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Singular Measures in Circle Dynamics

Jacek Graczyk[1] **and Grzegorz Świątek**[2]*

[1] Institute of Mathematics, Warsaw University, ul.Banacha 2, 02-097 Warszawa, Poland
[2] Institute for Mathematical Sciences, SUNY at Stony Brook, Stony Brook, NY 11794, USA

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Abstract. Critical circle homeomorphisms have an invariant measure totally singular with respect to the Lebesgue measure. We prove that singularities of the invariant measure are of Hölder type. The Hausdorff dimension of the invariant measure is less than 1 but greater than 0.

1. Preliminaries

1.1. Discussion of the Results. The long time behavior of nonlinear dynamical systems can be often characterized by means of invariant measures. A variety of "multifractal formalisms" have been developed recently to study statistical properties of singular measures (see [4, 2] for more details) which appear as a natural description of many physical phenomena. One of the characteristic quantities describing the multifractal structure of a singular measure μ is a singularity spectrum $g(\alpha)$ which is usually defined in an informal way (see [4, 2] and many others) as follows:

Cover the support of μ by small boxes L_i of size l. Then define the singularity strength α_i of μ in the i^{th} box by the relation:

$$\mu(L_i) \sim l^{\alpha_i} .$$

We count the number of boxes $N(\alpha)$ where μ has singularity strength between α and $\alpha + d\alpha$ (whatever that is to mean). Then $g(\alpha)$ is defined by the requirement that

$$N(\alpha) \sim l^{g(\alpha)} .$$

Unfortunately, many "multifractal formalisms" suffer from mathematical ambiguities (see [2] for a fuller discussion of this problem; for example, is $g(\alpha)$ a Hausdorff or a box dimension or something else?) even if they provide qualitative information on a given dynamical system. In the present paper we would like to propose a method of describing the dynamics of critical circle homeomorphisms.

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Our method is more general than the method relying on the scalings exponents (see [3]), and on the other hand, mathematically rigorous unlike the "multifractal formalism" in its present shape.

Description of the method. Unlike typical smooth diffeomorphisms, which were treated in [5], all critical circle homeomorphisms have singular invariant measures (see [7]). Moreover, it turns out that the unique normalized invariant measure is always completely singular with respect to the Lebesgue measure. We introduce two singularity exponents, the lower and the upper one, to measure the increments of distribution of the invariant measure in the logarithmic scale. We study these exponents with respect to two natural measures on the circle: the invariant measure μ and the Lebesgue measure λ. By ergodicity, these exponents are constants on sets of full measure μ or λ, respectively.

Our main achievement is to prove uniform bounds for the exponents in the class of circle maps with a critical point of polynomial type and an irrational rotation number of constant type.[1]

Universality. We should mention here that for critical maps with all critical points of polynomial type and rotation numbers of algebraical degree 2, the *universality conjecture* implies that the upper and the lower exponents coincide. The reader may consult [11] for more information about circle map universality and its consequences. There are strong computer-based arguments in favor of the conjecture (see [8], also for the list of other references). However, in the absence of a definite rigorous proof, we continue to regard the conjecture as just that, and will refrain from using it in our discussion.

Another important quantity which describes a singular measure μ is the Hausdorff dimension $\mathrm{HD}(\mu)$ of the measure theoretical support (i.e., the infimum of the dimensions of the sets of the full measure). Using the singularity exponents we immediately obtain universal bounds on $\mathrm{HD}(\mu)$ in our class of circle maps.

Hausdorff dimension. The renormalization group analysis applied to study high iterates of circle maps with special rotation numbers (like the golden mean) lead to several universality conjectures (see for example [4, 8, 11]).

We state one which is certainly true provided the *golden mean universality conjecture* holds.

Conjecture 1. $\mathrm{HD}(\mu)$ *is constant in any topological conjugacy class of cubic critical homeomorphisms with rotation number of algebraical degree* 2.

An intriguing question remains about universal properties for more general irrationals. We think that the same conjecture should be true for any irrational rotation number, even of Liouville type. However, the evidence for that is scarce and we leave this merely as an interesting open question.

1.2. Introduction

Assumptions. All results in this paper are true for C^3 smooth circle homeomorphisms with finitely many critical points of polynomial type and an irrational rotation number of constant type.

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[1] Constant type irrational number means that coefficients in the continued fraction representation are bounded

For simplicity of our presentation we will give detailed proofs of our results only for maps with exactly one critical point which after a C^2 change of coordinate system can be written in the proximity of the critical point $x = 0$ in the form $x_i \mapsto (x - x_i)^3 + \varepsilon$. As a consequence, the circle can be covered by two overlapping sets: in the vicinity of the critical point $x = 0$ by a symmetric interval U and a "remote" interval V on which the first derivative is bounded away from zero. On the interval V the map has strictly negative Schwarzian derivative. We reserve the letter f for maps from the class defined above. The real line is projected to the unit circle by means of the map

$$x \mapsto \exp(2\pi x i) .$$

Denote by $|x - y|$ the distance between points x and y on the circle in the metric induced by the projection.

Uniform Constants. Following the convention of [13] we will mean by a uniform constant a function on our class of maps which continuously depends on the quasisymmetric norm of the map, the logarithm the size of U, the lower bound of the derivative on the remote arc and the C^3 norm. Uniform constants will be always denoted by the letter K. Whenever confusion can arise we specify uniform constants by adding subscripts.

