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Construction of the Two-Dimensional sine-Gordon Model for $\beta < 8\pi$

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Abstract. We present a rigorous renormalization group construction of the two-dimensional massless and massive quantum sine-Gordon models in finite volume for the range $0 < \beta < 8\pi$. We prove analyticity in the coupling constant ζ, which implies the convergence of perturbation theory. The field correlation functions and their generating functional are analyzed and shown to have the short distance asymptotics of the free field theory. In the massive case the bounds are uniform in volume and we also obtain uniform estimates on the long distance decay of correlations.

1. Introduction

The Euclidean sine-Gordon field theory with mass $m \geqq 0$ has an action of the general form

$$\mathscr{A}(\phi) = \frac{1}{2\beta} \int \phi((-\Delta + m^2)\phi) - z \int \cos\phi$$

and is defined by the measure

$$\exp(-\mathscr{A}(\phi))d\phi\ . \tag{1}$$

It is of interest as a quantum field theory with a non-polynomial interaction. Then β is the field strength and z is the coupling constant. Moreover it is equivalent via the exact sine-Gordon transformation to the classical statistical mechanics of a gas at temperature β^{-1} and activity $z/2$ [Si, Ka, Mi]. The two-body potential is a Coulomb potential for $m = 0$ or a Yukawa potential for $m > 0$.

In two dimensions the model is especially interesting. As β goes through the values $0 < \beta < 4\pi$, $4\pi \leqq \beta < 8\pi$, $\beta = 8\pi$ and $8\pi < \beta$, the ultraviolet perturbation

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theory is finite (if Wick ordered), superrenormalizable, strictly renormalizable, and finally nonrenormalizable. Thus β plays a role like that of the dimension d in ϕ_d^4 models. It is known that for $\beta < 4\pi$ the massive model is equivalent to the massive Thirring–Schwinger model and so exhibits a boson-fermion equivalence [Co, FSe]. It is conjectured that its scattering amplitudes are exactly soluble and given by the closed form expressions which were derived by [Za] in the context of integrable field theories, [STF] in the context of quantum inverse scattering, and by [Ji, RS] in the context of quantum groups.

We are concerned here with the basic construction of the model for any $\beta < 8\pi$. For a full treatment this means we should introduce short distance (UV) and long distance (IR) cutoffs and attempt to find a limit as they are removed. In the main portion of the present paper we will consider the UV problem for the massless theory in a fixed unit volume. In Sect. 9, we extend our method to the massive model, and prove bounds on the theory which are uniform in the volume.

The original results for the (sine-Gordon)$_2$ model were proved only for $\beta < 4\pi$ and z small. In this regime the theory is finite, and no renormalization is needed. The earliest constructive treatment of the massive model was given in [Fr] and the most extensive results are in [FSe]. These include the complete construction of the model (both the UV and IR limits), a proof of existence of single particle states, and the construction of the scattering operator.

In a sequence of more recent papers on the massive model, UV-uniform bounds have been obtained on a *renormalized* partition function for $\beta = \alpha^2 < 8\pi$ [BGN, NRS]. These authors were concerned with the nature of a sequence of thresholds of renormalizability which occur at the values $\beta = \beta_n = 8\pi(1 - 1/2n), n = 1, 2, 3, \ldots$. For a general discussion of their methods, which are quite different from ours, see [Ga1].

Let us begin with a heuristic discussion of the renormalization group method (RG) for both the ultraviolet and infrared problems. It is based on studying iterations of a RG transformation, which is a map from a measure (1) to a measure $\exp(-\mathscr{A}'(\phi))d\phi$ obtained by integrating out some short distance modes leaving an effective measure for the larger length scales. The new measure has roughly the same form, but with new parameters (β', z'). The procedure is iterated and gives a flow of the parameters (β, z). The conventional wisdom (e.g. [Zi]) is that the flow diagram for the sine-Gordon/Coulomb-gas model for z small is as shown in Fig. 1.

For $\beta > 8\pi$ and $z = 0$ there is a line of attracting fixed points. The flow diagram in this neighbourhood was rigorously estabilished in [DH1]. (This is the

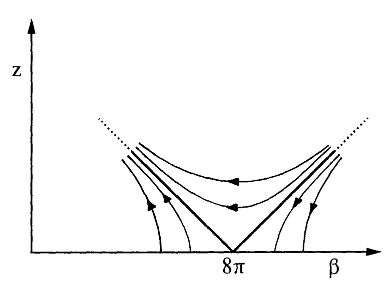

Fig. 1.

Kosterlitz–Thouless phase [KT]; for further results in this phase see [FSp, MKP].) These fixed points govern the long distance (IR) behaviour of the theory; on the other hand, the short distance (UV) behaviour is not illuminated by this flow.

Near the fixed point at $\beta = 8\pi$, $z = 0$ where the perturbation theory is strictly renormalizable, one expects the more complicated flow indicated in Fig. 1. We believe this flow can be established in a neighbourhood of $\beta = 8\pi$, $z = 0$ by the methods of [DH1] and the present paper. For a perturbation theory treatment of the UV problem see [NP].

For $\beta < 8\pi$ and $z = 0$ one has a line of repelling fixed points. In this paper we study the flow in a neighbourhood of this line, and we shall see these fixed points govern the UV behaviour of the theory in this region. It turns out for $\beta < 8\pi$ that we do not need to track changes in β, so we really study the projection of the flow onto the z-axis.

Let us first define the massless model more precisely. For any integer i let $\Lambda(i) = (\mathbf{R}/L^i\mathbf{Z})^2$ be the 2-tours with volume $|\Lambda(i)| = L^{2i}$, where $L \geqq 2$ is an integer scale factor we will fix later. For any $N \geqq i$, let $v_{i,N}$ be the inverse Laplace operator on $\Lambda(i)$, defined to be zero on constant functions, and with an ultraviolet cutoff at momentum scale L^N. The kernel of the operator is

$$v_{i,N}(x, y) = |\Lambda(i)|^{-1} \sum_{p \in \Lambda(i)^*} e^{ip(x-y)} \hat{v}_N(p) ,$$

where $\Lambda(i)^* = (2\pi L^{-i}\mathbf{Z})^2$ and where

$$\hat{v}_N(p) = p^{-2} e^{-L^{-4N}p^4}(1 - \delta_{p,0}) .$$

Let $d\mu_{\beta v_{i,N}}$ be the Gaussian measure with this covariance. This measure can be realized on the restricted Sobolev space

$$\mathscr{H}(\Lambda(i)) = \left\{ \phi \in \mathscr{H}_s(\Lambda(i)) : \int_{\Lambda(i)} \phi = 0 \right\}$$

of functions whose derivatives up to order $s \geqq 3$ are in $L^2(\Lambda(i))$. The massless model on the unit torus with UV cutoff L^N is defined by the measure on $\mathscr{H}(\Lambda(0))$

$$d\tilde{v}^N(\phi) = \exp(z_N \int \cos\phi(x) dx) d\mu_{\beta v_{0,N}}(\phi) . \tag{2}$$

The partition function is

$$Z^N = \int d\tilde{v}^N(\phi) .$$

The functional of the external field $\rho = \rho(x)$,

$$S^N(\rho) = (Z^N)^{-1} \int e^{i(\rho,\phi)} d\tilde{v}^N(\phi), \quad (\rho, \phi) = \int_{\Lambda(0)} \rho\phi$$

generates the field correlation functions

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle^N = (Z^N)^{-1} \int \phi(x_1) \ldots \phi(x_n) d\tilde{v}^N(\phi) .$$

The massive model, which we discuss in detail in Sect. 9, is defined in a similar way. We consider the theory with mass $m > 0$ on a finite torus $\Lambda(M)$ of side L^M for any $M \geqq 0$ (anticipating bounds which are uniform in M). For any integer $N \geqq 0$, let

$$v_{M,N}(x, y) = |\Lambda(M)|^{-1} \sum_{p \in \Lambda(M)^*} e^{ip(x-y)} \hat{v}_N(p) ,$$

where

$$\hat{v}_N(p) = (p^2 + m^2)^{-1} e^{-L^{-4N}p^4} .$$

Then the model is defined by the measure

$$d\tilde{v}_M^N(\phi) = \exp(z_N \int \cos\phi(x)dx) d\mu_{\beta v_{M,N}}(\phi) . \tag{3}$$

This is a measure on the ordinary Sobolev space $\mathscr{H}(\Lambda(M))$ without the restriction $\int \phi = 0$.

Let us return to the massless model on $\Lambda(0)$. It is convenient to scale the theory up to a volume $\Lambda(N)$ while taking the momentum cutoff down to $L^0 = 1$. This gives an equivalent measure on $\mathscr{H}(\Lambda(N))$,

$$dv^N(\phi) = \exp(\zeta_N \int \cos\phi(x)\, dx) d\mu_{\beta v_{N,0}}(\phi) ,$$

where $\zeta_N = z_N L^{-2N}$. We have $Z^N = \int dv^N(\phi)$ and

$$\langle \phi(x_1) \dots \phi(x_n)\rangle^N = (Z^N)^{-1} \int \phi(L^N x_1) \dots \phi(L^N x_n) dv^N(\phi) .$$

We want to choose z_N or ζ_N so that S^N and $\langle \phi(x_1) \dots \phi(x_n)\rangle^N$ have non-trivial limits as $N \to \infty$, i.e. we study the scaling limit.

It is here that the renormalization group enters. As we explain in detail in Sects. 2, 3 each RG transformation replaces a measure on $\Lambda(i)$ by an effective measure on $\Lambda(i-1)$. Starting on $\Lambda(N)$ and applying the transformation $N - i$ times yields an effective measure dv_i^N on $\mathscr{H}(\Lambda(i))$ which turns out to have the form (for ζ_N small)

$$dv_i^N(\phi) = \text{const} \times [\exp(\zeta_i \int \cos\phi(x)dx) + \mathscr{O}(\zeta_i^2)]\, d\mu_{\beta v_{i,0}}(\phi) ,$$

where

$$\zeta_i \sim L^{(2-\beta/4\pi)(N-i)} \zeta_N .$$

This is a reflection of the fact that $\cos\phi$ is the dominant relevant variable (besides constants). Now ζ_i is potentially N-dependent, but if we take

$$\zeta_N \sim L^{-(2-\beta/4\pi)N} \zeta_0 .$$

for some ζ_0 (i.e. $z_N = L^{\beta N/4\pi}\zeta_0$) we find

$$\zeta_i \sim L^{-(2-\beta/4\pi)i} \zeta_0 .$$

which is independent of N and small for all i (if $\beta < 8\pi$ and ζ_0 is small). This is the key to obtaining bounds uniform in N and the $N \to \infty$ limit.

This precise choice we make is $\zeta_N = L^{-2N} e^{\beta v_{N0}(0)/2} \zeta$ for some ζ in which case $\zeta_i = L^{-2i} e^{\beta v_{i0}/2} \zeta$, see Sect. 3. Since $v_{i0}(0) \sim i \log L/2\pi$ this is consistent with the above discussion. Then $z_N = e^{\beta v_{N0}(0)/2}\zeta$ and so on $\Lambda(0)$,

$$\begin{aligned} z_N \int \cos\phi(x)dx &= \zeta e^{\beta v_{N0}(0)/2} \int \cos\phi(x)dx \\ &= \zeta \int :\cos\phi(x):_{\beta v_{N,0}} dx . \end{aligned}$$

Thus renormalization is just Wick ordering, and ζ is the coupling constant for the Wick interaction.

Our main results are the following. For the massless model with $0 < \beta < 8\pi$, the effective measures dv_i^N, and the generating functional $S^N(\rho)$ for ρ small have limits as $N \to \infty$, except that for $4\pi \leqq \beta < 8\pi$ there is an overall divergent constant in $d\nu_i^N$. The partition function Z^N has a limit for $0 \leqq \beta < 4\pi$. These results imply that for all $0 < \beta < 8\pi$ the correlation functions $\langle \phi(x_1) \ldots \phi(x_n) \rangle^N$ have (distributional) limits as $N \to \infty$. For the limits we have a short distance regularity result: for $n > 2$, the truncated correlations $\langle \phi(x_1) \ldots \phi(x_n) \rangle^T$ are bounded functions of $\{x_i\}$, while for $n = 2$ they have the logarithmic singularity of the free theory ($\zeta = 0$). Finally all the above are analytic functions of ζ in a neighbourhood of the origin, and so perturbation theory is convergent. These results are new for $4\pi \leqq \beta < 8\pi$.

For the massive model on the torus $\Lambda(M)$, all of the above short distance results hold. Moreover, we can prove long-distance decay of field correlations for any power law (presumably the decay rate is exponential), uniformly in the volume $|\Lambda(M)|$. We have not yet addressed the proof of the infinite volume limit.

Our renormalization group method is of course influenced by other authors. The general philosophy is due to Wilson [Wi]. Early rigorous work can be found in [BCGNOPS, Ba, GK1, GK2]. For reviews see [GK3, Ga1]. Our treatment of the UV problem is particularly influenced by [GK4] who treat a hierarchial ϕ_4^4 model. Lesniewski [Le] treats (Yukawa)$_2$ by similar methods. Finally, the technical details of our method originate in a paper of Brydges and Yau [BY] on the dipole gas.

A final remark: there is the question of the long distance behaviour of the $m = 0$ theory for $\beta < 8\pi$. Does the infinite volume limit exist? Do these theories generate a mass dynamically? Formally we have $z \cos(\beta^{1/2}\phi) = z - 1/2(\beta z)\phi^2 + 1/4!\beta^2 z\phi^4 + \ldots$ which suggests a mass $(\beta z)^{1/2}$ for β sufficiently small. For $\beta \ll 1$, $z \ll 1$ this phenomenon, known as Debye screening, has been rigorously established by Brydges and Federbush [BF] and Yang [Ya]. It remains to be proven for all $\beta < 8\pi$, $z \ll 1$.

2. Renormalization Group Transformations

We now describe the Brydges–Yau formulation of the RG transformation. Until we reach Sect. 9 we consider only the massless sine-Gordon model. The RG transformation $\mathscr{R}$ is a map which takes a measure of the form $Z_i(\phi) d\mu_{\beta v_{i,0}}(\phi)$ on $\mathscr{H}(\Lambda(i))$ with $Z_i(\phi) \sim 1$ and yields a measure $Z_{i-1}(\phi) d\mu_{\beta v_{i-1,0}}(\phi)$ on $\mathscr{H}(\Lambda(i-1))$ still with $Z_{i-1} \sim 1$.

A key assumption is that the perturbation $Z_i(\phi)$ has an expression as a polymer gas. That is, it can be expressed as

$$Z_i(\phi) = \sum_{\{X_j\}} \prod_j K_i(X_j, \phi) . \tag{4}$$

The sum is over sets of disjoint polymers, where a polymer X is a union of *closed* unit squares Δ with centres on $\mathbf{Z}^2 \cap \Lambda(i)$. The polymer activity $K_i(X, \phi)$ is local in that it depends on ϕ only on the set X, $K_i(X, \phi) = K_i(X, \phi')$, if $\phi = \phi'$ on X. The transformation $\mathscr{R}$, a priori a map of measures, becomes a map of polymer activities $K_i \to K_{i-1}$.

Following [BY], Sect. 1, we use the "circle product" for polymer activities:

$$(K \circ K')(X) = \sum_{\substack{Y \cap Z = \emptyset \\ Y \cup Z = X}} K(Y) K'(Z)$$

which gives us a "circle exponential" representation for the polymer expansion (4):

$$Z_i(\phi) = \sum_{X \subset \Lambda(i)} \mathscr{E}\mathrm{xp}(K_i)(X, \phi)$$
$$= \mathscr{E}\mathrm{xp}(\Box + K_i(\phi))(\Lambda(i), \phi)\,. \tag{5}$$

The reader who prefers can replace circle exponentials by (4) throughout the paper.

The RG transformation $\mathscr{R}$ consists of three parts:

- a fluctuation integral in which modes with wavelength $< L$ are integrated out,
- an extraction step in which a vacuum energy part is factored out, and
- a reblocking/rescaling step in which the theory is scaled down by a factor L.

