (153) on x:

$$x \stackrel{\text{ad}}{\triangleright} Y^i{}_j = Y^k{}_l \langle x, S(A^i{}_k)\,A^l{}_j \rangle, \quad \text{for } \forall x \in U_q\mathfrak{g}\,. \tag{154}$$

At the expense of intuitive insight we can alternatively derive a more general formula directly from Eqs. (62), (69), and (82),

$$\begin{aligned} x \stackrel{\text{ad}}{\triangleright} Y_b &= x_{(1)}\, Y_b S(x_{(2)}) \\ &= (Y_b)^{(1)} \langle x_{(1)}, (Y_b)^{(2)'} \rangle\, x_{(2)}\, S(x_{(3)}) \\ &= (Y_b)^{(1)} \langle x_{(1)}, (Y_b)^{(2)'} \rangle\, \varepsilon(x_{(2)}) \\ &= (Y_b)^{(1)} \langle x, (Y_b)^{(2)'} \rangle \\ &= Y_{b_{(2)}} \langle x, S(b_{(1)})\, b_{(3)} \rangle\,; \end{aligned} \tag{155}$$

note the appearance of the (right) adjoint coaction [1] in Fun$(\mathfrak{G}_q)$,

$$\Delta^{\mathrm{Ad}}(b) = (b_{(2)} \otimes S(b_{(1)})\, b_{(3)}\ , \tag{156}$$

in this formula.

We have found exactly what we were looking for in a *quantum Lie algebra*; the adjoint action (154) or (155) – which is the generalization of the classical commutator – of elements of $U_q\mathrm{g}$ on elements in a certain subset of $U_q\mathrm{g}$ evaluates to a *linear* combination of elements of that subset. So we do not really have to use the whole universal enveloping algebra when dealing with quantum groups but can rather consider a subset spanned by elements of the general form $Y_b \equiv \langle \mathscr{Y}, b \otimes \mathrm{id} \rangle$, $\mathscr{Y} \in \mathrm{span}\{B_2\}$; we will call this subset the "quantum Lie algebra" g_q of the quantum group. Now we need to find a basis of generators with the right classical limit.

Let us first evaluate (154) in the case where x is a matrix element of Y. We introduce the shorthand,

$$\mathbb{A}^{(kl)}{}_{(ij)} \equiv S(A^i{}_k)\, A^l{}_j\ , \tag{157}$$

for the adjoint representation and find,

$$Y_A \overset{\mathrm{ad}}{\triangleright} Y_B = Y_C \langle Y_A, \mathbb{A}^C{}_B \rangle\ , \tag{158}$$

where, again, capital letters stand for pairs of indices. The evaluation of the inner product $\langle Y_A, \mathbb{A}^C{}_B \rangle =: C_A{}^C{}_B$ is not hard even though we do not have an explicit expression for the coproduct of Y; we simply use the commutation relation (86) of Y with A and the left and right vacua defined in Sect. 2.2:

$$\begin{aligned}
\langle Y_1, SA_2^T A_3 \rangle &= \langle Y_1 SA_2^T A_3 \rangle \\
&= \langle SA_2^T (R_{21}^{-1})^{T_2}\, Y_1 A_3 (R_{12}^{T_2})^{-1} \rangle \\
&= \langle SA_2^T (R_{21}^{-1})^{T_2}\, A_3 R_{31}\, Y_1\, R_{13} (R_{12}^{T_2})^{-1} \rangle \\
&= (R_{21}^{-1})^{T_2} R_{31} R_{13} (R_{12}^{T_2})^{-1}, \\
\Rightarrow C_{(ij)}{}^{(kl)}{}_{(mn)} &= ((R_{21}^{-1})^{T_2} R_{31} R_{13} (R_{12}^{T_2})^{-1})^{ikl}{}_{jmn}\ .
\end{aligned} \tag{159}$$