Continued Fractions and Dynamics. Let p_n/q_n be the n^{th} continued fraction approximant of the rotation number ρ of f. The numbers q_n and the coefficients a_n in the continued fraction representation of ρ are related by the recurrence formula:

$$q_{n+1} = a_n q_n + q_{n-1} , \quad n \geqq 2 , \quad q_0 = 1 , \quad q_1 = a_1 .$$

Dynamically q_n is that iterate of the rotation by ρ for which the orbit of any point makes the closest return so far to the point itself. According to the Yoccoz Theorem (see [15]) a homeomorphism from our class is conjugated to a rotation. In particular, it implies the same order of orbits both for f and the rotation by ρ. The numbers q_n are called *closest returns.*

Continued Fractions and Partitions. We will use the orbit of a critical point 0 to define a system of partitions of the circle. First, we define two sets of closed intervals of order n:

$$q_{n-1} \text{ "short" intervals:} \qquad (z, f^{q_n}(z)), \ldots, f^{q_{n-1}-1}(z, f^{q_n}(z)) ,$$

and

$$q_n \text{ "lengthy" intervals:} \qquad (z, f^{q_{n-1}}(z)), \ldots, f^{q_n - 1}(z, f^{q_{n-1}}(z)) .$$

The "lengthy" and "short" intervals are mutually disjoint except for the endpoints and cover the whole circle. The partition obtained by the above construction will be denoted by $\mathscr{B}(n;f)$ and called the dynamical partition of the n^{th} order.

We will briefly explain the structure of the dynamical partitions. Take two subsequent dynamical partitions of order n and $n + 1$. The latter is clearly a refinement of the former. All "short" intervals of $\mathscr{B}(n;f)$ become the "lengthy" intervals of $\mathscr{B}(n + 1;f)$ while all "lengthy" intervals of $\mathscr{B}(n)$ are split into a_n "lengthy" intervals and 1 "short" interval of the next partition $\mathscr{B}(n + 1;f)$. An interval of the n^{th} dynamical partition will be denoted by $\square^n(f)$ or by $\square^n_x(f)$ if we want to emphasize that the interval contains a given point x.

We will drop f in the notation when no confusion can arise.

Bounded Geometry. Let us quote a few basic results about the geometry of dynamical partitions which are commonly referred to as "bounded geometry" (see for the proof [6 and 13]).

- The ratio of lengths of two adjacent elements of any dynamical partition is bounded by a uniform constant.
- For any element of any dynamical partition, the ratios of its length to the lengths of extreme intervals of the next partition subdividing it are bounded by a uniform constant.

As a corollary we obtain that the elements of the n^{th} dynamical partition are exponentially small.

Fact 1.1. *There are uniform constants K_1, $K_2 \leqq 1$, $K_3 \leqq 1$ so that*

$$K_1 K_2^n \leqq |\Box^n| \leqq K_1 K_3^n$$

holds for all natural numbers n.

2. Technical Tools

Distortion Lemma. We will call a chain of intervals a sequence of intervals such that each is mapped onto the next by the map f. Denote by $\mathbf{Cr}(a, b, c, d)$ a cross-ratio of the quadruple (a, b, c, d), $a < b < c < d$ given by the formula

$$\mathbf{Cr}(a, b, c, d) = \frac{|b-a|\,|d-c|}{|c-a|\,|d-b|}\,.$$

Here is one possible stating of the Distortion Lemma for critical circle homeomorphisms:

Lemma 2.1. *Take a chain of disjoint intervals*

$$(a_0, b_0), \ldots, (a_m, b_m)$$

which do not contain a critical point of f. Then, for arbitrary points $x, y \in (a_1, b_1)$, the uniform estimate

$$\left|\log \frac{(f^m)'(x)}{(f^m)'(y)}\right| \leqq K\,\mathbf{Cr}(f^m(a_0), f^m(x), f^m(y), f^m(b_0))$$

holds.

The Pure Singularity Property. To have a "dynamical measure" of size of an interval we will make the following definition:

Definition 2.1. *An interval J will be said to be of the j^{th} order of size if*

$$j = \max\{i : \forall_{x \in J} f^{q_i}(x) \notin J\} + 1\,.$$

Note that each interval of a chain is of the same order of size.

Let us introduce a one form

$$\mathcal{N}f = \frac{f''}{f'}\,dx$$

called the nonlinearity of f. As opposed to diffeomorphisms, the nonlinearity of critical circle maps which measures the distortion on chains of disjoint intervals, is non-integrable.

One of the main achievements of [14] was that the distortion coming from parts of the circle far away from cirtical points can be neglected with an almost exponentially small error with the order of size of a given chain. It means that asymptotically only what happens in the small neighborhood of a critical point matters.

We pass to a detailed formulation of the **Pure Singularity Property**. Suppose we have a chain of disjoint intervals

$$(a_0, b_0), \ldots, (a_m, b_m)$$

of the k^{th} order of size and symmetric neighborhood U_j with size of the order j. Then

$$\left| \int_{C_j} \mathcal{N} f \right| \leqq K \exp(-\sqrt{k-j}),$$

where C_j is a union of these intervals of the chain which are not contained in U_j and a constant K is uniform

Integral Formula. We introduce another cross-ratio $\mathbf{Poin}(a, b, c, d)$ of a given quadruple (a, b, c, d), $a \leqq b \leqq c \leqq d$, by the following formula:

$$\mathbf{Poin}(a, b, c, d) = \frac{|b-c|\,|d-a|}{|c-a|\,|d-b|}.$$

By the distortion of the cross-ratio $\mathbf{Poin}(a, b, c, d)$ by f we mean

$$\mathbf{DPoin}(a, b, c, d; f) = \frac{\mathbf{Poin}(f(a), f(b), f(c), f(d))}{\mathbf{Poin}(a, b, c, d)}.$$

There is a very simple relation between cross-ratios **Cr** and **Poin**. Namely,

$$\mathbf{Cr} = \frac{1}{1 + \mathbf{Poin}}.$$

The logarithm of the distortion of the cross-ratio **Poin** can be expressed by the integral formula. The formula is due to Sullivan [12]:

$$-\log(\mathbf{Poin}(a, b, c, d)) = \iint_S \frac{dx\,dy}{(x-y)^2},$$

where $S = \{(x, y)\ a \leqq x \leqq b \text{ and } c \leqq y \leqq d\}$.

