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$\mathcal{W}$-Geometry

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Abstract. The geometric structure of theories with gauge fields of spins two and higher should involve a higher spin generalisation of Riemannian geometry. Such geometries are discussed and the case of $\mathcal{W}_\infty$-gravity is analysed in detail. While the gauge group for gravity in d dimensions is the diffeomorphism group of the space-time, the gauge group for a certain $\mathcal{W}$-gravity theory (which is $\mathcal{W}_\infty$-gravity in the case $d = 2$) is the group of symplectic diffeomorphisms of the cotangent bundle of the space-time. Gauge transformations for $\mathcal{W}$-gravity gauge fields are given by requiring the invariance of a generalised line element. Densities exist and can be constructed from the line element (generalising $\sqrt{\det g_{\mu\nu}}$) only if $d = 1$ or $d = 2$, so that only for $d = 1, 2$ can actions be constructed. These two cases and the corresponding $\mathcal{W}$-gravity actions are considered in detail. In $d = 2$, the gauge group is effectively only a subgroup of the symplectic diffeomorphism group. Some of the constraints that arise for $d = 2$ are similar to equations arising in the study of self-dual four-dimensional geometries and can be analysed using twistor methods, allowing contact to be made with other formulations of $\mathcal{W}$-gravity. While the twistor transform for self-dual spaces with one Killing vector reduces to a Legendre transform, that for two Killing vectors gives a generalisation of the Legendre transform.

1. Introduction

$\mathcal{W}$-gravity is a higher-spin generalisation of gravity which plays an important rôle in two-dimensional physics and has led to new generalisations of string theory [1–12] (for a review, see [13]). The gauge fields are the two-dimensional metric $h_{\mu\nu}$ together with a (possibly infinite) number of higher-spin gauge fields $h_{\mu\nu\ldots\rho}$. $\mathcal{W}$-gravity can be thought of as the gauge theory of local $\mathcal{W}$-algebra symmetries in the same sense that two-dimensional gravity can be thought of as the result of gauging the Virasoro algebra, and different $\mathcal{W}$-algebras lead to different $\mathcal{W}$-gravities. A $\mathcal{W}$-algebra is an extended conformal algebra containing the Virasoro algebra and is generated by a spin-two current and a number of other currents, including some of spin greater than two [22–26] (for a review, see [27]).

A matter system with $\mathscr{W}$-algebra symmetry can be coupled to $\mathscr{W}$-gravity in such a way that the conformal symmetry is promoted to a diffeomorphism symmetry and the whole $\mathscr{W}$-algebra symmetry is promoted to a local gauge symmetry. For chiral $\mathscr{W}$-algebras, the resulting coupling is always linear in the gauge fields [4, 5], but if both left and right handed $\mathscr{W}$-algebras are gauged, the theory is non-polynomial in the gauge fields of spin-two and higher [1]. For the coupling to pure gravity, the key to understanding the non-linear structure is Riemannian geometry. The spin-two gauge field is interpreted as a Riemannian metric and the non-linear action is then easily constructed using tensor calculus and the fundamental density, $\sqrt{-h}$, where $h = \det(h_{\mu\nu})$. This suggests that the non-polynomial structure of $\mathscr{W}$-gravity might be best understood in terms of some higher-spin generalisation of Riemannian geometry and the aim of this paper is to present just such an interpretation. The main results, which include the construction of the full non-linear action in closed form (without using auxiliary fields), were first summarised in [14], but here a more detailed account will be given and the geometry of the results will be discussed. Other approaches to the geometry of $\mathscr{W}$-algebras and $\mathscr{W}$-gravity are given in [15–21].

In Riemannian geometry, the line element for a manifold M is given in terms of the metric $h_{\mu\nu}(x)$ by

$$ds = (h_{\mu\nu}\, dx^\mu\, dx^\nu)^{1/2} . \tag{1.1}$$

An equally good line element can be defined using an n^{th} rank tensor field $h_{\mu_1\mu_2\ldots\mu_n}$:

$$ds = (h_{\mu_1\mu_2\ldots\mu_n}\, dx^{\mu_1}\, dx^{\mu_2} \ldots dx^{\mu_n})^{1/n} , \tag{1.2}$$

and this can be used to construct a geometry with almost all the features of the usual Riemannian geometry, although Pythagoras' theorem is replaced by a relation between the n^{th} powers of lengths.[1] In fact, the line element (1.2) was considered by Riemann [30], but rejected in favour of the simpler alternative (1.1).

A further generalization is to consider a line element

$$ds = N(x, dx) , \tag{1.3}$$

where N is some function which is required to satisfy the homogeneity condition

$$N(x, \lambda dx) = \lambda N(x, dx) \tag{1.4}$$

so that scaling a coordinate interval scales the length of that interval by the same amount. This defines a Finsler geometry [31] and (1.1) and (1.2) arise with special choices of the Finsler metric function N. The length of a curve $x^\mu(t)$ is given by $\int dt\, N(x, \dot{x})$ and this is invariant under reparameterizations $t \to t'(t)$ as a result of (1.4). It is possible to define Finsler geodesics, connections, curvatures etc. [31] and even to attempt a Finsler generalisation of general relativity (see [31] and references therein).

To describe $\mathscr{W}$-gravity, it is necessary to further generalise the geometry by adopting a general line element (1.3), without imposing the Finsler homogeneity condition (1.4). Then N is a real function on the tangent bundle TM that defines the

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[1] The line element (1.2) is invariant under the diffeomorphisms $\delta x^\mu = -k^\mu(x)$, $\delta h_{\mu_1\mu_2\ldots\mu_n} = \mathscr{L}_k h_{\mu_1\mu_2\ldots\mu_n}$, where $\mathscr{L}_k$ denotes the Lie derivative with respect to k^μ. The transformation of $h_{\mu_1\mu_2\ldots\mu_n}$ can be rewritten in a suggestive way as $\delta h_{\mu_1\mu_2\ldots\mu_n} = n\nabla_{(\mu_1} k_{\mu_2\ldots\mu_n)}$, where $k_{\mu_2\ldots\mu_n} = k^{\mu_1} h_{\mu_1\mu_2\ldots\mu_n}$ and ∇ is an affine connection constructed using $h_{\mu_1\ldots\mu_n}$

length of a tangent vector $y^\mu \in T_x M$ at $x \in M$ to be $|y| = N(x, y)$. It will prove convenient to work with the "metric function" $f(x, y) = N^2(x, y)$ instead of N. Expanding in y

$$f(x, y) = \ldots h_{\mu\nu}(x) y^\mu y^\nu + \ldots + h_{\mu_1 \mu_2 \ldots \mu_n}(x) y^{\mu_1} y^{\mu_2} \ldots y^{\mu_n} + \ldots \tag{1.5}$$

gives a series of coefficients $h_{\mu\nu}, h_{\mu_1\mu_2\ldots\mu_n}, \ldots$ and the line element will be coordinate-independent if these transform as tensors under diffeomorphisms of M. The gauge fields of $\mathcal{W}$-gravity will be given a geometric meaning by relating them to such tensors.