To define the fluctuation step, we write the covariance $v_{i,0}$ as the sum of a background covariance $v^{\#} = v_{i,-1}$ and a fluctuation covariance $C_i = v_{i,0} - v^{\#}$ whose Fourier transform is given by

$$\hat{C}_i(p) = p^{-2}[e^{-p^4} - e^{L^4p^4}][1 - \delta_{p,0}]\,. \tag{6}$$

Then $C_i(x, y)$ decays exponentially on the scale $|x - y| \sim \mathcal{O}(L)$. $\mathscr{E}$xpressing $v_{i,0}$ as a sum leads to a factorization of the Gaussian measure $d\mu_{\beta v_{i,0}}$. We write

$$\int d\mu_{\beta v_{i,0}}(\phi)\mathscr{E}\mathrm{xp}(\Box + K_i)(\phi)$$
$$= \int d\mu_{\beta v^{\#}}(\phi)\left[\int d\mu_{\beta C_i}(\eta)\mathscr{E}\mathrm{xp}(\Box + K_i)(\phi + \eta)\right] \tag{7}$$

and define $K^{\#} = \mathscr{F} K_i$ so the bracketed expression is $\mathscr{E}\mathrm{xp}(\Box + K^{\#})$. That is

$$\mathscr{E}\mathrm{xp}(\Box + K^{\#}) = \mu_{\beta C_i} * \mathscr{E}\mathrm{xp}(\Box + K_i)\,.$$

As shown in [BY], Sect. 8, $K^{\#}$ can be found by solving an integral equation: $K^{\#} = K(1)$, where $K(t)$ satisfies

$$K(t) = \mu_{t\beta C_i} * K^i + \frac{1}{2}\int_0^t ds\, \mu_{(t-s)\beta C_i} * \left[\int \beta C_i(x, y)\frac{\partial K(s)}{\partial \phi(x)} \circ \frac{\partial K(s)}{\partial \phi(y)} dx dy\right]. \tag{8}$$

Next we extract a constant $E_i(X) = E(K_i(X))$ from $K^{\#}$, and an overall factor from the measure. This vacuum energy renormalization removes the part of $K^{\#}$ which grows most under iteration of $\mathscr{R}$ (i.e. the most relevant piece). For other models additional extractions may be necessary: for example, quadratic extractions were required in [BY, DH1, DH2, DH3]. We specify $E_i(X)$ later: we always take $E_i(X) = 0$ unless X is a "small" set. A set X is called small ($X \in \mathscr{S}$) if X is connected and $|X| =$ (the number of unit squares in X) satisfies $|X| \leqq 4$. The factor extracted from the measure is $\exp(\sum_X E_i(X)) = \exp(E_i|\Lambda(i)|)$, where

$$E_i = \sum_{X \supset \Delta} \frac{E_i(X)}{|X|} \tag{9}$$

is independent of the unit square Δ.

We define $K^* = \mathscr{E}(K^{\#})$ so that

$$\mathscr{E}\mathrm{xp}(\Box + K^{\#}) = e^{E_i|\Lambda(i)|}\mathscr{E}\mathrm{xp}(\Box + K^*)\,. \tag{10}$$

Our definition of $\mathscr{E}$ adapts the arguments of [BY, Sect. 2] and [DH3, Sect. 5]. With $E(X) = E_i(X)$, let $R(X) = e^{E(X)} - 1$ ($= 0$, if $X \notin \mathscr{S}$), and define $J(X)$ by

$$J(X) = K^{\#}(X) - \sum_{\{X_i\} \to X} \prod_i R(X_i)\,. \tag{11}$$

The sum above is defined to be empty if X is not connected, and to be over all collections $\{X_i\}$ of distinct polymers with $\bigcup_i X_i = X$ if X is connected. Then one can prove that $\mathscr{E}K^{\#}$ defined by

$$\mathscr{E}K^{\#}(X) = \sum_{\{X_i\},\{Y_j\} \to X} \prod_i J(X_i) \prod_j (e^{-E(Y_j)} - 1) \tag{12}$$

satisfies (10). The sum above is over collections $\{X_i\}$, $\{Y_j\}$ such that $(\bigcup X_i) \cup (\bigcup Y_j) = X$, the $\{X_i\}$ are disjoint, the $\{Y_j\}$ are distinct elements of $\mathscr{S}$, each Y_j intersects some X_i, and the overlap graph for $\{X_i\}$, $\{Y_j\}$ is connected. (The overlap graph on a collection of polymers $\{Z_k\}$ is all pairs (k, k') such that $Z_k \cap Z_{k'} \neq \emptyset$.)

Finally, we reblock the 1-polymers K^* into L-polymers, and scale the theory down by L to a theory in terms of 1-polymers again. We have

$$\mathscr{E}\mathrm{xp}(\square + K^*(\phi)) = \mathscr{E}\mathrm{xp}(\square + \mathscr{S}K^*(\phi_L))\,,$$

where $\phi_L(x) = \phi(Lx)$ and

$$(\mathscr{S}K)(X, \phi_L) = \sum_{\{X_i\} \mapsto LX} \prod_i K(X_i, \phi)\,. \tag{13}$$

The sum in (13) is over collections of disjoint polymers $\{X_i\}$ such that $\bigcup_i \bar{X}_i = LX$ and the overlap graph on $\{\bar{X}_i\}$ is connected. ($\bar{X}_i$ denotes the smallest L-polymer containing X_i.) In addition, $v^{\#} = v_{i,-1}$ scales to $v_{i-1,0}$.

Putting it all together, with $K_{i-1} = \mathscr{S}K^*$, we have

$$\int d\mu_{\beta v_{i,0}}(\phi)\,\mathscr{E}\mathrm{xp}(\square + K_i)(\phi) = e^{E_i|\Lambda(i)|} \int d\mu_{\beta v_{i-1,0}}(\phi)\mathscr{E}\mathrm{xp}(\square + K_{i-1})(\phi)\,. \tag{14}$$

Thus $K_{i-1} = \mathscr{R}K_i \equiv \mathscr{S}\mathscr{E}\mathscr{F}K_i$, and the change in the measure is completely contained in the action $\mathscr{R}$ on the polymer activities K.

Now we specialize to the sine-Gordon model. We begin on $\Lambda(N)$ with

$$Z^N(\phi) = \exp\left[\zeta_N \int_{\Lambda(N)} \cos\phi(x)dx\right].$$

This has a polymer expansion $Z^N = \mathscr{E}\mathrm{xp}(\square + K^N)$, where

$$K^N(X, \phi) = \begin{cases} \prod_{\Delta \subset X}(\exp[\zeta_N \int_\Delta \cos\phi] - 1) & \text{if } X \text{ is connected} \\ 0 & \text{otherwise} \end{cases}. \tag{15}$$

Note an essential characteristic of the model: K^N is periodic in ϕ with period 2π.

Applying the RG transformation gives a sequence of polymer activities K_i^N on $\mathscr{H}(\Lambda(i))$ defined by $K_{i-1}^N = \mathscr{R}K_i^N$. Thus $K_i^N = \mathscr{R}^{N-i}K^N$. For the extraction we take

$$E_i^N(X) = E(K_i^N(X)) = \begin{cases} (2\pi)^{-1}\int_{-\pi}^{\pi}(K_i^N)^{\#}(X, \Phi)d\Phi & \text{if } X \in \mathscr{S} \\ 0 & \text{if } X \notin \mathscr{S} \end{cases}. \tag{16}$$

One can easily check that the periodicity of K is preserved by $\mathscr{R}$.

Going all the way down to $\Lambda(0)$ (a single unit square) by iterating (14), we have for the partition function:

$$\begin{aligned} Z^N &\equiv \int d\mu_{\beta v_{N,0}} \mathscr{E}\mathrm{xp}(\square + K^N) \\ &= \prod_{i=1}^{N} e^{E_i^N|\Lambda(i)|} \int d\mu_{\beta v_{0,0}} \mathscr{E}\mathrm{xp}(\square + K_0^N) , \end{aligned} \tag{17}$$

where $E_i^N = \sum_{X \supset \Delta} |X|^{-1} E_i^N(X)$.

3. Leading Behaviour of RG

Before entering into details, let us outline the analysis of the RG transformations $\mathscr{R}(K)$ on the polymer activities K. The acivities $K = K_i(X, \phi)$ are taken to be in a Banach space $\mathscr{K}_i$ of functionals on polymers in $\Lambda(i)$ and functions in $\mathscr{H}(\Lambda(i))$ (we define $\mathscr{K}_i$ in the next section). Then $\mathscr{R}$ maps elements of $\mathscr{K}_i$ to $\mathscr{K}_{i-1}$. In this setting the discussion falls into two parts: (i) an analysis of the linearization $\mathscr{R}_1 \equiv [\partial \mathscr{R}/\partial K]|_{K=0}$ around the fixed point $K = 0$, and (ii) control over the remainder $\mathscr{R}_{\geqq 2} = \mathscr{R} - \mathscr{R}_1$ when K is small.

For the analysis of the linearization the issue is to find the eigenvectors of $\mathscr{R}_1$ with eigenvalues larger than one. These correspond to the relevant variables under iteration (not counting those that are taken out in the extraction step, i.e. the constants).

The dominant term is the eigenvector of $\mathscr{R}_1$ with the largest eigenvalue and is given (in any $\mathscr{K}_i$) by

$$V(X, \phi) = \begin{cases} 0 & \text{if } |X| \geqq 2 \\ -\int_\Delta \cos\phi & \text{if } X = \Delta \end{cases} .$$

The next proposition shows that this has the eigenvalue $\lambda_i = L^2 e^{-\beta c_i/2}$, where $c_i = C_i(x, x)$. By an improvement of [DH1, lemma A.3] we have

$$c_i = C_i(x, x) = \frac{\log L}{2\pi} + \mathscr{O}(1) L^{-2(i-1)} .$$

Thus λ_i is only weakly dependent on i and satisfies

$$\lambda_i \sim L^{2-\beta/4\pi} .$$

In particular we have $\lambda_i > 1$ for $\beta < 8\pi$. (More generally if we replace $\int_\Delta \cos\phi$ by $\int_\Delta \cos n\phi$, $n \geqq 2$ we get an eigenvector with eigenvalue $L^2 \exp(-n^2 \beta c_i/2)$ which is greater than one for $\beta < 8\pi/n^2$: these terms play no special role in the present problem.)

Lemma 3.1.

$$\mathscr{R}_1 V = \lambda_i V . \tag{18}$$

Proof. We have $\mathscr{R}_1 = \mathscr{S}_1 \mathscr{E}_1 \mathscr{F}_1$, where $\mathscr{S}_1$ is the linearization of $\mathscr{S}$ at $K = 0$, etc. The linearized fluctuation operator $\mathscr{F}_1 K = \mu_{\beta C_i} * K$ can be evaluated explicitly by using

$$\mu_C * e^{i(\rho, \phi)} = \exp[-1/2(\rho, C\rho)] e^{i(\rho, \phi)} .$$

With ρ a δ-function, $\rho(x) = \delta(x - x')$, we find

$$(\mathscr{F}_1 V)(\Delta, \phi) = -e^{-\beta c_i/2} \int_{\Delta} \cos \phi .$$

Also $(\mathscr{F}_1 V)(X) = 0$ if $|X| > 1$ and thus V is an eigenvector of $\mathscr{F}_1$ with eigenvalue $\exp(-\beta c_i/2.)$

The linearized extraction operator $\mathscr{E}_1$ leaves V unchanged, since V has no vacuum energy part:

$$E(V(\Delta)) = -(2\pi)^{-1} \int_{-\pi}^{\pi} \cos \Phi \, d\Phi = 0 . \tag{19}$$

The linearized rescaling step $\mathscr{S}_1$ applied to V gives

$$\begin{aligned}(\mathscr{S}_1 V)(\Delta, \phi) &= \sum_{\Delta' \in L\Delta} V(\Delta', \phi_{L^{-1}}) \\ &= -\int_{L\Delta} \cos \phi(L^{-1}x) dx \\ &= L^2 V(\Delta, \phi)\end{aligned}$$

and $(\mathscr{S}_1 V)(X) = 0$ if $|X| > 1$. Thus V is an eigenvector of $\mathscr{S}_1$ with eigenvalue L^2.
Combining the above completes the proof. □

For the remainder $\mathscr{R}_{\geqq 2}(K)$ we start with a crude bound of the form $\|\mathscr{R}(K)\| \leqq \mathscr{O}(1)L^2 \|K\|$ for $\|K\|$ sufficiently small. This leads to $\|\mathscr{R}_{\geqq 2}(K)\| \leqq \mathscr{O}(1)L^2 \|K\|^2$, and it will follow that as long as $\|K\|$ is small, the true flow will stay close to the flow determined by $\mathscr{R}_1$.

Let us see how this works out for the specific activities $K_i^N = \mathscr{R}^{(N-i)}K^N$ defined in Sect. 2. The initial activity K^N given by (15) is quite close to the eigenvector V of $\mathscr{R}_1$: we have $K^N = \zeta_N V + \tilde{K}^N$, where $\tilde{K}^N$ is second order in the small quantity ζ_N. We will split each K_i^N similarly into a piece proportional to V and a remainder by:

$$K_i^N = \zeta_i V + \tilde{K}_i^N .$$

Since

$$\begin{aligned}K_{i-1}^N = \mathscr{R}(K_i^N) &= \mathscr{R}_1(K_i^N) + \mathscr{R}_{\geqq 2}(K_i^N) \\ &= \lambda_i \zeta_i V + \mathscr{R}_1(\tilde{K}_i^N) + \mathscr{R}_{\geqq 2}(K_i^N) ,\end{aligned} \tag{20}$$

we may take as the inductive definition of ζ_i, $\tilde{K}_i^N$,

$$\zeta_{i-1} = \lambda_i \zeta_i , \tag{21}$$

$$\tilde{K}_{i-1}^N = \mathscr{R}_1(\tilde{K}_i^N) + \mathscr{R}_{\geqq 2}(\zeta_i V + \tilde{K}_i^N) . \tag{22}$$

Note that $\tilde{K}_{i-1}^N$ is second order in ζ if $\tilde{K}_i^N$ is and so the term $\zeta_i V$ should be dominant.

Now as explained in the introduction we renormalize by choosing

$$\zeta_N = \left(\prod_{j=1}^{N} \lambda_j^{-1} \right) e^{\beta v_{0,0}(0)/2} \zeta = L^{-2N} e^{\beta v_{N,0}(0)/2} \zeta$$

for some ζ (the two expressions are equal since $v_{N,0}(L^N x) = \sum_{j=1}^{N} C_j(L^j x) + v_{0,0}(x)$). Then we have

$$\zeta_i = \left[\prod_{j=1}^{N} \lambda_j^{-1}\right] e^{\beta v_{0,0}(0)/2} \zeta \,,$$

where each $\lambda_j^{-1} < 1$. For ζ small and non-zero, ζ_i is also small and non-zero throughout the iteration.