Consequently,

$$\log \mathbf{DPoin}(a, b, c, d; f) = \iint_S d\mu - (f^* \times f^*) d\mu,$$

where μ is equal to $\dfrac{dx\,dy}{(x-y)^2}$. Calculating the integrand we get the

$$d\mu - (f^* \times f^*) d\mu = \left(1 - \frac{f'(x) f'(y)}{\left(\dfrac{f(x) - f(y)}{(x-y)} \right)^2} \right) \frac{dx\,dy}{(x-y)^2}.$$

For maps with negative Schwarzian derivative the integrand is positive and, as a consequence, the cross-ratio is not decreased by f. In the next paragraph we estimate how much the cross-ratio **Poin** is expanded by maps with strictly negative Schwarzian.

Expansion Lemma. *Let $a < b < c < d$. Suppose we have a chain of disjoint intervals*

$$(a_0, b_0), \ldots, (a_m, b_m)$$

of the n^{th} order of size which omit a critical point 0. Then

$$\log(\mathbf{DPoin}\,(a, b, c, d; f^m)) \geqq K_1 \sum_{f^i(a,d) \subset U} \frac{|f^i(a) - f^j(b)|\,|f^i(c) - f^j(d)|}{\max(|f^i(a)|, |f^i(d)|)^2} + K_2 \exp(-\sqrt{n}),$$

where K_1 and K_2 are uniform constants.

Proof. By the Pure Singularity Property we get that

$$\sum_V \log(\mathbf{DPoin}(a_i, b_i, c_i, d_i; f)) \leqq K_3 \exp(-\sqrt{n}),$$

where the sum $\sum_V$ is over all these indexes i for which $f^i(a, d)$ intersects V. Next we use Integral Formula to estimate the expansion of the cross-ratio for quadruples (a_i, b_i, c_i, d_i) contained in U,

$$\log(\mathbf{DPoin}(a_i, b_i, c_i, d_i; f)) = \iint_{S_i} \left(1 - \frac{9x^2y^2}{\left(\frac{x^3 - y^3}{x - y}\right)^2}\right) \frac{dx\,dy}{(x-y)^2}. \tag{1}$$

Here, S_i is defined by: $S_i = \{(x, y): f^i(a) \leqq x \leqq f^i(b), f^i(c) \leqq y \leqq f^i(d)\}$.
By algebra, the right-hand of Eq. (1) is rewritten as:

$$\iint_{S_i} \frac{(x^2 + xy + y^2)^2 - (3xy)^2}{(x^2 + xy + y^2)^2} \frac{dx\,dy}{(x-y)^2} = \iint_{S_i} \frac{x^2 + 4xy + y^2}{(x^2 + xy + y^2)^2}\,dx\,dy$$

$$\geqq \frac{|f^i(a) - f^j(b)|\,|f^i(c) - f^j(d)|}{3\max(|f^i(a)|, |f^i(d)|)^2} \tag{2}$$

which immediately gives the claim of the Expansion Lemma. The last inequality follows if we forget the numerator while dropping the power of the denominator by 1, and next estimate the denominator by

$$3\max(|f^i(a)|, |f^i(d)|)^2. \qquad \square$$

3. Singularity of the Invariant Measure

It is a well known fact that homeomorphisms of the circle have exactly one invariant measure μ. In this section we will investigate the properties of this measure for critical circle homeomorphisms. We will start with the following observation.

Proposition 1. *The invariant measure μ is totally singular with respect to the Lebesgue measure.*

Proof. Let ϕ be the conjugacy between f and a rotation ρ, $\rho \circ \phi = \phi \circ f$. It is enough to show that ϕ has the first derivative equal to zero on a set of full Lebesgue measure. To the contrary, suppose that at some point x the first derivative exists and is non-zero. Consider a first return q_n. The $q_{n+1} - 1$ images of $(x, f^{-q_n}(x))$ are disjoint. Clearly, there is an infinite sequence of first returns so that $f^{q_{n+1}}$ on this interval is not a diffeomorphism. By our conjugacy assumption, this map must be arbitrarily C^0 close to a linear map for large values of n. On the other hand, by bounded geometry, it is a composition of a few bounded distortion diffeomorphisms and a bounded number of critical iterates which are not diffeomorphisms. But maps of this type can not be arbitrarily C^0 close to linear. □

Another important property is ergodicity.

Proposition 2. *The map f is ergodic with respect to the Lebesgue measure λ.*

Proof. Suppose that there exist an invariant set A of positive but not full the Lebesgue measure $\lambda(A)$.

We fix $\varepsilon > 0$. Then by the Lebesgue Density Theorem we can find a point z and a number n_0 so that for all $n \geqq n_0$ the Lebesgue measure of an interval of n^{th} partition which contains z satisfies the inequality

$$\lambda(\square_z^n \cap A) \geqq (1 - \varepsilon)|\square_z^n|$$

or, equivalently,

$$\lambda(\square_z^n \cap A^c) \leqq \varepsilon|\square_z^n|,$$

where A^c denotes the complement of A.

Taking $q_{n+1} + q_n$ or $q_n + q_{n-1}$ images of $\square_z^n$ depending whether $\square_z^n$ is a "short" or a "long" interval of the n^{th} dynamical partition we obtain a cover of the circle. One can check that each point of the circle belongs to at most two intervals of this cover. We want to estimate $\lambda(f^k(\square_z^n) \cap f^k(A^c))$ for each interval of the cover.