The matrix Y becomes the identity matrix in the classical limit, so $X \equiv (I - Y)/\lambda$ is a better choice; it has the additional advantage that it has zero counit and its coproduct (144) resembles the coproduct of classical differential operators and therefore allows us to write the adjoint action (147) as a *generalized commutator*:

$$\begin{aligned}
Y_A \overset{\mathrm{ad}}{\triangleright} X_B &= Y_{A(1)} X_B S(Y_{A(2)}) \\
&= O_A{}^D X_B S(Y_D) \\
&= O_A{}^D X_B S(O_D{}^E)\, \underbrace{(I_E - \lambda X_E + \lambda X_E)}_{Y_E} \\
&= Y_A X_B + (O_A{}^E \overset{\mathrm{ad}}{\triangleright} X_B) \lambda X_E \\
&= Y_A X_B + \lambda \langle O_A{}^E, \mathbb{A}^D{}_B \rangle X_D X_E\ , \\
&\quad \text{with: } O_D{}^E I_E = Y_D,\ S(O_D{}^E)\, Y_E = I_D\ ; \\
\Rightarrow X_A \overset{\mathrm{ad}}{\triangleright} X_B &= X_A X_B - \langle O_A{}^E, \mathbb{A}^D{}_B \rangle X_D X_E\ .
\end{aligned} \tag{160}$$

Following the notation of reference [19] we introduce the $N^4 \times N^4$ matrix,

$$\hat{\mathbb{R}}^{DE}{}_{AB} := \langle O_A{}^E, \mathbb{A}^D{}_B \rangle\,, \tag{161}$$

$$\hat{\mathbb{R}}^{(mn)\,(kl)}{}_{(ij)\,(pq)} = ((R_{31}{}^{-1})^{T_3} R_{41} R_{24} (R_{23}{}^{T_3})^{-1})^{ilmn}{}_{kjpq}\,, \tag{162}$$

but realize when considering the above calculation that $\mathbb{R}$ is not the "R-matrix in the adjoint representation" – that would be $\langle \mathscr{R}, \mathbb{A}^E{}_A \otimes \mathbb{A}^D{}_B \rangle$ – but rather the R-matrix for the braided commutators of $\mathfrak{g}_q$, giving the commutation relations of the generators a form resembling an (inhomogeneous) quantum plane.

Now we can write down the generalized Cartan equations of a quantum Lie algebra $\mathfrak{g}_q$:

$$X_A \overset{\mathrm{ad}}{\triangleright} X_B = X_A X_B - \hat{\mathbb{R}}^{DE}{}_{AB} X_D X_E = X_C f_A{}^C{}_B\,, \tag{163}$$

where, from Eq. (159),

$$f_A{}^C{}_B = (I_A I^C I_B - C_A{}^C{}_B)/\lambda\,. \tag{164}$$

Equation (163) is strictly only valid for systems of N^2 generators with an $N^2 \times N^2$ matrix $\hat{\mathbb{R}}$ because $X \in M_N(\mathfrak{g}_q)$ in our construction. Some of these N^2 generators and likewise some of the matrix elements of $\hat{\mathbb{R}}$ could of course be zero, but let us anyway consider the more general case of Eq. (155). We will assume a set of n generators X_{b_i} corresponding to a set of n linearly independent functions $\{b_i \in \mathrm{Fun}(\mathfrak{G}_q) \mid i = 1, \ldots, n\}$ and an element of the pure braid group $\mathscr{X} \in \mathrm{span}(B_2)$ via:

$$X_{b_i} = \langle \mathscr{X}, b_i \otimes \mathrm{id} \rangle\,. \tag{165}$$

We will usually require that all generators have vanishing counit. A sufficient condition on the b_i's ensuring linear closure of the generators X_{b_i} under the adjoint action (155) is,

$$\Delta^{\mathrm{Ad}}(b_i) = b_j \otimes \mathbb{M}^j{}_i + k_l \otimes k_i^l\,, \tag{166}$$