4. Bounds on K and $\mathscr{R}$

In order to exhibit the renormalization cancellations which occur in the extraction step, it is convenient to consider K as a functional of ϕ and $\partial_\mu \phi$ rather than just ϕ. Precisely, the activity $K(X, \phi)$ will be replaced by a functional $\hat{K}(X, \psi)$ of two independent fields $\psi = (\psi_0, \psi_{1,\mu})$, $\mu = 1, 2$, such that

$$\hat{K}(X, \psi_\phi) = K(X, \phi) \,,$$

where $\psi_\phi = (\phi, \partial_\mu \phi)$. We replace $\mathscr{R}$ acting on the $K(X, \phi)$'s by $\hat{\mathscr{R}}$ acting on $\hat{K}(X, \psi)$'s defined so that

$$(\hat{\mathscr{R}} \hat{K})(X, \psi_\phi) = (\mathscr{R} K)(X, \phi) \,. \tag{23}$$

We define $\hat{\mathscr{R}} = \hat{\mathscr{S}} \hat{\mathscr{E}} \hat{\mathscr{F}}$, where $\hat{\mathscr{S}}, \hat{\mathscr{E}}, \hat{\mathscr{F}}$ each satisfy a condition like (23). For $\hat{\mathscr{S}}$, $\hat{\mathscr{E}}$, we take the natural lift of $\mathscr{S}, \mathscr{E}$ ($\hat{\mathscr{E}}$ is defined by taking $E(X)$ the same as before). For $\hat{\mathscr{F}}$, let $\mathscr{F}^0$ be the natural lift, i.e. define $(\mathscr{F}^0 K)\,(X) = K(t, X)|_{t=1}$, where

$$K(t) = \mu_{t\beta C} * K^i + \frac{1}{2}\int_0^t ds\, \mu_{(t-s)\beta C} * \left[\int \beta C(\xi, \xi') \frac{\partial K(s)}{\partial \psi(\xi)} \circ \frac{\partial K(s)}{\partial \psi(\xi')} d\xi d\xi'\right] . \tag{24}$$

Here $\xi = x$ or (x, μ) and define $\psi(x) = \psi_0(x)$ and $\psi(x, \mu) = \psi_{1,\mu}(x)$. Furthermore $C(x, \mu; x', \mu') = \partial_\mu \partial'_{\mu'} C(x, x')$ and we define

$$(\mu_{\beta C} * K)(X, \psi) = \int K(X, \psi_0 + \eta, \psi_1 + \partial\eta)\, d\mu_{\beta C}(\eta) \,.$$

For $X \notin \mathscr{S}$ we define $(\hat{\mathscr{F}} K)(X, \psi) = (\mathscr{F}^0 K)(X, \psi)$. For $X \in \mathscr{S}$ we have $(\mathscr{F}^0 K)(X, \psi) = \mu_{\beta C} * K(X, \psi)$. Instead of this we first replace K by the equivalent

$$K'(X, \psi_0, \psi_1) = K(X, \psi_0(x) + H\psi_1, \psi_1) \,,$$

where $H = H_X$ denotes the operator

$$(H\psi_1)(y) = \int_x^y \psi_{1,\mu}(z) dz^\mu$$

and x is a distinguished point in X. (Equivalent means equal when $\psi = \psi_\phi$.) Then for $X \in \mathscr{S}$ define $(\hat{\mathscr{F}} K)(X, \psi) = (\mathscr{F}^0 K')(X, \psi)$ and so $(\hat{\mathscr{F}} K)(X, \psi_\phi) = (\mathscr{F} K)(X, \phi)$. The rationale for this definition on small sets is that K' is periodic in the single variable $\psi_0(x)$ and can be Fourier analyzed. We will see that the non-zero Fourier coefficients get small when convolved with $\mu_{\beta C_i}$.

Throughout the remainder of the paper we use these new definitions of $\mathscr{R}$, K, etc. For simplicity we henceforth drop the "^"s on these quantities.

With our extended definition of $\mathscr{R}$ the functional $V = V(X, \psi_0)$ is no longer an eigenvector of $\mathscr{R}_1$. To recover this feature we further modify the definition and define $\mathscr{R}'$ by

$$\mathscr{R}'(K) = [\zeta_{i-1} V - \zeta_i \mathscr{R}_1 V] + \mathscr{R}(K)\,, \tag{25}$$

where the bracketed term vanishes at $\psi = \psi_\phi$. Now define again $K_{i-1}^N = \mathscr{R}'(K_i^N)$ starting with $K_N^N = K^N$ given by (15) with $\phi = \psi_0$. Also define $\tilde{K}_{i-1}^N$ by $K_i^N = \zeta_i V + \tilde{K}_i^N$ and then we have

$$\tilde{K}_{i-1}^N = \mathscr{R}_1(\tilde{K}_i^N) + \mathscr{R}_{\geqq 2}(K_i^N) \tag{26}$$

just as before. Although the actual flow is given by (25) it suffices to study (26).

We next define a norm of the functionals $K(X, \psi)$. We assume that the components of ψ_0 and $\psi_1 = (\psi_\mu)$ are elements of $C(\Lambda)$, the continuous complex-valued functions on Λ with sup norm. We also assume that $K(X, \psi)$ is analytic in ψ on an open strip around the real subspace $\psi = \psi_\phi$, $\phi \in \mathscr{H}(\Lambda)$. (See Appendix A for some basic facts about analytic functions on a complex Banach space.)

For $\mathbf{n} = (n_0, n_1)$ let $K_{\mathbf{n}}(X, \phi)$ be the (Frechet) derivative of $K(X, \psi)$ of order $\mathbf{n}$ with respect to ψ at $\psi = \psi_\phi$. This is a continuous multilinear functional on $C(\Lambda, \mathbf{R})^{n_0} \times C(\Lambda, \mathbf{R}^2)^{n_1}$, symmetric in each sector. We further assume that this functional is given by integration with respect to a signed bounded Borel measure on $\hat{\Lambda}^{\mathbf{n}} = \Lambda^{n_0 + n_1} \times$ indices. Formally we might represent the measure by

$$K_{\mathbf{n}}(X, \phi, \xi) = \left[\frac{\delta^n K(X, \psi)}{\delta\psi_0(\xi_1^0) \ldots \delta\psi_1(\xi_{n_1}^1)}\right]\Bigg|_{\psi = \psi_\phi}$$

with $\xi = (\xi_1^0, \ldots, \xi_{n_0}^0, \xi_1^1, \ldots, \xi_{n_1}^1)$ in $\hat{\Lambda}^{\mathbf{n}}$.

The basic locality assumption is that the measure $K_{\mathbf{n}}(X, \phi, \xi)$ has support in ξ in $\hat{X}^{\mathbf{n}} = X^{n_0 + n_1} \times$ indices (there are no collars around X as there are in [BY]).

First define the norm of $K_{\mathbf{n}}(X, \phi)$ to be the total variation norm

$$\|K_{\mathbf{n}}(X, \phi)\| = \sup_{\|F\| < 1} |K_{\mathbf{n}}(X, \phi; F)|\,,$$

where $F \in C(\hat{X}^{\mathbf{n}})$. Actually we usually consider the restriction of the measure to $\Delta_{\mathbf{n}} = \Delta_{0,1} \times \ldots \Delta_{0,n_0} \times \Delta_{1,1} \times \ldots \Delta_{1,n_1}$ with $\Delta_{ij} \subset X$ and so consider $\|K_{\mathbf{n}}(X, \phi) 1_{\Delta_{\mathbf{n}}}\|$.

Dependence on the variable ϕ is dominated by a large field regulator $G = G_L(X, \phi)$ which will have the form

$$G_L(X, \phi) = \exp\left(\kappa\left[\sum_{1 \leqq |\alpha| \leqq s} L^{2-2\alpha} \|\partial^\alpha \phi\|_X^2 + (Lc)^{-1} \|\partial\phi\|_{\partial X}^2\right]\right), \tag{27}$$

where κ and c are positive constants. We define

$$\|K_{\mathbf{n}}(X)\|_G = \sum_{\Delta_{\mathbf{n}}} \sup_{\phi \in \mathscr{H}(\Lambda)} \|K_{\mathbf{n}}(X, \phi) 1_{\Delta_{\mathbf{n}}}\| G^{-1}(X, \phi)\,.$$

Dependence on the set X is controlled by a large set regulator $\Gamma = \Gamma(X)$, which we will choose to be

$$\Gamma(X) = (2^8 L)^{2|X|} \Theta(X)\,, \tag{28}$$

$$\Theta(X) = \inf_\tau \prod_{b \in \tau} \theta(|b|)\,. \tag{29}$$

We will also have need of related regulators $\gamma\Gamma(X) = \gamma^{|X|}\Gamma(X)$ for constants γ. Here τ is a tree composed of bonds b connecting the centres of squares in X. Lengths such as $|b|$ are measured in an l^∞ metric on $\mathbf{R}^2$. The function θ is defined so that $\theta(s) = 1$ for $s = 0, 1$ and

$$\theta(\{s/L\}) = (2^8 L)^{-2}\theta(s), \quad s \geqq 2\,, \tag{30}$$

where $\{x\}$ denotes the smallest integer greater than x. Note that $\Gamma(\Delta) = (2^8 L)^2$. The large set regulator satisfies

$$\Gamma(X) \geqq 1$$

$$\Gamma(X_1 \cup X_2) \leqq \Gamma(X_1)\Gamma(X_2)\theta(d(X_1, X_2))\,,$$

where $d(X_1, X_2)$ is the length of the shortest bond from X_1 to X_2. In addition the scaling property

$$\Gamma(L^{-1}\bar{X}) \leqq \begin{cases} c\cdot 2^{-4|X|}\Gamma(X) & X \in \mathscr{S} \\ L^{-2}\cdot 2^{-4|X|}\Gamma(X) & X \notin \mathscr{S} \end{cases} \tag{31}$$

holds, where as before $\bar{X}$ denotes the smallest L-polymer containing X and $\mathrm{c} = 2^{16}$.

We define

$$\|K_{\mathbf{n}}\|_{G,\Gamma} = \sup_{\Delta} \sum_{X \supset \Delta} \Gamma(X)\|K_{\mathbf{n}}(X)\|_G\,. \tag{32}$$

We assume translation invariance of K so that the norm does not depend on the explicit pin at Δ and drop the supremum.

Finally, for $\mathbf{h} = (h_0, h_1)$, $\mathbf{h}^{\mathbf{n}} = h_0^{n_0} h_1^{n_1}$ and $\mathbf{n}! = n_1!n_2!$ we define

$$\|K\|_{G,\Gamma,\mathbf{h}} = \sum_{\mathbf{n}} (\mathbf{h}^{\mathbf{n}}/\mathbf{n}!)\,\|K\|_{G,\Gamma}\,.$$

For each $G, \Gamma, \mathbf{h}$ the space of all functionals $K = K(X, \psi)$ with $\|K\|_{G,\Gamma,\mathbf{h}} < \infty$ is denoted $\mathscr{K}_{G,\Gamma,\mathbf{h}}$. This is a Banach space [DH1]. We always work in the closed subspace of functionals which have period 2π in ψ_0.

To illustrate the norm, we bound the initial activity $K^N(X, \psi)$ defined by (15). We have $K^N = \zeta_N V + \tilde{K}^N$, where

$$\tilde{K}^N(X, \psi) = \begin{cases} [e^{\zeta_N V} - 1 - \zeta_N V](\Delta, \psi_0) & X = \Delta \\ \prod_{\Delta \subset X}[e^{\zeta_N V} - 1](\Delta, \psi_0) & X \text{ connected, } |X| > 1\,. \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4.1. *K^N, V, $\tilde{K}^N$ are analytic functionals of $\psi_0 \in C(\Lambda(N))$. For any $G \geqq 1$ and $\Gamma(\Delta)e^{h_0}|\zeta_N|$ sufficiently small, they are in $\mathscr{K}_{G,\Gamma,\mathbf{h}}$ and*

1. $\|V\|_{G,\Gamma,\mathbf{h}} \leqq \Gamma(\Delta)e^{h_0}$,
2. $\|\tilde{K}^N\|_{G,\Gamma,\mathbf{h}} \leqq \mathscr{O}(1)(\Gamma(\Delta)e^{h_0}|\zeta_N|)^2$,
3. $\|K^N\|_{G,\Gamma,\mathbf{h}} \leqq 2\Gamma(\Delta)e^{h_0}$.

Proof. The analyticity of $\psi \mapsto K^N(X, \psi)$ follows from the continuity of the functional and the analyticity on $\mathbf{C}$ of the function $\lambda \mapsto K^N(X, \psi + \lambda\chi)$ for any $\psi, \chi \in C(\Lambda(N))$; see Appendix A.

1. If sufficies to prove the bound with $G = 1$. We calculate the derivatives of $V(\Delta, \psi_0)$ at $\psi_0 = \phi$ and find

$$V_{n_0}(\Delta, \phi; F) = \int_{\Delta} \cos^{(n_0)}(\phi(x))F(x, \ldots, x)dx\,.$$

Therefore, $\| V_{n_0}(\Delta, \phi) \| \leqq 1$. This leads to $\| V(\Delta) \|_{G,\mathbf{h}} \leqq e^{h_0}$ and $\| V \|_{G,\Gamma,\mathbf{h}} \leqq \Gamma(\Delta) e^{h_0}$.

2. For $X = \Delta$ we have

$$\| \tilde{K}^N(\Delta) \|_{G,\mathbf{h}} \leqq \sum_{m=2}^{\infty} |\zeta_N|^m \| V(\Delta) \|_{G,\mathbf{h}}^m / m! \leqq (|\zeta_N| e^{h_0})^2 .$$

For any connected set X with $|X| > 1$,

$$\| \tilde{K}^N(X) \|_{G,\mathbf{h}} \leqq \prod_{\Delta \subset X} \| e^{\zeta_N V(\Delta)} - 1 \|_{G,\mathbf{h}} \leqq (2 |\zeta_N| e^{h_0})^{|X|} .$$

Thus

$$\| \tilde{K}^N \|_{G,\Gamma,\mathbf{h}} \leqq \sum_{\substack{X \supset \Delta_0 \\ X \text{ conn}}} \Gamma(X) \| \tilde{K}^N(X) \|_{G,\mathbf{h}} \leqq \mathcal{O}(1) (\Gamma(\Delta) |\zeta_N| e^{h_0})^2 ,$$

where the last estimate follows as in ([DH1], Proposition 3.4).

3. Follows from 1 and 2. □

We are now ready to state our main results for the activities K, $\tilde{K}$. We fix $0 < \beta < 8\pi$ and $0 < \varepsilon < 1/2$. Let L be sufficiently large (depending on β, ε) and choose $G = G_L(X, \phi)$ as in (27) with κ sufficiently small (independent of β, L). Take Γ as above and take

$$\mathbf{h}_i = (h_{0,i}, h_i) = (h_{0,0} + i\delta h, h_1) ,$$

where $h_{0,0} = c_{00} L$, $h_1 = c_1 L$ with constants c_{00}, c_1 to be specified, and where

$$\delta h = (\varepsilon/2)(2 - \beta/4\pi) \log L .$$

(h_{0i} must grow in i: we could arrange that it is bounded as in [DH1], but for Sect. 7 we need the linear growth. Note that $e^{h_{0i}} \zeta_i$ stays bounded in any case.) Let $\mathcal{K}_i = \mathcal{K}_{G,\Gamma,\mathbf{h}_i}$ with norm $\| \cdot \|_i = \| \cdot \|_{G,\Gamma,\mathbf{h}_i}$.

The next two results form the technical core of the method.

Proposition 4.2. *$\mathcal{R}_1$ maps $\mathcal{K}_i$ to $\mathcal{K}_{i-1}$ and*

$$\| \mathcal{R}_1 K \|_{i-1} \leqq \mathcal{O}(1) \lambda_i \| K \|_i . \tag{33}$$

The extraction $E(K)$ satisfies

$$|E(K)| \leqq \| K \|_i .$$

The next result gives analyticity in K and a crude bound which is satisfied even with no extraction operation $\mathcal{E}$.

Proposition 4.3. *$\mathcal{R}$ is an analytic functional of $K \in \mathcal{K}_i$ for $\| K \|_i$ sufficiently small and for such K,*

$$\| \mathcal{R}(K) \|_{i-1} \leqq \mathcal{O}(1) L^2 \| K \|_i .$$

We prove these two propositions in the next section. Taken together, they control the iteration of $\mathcal{R}$. This is the content of the following theorem which gives N-uniform bounds on the remainder term, $\tilde{K}_i^N$.

Theorem 4.4. *For $m = 0$, $0 < \beta < 8\pi$, $0 < \varepsilon < 1/2$, and L^{-1}, $|\zeta|$ sufficiently small:*

$$\|\tilde{K}_i^N\|_i \leqq |\zeta_i|^{2-\varepsilon}\,, \tag{34}$$

$$|E_i^N| \leqq |\zeta_i|^{2-\varepsilon}\,, \tag{35}$$

for all $0 \leqq i \leqq N$.

Proof of Theorem 4.4. The proof is by induction. For $i = N$, Lemma 4.1 and $e^{h_{0N}}|\zeta_N|^{\varepsilon/2} \leqq \mathcal{O}(1)e^{h_{00}}|\zeta_0|^{\varepsilon/2}$ imply

$$\begin{aligned}\|\tilde{K}^N\|_N &\leqq \mathcal{O}(1)[\Gamma(\Delta)e^{h_{0N}}|\zeta_N|]^2\\ &= \mathcal{O}(1)[\Gamma(\Delta)e^{h_{00}}|\zeta_0|^{\varepsilon/2}]^2|\zeta_N|^{2-\varepsilon}\\ &\leqq |\zeta_N|^{2-\varepsilon}\,,\end{aligned}$$

provided $|\zeta_0| \sim \mathcal{O}(1)|\zeta|$ is sufficiently small. The quantity

$$D = [\Gamma(\Delta)\, e^{h_{00}}|\zeta_0|^{\varepsilon/2}]$$

occurs frequently in what follows.