If $f^i(\square_z^n)$ contains a critical point then there is a uniform constant K_1 so that

$$\frac{\lambda(f^{i+1}(\square_z^n \cap A^c))}{|f^{i+1}(\square_z^n)|} \leqq K_1 \frac{\lambda(f^i(\square_z^n \cap A^c))}{|(f^i(\square_z^n)|}.$$

The above inequality and the Distortion Lemma implies that

$$\frac{\lambda(f^k(\square_z^n) \cap f^k(A^c)}{|f^k(\square_z^n)|} \leqq K_2 \frac{\lambda(\square_z^n \cap A^c)}{|(\square_z^n)|}.$$

Since A^c is invariant we obtain that

$$\lambda(A^c) \leqq \sum_k \lambda(f^k(\square_z^n) \cap f^k(A^c))$$

$$\leqq K_2 \sum_k |f^k(\square_z^n)| \frac{\lambda(\square_z^n \cap A^c)}{|\square_z^n|} \leqq K_2 \varepsilon,$$

in contradiction to our assumption that $\lambda(A^c)$ is positive. □

Singularity Exponents. We are going to study the nature of singularities of an invariant measure μ using some ideas underlying the concept of multifractal measures and multifractals, the objects which are intensively studied by physicists. Let us discuss briefly the concept of a singularity exponent of an invariant measure which can be loosely defined in the following way: Let $M(x) = \int_0^x d\mu$ be the distribution function of measure μ. If the increments in $M(x)$ between two close points x and $x + \varepsilon$ are of the order $\varepsilon^{\tau(x)}$ then we will say that the distribution $M(x)$ has in the point x an exponent of singularity $\tau(x)$.

For mathematical exactness we will introduce two exponents of singularity, the upper and the lower one.

Definition 3.1. *Let μ be a measure completely singular with respect to λ with distribution function $M(x)$. Then by the upper and the lower singularity exponents we mean respectively*

$$\bar{\tau}(x) = \limsup_{\varepsilon \to 0} \frac{\log(M(x+\varepsilon) - M(x))}{\log(|\varepsilon|)}$$

and

$$\underline{\tau}(x) = \liminf_{\varepsilon \to 0} \frac{\log(M(x+\varepsilon) - M(x))}{\log(|\varepsilon|)} .$$

Taking into consideration that the Lebesgue measure is the imge of μ by the conjugating homeomorphism ϕ, we can rewrite the exponents $\underline{\tau}$ and $\bar{\tau}$ in the language of the dynamical partitions[2]

$$\bar{\tau}(x) = \limsup_{n \to \infty} \frac{\log|\phi(\square_x^n)|}{\log|\square_x^n|}$$

and

$$\underline{\tau}(x) = \liminf_{n \to \infty} \frac{\log|\phi(\square_x^n)|}{\log|\square_x^n|} .$$

The Exponents are Constants. The Distortion Lemma immediately implies that

Lemma 3.1. *The exponents $\underline{\tau}(x)$ and $\bar{\tau}(x)$ are f invariant.*

By Proposition 2 and the uniqueness of the invariant measure μ we get that

- For almost all points with respect to the Lebesgue measure the exponents are constants. We will denote these constants by $\underline{\tau}(\lambda)$ and $\bar{\tau}(\lambda)$ respectively.
- The above statement holds verbatim if "the Lebesgue measure" is replaced by μ. Denote these new constants by $\underline{\tau}(\mu)$ and $\bar{\tau}(\mu)$ respectively.

We pass to the formulation of our Main Theorem.

The Main Theorem. *The singularities of the invariant measure μ are of Hölder type. It means that there exist uniform constants K_1 and K_2 so that for almost all x in the sense of the measure μ the following estimates*

$$0 < K_1 < \underline{\tau}(\mu) \leqq \bar{\tau}(\mu) < K_2 < 1$$

hold.

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[2] Here we use the fact that the rotation number ρ is of bounded type

Remark. We should mention here that $\underline{\tau}(\lambda)$ and $\bar{\tau}(\lambda)$ are uniformly greater than 1 and less than infinity.

The proof of the Main theorem will occupy the whole next section.

Reformulation of the Main Theorem. For technical reasons we introduce new exponents $\underline{\gamma}(x)$ and $\bar{\gamma}(x)$ which live in the phase space of the rotation ρ

$$\underline{\gamma}(x) = \bar{\tau}^{-1}(\phi^{-1}(x)) \quad \text{and} \quad \bar{\gamma}(x) = \underline{\tau}^{-1}(\phi^{-1}(x))$$

and state the Main Theorem in the following equivalent form:

There are uniform constants K_1 and K_2 so that for almost all points x with respect to the Lebesgue measure λ the estimates

$$1 < K_1 < \underline{\gamma}(x) \leqq \bar{\gamma}(x) < K_2 < \infty$$

hold.

4. Proof of the Main Theorem

4.1. Discrepancy. Our main object in this paragraph is to establish a quantity which would measure nonlinear behavior of critical maps. We want to show that critical maps stay away in a certain uniform distance from diffeomorphisms. To this end we will introduce a notion of discrepancy.

Discrepancy between Partitions. We always assume that the length of the interval being partitioned is less than 1.

Definition 4.1. *A partition of I, denoted with P_I, is a set (possibly infinite) of closed subintervals of I, disjoint except for the endpoints, whose union is I. In addition, we assume that the entropy $H(P_I)$ is finite.*

Given $J \subset I$. Partition P_I induces in natural way a partition of J denoted by $[\mathbf{P_I}:\mathbf{J}]$.

There is a probabilistic measure on P_I defined by

$$\mu(X) := \sum_{w \in X} \frac{|w|}{|I|}$$

for every $X \subset P_I$, where $|\cdot|$ stands for the Lebesgue measure.

Two partitions, P_J and P_I will be called *corresponding* if there is a homeomorphism h from I to J which maps each element of P_I onto an element of P_J.