where $\mathbb{M}^j{}_i \in M_n(\mathrm{Fun}(\mathfrak{G}_q))$ and $k_l, k_i^l \in \mathrm{Fun}(\mathfrak{G}_q)$ such that $\langle \mathscr{X}, k_l \otimes \mathrm{id} \rangle = 0$. The generators will then transform like,

$$\Delta_{\mathfrak{A}}(X_{b_i}) = X_{b_j} \otimes \mathbb{M}^j{}_i\,; \tag{167}$$

from $(\Delta_{\mathfrak{A}} \otimes \mathrm{id})\Delta_{\mathfrak{A}}(X_{b_i}) = (\mathrm{id} \otimes \Delta)\Delta_{\mathfrak{A}}(X_{b_i})$ and $(\mathrm{id} \otimes \varepsilon)\Delta_{\mathfrak{A}}(X_{b_i}) = X_{b_i}$ immediately follows[9] $\Delta(\mathbb{M}) = \mathbb{M} \dot{\otimes} \mathbb{M}$, $\varepsilon(\mathbb{M}) = I$ and consequently $S(\mathbb{M}) = \mathbb{M}^{-1}$. $\mathbb{M}$ is the *adjoint matrix representation.* We find,

$$X_{b_k} \overset{\mathrm{ad}}{\triangleright} X_{b_i} = X_{b_j} \langle X_{b_k}, \mathbb{M}^j{}_i \rangle\,, \tag{168}$$

as a generalization of (163) with structure constants $f_k{}^j{}_i = \langle X_{b_k}, \mathbb{M}^j{}_i \rangle$. Whether $X_{b_k} \overset{\mathrm{ad}}{\triangleright} X_{b_i}$ can be reexpressed as a deformed commutator should in general depend on the particular choice of $\mathscr{X}$ and $\{b_i\}$.

[9] This assumes that the X_{b_i}'s are linearly independent

Equations (153) and (157)–(164) apply directly to $GL_q(N)$ and $SL_q(N)$ and other quantum groups in matrix form with (numerical) R-matrices. Such quantum groups have been studied in great detail in the literature; see e.g. [6, 19, 20] and references therein. In the next subsection we would like to discuss the 2-dimensional quantum euclidean algebra as an example that illustrates some subtleties in the general picture.

4.1. Bicovariant generators for $e_q(2)$. In [21] Woronowicz introduced the functions on the deformed $E_q(2)$, the corresponding algebra $U_q(e(2))$ was explicitly constructed in [22]; here is a short summary: m, $\bar{m}$ and $\theta = \bar{\theta}$ are generating elements of the Hopf algebra $\mathrm{Fun}(E_q(2))$, which satisfy:

$$\begin{aligned} m\bar{m} &= q^2\bar{m}m, \quad e^{i\theta}m = q^2 m e^{i\theta}, \quad e^{i\theta}\bar{m} = q^2\bar{m}e^{i\theta}\,, \\ \Delta(m) &= m\otimes 1 + e^{i\theta}\otimes m, \quad \Delta(\bar{m}) = \bar{m}\otimes 1 + e^{-i\theta}\otimes\bar{m}\,, \\ \Delta(e^{i\theta}) &= e^{i\theta}\otimes e^{i\theta}, \quad S(m) = -e^{-i\theta}m, \quad S(\bar{m}) = -e^{i\theta}\bar{m}\,, \\ S(\theta) &= -\theta, \quad \varepsilon(m) = \varepsilon(\bar{m}) = \varepsilon(\theta) = 0\,. \end{aligned} \tag{169}$$

$\mathrm{Fun}(E_q(2))$ coacts on the complex coordinate function z of the euclidean plane as $\Delta_{\mathfrak{A}}(z) = z\otimes e^{i\theta} + 1\otimes m$; i.e. θ corresponds to rotations, m to translations. The dual Hopf algebra $U_q(e(2))$ is generated by $J = \bar{J}$ and $P_\pm = \overline{P_\mp}$ satisfying:

$$\begin{aligned} &[J, P_\pm] = \pm P_\pm, \quad [P_+, P_-] = 0\,, \\ &\Delta(P_\pm) = P_\pm\otimes q^J + q^{-J}\otimes P_\pm, \quad \Delta(J) = J\otimes 1 + 1\otimes J\,, \\ &S(P_\pm) = -q^{\pm 1}P_\pm, \quad S(J) = -J, \quad \varepsilon(P_\pm) = \varepsilon(J) = 0\,. \end{aligned} \tag{170}$$