Similarly, the inverse metric, which defines the squared length $|y|^2 = h^{\mu\nu} y_\mu y_\nu$ of a cotangent vector y_μ can be generalised by introducing a "cometric function" $F(x^\mu, y_\nu)$ on the cotangent bundle and defining the length of $y_\mu \in T^*_x M$ to be given by $|y|^2 = F(x^\mu, y_\nu)$. Expanding in y as in (1.5) gives

$$F(x^\mu, y_\mu) = \sum_n \frac{1}{n} h_{(n)}^{\mu_1 \ldots \mu_n}(x) y_{\mu_1} \ldots y_{\mu_n}\,, \tag{1.6}$$

where the coefficients $h_{(n)}^{\mu_1 \ldots \mu_n}(x)$ are contravariant tensors on M, so that the "length" of a cotangent vector is coordinate independent. For many purposes, we will find it convenient to work with a cometric function rather than a metric function.

We shall eventually want to regard the $h_{(n)}^{\mu_1 \ldots \mu_n}(x)$ as higher spin gauge fields on M with transformations of the form

$$\delta h_{(n)}^{\mu_1 \ldots \mu_n} = -\, n h_{(2)}^{\nu(\mu_1} \partial_\nu \lambda_{(n)}^{\mu_2 \ldots \mu_n)} + \ldots \tag{1.7}$$

plus higher order terms involving the gauge fields, where the infinitesimal parameter $\lambda_{(n)}^{\mu_1 \ldots \mu_{n-1}}(x)$ is a rank $n-1$ symmetric tensor. The cometric function (1.6) can be regarded simply as a generating function for these gauge fields, but as we shall see, the gauge transformations have a natural geometric interpretation on $T^* M$.

As the homogeneity condition (1.4) has been dropped, it is possible to consider a much larger group of transformations than the diffeomorphisms of M, Diff(M), namely the diffeomorphisms of the tangent bundle (Diff(TM)) or cotangent bundle (Diff(T^*M)), which in general mix x and y. It is natural to demand that f and F be invariant (scalar) functions on TM and $T^*(M)$, i.e. that $F'(x', y') = F(x, y)$ etc., and the transformation $F \to F'$ corresponds to variations under which the gauge fields $h^{\mu_1 \ldots \mu_n}$ of different spins transform into one another. These transformations turn out to be too general, however. Roughly speaking, this is because they do not preserve the important difference between the coordinates x on the base manifold M and the fibre coordinates y. More precisely, the action of Diff($T^* M$) leads to transformations of the $h_{(n)}^{\mu_1 \ldots \mu_n}(x)$ which depend on both x and y, and so are not of the desired form (1.7) of transformations of higher spin gauge fields on M whose transformations depend on x alone.

For this reason, we seek a natural subgroup of the bundle diffeomorphisms. For the cotangent bundle, we consider the symplectic diffeomorphism group $\mathrm{Diff}_0(T^* M)$ consisting of the subgroup of the diffeomorphism group that preserves the natural symplectic form $\Omega = dx^\mu_\wedge dy_\mu$. We shall discover the remarkable result that requiring the cometric function (restricted to certain natural sections of the bundle) to be invariant under symplectic diffeomorphisms leads to a natural set of transformations for the gauge fields $h^{\mu_1 \ldots \mu_n}$ that are independent of y. This is true for *any* dimension of M. A role for $\mathrm{Diff}_0(T^*M)$ has been suggested previously in the context of $\mathscr{W}$-gravity [17, 3]. As will be seen, $\mathrm{Diff}_0(T^*M)$ turns out to be the gauge group for one-dimensional $\mathscr{W}$-gravity, but for the two-dimensional case this is still too big and it is necessary to restrict further to a subgroup of $\mathrm{Diff}_0(T^*M)$.

In order to construct actions, we shall need some generalisation of the density $\sqrt{-h}$, $h = \det[h_{\mu\nu}]$. To construct an action for a scalar field ϕ, all that is needed is a tensor density $\tilde{h}^{\mu\nu}$, as the action

$$S = \frac{1}{2}\int d^d x\, \tilde{h}^{\mu\nu}\, \partial_\mu \phi \partial_\nu \phi \tag{1.8}$$

is then invariant. The tensor density can be regarded as an independent field, but as $\det[\tilde{h}^{\mu\nu}]$ is a scalar in two dimensions, one can consistently impose the constraint $\det[\tilde{h}^{\mu\nu}] = -1$ and this can then be solved in terms of an unconstrained metric $h_{\mu\nu}$ as $\tilde{h}^{\mu\nu} = \sqrt{-h}h^{\mu\nu}$, so that (1.8) becomes the standard minimal coupling. The quantity $\tilde{F}(x, y) = \frac{1}{2}\tilde{h}^{\mu\nu} y_\mu y_\nu$ changes by a total derivative on M under an infinitesimal diffeomorphism, $\delta\tilde{F} = \partial(k^\mu \tilde{F})/\partial x^\mu$ so that $\int_M d^d x \tilde{F}$ is invariant (with appropriate boundary conditions).

In order to construct $\mathscr{W}$-gravity actions, we shall need a "cometric density"

$$\tilde{F}(x^\mu, y_\mu) = \sum_n \frac{1}{n}\tilde{h}^{\mu_1 \dots \mu_n}_{(n)}(x) y_{\mu_1} \dots y_{\mu_n} \tag{1.9}$$

which transforms by a total derivative under an infinitesimal $\mathscr{W}$-gravity gauge transformation, so that $\int_M d^d x \tilde{F}$ is $\mathscr{W}$-invariant. In particular, invariance under $\mathrm{Diff}(M)$ will require that the $\tilde{h}^{\mu_1 \dots \mu_n}_{(n)}(x)$ transform as tensor densities under $\mathrm{Diff}(M)$. We will show that, with the gauge group $\mathrm{Diff}_0(T^*M)$, such cometric densities do not exist for dimensions $d > 2$, that they do exist for $d = 1$ and that they do not exist for $d = 2$, but that there are cometric densities in $d = 2$ for a certain subgroup $\mathscr{H}$ of $\mathrm{Diff}_0(T^*M)$. This means that $\mathscr{W}$-gravity actions of the type investigated in this paper exist only for $d = 1, 2$ and that the $\mathscr{W}$-gravity gauge group in $d = 1$ is $\mathrm{Diff}_0(T^*M)$ while that in $d = 2$ is the subgroup $\mathscr{H} \subset \mathrm{Diff}_0(T^*M)$. In one dimension, we give an explicit construction of a cometric density from a cometric, generalising the construction $\tilde{h}^{\mu\nu} = \sqrt{-h}h^{\mu\nu}$. In $d = 2$, we show that the constraint that generalises $\det[\tilde{h}^{\mu\nu}] = -1$ is

$$\det\left(\frac{\partial^2 \tilde{F}(x, y)}{\partial y_\mu \partial y_\nu}\right) = -1 \tag{1.10}$$

and give some evidence to support the conjecture that a cometric density satisfying this constraint can be written in terms of a cometric. This is the real Monge–Ampère equation [29] and is sometimes referred to as one of Plebanski's equations [28].