Definition 4.2. *The discrepancy between partitions P_I and P_J corresponding under h is denoted by $\delta(P_I, P_J)$ and defined to be*

$$\delta(P_I, P_J) = \int_{P_I} \log_+ \frac{dh}{d\mu},$$

where $\frac{dh}{d\mu}$ is the Jacobian of h, while $\log_+$ *means* $\max(0, \log)$.

The reader may note that $\delta(P_I, P_J)$ cannot be arbitrarily large regardless of the partitions involved.

The Discrepancy Lemma. *For any n and r the partitions*

$$[\mathscr{B}((n+1)r;f):\square^{nr}(f)] \overset{\phi}{\mapsto} [\mathscr{B}((n+1)r;\rho):\square^{nr}(\rho)]$$

are corresponding and the correspondence is given by the conjugation ϕ. As it turns out the discrepancy between these partitions is uniformly bounded away from zero.

Lemma 4.1. *We can choose r so that the inequality*

$$\delta([\mathscr{B}((n+1)r;f):\square^{nr}(f)], [\mathscr{B}((n+1)r;\rho):\square^{nr}(\rho)] \geqq K$$

is satisfied for large n and a uniform constant K.

Proof. The interval $\square^{nr}(f)$ contains at most two critical points of the map $f^{q_{nr}}$. Bounded Geometry implies that we can choose a number r in the definition of the refined dynamical partition $[\mathscr{B}((n+1)r;f)]$ so that:

- There exist three consecutive elements

$$(a,b), (b,c), (c,d) \quad \text{of} \quad [\mathscr{B}((n+1)r;f):\square^{nr}(f)]$$

  which do not contain a critical point of $f^{q_{nr}}$ and the length of the interval (a,d) is at least comparable to $|\square^{nr}(f)|$, i.e.

$$|(a,d)| > K|\square^{nr}(f)|$$

  with uniform K.
- The intervals $f^{q_{nr}}((a,b))$, $f^{q_{nr}}((b,c))$, $f^{q_{nr}}((c,d))$ belong to the partition $[\mathscr{B}((n+1)r;f):\square^{nr}(f)]$.

From the Expansion Lemma we have that

$$\log(\mathbf{DPoin}(a,b,c,d;f^{q_n})) \geqq K_1 \frac{|f^i(a)-f^i(d)|^2}{|f^i(d)|^2} + K_2 \exp(-\sqrt{n}),$$

where $f^i((a,b))$ is the closest interval to 0 amongst all q_{nr} images of (a,d) by f. Therefore, the distortion of the cross-ratio $\mathbf{Poin}(a,b,c,d)$ by $f^{q_{nr}}$ is by a definite amount greater than 1 since r which controls the relative size of the elements of $[\mathscr{B}((n+1)r;f):\square^{nr}(f)]$ is not too large.

But the distortion of the cross-ratio $\mathbf{Poin}(\phi(a),\phi(b),\phi(c),\phi(d))$ by any iterate of ρ is equal to 1 since ρ is an isometry. Hence, the discrepancy between partitions under consideration must be uniformly separated from zero, provided n is large enough. This concludes the proof. □

4.2. Partition Lemma. Here, we have a lemma about partitions:

Proposition 3. *Consider intervals I and J with corresponding partitions P_I and P_J respectively. Assume the following:*

$$\int \frac{|\log \mu(h(w))|}{|\log |J||} |\log \mu(w)| d\mu(w) \leqq K_3 \delta^2(P_I, P_J).$$

If

$$\frac{|\log |J||}{|\log |I||} \leqq \min\left(2, 1 + \frac{K_1}{H(P_I)} \delta^2(P_I, P_J)\right),$$

then

$$\sum_{w \in P_I} \frac{|\log|h(w)||}{|\log|w||}\,\mu(w) > \frac{|\log|J||}{|\log|I||}\left(1 + K_2\,\frac{\delta^2(P_I, P_J)}{|\log|I||}\right).$$

We will first work to approximate the sum

$$\sum_{w \in P_I} \frac{|\log|h(w)||}{|\log|w||}\,\mu(w) \tag{3}$$

by a sum easier to deal with. Let us consider an individual term:

$$\frac{|\log|h(w)||}{|\log|w||}\,\mu(w) = \frac{|\log|J||}{|\log|I||}\,\mu(w)\,\frac{1 + \dfrac{|\log\mu(h(w))|}{|\log|J||}}{1 + \dfrac{|\log\mu(w)|}{|\log|I||}}.$$

Now, an expression of the type

$$\frac{1+x}{1+y}$$

for positive x, y can be approximated with $1 + x - y$ so that

$$1 + x - y = \frac{1+x}{1+y} + \frac{y(x-y)}{1+y} > \frac{1+x}{1+y} + yx\,. \tag{4}$$

Inequality (4) allows us to bound a term of sum (3) from below by

$$\frac{|\log|J||}{|\log|I||}\,\mu(w)\left(1 + \frac{|\log\mu(h(w))|}{|\log|J||} - \frac{\mu(w)}{|\log|I||} + Q\right)$$

where the "quadratic correction" Q equals

$$\frac{|\log\mu(h(w))|}{|\log|J||}\;\frac{|\log\mu(w)|}{|\log|I||}.$$

Let us now bound the contribution of all quadratic corrections to the sum (3). It is equal to

$$\sum_{w \in P_I} \frac{|\log|J||}{|\log|I||}\,\mu(w)\,\frac{|\log\mu(h(w))|}{|\log|J||}\;\frac{|\log\mu(w)|}{|\log|I||}.$$

Now we use the first assumption of the proposition to see that this quantity is not greater than

$$\frac{|\log|J||}{|\log|I||}\,K_3\,\frac{\delta^2(P_I, P_J)}{|\log|I||}.$$

We can see that to prove Proposition 3 it is sufficient to show that

$$\sum_{w \in P_I}\left(\frac{|\log\mu(h(w))|}{|\log|J||} - \frac{|\log\mu(w)|}{|\log|I||}\right)\mu(w) > K_4\,\frac{\delta^2(P_I, P_J)}{|\log|I||}, \tag{5}$$

that is, to neglect the quadratic corrections. Indeed, we will just need to pick $K_3 := K_4/2$ to ensure that the quadratic corrections will not spoil the estimate.