Now, to prove the inductive step $i \to i-1$ we use the formula (26). By Lemma 4.1,

$$\|\zeta_i V\|_i \leqq \Gamma(\Delta)|\zeta_i|e^{h_{0i}} \leqq \mathcal{O}(1)D|\zeta_i|^{1-\varepsilon/2}\,. \tag{36}$$

By the inductive hypothesis, $\|\tilde{K}_i^N\|_i \leqq |\zeta_i|^{2-\varepsilon}$ which is smaller still, so we have

$$\|K_i^N\|_i \leqq \mathcal{O}(1)D|\zeta_i|^{1-\varepsilon/2}\,. \tag{37}$$

Now use the analyticity and write

$$\mathcal{R}_{\geqq 2}(K_i^N) = \frac{1}{2\pi i}\oint ds \frac{1}{s^2(s-1)}\mathcal{R}(sK_i^N)\,,$$

where the integral is over the circle $|s| = |\zeta_i|^{\varepsilon/2-1}$ (which is greater than 1). Then $\|sK_i^N\|_i \leqq \mathcal{O}(1)D$ is small for $|\zeta|$ small and by Proposition 4.3 ,

$$\|\mathcal{R}(sK_i^N)\|_{i-1} \leqq \mathcal{O}(1)L^2D\,,$$

which gives

$$\begin{aligned}\|\mathcal{R}_{\geqq 2}(K_i^N)\|_{i-1} &\leqq \mathcal{O}(1)L^2D|\zeta_i|^{2-\varepsilon}\\ &\leqq \frac{1}{2}|\zeta_{i-1}|^{2-\varepsilon}\end{aligned}$$

if $\mathcal{O}(1)L^2D < 1/2$.

By Proposition 4.2,

$$\begin{aligned}\|\mathcal{R}_1(\tilde{K}_i^N)\|_{i-1} &\leqq \mathcal{O}(1)\,\lambda_i|\zeta_i|^{2-\varepsilon}\\ &= \mathcal{O}(1)\lambda_i^{\varepsilon-1}|\zeta_{i-1}|^{2-\varepsilon}\\ &\leqq \frac{1}{2}|\zeta_{i-1}|^{2-\varepsilon}\end{aligned}$$

for L large enough, which gives the result $\|\tilde{K}_{i-1}^N\|_{i-1} \leqq |\zeta_{i-1}|^{2-\varepsilon}$.

Now consider

$$E_i^N = E(K_i^N) = E(\tilde{K}_i^N)$$

($E(V) = 0$ by (19)). Then by Proposition 4.2,

$$|E_i^N| \leqq \|\tilde{K}_i^N\|_i \leqq |\zeta_i|^{2-\varepsilon}\,. \qquad \square$$

5. Proof of Propositions 4.2 and 4.3

In this section we analyze in detail the maps $\mathscr{F}$, $\mathscr{E}$, $\mathscr{S}$ and their linearizations. We introduce the following intermediate regulators:

$$G^*(X, \phi) = G_{L=1}(X, \phi)\,, \tag{38}$$

$$\Gamma^*(X) = 2^{-3|X|}\Gamma(X)\,, \tag{39}$$

$$\mathbf{h}^* = (h_{0,i-1}, L^{-1}h_1)\,. \tag{40}$$

For the fluctuation map $\mathscr{F}$, the main technical result is due to Brydges and Yau. To state it we introduce the norm on covariances

$$\| C_\theta \| = \sup_{\Delta_1} \sum_{\Delta_2} C(\Delta_1, \Delta_2)\theta(d(\Delta_1, \Delta_2))\,, \tag{41}$$

$$C(\Delta_1, \Delta_2) = \sup_{\xi_i \in \Delta_i} |C(\xi_1, \xi_2)|\,.$$

For the covariances C_i in the present paper we always have $\| C_\theta \| = \mathcal{O}(L^\alpha)$ for some $\alpha > 1$.

Lemma 5.1. *Consider the one parameter family of Gaussian measures $\mu_{t\beta C}$, $0 \leqq t \leqq 1$, with $C(x, y)$ of the form* (6) *and the large field regulators*

$$g(t, X) = [2^{|X|}G^*(X)]^t [G(X)]^{1-t}\,.$$

Then for κ sufficiently small:

1. *For all $0 \leqq s \leqq t \leqq 1$*

$$\mu_{(t-s)\beta C} * g(s) \leqq g(t)\,.$$

2. *For any $\mathbf{h}$ and any functional $K(X)$ on a set X,*

$$\| \mu_{(t-s)\beta C} * K(X) \|_{g(t),\mathbf{h}} \leqq \| K(X) \|_{g(s),\mathbf{h}}\,. \tag{42}$$

3. *Let $\mathbf{h} = (h_0, h_1)$, $\mathbf{h}' = (h'_0, h'_1)$ be any regulators such that $\mathbf{h}' < \mathbf{h}$ and let K satisfy*

$$\| K \|_{G,\Gamma,\mathbf{h}} \leqq \frac{\delta}{\beta \| C_\theta \|} \min\{(h_0 - h'_0)^2, (h_1 - h'_1)^2\} \tag{43}$$

for some universal δ. Then there is a solution $K(t)$ of the flow equation (24), *and*

$$\| K(t) \|_{g(t),\Gamma,\mathbf{h}'} \leqq \| K \|_{g(0),\Gamma,\mathbf{h}}\,.$$

In particular at $t = 1$ we have

$$\| \mathscr{F}^0 K \|_{g(1),\Gamma,\mathbf{h}'} \leqq \| K \|_{g(0),\Gamma,\mathbf{h}}\,.$$

The original proof ([BY], Theorem B) was for G's which were not strictly localized and for $h_0 = h_1$, $h'_0 = h'_1$. The extension to strictly localized G's was given in ([DH3], Appendix). The extension to $h_0 \neq h_1$, $h'_0 \neq h'_1$ is given in Appendix B of the present paper.

The analysis of $\mathscr{E}$ is somewhat special, and is contained in the proofs which follow. The main result for the rescaling map $\mathscr{S}$ is

Lemma 5.2. *For any K with $\|K\|_{G^*,\Gamma^*,\mathbf{h}^*}$ sufficiently small, the reblocked and rescaled functional $\mathscr{S}K$ is bounded by*

$$\|\mathscr{S}K\|_{G,\Gamma,\mathbf{h}_{i-1}} \leqq \mathscr{O}(1)L^2\|K\|_{G^*,\Gamma^*,\mathbf{h}^*}.$$

Proof. This follows from [DH3], Propositions 4 and 5, where $\mathscr{S}$ is split into separate scaling and reblocking steps. Nevertheless we sketch the proof. From the definition (13) one can show

$$\|\mathscr{S}K(X)\|_{G,\Gamma,\mathbf{h}_{i-1}} \leqq \sum_{\{X_i\}} \prod_i \|K(X_i)\|_{G^*,\mathbf{h}^*}.$$

The sum over sets $\{X_i\}$ is done using their connectivity. We also use

$$\Gamma(L^{-1}\bar{X}) \leqq c2^{-4|X|}\Gamma(X) = c2^{-|X|}\Gamma^*(X),$$

which follows from (31) and then

$$\begin{aligned}\|\mathscr{S}K\|_{G,\Gamma,\mathbf{h}_{i-1}} &\leqq \sup_{\Delta} \sum_{\{X_i\}:\bigcup_i\bar{X}_i\supset L\Delta} \Gamma(L^{-1}(\textstyle\bigcup_i\bar{X}_i))\prod_i \|K(X_i)\|_{G^*,\mathbf{h}^*}\\ &\leqq \sup_{\Delta} \sum_{\{X_i\}:\bigcup_i\bar{X}_i\supset L\Delta} \prod_i c2^{-|X_i|}\Gamma^*(X_i)\|K(X_i)\|_{G^*,\mathbf{h}^*}\\ &\leqq \sum_{N\geqq 1} (4\cdot 3^2(\log 2)^{-1}cL^2\|K^*\|_{G^*,\Gamma^*,\mathbf{h}^*})^N.\end{aligned}$$

This gives the result. In the last step we sum over the $\{X_i\}$ by picking a tree on the index set and summing over $\{X_i\}$ such that $\{\bar{X}_i\}$ has the connectivity of the tree. If δ_i are the incidence numbers for the tree we have

$$\begin{aligned}&\sum_{X_i:\bar{X}_i\cap\bar{X}_j\neq\emptyset} |X_i|^{\delta_1-1}c2^{-|X_i|}\Gamma^*(X_i)\|K(X_i)\|_{G^*,\Gamma^*,\mathbf{h}^*}\\ &\qquad\leqq (\delta_i-1)!(\log 2)^{\delta_i-1}3^2cL^2|X_j|\,\|K\|_{G^*,\Gamma^*,\mathbf{h}^*}.\end{aligned}$$

Then one sums over trees. For details, see [BY] or [DH3]. □

Proof of Proposition 4.2. We have $\mathscr{R}_1K = \mathscr{S}_1K_1^*$, where $K_1^* \equiv \mathscr{E}_1\mathscr{F}_1K$. We will show

$$\|K_1^*(X)\|_{G^*,\mathbf{h}^*} \leqq \begin{cases} 2^{|X|}\|K(X)\|_{G,\mathbf{h}_i}, & \text{if } X\notin\mathscr{S}\\ \mathscr{O}(1)L^{-2}\lambda_i 2^{|X|}\|K(X)\|_{G,\mathbf{h}_i}, & \text{if } X\in\mathscr{S}.\end{cases} \tag{44}$$

Then, as in the proof of Lemma 5.2, we have for the linearized rescaling step

$$\begin{aligned}\|\mathscr{S}_1K_1^*\|_{G,\Gamma,\mathbf{h}_{i-1}} &\leqq \sup_{\Delta}\sum_{X:\bar{X}\supset L\Delta} \Gamma(L^{-1}\bar{X})\,\|K_1^*(X)\|_{G^*,\mathbf{h}^*}\\ &\leqq L^2\sup_{\Delta}\sum_{X\supset\Delta}\Gamma(L^{-1}\bar{X})\,\|K_1^*(X)\|_{G^*,\mathbf{h}^*}\end{aligned}$$

to which we apply the bounds (44). We kill the bad factor L^2 by using $\Gamma(L^{-1}\bar{X}) \leq 2^{-4|X|}L^{-2}\Gamma(X)$ when $X\notin\mathscr{S}$ (from (31)):

$$\begin{aligned}\|\mathscr{S}_1K_1^*\|_{i-1} &\leqq L^2\Bigg[\sup_{\Delta}\sum_{\substack{X\notin\mathscr{S}\\ X\supset\Delta}}(L^{-2}2^{-4|X|}\Gamma(X))2^{|X|}\|K(X)\|_{G,\mathbf{h}_i}\\ &\quad+\sup_{\Delta}\sum_{\substack{X\in\mathscr{S}\\ X\supset\Delta}}(c2^{-4|X|}\Gamma(X))(\mathscr{O}(1)L^{-2}\lambda_i2^{|X|})\|K(X)\|_{G,\mathbf{h}_i}\Bigg]\\ &\leqq \mathscr{O}(1)\lambda_i\|K\|_i\end{aligned}$$

which proves the first part of Proposition 4.2.

Thus we need to show (44). For $X \notin \mathscr{S}$, we have $K_1^*(X) = (\mathscr{F}_1 K)(X) = \mu_{\beta C_i} * K(X)$, ($E(X) = 0$ for $X \notin \mathscr{S}$). In this case the bound follows from (42), which gives

$$\| \mu_{\beta C_i} * K(X) \|_{G^*, \mathbf{h}^*} \leqq 2^{|X|} \| K(X) \|_{G, \mathbf{h}^*}$$

followed by $\| K(X) \|_{G, \mathbf{h}^*} \leqq \| K(X) \|_{G, \mathbf{h}_i}$ since $\mathbf{h}^* < \mathbf{h}_i$.

Now consider $X \in \mathscr{S}$. We study first $K_1^{\#}(X) \equiv (\mathscr{F}_1 K)(X) = \mu_{\beta C_i} * K'(X)$. First expand $K'(X, \psi) = K(X, \psi_0(x) + H\psi_1, \psi_1)$ in a Fourier series in $\psi_0(x)$,

$$K'(X, \psi) = \sum_{q \in \mathbf{Z}} e^{iq\psi_0(x)} k(X, q, \psi_1) ,$$

$$k(X, q, \psi_1) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-iq\Phi} K(X, \Phi + H\psi_1, \psi_1) d\Phi .$$

Convolution with $\mu_{\beta C_i}$ yields

$$K_1^{\#}(X, \psi) = \sum_{q \in \mathbf{Z}} e^{iq\psi_0(x)} k^{\#}(X, q, \psi_1) ,$$

$$k^{\#}(X, q, \psi_1) = \int e^{iq\eta(x)} k(X, q, \psi_1 + \partial\eta) d\mu_{\beta C_i}(\eta) .$$

Then we have $K_1^*(X) = \mathscr{E}_1 K_1^{\#}(X) = K_1^{\#}(X) - E(X)$, where

$$E(X) = (2\pi)^{-1} \int_{-\pi}^{\pi} K_1^{\#}(X, \Phi, 0) d\Phi = k^{\#}(X, 0, 0) .$$

Thus finally for $X \in \mathscr{S}$,

$$K_1^*(X, \psi) = \sum_{q \in \mathbf{Z}} e^{iq\psi_0(x)} [k^{\#}(X, q, \psi_1) - \delta_{q,0} k^{\#}(X, 0, 0)] .$$

This is estimated by

$$\| K_1^*(X) \|_{G^*, \mathbf{h}^*} \leqq \| k^{\# \geqq 2}(X) \|_{G^*, h_1^*} + \sum_{q \neq 0} e^{|q| h_0^*} \| k^{\#}(X, q) \|_{G^*, h_1^*} , \tag{45}$$

where $k^{\# \geqq 2}(X, \psi_1) \equiv k^{\#}(X, 0, \psi_1) - k^{\#}(X, 0, 0)$. For the $q \neq 0$ terms above we have used the result $\| e^{iq\psi_0(x)} \|_{G = 1, \mathbf{h}} \leqq e^{|q| h_0}$ (see [DH1, Lemma 4.2]).

We use a bound which shows the beneficial effect of extracting a constant from the $q = 0$ term,

$$\| k^{\# \geqq 2}(X) \|_{G^*, h_1^*} \leqq \mathcal{O}(1) L^{-2} \| k^{\#}(X, 0) \|_{G^*, L h_1^* = h_1} . \tag{46}$$

This follows from [BY], Lemma 4.3 and needs $(\kappa h_1^{*2})^{-1} \leqq \mathcal{O}(1)$.

Bounds on $k^{\#}$ are proved in [DH1, Sect. 6] and say for any h_1:

$$\| k^{\#}(X, q) \|_{G^*, h_1} \leqq \begin{cases} 2^{|X|} \| k(X, 0) \|_{G, h_1}, & q = 0 \\ 2^{|X|} e^{-(|q| - 1/2)\beta c_i} \| k(X, q) \|_{G, h_1 + \beta |\partial C|_\infty} & q \neq 0 \end{cases} \tag{47}$$

($|\partial C|_\infty = \mathcal{O}(1)$). The Fourier components $k(X, q, \psi_1)$ exhibit decay for $|q|$ large, as shown in [DH1, Lemma 4.1]. This says there is a constant B so that if $h_0 > B h_1$,

$$\| k(X, q) \|_{G, h_1} \leqq e^{-(h_0 - B h_1)|q|} \| K(X) \|_{G, \mathbf{h}} . \tag{48}$$

Combining (46), (47), (48) gives the required bound for the $q = 0$ term of (45),

$$\| k^{\# \geqq 2}(X) \|_{G^*, h_1^*} \leqq \mathcal{O}(1) L^{-2} 2^{|X|} \| K(X) \|_{G, \mathbf{h}_i} .$$

By (47) and (48), the $q \neq 0$ terms of (45) are bounded by

$$\sum_{q \neq 0} 2^{|X|} \exp[-(|q| - 1/2)\beta c_i - |q|(h_{0,i} - h_0^* - B(h_1^* + \beta|\partial C|_\infty))]$$
$$\times \| K(X) \|_{G, h_{0,i}, h_1^* + \beta|\partial C|_\infty} .$$

Since $h_1^* + \beta|\partial C|_\infty = \mathcal{O}(1) \leqq h_1$ and $h_{0,i} > h_0^*$ and $e^{-\beta c_i/2} = L^{-2}\lambda_i = \mathcal{O}(L^{-\beta/4\pi})$, the sum over q is also bounded by $\mathcal{O}(1) 2^{|X|} L^{-2} \lambda_i \| K(X) \|_{G, \mathbf{h}_i}$ if L is large. Thus (44) is proved.