The duality between $\mathrm{Fun}(E_q(2))$ and $U_q(e(2))$ is given by:

$$\begin{aligned} &\langle P_+{}^k P_-{}^l q^{mJ}, e^{i\theta a}m^b\bar{m}^c\rangle \\ &= (-1)^l q^{-1/2(k-l)(k+l-1)+l(k-1)} q^{(k+l-m)a}[k]_q!\,[l]_{q^{-1}}!\,\delta_{lb}\delta_{kc}\,, \end{aligned} \tag{171}$$

where $k, l, b, c \in \mathbb{N}_0$, $m, a \in \mathbb{Z}$, and,

$$[x]_q! = \prod_{y=1}^{x}\frac{q^{2y}-1}{q^2-1}, \quad [0]_q! = [1]_q! = 1\,.$$

Note that P_+P_- is central in $U_q(e(2))$; i.e. it is a Casimir operator. $U_q(e(2))$ does not have a (known) universal $\mathcal{R}$, so we have to construct an element $\mathcal{X}$ of span (B_2) from the Casimir P_+P_-:

$$\begin{aligned} \mathcal{X} &:= \frac{1}{q-q^{-1}}\{\Delta(P_+P_-) - (P_+P_-\otimes 1)\} \\ &= \frac{1}{q-q^{-1}}\{P_+P_-\otimes(q^{2J}-1) + P_+q^{-J}\otimes q^J P_- \\ &\quad + P_-q^{-J}\otimes q^J P_+ + q^{-2J}\otimes P_+P_-\}\,. \end{aligned} \tag{172}$$

$\mathcal{X}$ commutes with $\Delta(x)$ for all $x \in U_q(e(2))$ because P_+P_- is a Casimir. We introduced the second term $(P_+P_-\otimes 1)$ in $\mathcal{X}$ to ensure $(\mathrm{id}\otimes\varepsilon)\mathcal{X} = 0$ so that we are guaranteed to get bicovariant generators with zero counit. Now we need a set of functions which transform like (166). A particular simple choice is $a_0 := e^{i\theta} - 1$, $a_+ := m$, and $a_- := e^{i\theta}\bar{m}$. These functions transform under the adjoint coaction as:

$$\Delta^{\mathrm{Ad}}(a_0, a_+, a_-) = (a_0, a_+, a_-)\dot{\otimes}\begin{pmatrix} 1 & e^{-i\theta}m & -e^{i\theta}\bar{m} \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix}. \tag{173}$$

Unfortunately we notice that a_0 and thereby X_{a_0} are invariant, forcing X_{a_0} to be a Casimir independent of the particular choice of $\mathscr{X}$. Indeed we find $X_{a_0} = qP_+P_-$, $X_{a_+} = -\sqrt{q}/(q - q^{-1})q^J P_+$, and $X_{a_-} = q/(q - q^{-1})q^J P_-$, making this an incomplete choice of bicovariant generators for $e_q(2)$. An ansatz with four functions $b_0 := (e^{i\theta} - 1)^2$, $b_1 := -me^{i\theta}\bar{m}$, $b_+ := -(e^{i\theta} - 1)m$, and $b_- := q^{-2}(e^{i\theta} - 1)e^{i\theta}\bar{m}$ gives:

$$\begin{aligned} &\Delta^{\mathrm{Ad}}(b_0, b_1, b_+, b_-) \\ &= (b_0, b_1, b_+, b_-)\dot{\otimes}\begin{pmatrix} 1 & \bar{m}m & -e^{-i\theta}m & -q^{-2}e^{i\theta}\bar{m} \\ 0 & 1 & 0 & 0 \\ 0 & -\bar{m} & e^{-i\theta} & 0 \\ 0 & -m & 0 & e^{i\theta} \end{pmatrix}. \end{aligned} \tag{174}$$