The plan of the paper is as follows. In Sect. 2, classical $\mathscr{W}$-algebras and linearised $\mathscr{W}$-gravity will be reviewed and in Sect. 3 the construction of $\mathscr{W}$-gravities involving auxiliary variables [2] will be reviewed. In Sect. 4, d-dimensional $\mathscr{W}$-gravity and symplectic diffeomorphisms are introduced, cometrics and cometric densities are analysed in Sect. 5 and actions are constructed in Sect. 6. In Sect. 7, the relation between Eq. (1.10) and self-dual geometry in four dimensions is used to motivate a twistor-transform solution of (1.10) which leads to a recovery of the auxiliary variable formulation of Sect. 3. In Sect. 8, the solution of (1.10) that generalises the construction $\tilde{h}^{\mu\nu} = \sqrt{-h}h^{\mu\nu}$ is considered and $\mathcal{W}$-Weyl symmetry discussed. Section 9 summarises the results and discusses some generalisations.

2. Classical $\mathcal{W}$-Algebras and Linearised $\mathcal{W}$-Gravity

The $\mathcal{W}_\infty$ algebra [25] is a Lie algebra generated by an infinite set of currents $W^r = \{W^2, W^3, W^4, \ldots\}$, where W^r has spin r. Expanding W^r in modes W^r_n, the algebra can be written as

$$[W^r_m, W^s_n] = \{(r-1)n - (s-1)m\} W^{r+s}_{m+n} \tag{2.1}$$

with $r, s \geqq 2$. (Note that this algebra is sometimes referred to as w_∞. Throughout this paper, $\mathcal{W}_\infty$ will be used to denote the algebra (2.1).) Expanding the range of r, s to include a spin-one field W^1 gives the algebra $\mathcal{W}_{1+\infty}$ [3], which is the algebra of symplectic diffeomorphisms of the cylinder, $\mathbb{R} \times S^1$. Note that the spin-two current W^2 generates a Virasoro algebra (without central charge).

This algebra can be realised as the symmetry algebra of the non-linear sigma-model with action

$$S_0 = \frac{1}{2}\int d^2x\, g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j\,, \tag{2.2}$$

where the fields $\phi^i(x^\mu)$ are maps from two-dimensional flat space-time[2] (with coordinates x^μ) to a suitable target space $\mathcal{M}$, which is some D-dimensional manifold with coordinates ϕ^i $(i = 1, \ldots, D)$ and metric $g_{ij}(\phi^i)$. It is useful to introduce null coordinates, $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, so that the flat metric is $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = 2dx^+dx^-$. Then any symmetric rank-s tensor $T_{\mu_1\ldots\mu_s}$ is traceless ($\eta^{\mu\nu}T_{\mu\nu\rho\sigma\ldots} = 0$) if $T_{+-\rho\sigma\ldots} = 0$, so that it has only two non-vanishing components, $T_{++\ldots+}$ and $T_{--\ldots-}$, which both have spin s, but have helicities s and $-s$ respectively.

The spin-two currents

$$W_{(\pm 2)} = T_{\pm\pm} = \frac{1}{2}g_{ij}\partial_\pm\phi^i\partial_\pm\phi^j \tag{2.3}$$

are the components of the traceless stress-energy tensor $T_{\mu\nu}$ and satisfy the conservation laws $\partial_\mp T_{\pm\pm} = 0$. They generate the conformal transformations $\delta\phi^i = k_\mp\partial_\pm\phi^i$, where the parameters satisfy $\partial_\pm k_\pm = 0$; these conformal transformations are a symmetry of (2.2). Any symmetric tensor $d^{(s)}_{i_1\ldots i_s}(\phi)$ on $\mathcal{M}$ can be used to define the spin-s currents

$$W_{(\pm s)} = \frac{1}{s}d^{(s)}_{i_1 i_2\ldots i_s}\partial_\pm\phi^{i_1}\partial_\pm\phi^{i_2}\ldots\partial_\pm\phi^{i_s} \tag{2.4}$$

which are conserved if the tensor is covariantly constant, i.e. $\partial_\mp W_{(\pm s)} = 0$ if

$$\nabla_j d_{i_1\ldots i_s} = 0\,, \tag{2.5}$$

[2] Throughout this paper, the two-dimensional space-time or world-sheet will be taken to have Lorentzian signature. The conversion of formulae to the Euclidean case is straightforward and given explicitly in [14]

where ∇_j is the covariant derivative involving the Christoffel connection for the metric g_{ij}. If (2.5) is satisfied, these currents generate the following semi-local symmetries of S_0:

$$\delta\phi^j = \sum_s \lambda^{(\pm s)} d^j_{i_1 i_2 \ldots i_{s-1}} \partial_\pm \phi^{i_1} \partial_\pm \phi^{i_2} \ldots \partial_\pm \phi^{i_{s-1}}, \tag{2.6}$$

where $\lambda^{(\pm s)}$ are parameters of helicity $\pm 1 \mp s$ satisfying $\partial_\mp \lambda^{(\pm s)} = 0$. These non-linear transformations are higher-spin generalisations of the spin-two conformal transformation [1].