For the bound on the energy extraction note that

$$|E(X)| = |k^{\#}(X, 0, 0)| \leqq \| k^{\#}(X, 0) \|_{G^*, h_1} \leqq 2^{|X|} \| K(X) \|_{G, \mathbf{h}_i} . \tag{49}$$

Since $2^{|X|} \leqq \Gamma(X)$ the bound $|E(K)| \leqq \| K \|_i$ follows. □

Proof of Proposition 4.3. Let $K^* = \mathscr{E}\mathscr{F} K$. Then by Lemma 5.2,

$$\| \mathscr{R}(K) \|_{i-1} = \| \mathscr{S} K^* \|_{G, \Gamma, \mathbf{h}_{i-1}} \leqq \mathcal{O}(1) L^2 \| K^* \|_{G^*, \Gamma^*, \mathbf{h}^*} .$$

To bound K^* we need a bound on $\| J \|_{G^*, \gamma\Gamma^*, \mathbf{h}^*}$, where $\gamma(X) = 2^{|X|}$. Since $E(X) = 0$ for $X \notin \mathscr{S}$ and $\mathscr{F} = \mathscr{F}_1$ for small sets we may write for $J(X)$,

$$J(X) = \begin{cases} (\mathscr{F} K)(X) & \text{if } X \notin \mathscr{S} \\ ((\mathscr{F}_1 K)(X) - E(X)) + (E(X) - R(X)) - R^+(X) & \text{if } X \in \mathscr{S} \end{cases},$$

where $R(X) = e^{E(X)} - 1$ and

$$R^+(X) = \sum_{\substack{\{X_i\} \to X \\ \geqq 2 \text{ sets}}} \prod_i R(X_i) .$$

For the case $X \notin \mathscr{S}$ we have $(\mathscr{F} K)(X) = (\mathscr{F}^0 K)(X)$ and we use the bound from Lemma 5.1,

$$\| (\mathscr{F} K) 1_{\bar{\mathscr{S}}} \|_{G^*, \gamma\Gamma^*, \mathbf{h}^*} \leqq \| \mathscr{F}^0 K \|_{G^*, \gamma\Gamma^*, \mathbf{h}^*} = \| \mathscr{F}^0 K \|_{g(1), \gamma^2\Gamma^*, \mathbf{h}^*}$$
$$\leqq \| K \|_{g(0), \gamma^2\Gamma^*, \mathbf{h}_i} \leqq \| K \|_{G, \Gamma, \mathbf{h}_i} . \tag{50}$$

Here $1_{\bar{\mathscr{S}}}$ is the characteristic function of $\{X : X \notin \mathscr{S}\}$. Note that $\mathbf{h}_i - \mathbf{h}^* = (\delta h, h_1(1 - L^{-1}))$ and so the condition (43) will be satisfied for $\| K \|_i$ sufficiently small.

For $X \in \mathscr{S}$, we have from (44):

$$\| (\mathscr{F}_1 K - E) 1_{\mathscr{S}} \|_{G^*, \gamma\Gamma^*, \mathbf{h}^*} \leqq \| K_1^* \|_{G^*, \gamma\Gamma^*, \mathbf{h}^*} \leqq \| K \|_{G, \gamma^2\Gamma^*, \mathbf{h}_i} .$$

For the remaining terms we need

Lemma 5.3. *For* $\| K \|_{G, \gamma^2\Gamma, \mathbf{h}}$ *sufficiently small and* $|E(X)| \leqq 2^{|X|} \| K(X) \|_{G, \mathbf{h}}$,

1.
$$\| e^{\pm E} - 1 \|_{1, \Gamma, 0} \leqq 2 \| K \|_{G, \gamma\Gamma, \mathbf{h}} . \tag{51}$$

2.
$$\| R - E \|_{1, \Gamma, 0} \leqq \frac{1}{2} \| K \|_{G, \gamma\Gamma, \mathbf{h}} .$$

3.
$$\| R^+ \|_{1, \Gamma, 0} \leqq \frac{1}{2} \| K \|_{G, \gamma^2\Gamma, \mathbf{h}} . \tag{52}$$

Proof. The first two are immediate. For (52), we note

$$\|R^+\|_{1,\Gamma,0} = \sup_{\Delta} \sum_{X \supset \Delta} \Gamma(X)|R^+(X)|$$
$$\leqq \sup_{\Delta} \sum_{\{X_i\}:\bigcup_i X_i \supset \Delta} \prod_i 2^{-|X_i|}(\gamma\Gamma)(X_i)|R(X_i)| .$$

The sum over sets is done as in Lemma 5.2 and together with (51) yields the bound

$$\sum_{N \geqq 2} (4 \cdot 3^2/\log 2)^{N-1}(2\|K\|_{G,\gamma^2\Gamma,\mathbf{h}})^N$$

which gives the required bound for K small. □

Then, from all of the above and $\gamma^3\Gamma^* = \Gamma$,

$$\|J\|_{G^*,\gamma\Gamma^*,\mathbf{h}^*} \leqq \|\mathscr{F}K1_{\bar{\mathscr{S}}}\|_{G^*,\gamma\Gamma^*,\mathbf{h}^*} + \|(\mathscr{F}_1K - E)1_{\mathscr{S}}\|_{G^*,\gamma\Gamma^*,\mathbf{h}^*}$$
$$+ \|E - R\|_{1,\gamma\Gamma^*,0} + \|R^+\|_{1,\gamma\Gamma^*,0}$$
$$\leqq 3\|K\|_{G,\Gamma,\mathbf{h}_i} . \tag{53}$$

This bound and the bound (51) for $e^{-E} - 1$ are used in formula (12) for $\mathscr{E}\mathscr{F}K$. Then as in the proof of [DH3, Proposition 3] we find

$$\|K^*\|_{G^*,\Gamma^*,\mathbf{h}^*} \leqq \sum_{N \geqq 1, M \geqq 0} \frac{(N+M)!}{N!M!} \cdot \left(\frac{4 \cdot 3^2}{\log 2}\right)^{N+M-1} \|J\|^N_{G^*,\gamma\Gamma^*,\mathbf{h}^*} \|e^{-E} - 1\|^M_{1,\gamma\Gamma^*} ,$$
$$\leqq 6\|K\|_{G,\Gamma,\mathbf{h}_i} . \tag{54}$$

This completes the bound in Proposition 4.3.

The analyticity of the function $\mathscr{R}$ follows from the analyticity of $\lambda \mapsto \mathscr{R}(K_0 + \lambda K_1)$ and local boundedness; see Appendix B. Both these properties can be established using our estimate on $\mathscr{R}$. □

6. The $N \to \infty$ Limit

We write $K_i = \lim_{N\to\infty} K_i^N$ as a telescoping sum

$$K_i = K_i^i + \sum_{N=i}^{\infty} (K_i^{N+1} - K_i^N) \tag{55}$$

which we must show converges. The increments $\delta K_i^N \equiv K_i^{N+1} - K_i^N = \tilde{K}_i^{N+1} - \tilde{K}_i^N$ give the change in the activity at scale i if the UV cutoff is changed from N to $N+1$. They satisfy the recurrence relation

$$\delta K_{i-1}^N = \mathscr{R}_1 \delta K_i^N + \delta\mathscr{R}_{\geqq 2} , \tag{56}$$

$$\delta\mathscr{R}_{\geqq 2} \equiv \ulcorner \mathscr{R}_{\geqq 2}(K_i^{N+1}) - \mathscr{R}_{\geqq 2}(K_i^N) \urcorner . \tag{57}$$

We can express the second term as a double contour integral:

$$\delta\mathscr{R}_{\geqq 2} = \left(\frac{1}{2\pi i}\right)^2 \oint dt \oint ds \frac{1}{t(t-1)s^2(s-1)} \mathscr{R}(s(K_i^N + t\delta K_i^N)) , \tag{58}$$

where the contours are circles of radius greater than 1.

Lemma 6.1. *Under the hypotheses of Theorem* 4.4,

$$\|\delta K_i^N\|_i \leqq 2|\zeta_i|^{3/2-\varepsilon}|\zeta_N|^{1/2} .$$

Proof of Lemma 6.1. The proof is by induction on i, very much in parallel with the proof of Theorem 4.4. For $i = N$, we have by Theorem 4.4,

$$\|\delta K_N^N\|_N \leqq \|\tilde{K}_N^{N+1}\|_N + \|\tilde{K}_N^N\|_N \leqq 2|\zeta_N|^{2-\varepsilon} ,$$

as required.

Now we bound the inductive step (56). We use the bound (37) for $\|K_i^N\|_i$, and the inductive bound for $\|\delta K_i^N\|_i$ and find

$$\|K_i^N + t\delta K_i^N\|_i \leqq \mathcal{O}(1)D|\zeta_i|^{1-\varepsilon/2}$$

for any value t with $|t| \leqq |\zeta_i|^{1/2}|\zeta_N|^{-1/2}$. As in Theorem 4.4, we assume that $D = \Gamma(\Delta)e^{h_{00}}|\zeta_0|^{\varepsilon/2}$ is sufficiently small.

We choose the circles $|s| = |\zeta_i|^{\varepsilon/2-1}$, $|t| = |\zeta_i|^{1/2}|\zeta_N|^{-1/2}$ as the contour for (58). Then on this contour, by Proposition 4.3,

$$\|\mathscr{R}(s(K_i^N + t\delta K_i^N))\|_{i-1} \leqq \mathcal{O}(1)L^2D ,$$

and so

$$\begin{aligned}\|\delta\mathscr{R}_{\geqq 2}\|_{i-1} &\leqq \mathcal{O}(1)L^2D|\zeta_i|^{3/2-\varepsilon}|\zeta_N|^{1/2} \\ &\leqq |\zeta_{i-1}|^{3/2-\varepsilon}|\zeta_N|^{1/2} .\end{aligned}$$

By Proposition 4.2,

$$\begin{aligned}\|\mathscr{R}_1\delta K_i^N\|_{i-1} &\leqq \mathcal{O}(1)\lambda_i|\zeta_i|^{3/2-\varepsilon}|\zeta_N|^{1/2} \\ &\leqq |\zeta_{i-1}|^{3/2-\varepsilon}|\zeta_N|^{1/2} ,\end{aligned}$$

which yields the desired bound for $\|\delta K_{i-1}^N\|_{i-1}$. □

This propostion leads immediately to the main result:

Theorem 6.2. 1. *For* $m = 0, 0 < \beta < 8\pi$, *and* $L^{-1}, |\zeta|$ *sufficiently small, the following limits exist and are analytic in* ζ:

$$K_i = \lim_{N\to\infty} K_i^N , \tag{59}$$

$$E_i = \lim_{N\to\infty} E_i^N . \tag{60}$$

2. *For* $\beta < 4\pi$, *the partition function has a limit*:

$$Z = \lim_{N\to\infty} Z^N ,$$

which is analytic in ζ.

Proof.

1. The sum (55) defining K_i converges in $\mathscr{K}_i$ by the bound on $\|\delta K_i^N\|_i$. Each K_i^N and δK_i^N is analytic in ζ by an inductive argument using the analyticity of $\mathscr{R}$ and the analyticity of K_i follows. The same argument shows E_i^N has an analytic limit since $|\delta E_i^N| \leqq \|\delta K_i^N\|_i$

2. The partition function from (17) can be written

$$Z^N = \exp\left[\sum_{i=1}^{N} E_i^N |\Lambda(i)|\right] \int d\mu_{\beta v_{0,0}}(1 + K_0^N(\Delta)) . \tag{61}$$

For the first factor we note:

$$|E_i^N|\,|\Lambda(i)| \leqq |\zeta_i|^{2-\varepsilon} L^{2i} \leqq \mathcal{O}(1)|\zeta|^{2-\varepsilon} L^{i[(2-\varepsilon)(\beta/4\pi-2)+2]} .$$

This is summable for $\beta < 4\pi$ and ε close enough to zero (depending on β). Thus the first factor has a limit as $N \to \infty$ by dominated convergence. For the second factor we use $|K_0^N(\Delta, \phi)| \leqq G(\phi)\Gamma(\Delta)^{-1} \|K_0^N\|_0$ and dominated convergence to obtain the limit. Z is given by (61) with $N = \infty$.

Remarks. The bound on the partition function for $\beta < 4\pi$ says that the free energy per unit volume for the Coulomb gas is finite. If the model is regarded as a quantum field theory, it says that the vacuum energy density is finite.

If $4\pi \leqq \beta < 8\pi$, the argument breaks down and the energy density is infinite. In [BGN] this is interpreted as a collapse of dipoles to zero radius. Furthermore, they find a sequence of thresholds at $\beta = 8\pi(1 - 1/2n)$, $n = 2, 3, 4, \ldots$ at which values the vacuum renormalization is taken to the next higher order in ζ. They identify these thresholds as the collapse of higher multipoles. This phenomenon does not affect the correlation functions which we study next.

7. The Generating Functional

To control the field correlation functions, we follow [DH3] and extend our analysis to the unnormalized generating functional

$$Z^N(\rho) = \int e^{i(\rho,\phi)} d\tilde{v}^N(\phi) ,$$

where $\rho = \rho(x)$ is an external field. The measure is on the unit box $\Lambda(0)$; after scaling up to $\Lambda(N)$ we have

$$\begin{aligned} Z^N(\rho) &= \int e^{i(\rho^N,\phi)} dv(\phi) \\ &= \int e^{i(\rho^N,\phi)} \mathscr{E}\mathrm{xp}(\square + K^N(\phi)) d\mu_{\beta v_{N,0}}(\phi) , \end{aligned}$$

where $\rho^N(x) = L^{-2N}\rho(L^{-N}x)$.

The renormalization group transformations will leave us with similar expressions on $\Lambda(i)$:

$$\begin{aligned} Z^N(\rho) = \exp&\left(\sum_{k=i+1}^{N} -\beta/2(\rho^k, C_k\rho^k) + E_k^N|\Lambda_k|\right) \\ &\times \left[\int e^{i(\rho^i,\phi)} \mathscr{E}\mathrm{xp}(\square + K_i^N(\phi, \rho^i)) d\mu_{\beta v_{i,0}}(\phi)\right] . \end{aligned} \tag{62}$$

The activity $K_i^N = K_i^N(X, \phi, \rho)$ will depend on ρ, but will reduce to the previous activity $K_i^N(X, \phi)$ at $\rho = 0$.

We now explain the RG step $i \to i - 1$, modified to account for the ρ-dependence. First the presence of the $e^{i(\rho^i,\phi)}$ alters the fluctuation integral. After a contour shift $\eta \to \eta + i\beta C_i\rho$ in this integral, which is permissible for ρ not too large, we find

$$\begin{aligned} &\int e^{i(\rho^i,\phi)} \mathscr{E}\mathrm{xp}(\square + K_i^N(\phi, \rho^i)) d\mu_{\beta v_{i,0}}(\phi) \\ &\quad = e^{-\beta/2(\rho^i, C_i\rho^i)} \int e^{i(\rho^i,\phi)} \mathscr{E}\mathrm{xp}(\square + K^{\#}(\phi, \rho^i)) d\mu_{\beta v^{\#}}(\phi) . \end{aligned} \tag{63}$$

Here $K^{\#}(X, \phi, \rho^i) = (\mathscr{T}\mathscr{F}K_i^N)(X, \phi, \rho^i)$, where $\mathscr{F}$ is as before and where

$$(\mathscr{T}K)(X, \phi, \rho) = K(X, \phi + i\beta C_i\rho, \rho)$$

is the operation which introduces the ρ dependence.