We claim that estimate (5) follows from the following:

$$\sum_{w\in P_I} |\log \mu(h(w))|\mu(w) - \sum_{w\in P_I} |\log \mu(w)|\mu(w) \geqq K_5\delta^2(P_I, P_J)\,. \tag{6}$$

Indeed, assume that (6) holds. The left-hand side of estimate (5) is

$$\sum_{w\in P_I}\left(\frac{|\log \mu(h(w))|}{|\log|J||} - \frac{|\log \mu(w)|}{|\log|I||}\right)\mu(w)$$
$$= \frac{1}{|\log|I||}\left(\frac{|\log|I||}{|\log|J||}\sum_{w\in P_I}|\log \mu(h(w))| - \sum_{w\in P_I}|\log \mu(w)|\right). \tag{7}$$

We know by hypotheses of Proposition 3 that

$$\frac{|\log|J||}{|\log|I||} = 1 + K_2\frac{\delta^2(P_I, P_J)}{H(P_I)}\,,$$

where K_6 is not greater than a certain constant K_1 which we are free to specify, and, in addition, this quantity is not greater than 2.

From this and estimate (6) we can bound expression (7) from below by

$$\frac{1}{|\log|I||}\;\frac{H(P_I) + K_5\delta^2(P_I, P_J) - H(P_I) - K_6\delta^2(P_I, P_J)}{2}\,.$$

It is evident that if we choose $K_6 \leqq K_1 < K_5$, estimate (5) follows.

Proof of Estimate (6). We need to show that

$$\sum_{w\in P_I} |\log \mu(h(w))|\mu(w) - \sum_{w\in P_I} |\log \mu(w)|\mu(w) \geqq K_5\delta^2(P_I, P_J)\,.$$

Here, we notice that it is a well-known fact that the difference on the left-hand side is non-negative. It can be checked directly by calculus, or deduced from the variational principle for Gibbs measures (see [1]).

Thus, we are trying to prove that this is a "sharp" inequality.

The idea is to split P_I between two sets, called E and C, so that h expands on E and contracts on C. We define

$$E = \left\{w \in P_I : \frac{dh}{d\mu}(w) > 1\right\},$$

then C is the complement of E.

By Jensen's inequality

$$\frac{\int_E \log \frac{dh}{d\mu}}{\mu(E)} \leqq \log\frac{\mu(h(E))}{\mu(E)}\,.$$

This allows an estimate of the average rate of expansion of h on E:

$$\frac{\mu(h(E))}{\mu(E)} \geqq \exp\frac{\delta(P_I, P_J)}{\mu(E)}\,. \tag{8}$$

Let us now look at the sum

$$\sum_{w \in P_I} |\log \mu(h(w))| \mu(w) .$$

Its value given P_I as well as sets C, E, $h(C)$, $h(C)$ will be the smallest if the Jacobian of h is constant on both A and C. Hence,

$$\sum_{w \in P_I} |\log \mu(h(w))| \mu(w) - \sum_{w \in P_I} |\log \mu(w)| \mu(w)$$
$$\geqq \mu(E) |\log \mu(h(E))| + (1 - \mu(E)) |\log(1 - \mu(h(E)))|$$
$$- \mu(E) |\log \mu(E)| - (1 - \mu(E)) |\log(1 - \mu(E))| .$$

To finish the proof of estimate (6), we need to compare the value of this difference (which must be non-negative) with $\delta^2(P_I, P_J)$.

Until the end of this proof we adopt notations $x := \mu(E)$ and $y := \mu(h(E))$. We have $y > x$. First of all, we see that

$$x \log x + (1 - x) \log(1 - x) - x \log y - (1 - x) \log(1 - y) \geqq x \left(\frac{y}{x} - 1 - \log \frac{y}{x} \right)$$

provided that $y \geqq x$. To see this, we notice that the equality holds when $y = x$, and next we compare derivatives with respect to y. As x is fixed, the right-hand side of the preceding inequality grows with y/x. This enables us to use estimate (8) and bound the right-hand side of last inequality by

$$x \exp \frac{\delta(P_I, P_J)}{x} - x - \delta(P_I, P_J) .$$

As we neglect the terms of the exponential higher than the quadratic, we get another estimate from below by

$$\frac{\delta^2(P_I, P_J)}{2x}$$

which is what was needed to prove estimate (6).

4.3. The Upper Exponent $\bar{\gamma}$. We begin with the observation that Fact 1.1 implies that the upper exponent $\bar{\gamma}(x)$ is bounded from above by a uniform constant. Here is the main result of this subsection.

Proposition 4. *For almost all points of the circle the upper exponent* $\bar{\gamma}(x)$ *is greater than* 1 *and the estimate is uniform for maps from our class.*

Checking Procedure. Consider a sequence of nested partitions $\mathscr{B}(nr; f)$ and $\mathscr{B}(nr; \rho)$. Take an arbitrary interval $\square_f^{nr}$ of the nr^{th} dynamical partition. We will apply Proposition 3 to partitions $\mathscr{B}((n+1)r; f)$ and $\mathscr{B}((n+1)r; \rho)$ restricted to $\square^{nr}(f)$ and $\square^{nr}(\rho)$ respectively.