Any set of covariantly constant symmetric tensors on $\mathscr{M}$ gives in this way a set of conserved currents W^A, each of which generates a semi-local symmetry of S_0. The symmetry algebra and corresponding current algebra will then only close if the tensors satisfy certain non-linear algebraic identities [1, 5]. If the current algebra is non-linear (i.e. not a Lie algebra), as will usually be the case for algebras generated by a finite number of currents, then the corresponding symmetry algebra has field-dependent structure functions instead of structure constants [1]. For $\mathscr{W}_\infty$, $d^{(2)}_{ij} = g_{ij}$ and a rank-s symmetric tensor is needed for each $s = 3, 4, \ldots$. The algebra generated by the currents (2.4) closes to give the algebra (2.1) provided the tensors satisfy the following algebraic constraint [5]:

$$d^{(s)}_{i(j_1 j_2 \ldots j_{s-2}} {}^i d^{(t)}_{j_{s-1} j_s \ldots j_{s+t-3}) j_{s+t-2} l} = d^{(s+t-2)}_{j_1 j_2 \ldots j_{s+t-2} l} \tag{2.7}$$

for all s, t. For flat $\mathscr{M}$, there is a solution to this corresponding to any Jordan algebra, with $d^{(3)}$ proportional to the structure constants of that algebra [26]. For $D = 1$, there is a solution with $d^{(s)}_{11\ldots 1} = 1$ for all s, (corresponding to the one-dimensional Jordan algebra $\mathbb{R}$). For Jordan algebras of order N (i.e. those with a norm which is an N^{th} degree polynomial), then, as in [3], the algebra "telescopes," i.e. all currents $W_{\pm s}$ of spin $s > N$ can be written as products of the currents of spin $s \leqq N$ [5]. Then the algebra can be regarded as closing on the finite set of currents of spin $\leqq N$, giving the non-linear $\mathscr{W}_N$ algebra, which is a certain classical limit of the $\mathscr{W}_N$ algebras found in [23, 24]. For example, for Jordan algebras with cubic norm [26], the spin-four current can be written locally in terms of the spin-two current as $W_{(\pm 4)} = W_{(\pm 2)} W_{(\pm 2)}$ and all higher currents can be written in terms of $W_{(\pm 2)}, W_{(\pm 3)}$. This leads to (two copies of) the algebra $\mathscr{W}_3$, generated by $W_{(\pm 2)}, W_{(\pm 3)}$, with classical commutation relations given by (2.1) for $r, s \leqq 3$ with $W_{(\pm 4)} = (T_{\pm\pm})^2$. This algebra is a classical limit of Zamolodchikov's quantum operator algebra [22].

The chiral semi-local $\mathscr{W}$-algebra symmetry can be promoted to a fully local one (with parameters $\lambda^{(\pm s)}(x^+, x^-)$ depending on both x^+ and x^-) by coupling to gauge fields $h^{(\pm s)}$ which transform as

$$\delta h^{(\pm s)} = \partial_\mp \lambda^{(\pm s)} + O(h) \tag{2.8}$$

plus terms of higher order in the gauge fields [1, 5]. The linearised action is then given by adding the Noether coupling to S_0, giving

$$S_0 + S_1 = S_0 + 2\int d^2x \sum_{s=2}^{\infty} [h^{(+s)} W_{(+s)} + h^{(-s)} W_{(-s)}] \tag{2.9}$$

which is invariant under the linearised transformations (2.6), (2.8) for general local parameters λ, up to terms dependent on the gauge fields. These can be cancelled by adding terms of higher order in the gauge fields and the full gauge-invariant action, which can be constructed perturbatively to any given order in the gauge fields using the Noether method, is non-polynomial in the gauge fields [1]. The aim of this paper is to investigate the full non-linear structure of this theory and give it a geometric interpretation. Note that the linearised field equations given by varying (2.9) with respect to the gauge fields imply the W_∞ constraints $W_{\pm s} = 0$.

Although only the bosonic realisation of $\mathscr{W}_\infty$ gravity will be considered here, other realisations and other $\mathscr{W}$-algebras can be treated in a similar way. For bosonic realisations, choosing different sets of d-tensors satisfying different algebraic constraints gives different $\mathscr{W}$-algebras [5]. Similar $\mathscr{W}$-algebra realisations are obtained in many other models, including free-fermion theories, Wess–Zumino–Witten models and Toda field theories. In each case, the symmetry can be gauged by coupling to an appropriate $\mathscr{W}$-gravity, with gauge fields corresponding to each current [5]. For any model with a classical $\mathscr{W}$-algebra symmetry, the chiral gauging of the right-handed $\mathscr{W}$-symmetry, i.e. the coupling to the gauge fields $h^{(+s)}$, is given completely by the Noether coupling (2.9) with $h^{(-s)}$ set equal to zero, and no higher order terms in the gauge fields are needed [4, 5]. For models with a $\mathscr{W}_\infty$ symmetry which telescopes down to a $\mathscr{W}_N$ symmetry, the coupling to linearised $\mathscr{W}_N$ gravity is obtained by setting all the gauge fields of spin $s > N$ to zero in the coupling to $\mathscr{W}_\infty$-gravity and modifying the transformations, as in [3]; however, the coupling to non-linear $\mathscr{W}_N$ gravity is rather more subtle [37].

3. Non-Linear Gravity and $\mathscr{W}$-Gravity

Consider first the coupling of the sigma-model (2.2) to two dimensional gravity. The conformal invariance implies that the only components of the stress-energy tensor are $T_{\pm\pm} = g_{ij}\partial_\pm\phi^i\partial_\pm\phi^j$ and the linearised Noether coupling to the spin-two gauge fields $h_{\pm\pm}$ is given by

$$S_{lin} = \frac{1}{2}\int d^2x(g_{ij}\partial_+\phi^i\partial_-\phi^j - h_{--}T_{++} - h_{++}T_{--})\,. \tag{3.1}$$

The Noether method gives the higher order terms, which can be summed to give

$$S_n = \frac{1}{2}\int d^2x\frac{1}{1-h_{--}h_{++}}([1+h_{--}h_{++}]g_{ij}\partial_+\phi^i\partial_+\phi^j \\ - h_{--}T_{++} - h_{++}T_{--})\,. \tag{3.2}$$

This non-polynomial action is invariant under diffeomorphisms, with $\delta\phi^i = k^\mu\partial_\mu\phi^i$ and δh as in [32]. Following [2], it can be re-written in a polynomial "first-order" form as

$$S_a = 2\int d^2x\, g_{ij}\Bigg[\pi^j_+\partial_-\phi^i + \pi^j_-\partial_+\phi^i - \pi^i_+\pi^j_- - \frac{1}{2}\partial_+\phi^i\partial_-\phi^j \\ - \frac{1}{2}h_{--}\pi^i_+\pi^j_+ - \frac{1}{2}h_{++}\pi^i_-\pi^j_-\Bigg]\,. \tag{3.3}$$

Solving the algebraic field equations for the auxiliary fields $\pi^i_\pm$ and substituting the solutions into (3.3) gives back the action (3.2).

Although the actions (3.2) and (3.3) give a gauge-invariant coupling to spin-two gauge fields, they give little insight into the geometric structure and most would prefer to use the geometric coupling

$$S_{\rm geom} = \frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j g_{ij} \tag{3.4}$$

to a metric $g_{\mu\nu}$. This is invariant under diffeomorphisms and under the Weyl transformation $\delta g_{\mu\nu} = \sigma(x) g_{\mu\nu}$. Choosing the parameterization

$$g_{\mu\nu} = \Omega \begin{pmatrix} 2h_{++} & 1 + h_{++}h_{--} \\ 1 + h_{++}h_{--} & 2h_{--} \end{pmatrix} \tag{3.5}$$

for the metric $g_{\mu\nu}$, the conformal factor Ω drops out of (3.4) as a result of Weyl invariance and the action becomes (3.2), with the singularity of (3.2) at $h_{++}h_{--} = 1$ corresponding to the singularity of (3.4) when $g = \det(g_{\mu\nu})$ vanishes.