For the extraction step we only extract the energy at $\rho = 0$. Thus just as before we take out $E_i^N = E(K_i^N)$ and $\mathscr{E}K$ is again given by (11), (12) with this E. This gives for (63),

$$(e^{-\beta/2(\rho^i, C_i\rho^i) + E_i^N|\Lambda_i|} \int e^{i(\rho^i, \phi)} \mathscr{E}\mathrm{xp}(\square + K^*(\phi, \rho^i)) d\mu_{\beta v^*}(\phi)$$

with $K^* = \mathscr{E}K^{\#}$.

Finally we again reblock and rescale defining $K_{i-1}^N = \mathscr{S}K^*$, where

$$(\mathscr{S}K)(X, \phi_L, L^2\rho_L) = \sum_{\{X_i\} \to X} \prod_i K(X_i, \phi, \rho) .$$

Then we obtain (62) for $i - 1$, with $K_{i-1}^N = \mathscr{R}_{\mathrm{ex}} K_i^N$ and $\mathscr{R}_{\mathrm{ex}} = \mathscr{S}\mathscr{E}\mathscr{T}\mathscr{F}$.

As before we replace the functionals $K(X, \phi, \rho)$ with $K(X, \psi, \rho)$ which have the proper restriction to $\psi = \psi_\phi$. If $\mathscr{F}^0$ is the natural lift defined by (24) we define the extended $\mathscr{F}$ by

$$(\mathscr{F}K)(X, \psi, \rho) = (\mathscr{F}(K(\cdot, \cdot, 0)))(X, \psi) + [(\mathscr{F}^0K)(X, \psi, \rho) - (\mathscr{F}^0K)(X, \psi, 0)] ,$$

where on the right, $\mathscr{F}$ has already been defined for $\rho = 0$ in Sect. 4. We define $\mathscr{T}$ as the natural lift:

$$(\mathscr{T}K)(X, \psi, \rho) = K(X, \psi_0 + i\beta C_i\rho, \psi_1 + i\beta\partial C_i\rho, \rho) .$$

$\mathscr{S}$ and $\mathscr{E}$ have natural lifts as before and thus $\mathscr{R}_{\mathrm{ex}}$ has an extended definition. It still preserves the periodicity of K in ψ_0. Also we have $(\mathscr{R}_{\mathrm{ex}}K)(X, \psi, 0) = (\mathscr{R}(K(\cdot, \cdot, 0)))(X, \psi)$.

Finally we define $K_{i-1}^N = \mathscr{R}'_{\mathrm{ex}}(K_i^N)$, where

$$\mathscr{R}'_{\mathrm{ex}}(K_i^N) = [\zeta_{i-1}V - \zeta_i\mathscr{R}_1V] + \mathscr{R}_{\mathrm{ex}}(K_i^N) .$$

The bracketed term vanishes for $\psi = \psi_\phi$ and this reduces to (25) at $\rho = 0$. If we define $\tilde{K}_i^N$ by $K_i^N = \zeta_iV + \tilde{K}_i^N$, then we have the recursion:

$$\tilde{K}_{i-1}^N = \mathscr{R}_{\mathrm{ex},1}(\zeta_iV) - \mathscr{R}_1(\zeta_iV) + \mathscr{R}_{\mathrm{ex},1}\tilde{K}_i^N + \mathscr{R}_{\mathrm{ex},\geq 2}(K_i^N) . \tag{64}$$

All the activities K are analytic in $\psi_0 \in C(\Lambda)$, $\psi_1 \in C(\Lambda, R^2)$ in a strip as before, and now are required to be analytic in ρ in a ball around $\rho = 0$ in $C'(\Lambda)$, the bounded Borel measures on Λ. The derivatives are

$$K_{\mathbf{n},p}(X, \phi, \xi, \mathbf{x}) = \left[\frac{\delta^{n+p}K(X, \psi, \rho)}{\delta\rho(x_1) \dots \delta\rho(x_p)\delta\psi_0(\xi_1^0) \dots \delta\psi_1(\xi_{n_1}^1)}\right]\Bigg|_{\psi = \psi_\phi, \rho = 0}$$

and are assumed to be given by functions $\mathbf{x} \mapsto K_{\mathbf{n},p}(X, \phi; \xi, \mathbf{x})$ from $\mathbf{x} \in \Lambda^p$ to measures on $\hat{\Lambda}^{\mathbf{n}}$.

Let $\mathscr{K}_{G,\Gamma,\mathbf{h},u}$ be the Banach space of all functionals $K(X, \psi, \rho)$ of this form with $\|K\|_{G,\Gamma,\mathbf{h},u} < \infty$ where the norms are now

$$\|K\|_{G,\Gamma,\mathbf{h},u} = \sum_{\mathbf{n},p} \frac{\mathbf{h}^{\mathbf{n}}u^p}{\mathbf{n}!p!} \|K_{\mathbf{n},p}\|_{G,\Gamma} , \tag{65}$$

$$\|K_{\mathbf{n},p}\|_{G,\Gamma} = \begin{cases} \sup_\Delta \sum_{X \supset \Delta} \Gamma(X) \|K_{\mathbf{n}}(X)\|_G & p = 0 \\ \sup_{\mathbf{x}} \sum_X \Gamma(X, \mathbf{x}) \|K_{\mathbf{n},p}(X, \mathbf{x})\|_G & p \neq 0 \end{cases} . \tag{66}$$

For $p \neq 0$,

$$\Gamma(X, \mathbf{x}) = \Gamma(X \cup \delta_{\mathbf{x}}), \tag{67}$$

where $\delta_{\mathbf{x}} = \bigcup_i \delta_{x_i}$ and δ_x is the semi-open unit square containing x selected from a disjoint covering of Λ.

For $0 < \beta < 8\pi$, $0 < \varepsilon < 1/2$, we pick L large and $G, \Gamma, \mathbf{h}$ as before (see Sect. 4). Now we also take

$$u \leqq (\beta \| C_\theta \|)^{-1} \min\{\delta h/2, 2h_1/L\},$$

where we recall $\delta h = (\varepsilon/2)(2 - \beta/4\pi)\log L$ and $h_1 = \mathcal{O}(1)L$. Let $\mathcal{K}_{i,u} = \mathcal{K}_{G,\Gamma,h_i,u}$ with norm $\|\cdot\|_{i,u}$.

Now we can state the extension of Propositions 4.2 and 4.3 to ρ-dependent activities. We split any K into $K_{p=0}$ (the value at $\rho = 0$) and $K_{p>0}$ (the remainder).

Proposition 7.1. *$\mathcal{R}_{\mathrm{ex},1} \equiv [\delta\mathcal{R}_{\mathrm{ex}}/\delta K]|_{K=0}$ is a linear map from $\mathcal{K}_{i,u}$ to $\mathcal{K}_{i-1,u}$ and*

$$\|[\mathcal{R}_{\mathrm{ex},1}K]_{p=0}\|_{i-1} \leqq \mathcal{O}(1)\lambda_i \|K_{p=0}\|_i,$$

$$\|[\mathcal{R}_{\mathrm{ex},1}K]_{p>0}\|_{i-1,u} \leqq \mathcal{O}(1)\|K\|_{i,u}.$$

Proposition 7.2. *$\mathcal{R}_{\mathrm{ex}}$ is an analytic functional of $K \in \mathcal{K}_{i,u}$ for $\|K\|_{i,u}$ sufficiently small, and for such K*

$$\|\mathcal{R}_{\mathrm{ex}}K\|_{i-1,u} \leqq \mathcal{O}(1)L^2\|K\|_{i,u}.$$

These lead to our main technical result which gives N-uniform bounds on K_i^N, generalizing Theorem 4.4, and the $N \to \infty$ limit, generalizing Theorem 6.2.

Theorem 7.3. *Under the above hypotheses, for $|\zeta|$ sufficiently small*

$$\|\tilde{K}_i^N\|_{i,u} \leqq |\zeta_i|^{1-\varepsilon}$$

for all $0 \leqq i \leqq N$. Furthermore $K_i = \lim_{N\to\infty} K_i^N$ exists in $\mathcal{K}_{i,u}$ for all i.

Proof. The bound for $i = N$ is the same as for $\rho = 0$. We proceed by induction, assuming the bound for i, and using (64) to prove it for $i-1$. The first term of (64) is $[\mathcal{R}_{\mathrm{ex},1}(\zeta_i V)]_{p>0}$, and so by Proposition 7.1 and (36),

$$\|\mathcal{R}_{\mathrm{ex},1}(\zeta_i V)_{p>0}\|_{i-1,u} \leqq \mathcal{O}(1)\|\zeta_i V\|_i \leqq \frac{1}{3}|\zeta_{i-1}|^{1-\varepsilon}.$$

For the second term of (64), Proposition 7.1 and Theorem 4.4 give

$$\begin{aligned}\|\mathcal{R}_{\mathrm{ex},1}(\tilde{K}_i^N)\|_{i-1,u} &\leqq \mathcal{O}(1)\lambda_i\|\tilde{K}^N_{i,p=0}\|_i + \mathcal{O}(1)\|\tilde{K}_i^N\|_{i,u}\\ &\leqq \mathcal{O}(1)\lambda_i|\zeta_i|^{2-\varepsilon} + \mathcal{O}(1)|\zeta_i|^{1-\varepsilon}\\ &\leqq \frac{1}{3}|\zeta_{i-1}|^{1-\varepsilon}.\end{aligned}$$

For the third term we use Proposition 7.2 and (36) to obtain

$$\|\mathcal{R}_{\mathrm{ex}}(K_i^N)\|_{i-1,u} \leqq \mathcal{O}(1)L^2\|K_i^N\|_{i,u} \leqq \mathcal{O}(1)L^2|\zeta_i|^{1-\varepsilon},$$

Using the analyticity in K as in the proof of Theorem 4.4, say with $|s| = |\zeta_i|^{-1+2\varepsilon}$ this gives

$$\|\mathcal{R}_{\mathrm{ex},\geqq 2}(K_i^N)\|_{i-1,u} \leqq \frac{1}{3}|\zeta_{i-1}|^{1-\varepsilon}$$

to complete the bound $\|\tilde{K}_i^N\|_{i-1,u} \leqq |\zeta_{i-1}|^{1-\varepsilon}$.

For the $N \to \infty$ limit we prove $\delta K_i^N \equiv K_i^{N+1} - K_i^N = \tilde{K}_i^{N+1} - \tilde{K}_i^N$ satisfies

$$\|\delta K_i^N\|_{i,u} \leqq 2|\zeta_i|^{1/2-\varepsilon}|\zeta_N|^{1/2} .$$

The proof is again by induction. We have

$$\delta K_{i-1}^N = \mathscr{R}_{\mathrm{ex},1}(\delta K_i^N) + \delta \mathscr{R}_{\mathrm{ex},\geqq 2} ,$$

$$\delta \mathscr{R}_{\mathrm{ex},\geqq 2} = \mathscr{R}_{\mathrm{ex},\geqq 2}(K_i^{N+1}) - \mathscr{R}_{\mathrm{ex},\geqq 2}(K_i^N) .$$

The latter term is estimated by an integral like (58). For $|t| \leqq |\zeta_i|^{1/2}|\zeta_N|^{-1/2}$,

$$\|K_i^N + t\delta K_i^N\|_{i,u} \leqq \mathcal{O}(1)|\zeta_i|^{1-\varepsilon} .$$

So for $|s| \leqq |\zeta_i|^{2\varepsilon-1}$, by Proposition 7.2

$$\|\mathscr{R}_{\mathrm{ex}}(s(K_i^N + t\delta K_i^N)\|_{i-1,u} \leqq \mathcal{O}(1)L^2|\zeta_i|^{\varepsilon} ,$$

and hence

$$\|\delta\mathscr{R}_{\mathrm{ex},\geqq 2}\|_{i-1,u} \leqq \mathcal{O}(1)L^2|\zeta_i|^{3/2-3\varepsilon}|\zeta_N|^{1/2} \leqq \frac{1}{2}|\zeta_{i-1}|^{1/2-\varepsilon}|\zeta_N|^{1/2} .$$

The other term is bounded by Proposition 7.1 and Lemma 6.1,

$$\begin{aligned}\|\delta\mathscr{R}_{\mathrm{ex},1}(\delta K_i^N)\|_{i-1,u} &\leqq \mathcal{O}(1)\lambda_i\|\delta K_{i,p=0}^N\|_i + \mathcal{O}(1)\|\delta K_i^N\|_{i,u} \\ &\leqq \mathcal{O}(1)\lambda_i|\zeta_i|^{3/2-\varepsilon}|\zeta_N|^{1/2} + \mathcal{O}(1)|\zeta_i|^{1/2-\varepsilon}|\zeta_N|^{1/2} \\ &\leqq \frac{1}{2}|\zeta_{i-1}|^{1/2-\varepsilon}|\zeta_N|^{1/2}\end{aligned}$$

to complete the proof of Theorem 7.3. □

Proof of Proposition 7.1. The first bound is just a restatement of Proposition 4.2.

For the second we have $(\mathscr{R}_{\mathrm{ex},1}K)_{p>0} = \mathscr{S}_1(K^*_{1,p>0})$ where $K^*_{1,p>0} = (\mathscr{E}_1\mathscr{T}\mathscr{F}_1K)_{p>0} = (\mathscr{T}\mathscr{F}_1K)_{p>0}$. As in the proof of Proposition 4.2, but now for $p>0$ and any K:

$$\begin{aligned}\|(\mathscr{S}_1K)_{\mathbf{n},p}\|_{G,\Gamma} &\leqq L^{-n_1}\sup_{\mathbf{x}}\sum_X \Gamma(X,\mathbf{x}) \sum_{X':\bar{X}'=LX} \|K_{\mathbf{n},p}(X',L\mathbf{x})\|_{G^*} \\ &\leqq L^{-n_1}\sup_{\mathbf{x}'}\sum_{X'}\Gamma(L^{-1}\bar{X}',L^{-1}\mathbf{x}')\|K_{\mathbf{n},p}(X',\mathbf{x}')\|_{G^*} \\ &\leqq cL^{-n_1}\sup_{\mathbf{x}'}\sum_{X'}\Gamma^*(X',\mathbf{x}')\|K_{\mathbf{n},p}(X',\mathbf{x}')\|_{G^*} \\ &\leqq cL^{-n_1}\|K_{\mathbf{n},p}\|_{G^*,\Gamma^*} .\end{aligned}$$

(Note that no factor L^2 arises, since the sum on X in the norm has no pin for $p>0$.) It follows that

$$\|(\mathscr{S}_1K_1^*)_{p>0}\|_{i-1,u} \leqq \mathcal{O}(1)\|K^*_{1,p>0}\|_{G^*,\Gamma^*,\mathbf{h}^*,u} .$$

The mapping $\mathscr{T}$ was analyzed in [DH2, Proposition 2] and satisfies:

$$\begin{aligned}\|(\mathscr{T}\mathscr{F}_1K)_{p>0}\|_{G^*,\Gamma^*,\mathbf{h}^*,u} &\leqq \|\mathscr{F}_1K\|_{G^*,\Gamma^*,\mathbf{h}^*+\beta\|C_\theta\|u,u} \\ &\leqq \|\mathscr{F}_1K\|_{G^*,\Gamma^*,\hat{\mathbf{h}},u} ,\end{aligned}$$

where we use our bound on u and define

$$\hat{\mathbf{h}} = (h_{0i} - \delta h/2, 2h_1/L) .$$

Note that $\hat{\mathbf{h}} < \mathbf{h}_i$. It is the growth in $\mathbf{h}$ in this step which forces our choice of $\mathbf{h}_i$.