The corresponding bicovariant generators are:

$$\begin{aligned} &X_{b_0} = q(q^2 - 1)P_+P_-, \quad X_{b_1} = (q - q^{-1})^{-1}(q^{2J} - 1), \\ &X_{b_+} = q^J P_+, \quad X_{b_-} = qq^J P_-. \end{aligned} \tag{175}$$

In the classical limit $(q \to 1)$ these generators become "zero," J, P_+ and P_- respectively. The same generators and their transformation properties can alternatively be obtained by contracting the bicovariant calculus on $SU_q(2)$. The commutation relations of the generators follow directly from (170), their adjoint actions are calculated from (168), (171), and (174) and finally the commutation relations of the generators with the functions can be obtained from (52), (169) and (170).

5. Conclusion

In the first two sections we generalized the classical concept of an algebra of differential operators to quantum groups, combining the "functions on the quantum group" $\mathrm{Fun}(\mathfrak{G}_q)$ and the universal enveloping algebra $U_q\mathfrak{g}$ into a single algebra. This structure, called the cross product $\mathrm{Fun}(\mathfrak{G}_q) \rtimes U_q\mathfrak{g}$, is a Hopf algebra version of the classical semidirect product of two algebras. We proceeded by extending the natural coaction of $\mathrm{Fun}(\mathfrak{G}_q)$, i.e. its coproduct, to the combined algebra $\mathrm{Fun}(\mathfrak{G}_q) \rtimes U_q\mathfrak{g}$, introducing a left and right $\mathrm{Fun}(\mathfrak{G}_q)$-coaction on $U_q\mathfrak{g}$. These coactions are to be interpreted as giving the transformation properties of the elements of $U_q\mathfrak{g}$. In our construction we choose all elements of $U_q\mathfrak{g}$ to be left invariant (${}_{\mathfrak{A}}\Delta(x) = 1 \otimes x$) and give a general formula (72) for the right coaction $\Delta_{\mathfrak{A}}$. The problem with the right coaction is that it is hard to compute as it will generally give infinite power series in the generators of $U_q\mathfrak{g}$ and $\mathrm{Fun}(\mathfrak{G}_q)$. At the end of Sect. 2 we showed how a large subset of $U_q\mathfrak{g}$ with "nice" transformation properties arises via the use of invariant maps from $\mathrm{Fun}(\mathfrak{G}_q)$ to $U_q\mathfrak{g}$, which are given by polynomials in elements of the pure braid group. In this article we were not interested in a possible extension of the $U_q\mathfrak{g}$-coaction from $U_q\mathfrak{g}$ to $\mathrm{Fun}(\mathfrak{G}_q) \rtimes U_q\mathfrak{g}$. Such a program would likely lead to braided linear algebras as they are considered in [12]. In Sect. 3 we utilized the invariant maps to translate (matrix) expressions known for $\mathrm{Fun}(\mathfrak{G}_q)$ to corresponding relations in $U_q\mathfrak{g}$ that would be very hard to obtain directly. The subset of elements of $U_q\mathfrak{g}$ that we obtained through the use of invariant maps turns out to close onto itself under adjoint actions and this leads naturally to the introduction of a class of generalized Lie algebras in Sect. 4. The adjoint action in $U_q\mathfrak{g}$ is directly related to the transformation properties of its elements and so it comes as no surprise that a finite set of *bicovariant* generators can generate a closed quantum Lie algebra. It is the *adjoint action* that is important for physical applications as e.g. deformed gauge theories. A general feature of these quantum Lie algebras is that they typically contain more generators than their classical counterparts. These extra generators are Casimir operators that only decouple in the classical limit ($q \to 1$) as we illustrated in the example of the 2-dimensional quantum euclidean group.

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