Consider now the coupling to $\mathscr{W}_\infty$-gravity. The linearised coupling is given by (2.9) and the higher-order terms can be constructed perturbatively, but no obvious pattern emerges and no closed form summation analogous to (3.2) appears feasible. The approach of [2] gives a generalisation of the action (3.3) that is fully invariant under local $\mathscr{W}$-symmetries [3, 7, 5]. The action is

$$\begin{aligned} S &= S_0 + S_n\,, \\ S_0 &= 2 \int d^2x\, g_{ij} \left[\pi^j_+ \partial_- \phi^i + \pi^j_- \partial_+ \phi^i - \pi^i_+ \pi^j_- - \frac{1}{2} \partial_+ \phi^i \partial_- \phi^j \right], \\ S_n &= 2 \int d^2x \sum_{s=2}^{\infty} [h^{(+s)} W_{(+s)}(\pi) + h^{(-s)} W_{(-s)}(\pi)]\,, \end{aligned} \tag{3.6}$$

where

$$W_{(\pm s)}(\pi) = \frac{1}{s} d^{(s)}_{i_1 i_2 \ldots i_s} \pi^{i_1}_\pm \pi^{i_2}_\pm \ldots \pi^{i_s}_\pm\,. \tag{3.7}$$

However, the polynomial field equations for the auxiliary fields $\pi^i_\pm$ cannot be solved in closed form, so that the fields $\pi^i_\pm$ cannot be eliminated. Nevertheless, these field equations can be solved perturbatively to any given order in the gauge fields, and the perturbative solution can then be used to reproduce the Noether-method perturbative action to that order in the gauge fields.

It is clearly desirable to find a geometric approach which gives a closed-form action to all orders in the gauge fields without using auxiliary fields. In the coupling to gravity, the Noether approach led to two gauge fields $h_{\pm\pm}$, which could be assembled into a symmetric tensor $h_{\mu\nu}$ satisfying $\eta^{\mu\nu} h_{\mu\nu} = 0$, where $\eta_{\mu\nu}$ is the flat metric. In the covariant approach, all reference to the flat metric $\eta_{\mu\nu}$ is avoided by dropping the tracelessness condition on $h_{\mu\nu}$. The extra component of $h_{\mu\nu}$ then decouples from the theory as a result of Weyl invariance.

For $\mathscr{W}$-gravity, for each s, the two gauge fields $h^{(+s)}$ and $h^{(-s)}$ can be assembled into a symmetric tensor $h_{\mu_1 \mu_2 \ldots \mu_s}$ which is traceless, $\eta^{\mu\nu} h_{\mu\nu \ldots \rho} = 0$. This suggests that the covariant theory might be written in terms of unconstrained symmetric tensor gauge fields $h_{\mu_1 \mu_2 \ldots \mu_s}$, provided that there are higher spin generalisations of the Weyl symmetry which can be used to eliminate the traces of the gauge fields, so that for each s all but two of the components of the gauge field decouple. An example of such a higher-spin Weyl symmetry, which was suggested in [5], is

$$\delta h^{(s)}_{\mu_1 \mu_2 \ldots \mu_s} = h_{(\mu_1 \mu_2} \sigma^{(s)}_{\mu_3 \ldots \mu_s)}\,, \tag{3.8}$$

where the parameter of the Weyl-transformations for a spin-s gauge field is a rank-$(s-2)$ tensor $\sigma^{(s)}$. It will be seen later that for a large class of models, the covariant action can indeed be written in such a way, with a $\mathcal{W}$-Weyl symmetry which is similar to (3.8), but in which the spin-two gauge field has no preferred role.

We shall need a covariant generalisation of the spin-s transformation (2.6) which does not involve any reference to a background world-sheet metric. A natural guess for this is (cf. [17, 3])

$$\delta\phi^i = \sum_s \lambda_{(s)}^{\mu_1\mu_2\ldots\mu_{s-1}} d^i_{j_1 j_2 \ldots j_{s-1}} \partial_{\mu_1}\phi^{j_1}\partial_{\mu_2}\phi^{j_2}\ldots\partial_{\mu_{s-1}}\phi^{j_{s-1}} . \tag{3.9}$$

However, this corresponds to too many gauge transformations, as in the linearised theory there are only two parameters, $\lambda^{(+s)}$ and $\lambda^{(-s)}$, for each spin s. In the linearised theory, the transformations (2.6) can be rewritten in terms of symmetric tensors $\lambda_{(s)}^{\mu_1\mu_2\ldots\mu_{s-1}}$ subject to the condition

$$\eta_{\mu\nu}\lambda_{(s)}^{\mu\nu\ldots\rho} = 0 \tag{3.10}$$

which implies that, for each spin s, the symmetric tensor $\lambda_{(s)}^{\mu_1\mu_2\ldots\mu_{s-1}}$ has only two non-vanishing components, $\lambda^{(\pm s)}$. In the full non-linear theory, it will be seen that the transformations can be written as in (3.9) but with the parameters satisfying a non-linear generalisation of the tracelessness condition which is independent of the flat metric and which reduces to (3.10) in the linearised theory. These constraints can be solved in terms of some unconstrained tensors $k^{\mu\nu\cdots}$ in such a way that all but two of the components of the gauge parameters $k^{\mu\nu\cdots}$ drop out of the gauge transformation. When expressed in terms of the unconstrained $k^{\mu\nu\cdots}$ parameters, the symmetry is reducible, in the sense of [33].

4. Geometry, Gravity and $\mathcal{W}$-Gravity

Instead of restricting attention to two dimensions, it is of interest to attempt to formulate $\mathcal{W}$-gravity in d-dimensions. Consider, then, the d-dimensional sigma-model or $(d-1)$-brane in which a configuration is a map $\phi^i(x^\mu)$ from an d-dimensional space-time or world-volume $\mathcal{N}$, with coordinates x^μ, to a D-dimensional target-space $\mathcal{M}$ with coordinates ϕ^i. The cotangent bundle $T^*\mathcal{N}$ has coordinates (x^μ, y_μ), where y_μ are fibre coordinates. The map $\phi^i(x^\mu)$ can be used to pull-back a metric $g_{ij}(\phi)$ on $\mathcal{M}$ to an induced metric $G_{\mu\nu}(x) = g_{ij}(\phi(x))\partial_\mu\phi^i\partial_\nu\phi^j$ on $\mathcal{N}$. This transforms as a tensor under Diff($\mathcal{N}$) and can be used to define actions that are invariant under Diff($\mathcal{N}$), such as

$$S_{\text{Nambu-Goto}} = \int d^d x\sqrt{-\det(G_{\mu\nu})}, \quad S_{\text{cov}} = \int d^d x\sqrt{-h}(h^{\mu\nu}G_{\mu\nu} + \mu) , \tag{4.1}$$

where $h_{\mu\nu}$ is a metric on $\mathcal{N}$ and μ is a constant.