For the last step we break $\mathscr{F}_1 K$ into $(\mathscr{F}_1 K)_{p>0} = (\mathscr{F}_1^0 K)_{p>0}$ and $(\mathscr{F}_1 K)_{p=0}$. For the $p = 0$ part we write

$$(\mathscr{F}_1 K)_{p=0}(X) = (\mathscr{E}_1 \mathscr{F}_1 K)_{p=0}(X) + E(K_{p=0}(X))$$

and use the bounds (44), (49); the $p > 0$ part is bounded using a version of Lemma 5.1 again. Thus,

$$\begin{aligned} \|\mathscr{F}_1 K\|_{G^*, \Gamma^*, \hat{\mathbf{h}}, u} &\leqq \|(\mathscr{E}_1 \mathscr{F}_1 K)_{p=0}\|_{G^*, \Gamma^*, \hat{\mathbf{h}}} \\ &\quad + \sum_X \Gamma^*(X) E(K_{p=0}(X)) + \|(\mathscr{F}_1^0 K)_{p>0}\|_{G^*, \Gamma^*, \hat{\mathbf{h}}, u} \\ &\leqq \mathscr{O}(1) \|K\|_{i,u} , \end{aligned} \tag{68}$$

which completes the proof of Proposition 7.1. □

Proof of Proposition 7.2. We follow the proof of Proposition 4.3. Extending Lemma 5.2 to the ρ-dependent case, we have

$$\|\mathscr{R}_{\mathrm{ex}} K\|_{i-1,u} = \|\mathscr{S} K^*\|_{i-1,u} \leqq \mathscr{O}(1) L^2 \|K^*\|_{G^*, \Gamma^*, \mathbf{h}^*, u}$$

if $K^* = \mathscr{E}\mathscr{T}\mathscr{F} K$ is sufficiently small. (See Propositions 4 and 5 in [DH3] for details.)

To bound the extracted activity $K^* = \mathscr{E}\mathscr{T}\mathscr{F} K$ we note that since the extracted parts are ρ-independent, the ρ derivatives of J are given by

$$J_p(X, \psi, \mathbf{x}) = \begin{cases} (\mathscr{T}\mathscr{F} K)_p(X, \psi, \mathbf{x}) & \text{if } p > 0 \\ J(X, \psi, 0) & \text{if } p = 0 \end{cases} .$$

Now by [DH3, Proposition 2] and an adaptation of Lemma 5.1 ([DH3], Proposition 1) we have for $\|K\|_{i,u}$ sufficiently small,

$$\begin{aligned} \|\mathscr{T}\mathscr{F} K 1_{\bar{\mathscr{S}}}\|_{G^*, \gamma\Gamma^*, \mathbf{h}^*, u} &\leqq \|\mathscr{F} K 1_{\bar{\mathscr{S}}}\|_{G^*, \gamma\Gamma^*, \hat{\mathbf{h}}, u} \\ &\leqq \|\mathscr{F}^0 K\|_{G^*, \gamma\Gamma^*, \hat{\mathbf{h}}, u} \\ &\leqq \|K\|_{i,u} \end{aligned}$$

(see (50) for the last step). On the other hand, since $\mathscr{F} = \mathscr{F}_1$ for small sets,

$$\begin{aligned} \|\mathscr{T}\mathscr{F} K 1_{\mathscr{S}}\|_{G^*, \gamma\Gamma^*, \mathbf{h}^*, u} &\leqq \|\mathscr{F} K 1_{\mathscr{S}}\|_{G^*, \gamma\Gamma^*, \hat{\mathbf{h}}, u} \\ &\leqq \|\mathscr{F}_1 K\|_{G^*, \gamma\Gamma^*, \hat{\mathbf{h}}, u} \\ &\leqq \mathscr{O}(1) \|K\|_{i,u} \end{aligned}$$

by the bound (68) in Proposition 7.1.

Combining these bounds with our previous bound on $J(X, \psi, 0)$ in (53) we find

$$\|J\|_{G^*, \gamma\Gamma^*, \mathbf{h}^*, u} \leqq \mathscr{O}(1) \|K\|_{i,u} .$$

Then it follows by an adaptation of (54) that for $\|K\|_{i,u}$ small

$$\|K^*\|_{G^*, \Gamma^*, \mathbf{h}^*, u} \leqq \mathscr{O}(1) \|K\|_{i,u}$$

(for a proof see [DH3], Proposition 3). This completes our proof. □

8. Correlation Functions

Now we are ready to remove the ultraviolet cutoff in the correlation functions. After iterating the RG transformation N times we have

$$Z^N(\rho) = \exp\left(\sum_{k=1}^{N} (-\beta/2(\rho^k, C_k\rho^k) + E_k^N|\Lambda_k|)\right) \times [\textstyle\int e^{i(\rho,\phi)}\mathscr{E}\mathrm{xp}(\square + K_0^N(\phi,\rho))d\mu_{\beta v_{0,0}}(\phi)]\,, \tag{69}$$

where the last integral is over functions on $\Lambda(0) = \Delta$ (a unit square). We perform one last fluctuation and complex translation step (with $C_0 = v_{0,0}$). Then there are no fields left to integrate and the bracketed expression above is

$$\begin{aligned}[\cdots] &= \exp(-\beta/2(\rho, v_{0,0}\rho))\mathscr{E}\mathrm{xp}(\square + K_0^{N,\#})(\Delta,\rho)\\ &= \exp(-\beta/2(\rho, v_{0,0}\rho))(1 + K_0^{N,\#}(\Delta,\rho))\,,\end{aligned}$$

where

$$(K_0^{N,\#})(\Delta,\rho) = (\mathscr{T}\mathscr{F}K_0^N)(\Delta,\rho) = \int K_0^N(\Delta, \eta + i\beta v_{0,0}\rho, \rho)d\mu_{\beta v_{0,0}}(\eta)\,. \tag{70}$$

We reassemble the fluctuation covariances in (69)

$$\sum_{k=1}^{N} (\rho^k, C_k\rho^k) + (\rho, v_{0,0}\rho) = (\rho, v_{0,N}\rho)$$

to obtain

$$Z^N(\rho) = \exp\left(-\beta/2(\rho, v_{0,N}\rho) + \sum_{k=1}^{N} E_k^N|\Lambda(k)|\right)(1 + K_0^{N,\#}(\Delta,\rho))\,.$$

This shows the leading behaviour $\exp(-\beta/2(\rho, v_{0,N}\rho))$ valid for the free field theory with $\zeta = 0$, with corrections for $\zeta \neq 0$ contained in E_k^N and $K_0^{N,\#}(\Delta,\rho)$. The normalized generating functional is given by:

$$\begin{aligned}S^N(\rho) &= Z^N(\rho)/(Z^N(0)\\ &= \exp(-\beta/2(\rho, v_{0,N}\rho))(1 + K_0^{N,\#}(\Delta,0))^{-1}(1 + K_0^{N,\#}(\Delta,\rho))\,.\end{aligned} \tag{71}$$

The expression $(\rho, v_{0,N}\rho)$ may not have a limit as $N \to \infty$, but this will be true if we impose the additional regularity condition on ρ:

$$(\rho, v_{0,\infty}\rho) = \sum_{p\neq 0}^{\infty} |\tilde{\rho}(p)|^2 p^{-2} < \infty\,.$$

The n-point correlation function is given by

$$\langle\phi(\rho_1)\ldots\ \phi(\rho_n)\rangle^N = \frac{\partial^n}{\partial s_1\ldots\partial s_n} S^N\left(\textstyle\sum s_i\rho_i\right)|_{s_i=0}\,.$$

We also consider

$$\log S^N(\rho) = -\beta/2(\rho, v_{0,N}\rho) + \log(1 + K_0^{N,\#}(\Delta,\rho)) - \log(1 + K_0^{N,\#}(\Delta,0)) \tag{72}$$

and the truncated correlation functions

$$\langle\phi(\rho_1)\ldots\ \phi(\rho_n)\rangle^{N,T} = \frac{\partial^n}{\partial s_1\ldots\partial s_n} \log S^N\left(\textstyle\sum s_i\rho_i\right)|_{s_i=0}\,.$$

If $n > 2$, the term $-\beta/2(\rho, v_{0,N}\rho)$ in $\log S^N$ makes no contribution and we can drop the condition $(\rho, v_{0,\infty}\rho) < \infty$.

Theorem 8.1. *For $m = 0$, $0 < \beta < 8\pi$ and ζ sufficiently small:*

1. *If $\rho \in C'(\Lambda(0))$, $\|\rho\| < u$ and $(\rho, v_{0,\infty}\rho) < \infty$, then the generating functional has a limit:*

$$S(\rho) = \lim_{N\to\infty} S^N(\rho) .$$

2. *If $\rho_i \in C'(\Lambda(0))$ and $(\rho, v_{0,\infty}\rho_i) < \infty$, then the correlation functions have a limit:*

$$\langle\phi(\rho_1)\dots\phi(\rho_n)\rangle = \lim_{N\to\infty} \langle\phi(\rho_1)\dots\phi(\rho_n)\rangle^N .$$

3. *If $\rho_i \in C'(\Lambda(0))$ and $n > 2$ then the truncated correlation functions have a limit:*

$$\langle\phi(\rho_1)\dots\phi(\rho_n)\rangle^T = \lim_{N\to\infty} \langle\phi(\rho_1)\dots\phi(\rho_n)\rangle^{N,T} .$$

4. *All the above are analytic in ζ in a neighbourhood of the origin.*

Proof. By Theorem 7.3, $K_0^N(\Delta, \rho)$ is analytic in $\rho \in C'(\Delta)$ for $\|\rho\| < u$ with bounds uniform in N and the limit as $N \to \infty$ exists. The same is true for $K_0^{N,\#}(\Delta, \rho)$: use (70) and the bound

$$|K_0^N(\Delta, \phi, \rho)| \leqq G(\Delta, \phi)\Gamma(\Delta)^{-1}(1 - \|\rho\|/u)^{-1}\|K_0^N\|_{0,u}$$

for dominated convergence. The convergence of $K_0^{N,\#}$ and $(\rho, v_{0,N}\rho)$ give the convergence of $S^N(\rho)$.

The convergence of the correlation functions for $\|\rho_i\| < u$ follows from convergence and uniform bounds for $S^N(\rho)$, the analyticity of $s_i \mapsto S^N(\sum s_i\rho_i)$ and the representation

$$\frac{\partial^n}{\partial s_1 \dots \partial s_n} S^N(\textstyle\sum s_i\rho_i)|_{s_i=0} = \displaystyle\oint \left[\prod_{i=1}^n (2\pi i)^{-1} s_i^{-2} ds_i\right] S^N(\textstyle\sum s_i\rho_i) .$$

Here the integral is taken over the circles with radius $|s_i| = \frac{1}{2}(n\|\rho_i\|)^{-1}u$. Since the correlation functions are linear the condition $\|\rho_i\| < u$ may be dropped.

The convergence of the truncated correlation functions follows similarly (use the smallness of $K_0^{N,\#}(\Delta, \rho)$ in (72) to get the analyticity of $s_i \mapsto \log S^N(\sum s_i\rho_i)$).

Analyticity in ζ follows from the analyticity for $N < \infty$ and bounds in ζ uniform in N. □

Remarks. (a) The condition $(\rho_i, v_{0,\infty}\rho_i) < \infty$ in (1), (2) does not allow δ-functions, but they are allowed in (3). Thus the truncated n-point functions $\langle\phi(x_1)\dots\phi(x_n)\rangle^T$ exist as bounded functions. This means the only short distance singularity in the theory is the logarithmic singularity of the two-point function $\langle\phi(x)\phi(y)\rangle \sim v_{0,\infty}(x, y)$ at $x = y$.
(b) We might also consider the Wick ordered field

$$:e^{i\phi(\rho)}: = \exp(\beta/2(\rho, v_{0,N}\rho))e^{i\phi(\rho)} .$$

Then

$$\langle :e^{i\phi(\rho)}:\rangle^N = (1 + K_0^{N,\#}(\Delta, 0))^{-1}(1 + K_0^{N,\#}(\Delta, \rho))$$

and we have a limit for all $\rho \in C'(\Lambda(0))$ with $\|\rho\| < u$; δ-functions are allowed. Similarly

$$\langle :e^{i\phi(\rho_1)}::e^{i\phi(\rho_2)}:\rangle^N = \exp(-\beta(\rho, v_{0,N}\rho_2))\frac{1 + K_0^{N,\#}(\Delta, \rho_1 + \rho_2)}{1 + K_0^{N,\#}(\Delta, 0)} \tag{73}$$

will have a limit for $\|\rho_i\| < u/2$.

In the Coulomb gas model the introduction of $:e^{i\phi(\rho)}:$ corresponds to adding an external charge density ρ in the Gibbs measure (with a self-energy subtraction). Equation (73) above measures the correlations between two such charges. Our condition $\|\rho\| < u \ll 1$ means these are fractional charges. The behaviour for integral charges may be quite different.

(c) The analyticity of the correlation functions in ζ means perturbation expansions around $\zeta = 0$ converge. One can compute for example

$$S(\rho) = e^{-\beta/2(\rho, v_{0\infty}\rho)}\,[1 + \zeta(\textstyle\int \cosh(\beta v_{0\infty}\rho)(x)dx - 1) + \mathcal{O}(\zeta^2)]\,.$$

This formula leads to first order expressions for the correlation functions.

9. Volume-Uniform Estimates for the Massive Model

In this final section, we dicuss in more detail the massive model formulated in the introduction. For convenience, we take $m = 1$ (this may be achieved by an a priori rescaling of the model). We begin with the measure $d\tilde{v}_M^N(\phi)$ of Eq. (3) on the torus $\Lambda(M)$ for $M \geqq 0$. (In the following, we often omit M labels.) After a rescaling up to a volume $\Lambda(M + N)$ one has an equivalent measure on $\mathscr{H}(\Lambda(M + N))$

$$\begin{aligned} dv^N(\phi) &= \exp(\zeta_N \textstyle\int \cos\phi(x)dx)d\mu_{\beta v_{M+N,0}}(\phi) \\ &= \mathscr{E}\mathrm{xp}(\Box + K^N)(\phi)d\mu_{\beta v_{M+N,0}(\phi)}\,, \end{aligned}$$

where

$$\hat{v}_{M+N,0}(p) = (p^2 + L^{-2N})^{-1}e^{-p^4}\,.$$

Just as for the massless model, the coupling constants are given the following N-dependence:

$$\zeta_N = L^{-2N}z_N = L^{-2N}\exp[\beta v_{N+M,0}(0)/2]\zeta\,.$$

We first note that we can reproduce the results previously obtained for $m = 0$, $M = 0$. Starting with the renormalized generating functional,

$$S^N(\rho) = \textstyle\int e^{i(\rho^N,\phi)}\mathscr{E}\mathrm{xp}(\Box + K^N)(\phi)d\mu_{\beta v_{N+M,0}}/[\rho = 0]\,,$$

(still with $\rho^N(x) = L^{-2N}\rho(L^{-N}x)$) we apply renormalization group transformations to obtain expressions on $\Lambda(M + i)$ in terms of polymer activities K_i^N and fluctuation covariances

$$\hat{C}_i(p) = (p^2 + L^{-2i})^{-1}(e^{-p^4} - e^{-L^4p^4})\,.$$

After N steps we have on $\Lambda(M)$

$$S^N(\rho) = \exp(-\beta/2\sum_{k=1}^{N}(\rho^k, C_k\rho^k))\textstyle\int e^{i(\rho,\phi)}\mathscr{E}\mathrm{xp}(\Box + K_0^N)(\phi,\rho)d\mu_{\beta v_{M,0}}/[\rho = 0]\,.$$

We make one last fluctuation and complex translation step with $v_{M,0}$ instead of C_0. This removes all the ϕ dependence and gives

$$S^N(\rho) = \exp(-\beta/2(\rho, v_{M,N}\rho))\mathscr{E}\mathrm{xp}(\Box + K_0^{N,\#})(\Lambda(M),\rho)/[\rho = 0]\,.$$

The analysis of these transformations goes more or less as before. We obtain the same bounds uniform in N and the $N \to \infty$ limit provided $|\zeta|$ is sufficiently small.

Now we look to obtain bounds which are uniform in M as well as N. These then hold in the $N \to \infty$ limit which we just proved, and would hold in the $M \to \infty$ limit oince that is obtained.