For rotation numbers of constant type Bounded Geometry implies that the logarithms of conditional measures of atoms of our partitions are bounded by a uniform constant. The same is with the logarithm of the Jacobian of the isomorphism. So the hypothesis of Proposition 3 is verified. We will keep the following scheme of *checking* the elements of the partitions $[\mathscr{B}((n+1)r;\rho):\square^{nr}(\rho)]$:

- If the hypothesis of the implication in the thesis of Proposition 3 is not satisfied for an element of $[\mathscr{B}((n+1)r;\rho):\square^{nr}(\rho)]$ then we will call this element a "good" one. We stop *checking.*
- Otherwise, we call an element of $[\mathscr{B}((n+1)r;\rho):\square^{nr}(\rho)]$ a "bad" one, denote by $I^{(n+1)r}$, and pass to the subdivision of this interval by the next partition $\mathscr{B}((n+2)r;\rho)$. We repeat the whole procedure.

Denote by A a set of points which are covered infinitely many times by "bad" elements of partitions $\mathscr{B}(nr;\rho)$.

Lemma 4.2. *The Lebesgue measure of A must be zero.*

Proof. Suppose that the assertion of the lemma is false. Then there is an arbitrary fine cover of the set A by "bad" elements of the partition $\mathscr{B}(nr;\rho)$ (i.e. n is large) which total length is greater than $\lambda(A)>0$. We will apply Proposition 3 step by step to the partitions $\mathscr{B}((n+1)r;\rho)$ restricted to elements I^{nr}. However, first we will make some preparation.

From Fact 1.1 it follows easily that

$$\max_{I^{jr}\in\mathscr{B}(jr;\rho)} \frac{|\log|\phi^{-1}(I^{jr})||}{|\log|I^{jr}||^2}$$

decreases up to a uniform constant as $1/j$.

By the Discrepancy Lemma,

$$\delta^2([\mathscr{B}((j+1)r;\rho):I^{jr}],[\mathscr{B}((j+1)r;f):\phi^{-1}(I^{jr})])\geqq K\,.$$

Finally, repeated application of Proposition 3 yields

$$\sum_{I^{nr}\in\mathscr{B}(nr;\rho)} \frac{|\log|\phi^{-1}(I^{nr})||}{|\log|I^{nr}||}|I^{nr}|\geqq K_1\lambda(A)\sum_{I^1\in B(1,r;\phi)}\frac{|\log|\phi^{-1}(I^r)||}{|\log|I^r||}+K_2\sum_{j=1}^{n-1}\frac{1}{j}\,.$$

The right-hand side of the above inequality tends to infinity with n while the left-hand side is bounded as we noticed at the beginning of this subsection. This contradiction completes the proof. □

As a consequence, we see that the total length of "good" intervals of the partitions $\mathscr{B}(nr;\rho)$ is equal to 1. Since now we will refer to "good" intervals as "good" intervals of the first generation. We pass to a subdivision of each "good" interval of the first generation and repeat the procedure of checking for all intervals of the subdivision. By the same way as above we find "good" intervals of second generation which occupy again the whole space up to a set of the Lebesgue measure zero. Repeating the procedure of *checking* countably many times we will obtain a sequence of sets of "good" intervals of different generations. By the construction a "good" interval of n^{th} generation must be finer than any element of the partition $\mathscr{B}((n-1)r;\rho)$.

Denote by G_x^{nr} a "good" interval which belong to $\mathscr{B}(nr;\rho)$ and contains a point x of the circle. Let B be a set of points which belong to infinitely many "good" intervals. Then for any $x \in$ and infinitely many n Proposition 3 implies the following estimate:

$$\bar{\gamma}(x) \geqq \min\left(2, 1 + \frac{K}{H([\mathscr{B}((n+1)r; \rho) : G_x^{nr}])}\right).$$

But the entropy $H([\mathscr{B}((n+1)r; \rho) : G_x^n])$ is bounded from above by a uniform constant. Hence,

$$\bar{\gamma}(x) \geqq 1 + K_2 ,$$

where $K > 0$ is an uniform constant.

4.4. Lower Exponent. Statement. Now we are in a position to prove

Proposition 5. *For a constant $K > 1$, we have*

$$\underline{\gamma}(x) \geqq K$$

for a full Lebesgue measure set of points x.

Preliminaries of the Proof. Since

$$\delta^2([\mathscr{B}((n+1)r; \rho : \square^{nr}(\rho)], [\mathscr{B}((n+1)r; f) : \square^{nr}(f)]) \geqq K_1$$

and the entropy of $[\mathscr{B}((n+1)r; \rho) : \square^{nr}(\rho)]$ is uniformly bounded away from 0, it follows that whenever

$$\frac{\log \phi^{-1}(|J|)}{\log |J|} < 1 + K_2$$

for uniform K_2, the assumptions of Proposition 3 are fulfilled for subpartitions generated by $\mathscr{B}(n+1, r; \rho)$ and $\mathscr{B}(n+1, r; \rho)$ on J and $\phi^{-1}(J)$ respectively. Now choose a number a which is less than the a.e. upper exponent and does not exceed $1 + K_2$ either. Almost every trajectory will spend an infinite amount of time above a.

Suppose that the lower exponent less or equal to $b - \varepsilon$ for some b and $\varepsilon > 0$ on a positive measure set B. Our proof will consist in showing that $b \geqq a$.

The Exponent as a Random Process. We define a random process $(\tilde{Y}_n)_{n=1,\ldots,\infty}$ so that each $\tilde{Y}_n$ is measurable with respect to $\mathscr{B}(nr; \rho)$. If J is an element of $\mathscr{B}((n+1)r; \rho)$, $\tilde{Y}_n$ is constant on J and equal to

$$\frac{\log|\phi^{-1}(J)|}{\log|J|}.$$

Then $\tilde{X}_n$ will be the increments of $\tilde{Y}_n$, i.e.

$$\tilde{X}_n = \tilde{Y}_n - \tilde{Y}_{n-1} .$$

We will use the following information about $\tilde{Y}_n$:

1. $\tilde{X}_n$ is uniformly bounded by K/n. This follows immediately from the definition of $\tilde{Y}_n$ and bounded geometry.
2. $E(\tilde{X}_n | \tilde{Y}_{n-1}) \geqq \frac{K'}{n}$ for a positive K' provided that $\tilde{Y}_{n-1}$ is less than a. This

follows from Proposition 3.