In a similar way, a $\mathcal{W}$-metric function given by $f(\phi, d\phi) = g_{ij}(\phi)d\phi^i d\phi^j + d_{ijk}\, d\phi^i\, d\phi^j\, d\phi^k + \ldots$ on $\mathcal{M}$ can be pulled back to one on $\mathcal{N}$,

$$\begin{aligned}\hat{f}(x, dx) &= f(\phi(x), \partial_\mu\phi dx^\mu)\\ &= g_{ij}(\phi(x))\partial_\mu\phi^i\partial_\nu\phi^j dx^\mu\, dx^\nu\\ &\quad + d_{ijk}(\phi(x))\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k dx^\mu\, dx^\nu\, dx^\rho + \ldots .\end{aligned} \tag{4.2}$$

This can then be used to define an action, such as

$$S = \int d^d x \, [\tilde{h}^{\mu\nu}_{(2)} \, g_{ij} \partial_\mu \phi^i \partial_\nu \phi^j + \tilde{h}^{\mu\nu\rho}_{(3)} \, d_{ijk} \partial_\mu \phi^i \partial_\nu \phi^j \partial_\rho \phi^k + \ldots] \,, \tag{4.3}$$

where $\tilde{h}^{\mu_1 \ldots \mu_n}_{(n)}$ are some tensor densities on $\mathcal{N}$. It will be useful to introduce a generating function $\tilde{F}(x^\mu, y_\mu)$ for these,

$$\tilde{F}(x^\mu, y_\mu) = \sum_n \frac{1}{n} \tilde{h}^{\mu_1 \ldots \mu_n}_{(n)}(x) \, y_{\mu_1} \ldots y_{\mu_n} \tag{4.4}$$

which can be thought of as a variant of the cometric function and so will be referred to as a "cometric density function." In pure gravity, the tensor density $\tilde{h}^{\mu\nu}_{(2)}$ can be written in terms of a metric tensor $h^{\mu\nu}_{(2)}$ by $\tilde{h}^{\mu\nu}_{(2)} = \sqrt{-h_{(2)}} h^{\mu\nu}_{(2)}$, suggesting that the cometric density might in turn be related in some complicated way to some cometric function

$$F(x^\mu, y_\mu) = \sum_n \frac{1}{n} h^{\mu_1 \ldots \mu_n}_{(n)}(x) y_{\mu_1} \ldots y_{\mu_n} \,, \tag{4.5}$$

where the coefficients $h^{\mu_1 \ldots \mu_n}_{(n)}$ are tensors on $\mathcal{N}$.

An important special case is that in which $\mathcal{M}$ is one-dimensional, with $g_{11} = 1, d_{11 \ldots 1} = 1, \ldots$, etc. Then a real-valued function $\phi(x)$ on $\mathcal{N}$ defines a section of the cotangent bundle, $y_\mu(x) = \partial_\mu \phi$, and the lagrangian (4.3) becomes the cometric density $\tilde{F}$ evaluated on the section, $\tilde{F}(x^\mu, \partial_\mu \phi(x))$.

If $g_{ij}, d_{ijk}, \ldots$ transform as tensors under Diff($\mathcal{M}$), then the line element $f(\phi, d\phi)$ is invariant under Diff($\mathcal{M}$) and its pull-back $\hat{f}(x, dx)$ is invariant under Diff($\mathcal{N}$), as is the action (4.3), provided that the $\tilde{h}^{\mu\nu \cdots}$ transform as tensor densities. However, much larger non-linear symmetries can be considered which transform tensors of different rank into one another. The Diff($\mathcal{M}$) transformation $\delta\phi^i = \xi^i(\phi)$ can be generalised to a Diff($T\mathcal{M}$) transformation $\delta\phi^i = \xi^i(\phi, d\phi)$ and the metric function $f(\phi, d\phi)$ will be invariant if it is a scalar function on the tangent bundle. The pull-back $\hat{f}(x, dx)$ will then be invariant under Diff($T\mathcal{N}$). Unfortunately, this does not lead to a natural set of transformations for the gauge fields.

In a similar way, the cometric function (4.5) can be taken to be a scalar under Diff($T^*\mathcal{N}$), so that under $x \to x'(x, y), y \to y'(x, y)$, the cometric (4.5) is invariant, $F'(x', y') = F(x, y)$. However, this group is not useful as the gauge group of $\mathcal{W}$-gravity, as it has no natural action on the gauge fields. The relation between $\mathcal{W}_\infty$ and symplectic diffeomorphisms [25] suggests restricting to these and, as we shall see, this does lead to useful results.

The symplectic diffeomorphisms of the cotangent bundle, Diff$_0(T^*\mathcal{N})$, preserve the two-form $dx^\mu_\wedge dy_\mu$ and the infinitesimal transformations take the form

$$\delta x^\mu = -\frac{\partial}{\partial y_\mu} \Lambda(x, y), \quad \delta y_\mu = \frac{\partial}{\partial x^\mu} \Lambda(x, y) \tag{4.6}$$

for some function Λ. The transformations (4.6) satisfy the algebra

$$[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\{\Lambda, \Lambda'\}} \,, \tag{4.7}$$

where the Poisson bracket for functions $\Lambda(x, y), \Lambda'(x, y)$ on $T^*\mathcal{N}$ is

$$\{\Lambda, \Lambda'\} = \frac{\partial \Lambda}{\partial x^\mu} \frac{\partial \Lambda'}{\partial y_\mu} - \frac{\partial \Lambda'}{\partial x^\mu} \frac{\partial \Lambda}{\partial y_\mu} \,. \tag{4.8}$$

This symmetry algebra is isomorphic to the $\mathcal{W}_\infty$-algebra [25]. Strictly speaking, this is the $\mathcal{W}_\infty$ algebra if the functions Λ are restricted to have the Taylor expansion

$$\Lambda(x, y) = \sum_{s=2}^{\infty} \lambda_{(s)}^{\mu_1 \dots \mu_{s-1}}(x) y_{\mu_1} \dots y_{\mu_{s-1}} \tag{4.9}$$

on $T^*\mathcal{N}$, while if this sum is extended to include a spin-one transformation with $s = 1$, then the algebra is $\mathcal{W}_{1+\infty}$ [3]. A function $F(x, y)$ transforms under these transformations as $F(x, y) \to F(x', y')$, which implies

$$\delta F = \delta x^\mu \frac{\partial F}{\partial x^\mu} + \delta y_\mu \frac{\partial F}{\partial y_\mu} = \{\Lambda, F\} . \tag{4.10}$$