The above analysis is all uniform in M except possibly the last step. However since the covariance $v_{M,0}$ has unit mass, it has exponential decay and so the essential bound $\|(v_{M,0})_\theta\| \leqq \mathcal{O}(1)$ holds uniformly in M. This makes it possible to apply Propositions 1 and 2 from [DH3] to obtain

$$\begin{aligned}\|K_0^{N,\#}\|_{\gamma^{-1}\Gamma,u} &= \|\mathcal{T}\mathcal{F}_0 K_0^N\|_{\gamma^{-1}\Gamma,u} \\ &\leqq \|K_0^N\|_{G,\Gamma,h_0,u} \\ &\leqq |\zeta_0|^{1-\varepsilon} .\end{aligned}$$

Hereafter we shorten the notation to $K^{\#} = K_0^{N,\#}$.

To analyze the correlation functions we apply a polymer expansion to obtain:

$$\log[\mathcal{E}\mathrm{xp}(\square + K^{\#})(\Lambda(M),\rho)] = \sum_{k=1}^{\infty} \sum_{\vec{X} \in C_k} \frac{n(\vec{X})}{k!} \prod_i K^{\#}(X_i,\rho) ,$$

where C_k denotes the set of connected k-tuples $\vec{X} = (X_1, \ldots, X_k)$ in $\Lambda(M)$. The index is $n(\vec{X}) = \sum_G (-1)^{l(G)}$, where G is a connected graph on $\vec{X}$ and $l(G) =$ number of lines in G. (See for example [GJ], Theorem 20.2.1.)

We conclude

$$\log S^N(\rho) = -\beta/2(\rho, v\rho) + \sum_{k=1}^{\infty} \sum_{\vec{X} \in C_k} \frac{n(\vec{X})}{k!} \prod_i K^{\#}(X_i,\rho) - [\rho = 0] .$$

Now we are in a position to prove our main result of this section. Define

$$\Gamma'(x_1, \ldots, x_n) = \inf_X (\gamma^{-2}\Gamma)(X, x_1, \ldots, x_n) .$$

Thorem 9.1. *For $m = 1, 0 < \beta < 8\pi$, and $|\zeta|$ sufficiently small:*

$$|\langle \phi(x_1) \ldots \phi(x_n)\rangle^T| \leqq \mathcal{O}(1) n! u^{-n} 2^n |\zeta_0|^{1-\varepsilon} \Gamma'(x_1, \ldots, x_n)^{-1}, \quad n > 2 ,$$
$$|\langle \phi(x_1)\phi(x_2)\rangle^T - \beta v(x_1,x_2)| \leqq \mathcal{O}(1) u^{-2} |\zeta_0|^{1-\varepsilon} \Gamma'(x_1,x_2)^{-1}, \quad n = 2 ,$$

uniformly in all cutoffs.

Remarks. This proves polynomial tree decay for correlations at a rate which depends on the initial choice (67), (28) of large set regulator Γ. It turns out that Γ may be chosen with any polynomial rate. The natural result, namely an exponential decay rate, does not follow from our analysis in its present form.

Proof. The required quantities are given by

$$I = \frac{\partial}{\partial\rho(x_1)} \cdots \frac{\partial}{\partial\rho(x_n)} \log S^N(\rho)|_{\rho=0}, \quad n \geqq 2 .$$

The repeated Leibniz rule gives

$$I = \sum_{k \geqq 1} \frac{1}{k!} \sum_{\vec{\pi} \in \mathcal{P}_k(n)} \sum_{\vec{X} \in C_k} n(\vec{X}) \prod_{i=1}^{k} K_{|\pi_i|}^{\#}(X_i, x_{\pi_i}) ,$$

where $\mathscr{P}_k(n)$ denotes the ordered partitions of the set $\{1, \ldots, n\}$ into k (possibly empty) subsets. By a standard estimate

$$n(\vec{X}) \leqq \sum_{\tau:\ \text{trees on }(1,\ldots,k)} 1 .$$

Introduction of coordination numbers $\{d_i\}_{i=1}^k$, $d_i \geqq 1$, for the trees τ gives

$$|I| \leqq \sum_{k,\vec{\pi},\vec{d},\tau} \frac{1}{k!} \sum_{\vec{X}\in \mathscr{C}_k(\tau)} \prod_i |K^{\#}_{|\pi_i|}(X_i, x_{\pi_i})| ,$$

where now τ has the specified coordination numbers, and the connectedness of $\vec{X}$ is to be compatible with τ. Now $\Gamma'(x_1, \ldots, x_n) \leqq \prod_i (\gamma^{-2}\Gamma)(X_i, x_{\pi_i})$. Thus

$$\Gamma'(x_1, \ldots, x_n)|I| \leqq \sum_{k,\vec{\pi},\vec{d},\tau} \frac{1}{k!} \sum_{\vec{X}\in \mathscr{C}_k(\tau)} \prod_i (\gamma^{-2}\Gamma)(X_i, x_{\pi_i})|K^{\#}_{|\pi_i|}(X_i, x_{\pi_i})| .$$

In the usual way (see the proof of Lemma 5.2 or the proof of Lemma 3 in [DH 3]), we sum over X_i's in the order given by the tree, leaving till last an X_{i_0} chosen with π_{i_0} nonempty. We obtain

$$\Gamma'(x_1, \ldots, x_n)|I| \leqq \sum_{k,\vec{\pi},\vec{d},\tau} \frac{1}{k!} d_{i_0}! \prod_{i \neq i_0} (d_i - 1)! \left(\frac{3^2}{\log \gamma}\right)^{k-1} \prod_i \|K^{\#}_{|\pi_i|}\|_{\gamma^{-1}\Gamma} .$$

Using Cayley's theorem, $\sum_{\tau,\vec{d}} \prod_i (d_i - 1)! \leqq (k-2)! 4^{k-1}$ (and also $d_{i_0}! \leqq (k-1)(d_{i_0} - 1)!$),

$$\Gamma'(x_1, \ldots, x_n)|I| \leqq \sum_{k,\vec{\pi}} \frac{1}{k} \left(\frac{4\cdot 3^2}{\log \gamma}\right)^{k-1} \prod_i \|K^{\#}_{|\pi_i|}\|_{\gamma^{-1}\Gamma} .$$

Now

$$\|K^{\#}_p\|_{\gamma^{-1}\Gamma} \leqq u^{-p} p! \|K^{\#}\|_{\gamma^{-1}\Gamma, u} \leqq u^{-p} p! |\zeta_0|^{1-\varepsilon} ,$$

and

$$\sum_{\vec{\pi}} \prod_i |\pi_i|! = \frac{(k+n-1)!}{(k-1)!} \leqq n! 2^{n+k-1} .$$

Thus we have the required bound

$$\begin{aligned}\Gamma'(x_1, \ldots, x_n)|I| &\leqq (2/u)^n n! \sum_{k \geqq 1} \left(\frac{2\cdot 4\cdot 3^2}{\log \gamma}\right)^{k-1} |\zeta_0|^{(1-\varepsilon)k} \\ &\leqq 2(2/u)^n n! |\zeta_0|^{1-\varepsilon} .\end{aligned}$$

A. Analytic Functions on a Banach Space

We collect some facts about analytic functions on a Banach space (see [HP]). Let E, F be complex Banach spaces and U an open set in E. A function $f: U \to F$ is *Gateaux-analytic* or *G-analytic* if for each $u \in U$, $x \in E$ the function $\lambda \mapsto f(u + \lambda x)$ is analytic on a neighourhood of the origin in $\mathbf{C}$. Equivalently, f restricted to $U \cap E^0$ is analytic for any finite dimensional subspace E^0 of E (by Hartog's theorem).

If f is G-analytic then we define

$$f_n(u; x) = d^n/d\lambda^n f(u + \lambda x)|_{\lambda=0} .$$

For each $u \in U$ the function $x \mapsto f_n(u; x)$ from E to F is a homogeneous polynomial of degree n, i.e. the restriction to the diagonal of a symmetric multilinear functional f_n: $E^n \to F$. We have the expansion

$$f(u + x) = \sum_{n=0}^{\infty} 1/n!\, f_n(u; x)$$

for all x in a neighbourhood of zero. We also have the Cauchy bound for $u \in U$,

$$\| f_n(u) \| = \sup_{\|x\| < 1} \| f_n(u; x) \| \leqq \frac{n!}{R^n} \sup_{v \in B(u, R)} \| f(v) \| ,$$

provided $B(u, R) \subset U$.

The function f: $U \to F$ is defined to be *analytic* if one of the following equivalent conditions are satisfied:

1. f is G-analytic and continuous.
2. f is G-analytic and locally bounded.
3. f is Frechet-differentiable on U.
4. For each $u \in U$ there are homogeneous polynomials $p_n(u)$; $E \to F$ such that $f(u + x) = \sum_{n=0}^{\infty} p_n(u; x)$ with uniform convergence for x in a neighbourhood of zero. In this case $p_n(u; x) = 1/n!\, f_n(u; x)$.

The composition of two analytic functions is again analytic.

B. Proof of Lemma 5.1

Lemma 5.1 says that a solution $K(t)$ of (24) satisfies

$$\| K(t) \|_{g(t), \Gamma, \mathbf{h}'} \leqq \| K \|_{g(0), \Gamma, \mathbf{h}} ,$$

if $0 < h_0' < h_0, 0 < h_1' < h_1$ and for some constant δ

$$\| K \|_{G, \Gamma, \mathbf{h}} \leqq \frac{\delta}{\beta \| C_\theta \|} \Delta(\mathbf{h}, \mathbf{h}')^2 ,$$

where

$$\Delta(\mathbf{h}, \mathbf{h}') = \min(h_0 - h_0', h_1 - h_1') .$$

This is proved in [BY] with the assumption $h_0 = h_1, h_0' = h_1'$. We explain here the modifications needed for the general case $h_0 \neq h_1$.

As in [BY] one shows that $k(t, \mathbf{h}) = \| K(t) \|_{g(t), \Gamma, \mathbf{h}}$ is dominated by the solution $u(t, \mathbf{h})$ of the Hamilton–Jacobi equation ($c = \beta \| C_\theta \|$):

$$\frac{\partial u}{\partial t} = c |\nabla u|^2 ,$$

$$u(0, \mathbf{h}) = k(\mathbf{h}) \equiv k(0, \mathbf{h}) .$$

We must show that solutions exist, are analytic, and that $u(t, \mathbf{h}') \leqq k(\mathbf{h})$ when $k(\mathbf{h}) \leqq (\delta/c) \Delta(\mathbf{h}, \mathbf{h}')^2$.

The Hamilton–Jacobi equation can be solved by the method of characteristics and one finds

$$u(t, \xi) = -c(\mathbf{z} \cdot \mathbf{z})t + k(\xi + 2ct\mathbf{z}) ,$$

where $\mathbf{z} = \mathbf{z}(t, \xi)$ is the solution of

$$\mathbf{z} - (\nabla k)(\xi + 2ct\mathbf{z}) = 0 .$$

If the last equation has a unique solution, analytic in ξ and it will give a (unique) analytic solution $u(t, \xi)$. Furthermore, $\mathbf{z}$ and u will be real if ξ is real.

We will find a solution for $|\xi_0| < h'_0$, $|\xi_1| < h'_1$ in the region $|\mathbf{z}| < \Delta(\mathbf{h}, \mathbf{h}')/4c$. First note that $\mathbf{h}' + 2ct|\mathbf{z}| < \mathbf{h}' + \Delta(\mathbf{h}, \mathbf{h}')/2$ and this is at least $\Delta(\mathbf{h}, \mathbf{h}')/2$ from the boundary of the polydisc of radius $\mathbf{h}$ around 0. Furthermore the maximum value of k on this polydisc is $k(\mathbf{h})$. Thus by the Cauchy bounds for the analytic function $k(\mathbf{h})$

$$|(\nabla k)(\xi + 2ct\mathbf{z})| \leqq [4/\Delta(\mathbf{h}, \mathbf{h}')]k(\mathbf{h}) .$$

Now we use a version of the analytic implicit function theorem due to Gallavotti [Ga2] which says that $\mathbf{z} - \mathbf{g}(\mathbf{z}) = 0$ has a unique solution in the region $|\mathbf{z}| < \rho$ if $|\mathbf{g}(\mathbf{z})| < \gamma^{-1}\rho$ for all $|\mathbf{z}| < \rho$, where γ is a (large) universal constant. Here we have $\rho = \Delta(\mathbf{h}, \mathbf{h}')/4c$ and $\mathbf{g}(\mathbf{z}) = (\nabla k)(\xi + 2ct\mathbf{z})$. Using our bound $|\mathbf{g}(\mathbf{z})| \leqq (c\rho)^{-1}k(\mathbf{h})$ from above and our hypothesis $k(\mathbf{h}) \leqq 16\delta c\rho^2$ we have $|\mathbf{g}(\mathbf{z})| \leqq 16\delta\rho < \gamma^{-1}\rho$ if we take $\delta < (16\gamma)^{-1}$. Thus $\mathbf{z} = \mathbf{z}(t, \xi)$ exists, can be shown to be analytic in ξ, and gives the solution.

Finally, then $u(t, \mathbf{h}') \leqq k(\mathbf{h}' + 2ct\mathbf{z}) \leqq k(\mathbf{h})$ as required.

C. Erratum to [DH1]

We take this opportunity to correct some errors in the companion paper [DH1]. These are due to an insufficiently careful choice of the analyticity parameters $\mathbf{h}$ and the large field regulators G_κ. One mistake, pointed out to us by Dr. D.H.U. Marchetti, was that (4.4) in [DH1] only holds when $h_0 - Bh_1 > 0$, but was applied when $h_0 = h_1$ and so $h_0 - Bh_1 < 0$. A second mistake was that the application of Lemma 4.3 of [BY] in Lemma 7.1 of [DH1] needed $(\kappa_0 h_1^{*2})^{-1} \leqq \mathcal{O}(1)$ and $(\kappa_0 h_2^{*2})^{-1} \leqq \mathcal{O}(1)$ independent of L. But κ_0 was forced to be $\mathcal{O}(L^{-2})$ by the requirement that the homotopy property hold and we chose $h_1^* = \mathcal{O}(L^{-1})$ and $h_2^* = \mathcal{O}(L^{-2})$. Thus we have incompatible conditions for L large.

Both errors can be corrected by taking $G^j, G^\# = G^*$ as in the present paper (allowing $\kappa_0 = \mathcal{O}(1)$) and making the choices

$$\begin{aligned} \mathbf{h}^j &= h^j(aL, L, L^2) , \\ \mathbf{h}^\# &= h^{j+1}(aL, L/2, L^2/2) , \\ \mathbf{h}^* &= h^{j+1}(aL, 1, 1) , \end{aligned}$$

with $a > B$ so $h_0^j - Bh_1^j > 0$.

The proof proceeds as before with the following modifications. In Proposition 1(i) we must use the bound on the fluctuation transformation $\mathcal{F}$ of the present paper (see Appendix B). This allows unequal components in $\mathbf{h}$. Proposition 1(ii) should read for $q \neq 0$,

$$\|k^\#(q)\|_{G^\#, \Gamma^\#, \hat{\mathbf{h}}^*} \leqq \exp(-(|q| - 1/2)\beta^* - [h_0^j - B(h_1^* + |\partial f|_\infty)]|q|)\delta^j .$$

For Proposition 2 we have the same bound as above for $\|i(q)\|_{G^\#, \Gamma^\#, \hat{\mathbf{h}}^*}$ and estimate $\exp(B(h_1^* + |\partial f|_\infty) \leqq \mathcal{O}(1)$ so the bound is $\mathcal{O}(1)\exp(-(|q| - 1/2)\beta^* - h_0^j|q|)$. This combined with $\|\exp(iq\psi_0(x))\|_{1, h_0^*} \leqq \exp(h_0^*|q|)$ and $h_0^* < h_0^j$ bounds the $q \neq 0$ terms in I. For $q = 0$ we have by Lemma 4.3 of [BY] and a separate bound on $F - R$ that $\|i(0)\|_{G^{\#}, \Gamma^{\#}, \hat{\mathfrak{h}}^*} \leqq \mathcal{O}(1) L^{-3} \delta^j$. Altogether this yields

$$\|I\|_{G^{\#}, \Gamma^{\#}, \hat{\mathfrak{h}}^*} \leqq \mathcal{O}(1)(L^{-3} + L^{-\beta/4\pi})\delta^j$$

in Lemma 7.1. The bound $\|K^*\|_{g^*, \Gamma^*, \mathfrak{h}^*} \leqq \delta^{j+1}$ follows as before.

With these changes the main results of the paper still hold.

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