The Beginning of the Proof. Suppose that $b < a$. Almost every trajectory of $(\tilde{Y}_n)$ on B must oscillate infinitely many times between a and b. For a time k, define an event $\tilde{A}_k$ as follows: $\tilde{Y}_k > \frac{a+b}{2}$ and $\tilde{Y}_{k+1} \leqq \frac{a+b}{2}$ and the trajectory hits b before hitting a. We will show that the series of probabilities

$$\sum_{k=1}^{\infty} P(\tilde{A}_k)$$

is summable which will immediately give us the desired contradiction.

A Supermartingale. We modify the process $(\tilde{Y}_n)$ by making it constant after it hits a for the first time with $n \geqq k$ (k is fixed). The new process will be denoted with $(Y_n)_{n \geqq k}$, with increments $(X_n)_{n \geqq k}$. Formally, $(Y_n)_{n \geqq k}$ also depends on k, but since k is fixed in our discussion we choose not to emphasize that in our notation. The event A_k is defined analogously to $\tilde{A}_k$ with $\tilde{Y}$ replaced by Y. We observe that the probability of A_k remains the same as the probability of $\tilde{A}_k$.

The increments X_n are still bounded by K/n, and Y_n becomes a submartingale (increasing conditional mean).

Definition 4.3. *We define a family of processes* $(M(C, k, n))_{n=k,\ldots,\infty}$ *indexed by* k *by*

$$M(C, k, n) = \exp\left(\sqrt{n}(c - Y_n) - kC \sum_{j=k+1}^{n} \frac{1}{j^2}\right),$$

where c *was used to denote* $(a + b)/2$.

Lemma 4.3. *One can choose uniform constants* K_2 *and* K_3 *so that for all* $k \geqq K_2$ *the process* $M(K_3, k, n)$ *is a supermartingale.*

Proof. We compute:

$$E(M(C, k, n)| Y_{n-1}) = E(M(C, k, n-1))\, E(\exp(-\sqrt{k}X_n)| Y_{n-1}) \exp(-kC/n^2)\,.$$

One has to show that

$$\log E(\exp(-\sqrt{k}X_r)) \leqq \frac{kC}{n^2} \tag{9}$$

if k and C are large. Since X_n is of the order of $1/n \leqq 1/k$, one can bound the exponent from above for large k by

$$1 - \sqrt{k}X_n + k(X_n)^2 \leqq 1 - \sqrt{k}X_n + kK_3/n^2\,.$$

Since $E(X_n| Y_{n-1}) \geqq 0$ we get

$$E(\exp(-\sqrt{k}X_n)) \leqq 1 + kK_3/n^2\,.$$

Thus, whenever k is large and $C \geqq K_3$, Estimate (9) holds true, and the lemma immediately follows. □

The Bound for $P(A_k)$. We substitute A_k with a larger event B_k which occurs when $\tilde{Y}_k \geqq c$ and the trajectory by $(Y_n)_{n \geqq k}$ eventually hits b. We define the stopping time

j as the time of the first crossing of b by Y_n, $n > k$. By the optional sampling theorem, (see [9])

$$\int_{B_k} M(C, k, j) \leqq \int_{B_k} M(C, k, k) \leqq 1$$

since $M(C, k, k) \leqq 1$ everywhere on B_k. On the other hand,

$$M(C, k, j) \geqq \exp\left(\sqrt{k}(c - b) - nC \sum_{l=k+1}^{\infty} \frac{1}{i^2}\right) \geqq \exp(\sqrt{k}(c - b) - C)$$

on B_k. Thus, the measure of B_k decreases like $K^{-\sqrt{k}}$ which is summable.

In the consequence Proposition 5 follows and completes the proof of the *Main Theorem*. Changing the roles of ρ and f in the proof we immediately obtain the claim of the *Remark*.

5. Hausdorff Dimension of μ

The Hölder type of singularity implies natural bounds on the Hausdorff dimension of the measure μ.

Proposition 6. *The Hausdorff dimension of the invariant measure μ is equal to the lower exponent $\underline{\tau}(\mu)$ and, consequently, is uniformly bounded away from* 0 *and* 1.

Proof. The proof the Proposition 6 is based on the following Frostman's Lemma (see [10]):

Fact 5.1. *Suppose that ν is a probabilistic Borel measure on the interval and for ν-a.e. x*

$$\liminf_{\varepsilon \to 0} \log(\nu(x - \varepsilon, x + \varepsilon)/\log(\varepsilon) = \kappa \, .$$

Then the Hausdorff dimension of ν is equal to κ.

By the Main Theorem follows that

$$\kappa = \underline{\tau} \, ,$$

which completes the argument.

6. Open Questions

In the end of our presentation we would like to pose a few open questions which we believe to be of natural interest and importance.

- Assuming that the rotation number is algebraic of degree 2, prove that the lower exponent is equal to its upper counterpart. This should hold for the exponents related to λ as well as μ and would give us just one exponent with respect to each measure.
- In the same situation, establish a relation between exponents $\tau(\mu)$ and $\tau(\lambda)$.
- Prove that $\tau(\mu)$ and $\tau(\lambda)$ are universal given the rotation number (algebraic of degree 2? any irrational?).

- Do there exist critical circle homeomorphisms with a rotation number of constant type for which $\underline{\tau}(\mu) \neq \overline{\tau}(\mu)$ and $\underline{\tau}(\lambda) \neq \overline{\tau}(\lambda)$? We suspect so.
- What is the situation for unbounded rotation numbers? Are the main results of this paper still valid? We suspect not.

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