Consider a section Σ of $T^*\mathcal{N}$ in which the fibre coordinate y_μ is set equal to some cotangent vector field, $y_\mu|_\Sigma = y_\mu(x)$. On restricting functions $\Lambda(x, y)$ on $T^*\mathcal{N}$ to functions $\Lambda|_\Sigma = \Lambda(x, y(x))$ on the section, the Poisson bracket has the property that

$$\{\Lambda|_\Sigma, \Lambda'|_\Sigma\} = \{\Lambda, \Lambda'\}|_\Sigma + 2 \left.\frac{\partial \Lambda}{\partial y_\mu}\right|_\Sigma \left.\frac{\partial \Lambda'}{\partial y_\nu}\right|_\Sigma \partial_{[\mu}(y_{\nu]}|_\Sigma) , \tag{4.11}$$

where

$$\{\Lambda|_\Sigma, \Lambda'|_\Sigma\} \equiv \frac{\partial \Lambda|_\Sigma}{\partial x^\mu} \frac{\partial \Lambda'|_\Sigma}{\partial y_\mu(x)} - \frac{\partial \Lambda'|_\Sigma}{\partial x^\mu} \frac{\partial \Lambda|_\Sigma}{\partial y_\mu(x)} . \tag{4.12}$$

Note $\partial_\mu y_\nu = 0$, so that there are no $\partial_\mu y_\nu$ terms in $\{\Lambda, \Lambda'\}|_\Sigma$, but $\partial/\partial x^\mu(y_\nu|_\Sigma) \neq 0$. For sections corresponding to vector fields of the form $y_\mu(x) = \partial_\mu \phi$ for some function $\phi(x)$ on $\mathcal{N}$, $\partial_{[\mu} y_{\nu]} = 0$ and the Poisson brackets have the important property $\{\Lambda|_\Sigma, \Lambda|_\Sigma'\} = \{\Lambda, \Lambda'\}|_\Sigma$, so that for such vector fields it will not be necessary to differentiate between y_μ and $y_\mu|_\Sigma = y_\mu(x)$. Furthermore, for such vector fields the transformation (4.6) on y_μ is

$$\delta(\partial_\mu \phi(x)) = \frac{\partial}{\partial x^\mu} \Lambda(x, \partial\phi) \tag{4.13}$$

and this can be consistently rewritten in terms of a transformation of $\phi(x)$, so that

$$\delta\phi = \Lambda(x, \partial_\mu \phi) = \sum_s \lambda_{(s)}^{\mu_1 \dots \mu_{s-1}} \partial_{\mu_1} \phi \dots \partial_{\mu_{s-1}} \phi , \tag{4.14}$$

and this induces the following transformation on any function $F(x^\mu, \partial_\mu \phi(x))$:

$$\begin{aligned} \delta F &\equiv F(x^\mu, \partial_\mu \phi(x) + \partial_\mu \delta\phi(x)) - F(x^\mu, \partial_\mu \phi(x)) \\ &= \frac{\partial F}{\partial(\partial_\mu \phi)} \partial_\mu \Lambda = \{\Lambda, F\} - \delta x^\mu \partial_\mu F . \end{aligned} \tag{4.15}$$

The last term in (4.15), $-\delta x^\mu \partial_\mu F$, would be cancelled if in addition x were varied as in (4.6), in which case the result (4.10) would be recovered. Transformations in which the coordinates x^μ, y_μ transform as in (4.6) will be referred to as passive transformations, while transformations such as (4.15) in which x^μ is inert but the fields transform as in (4.14) will be referred to as active transformations. Both satisfy an algebra isomorphic to the symplectic diffeomorphism algebra. The transformation (4.14) is precisely the $D = 1$ form of the transformation (3.9), with $\phi(x)$ the bosonic field, so that these transformations indeed satisfy the algebra (4.7), which means that the one-boson realisation of $\mathscr{W}$-symmetry has a geometric interpretation in terms of symplectic diffeomorphisms. Note that no natural transformations can be obtained for the fields ϕ under the full diffeomorphism group of the cotangent bundle.

5. Scalars and Densities

We now turn to the search for natural geometric transformations for the gauge fields that arise in $\mathscr{W}$-gravity. Before doing this, it will be useful to review the derivation of the transformation of the metric in ordinary gravity. In Riemannian geometry, a central role is played by the line element $ds^2 = h_{\mu\nu} dx^\mu dx^\nu$. Under an infinitesimal passive diffeomorphism, $\delta x^\mu = -k^\mu(x)$ and the transformation of the metric $h_{\mu\nu}$ under diffeomorphisms can be determined by requiring that the line element ds^2 be invariant, which will be the case if $h_{\mu\nu}$ transforms as a second rank tensor, $\delta h_{\mu\nu} = 2\nabla_{(\mu} k_{\nu)}$. Then $f(x) = h_{\mu\nu} y^\mu y^\nu$ is an invariant for all vector fields $y^\mu(x)$, i.e. under the transformation $x^\mu \to x'^\mu(x)$ one has $f'(x') = f(x)$. Equivalently, the transformation of the inverse metric $h^{\mu\nu}$ can be determined by requiring the invariance of $h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ for all functions ϕ, or of $F(x) = h^{\mu\nu} y_\mu y_\nu$ for all cotangent vector fields $y_\mu(x)$.

Instead of asking for an invariant function $F(x) = F'(x')$, it is sometimes of interest (for example in constructing actions) to seek a density $\tilde{F}(x)$ such that the integral $I = \int d^d x \tilde{F}(x)$ is invariant, which will be the case if $\tilde{F}'(x') = |\partial x/\partial x'| \tilde{F}(x)$. Then $\tilde{F}(x) = \tilde{h}^{\mu\nu} y_\mu y_\nu$ is a density for all cotangent vector fields $y_\mu(x)$ if $\tilde{h}^{\mu\nu}$ transforms as a tensor density. So far, $\tilde{h}^{\mu\nu}$ and $h^{\mu\nu}$ are independent; it is a remarkable fact that given any tensor $h^{\mu\nu}$, one can construct a tensor density by writing $\sqrt{-h} h^{\mu\nu}$, where $h = \det[h_{\mu\nu}]$. If $d \neq 2$, one can invert this and obtain $h^{\mu\nu}$ from $\tilde{h}^{\mu\nu}$, while if $d = 2$, one can only obtain $h^{\mu\nu}$ up to a local Weyl transformation. While $h^{\mu\nu}$ is the fundamental quantity for the discussion of geometry, it is $\tilde{h}^{\mu\nu}$ which is crucial for the construction of actions; nevertheless for Riemannian geometry the two concepts are equivalent (modulo Weyl transformations if $d = 2$).

Note that instead of dealing with passive transformations under which the coordinates x^μ transform, the above can be formulated in terms of active transformations under which the coordinates remain fixed and the fields transform. Under active transformations, we demand that $F(x)$ transform as a scalar, $\delta F = k^\mu \partial_\mu F$ and that $\tilde{F}$ transform as a scalar density, $\delta \tilde{F} = \partial_\mu(k^\mu \tilde{F})$, so that the integral $I = \int d^d x \tilde{F}(x)$ changes by a surface term under diffeomorphisms.

The purpose of this section is to generalise this to obtain $\mathscr{W}$-transformations of the gauge fields $h^{\mu\nu}, h^{\mu\nu\rho}, \ldots$ occurring in the y expansion of some $F(x, y)$ by requiring that F transform in an appropriate fashion. The first case to be considered will be that in which F is a $\mathscr{W}$-scalar, i.e. it is invariant under (passive) $\mathscr{W}$-transformations, and the gauge fields $h^{\mu\nu}, h^{\mu\nu\rho}, \ldots$ are all tensors. The second case will be that in which $\tilde{F}(x, y)$ is a $\mathscr{W}$-density, i.e. $\tilde{F}(x, y)$ changes in such a way that $\int d^d x \tilde{F}$ is $\mathscr{W}$-invariant, which will give a different set of $\mathscr{W}$-transformations for the gauge fields $\tilde{h}^{\mu\nu}, \tilde{h}^{\mu\nu\rho}, \ldots$ occurring in the y expansion of $\tilde{F}(x, y)$, which will be tensor densities. $\mathscr{W}$-densities will be used to construct invariant actions in the following sections. We will concentrate on the case in which the matter system is a single free boson, as this has a natural relation to the symplectic diffeomorphisms. However, many of the results generalise to other matter systems and we will comment further on this in Sect. 9.

Consider first a cometric function $F(x^\mu, y_\mu)$ ($\mu = 1, \ldots, d$) with the y-expansion (4.5). Invariance of F under the action of Diff($\mathscr{N}$) (with y_μ transforming as a covariant vector) implies that the coefficients $h^{\mu_1 \cdots \mu_m}$ in (4.5) transform as contravariant tensors under Diff($\mathscr{N}$). Similarly, the requirement that F be invariant under general reparameterizations of $T^*\mathscr{N}$, i.e. the requirement that $F'(x', y') = F(x, y)$ under $x \to x'(x, y)$, $y \to y'(x, y)$, can be used to obtain transformations of the coefficients $h^{\mu_1 \cdots \mu_m}$. However, in general the transformations of the tensors $h^{\mu_1 \cdots \mu_m}$ obtained in this way will be y-dependent and this is unsatisfactory for the application to $\mathscr{W}$-gravity. We shall want to interpret the cometric as a generating functional for an infinite number of gauge fields $h_{(n)}(x)$ which are defined on $\mathscr{N}$ and which transform into functions of the gauge fields, the gauge parameters and their derivatives that are independent of y. This is certainly true of the gauge fields that arise in the Noether approach and is necessary if it is to be possible to couple the same gauge fields to other realisations of the $\mathscr{W}$-algebra. We will now show that if we restrict our attention to the symplectic diffeomorphisms of $\mathscr{N}$, then it is possible to find y-independent transformations for the gauge fields $h_{(n)}$.

For any vector field $y_\mu(x)$, the variation of $F(x, y(x))$ under the (passive) action (4.6), (4.9) of the symplectic diffeomorphisms on x and y is given by (4.10), which can be rewritten as

$$\delta F = \sum_{m,n=2}^{\infty} \left[-\frac{m-1}{n} \lambda_{(m)}^{\nu(\mu_1 \cdots} \partial_\nu h_{(n)}^{\cdots \mu_{m+n-2})} + h_{(n)}^{\nu(\mu_1 \cdots} \partial_\nu \lambda_{(m)}^{\cdots \mu_{m+n-2})} \right] y_{\mu_1} y_{\mu_2} \cdots y_{\mu_{m+n-2}} + 2 \frac{\partial F}{\partial y_\mu} \frac{\partial \Lambda}{\partial y_\nu} \partial_{[\nu} y_{\mu]} \,. \tag{5.1}$$

(Note that for general $y(x)$, the x variation in (4.6) induces an extra transformation of $y(x)$, $\delta y = \delta x^\mu \partial_\mu y$.) If the tensor fields $h_{(n)}(x)$ transform under Diff$_0(T^*\mathscr{N})$ in the following y-independent fashion:

$$\delta h_{(p)}^{\mu_1 \cdots \mu_p} = p \sum_{m,n} \delta_{n+m,p+2} \left[\frac{m-1}{n} \lambda_{(m)}^{\nu(\mu_1 \cdots} \partial_\nu h_{(n)}^{\cdots \mu_p)} - h_{(n)}^{\nu(\mu_1 \cdots} \partial_\nu \lambda_{(m)}^{\cdots \mu_p)} \right], \tag{5.2}$$

then the cometric function is not quite a scalar, but transforms under Diff$_0(T^*\mathscr{N})$ as

$$\delta F(x, y) \equiv F(x + \delta x, y + \delta y, h + \delta h) - F(x, y, h) = 2 \frac{\partial F}{\partial y_\mu} \frac{\partial \Lambda}{\partial y_\nu} \partial_{[\nu} y_{\mu]} \,. \tag{5.3}$$

If the dimension d of $\mathscr{N}$ is one, the right-hand-side vanishes and F is invariant, $F'(x', y') = F(x, y)$, where $F'(x, y)$ is given by replacing $h_{(s)}$ by $h'_{(s)} = h_{(s)} + \delta h_{(s)}$ in (4.5), i.e. $F'(x, y, h_{(s)}) \equiv F(x, y, h_{(s)} + \delta h_{(s)}) - F(x, y, h_{(s)})$. For general dimension d of $\mathscr{N}$, the right-hand-side vanishes for sections in which $y_\mu = \partial_\mu \phi$ for some ϕ, so that $F(x^\mu, \partial_\mu \phi)$ is invariant under the transformations (4.6)–(5.2), restricted to the section $y_\mu = \partial_\mu \phi$. For any dimension d, this gives a realisation of an infinite group of higher-spin gauge transformations acting on scalar fields and gauge fields. The spin-two $\lambda_{(2)}$ transformations are just the diffeomorphisms of $\mathscr{N}$, with $h_{(2)}^{\mu\nu}$ the corresponding metric gauge-field, while the $\lambda_{(s)}$ transformations give higher spin analogues for which the gauge fields are $h_{(s)}^{\mu\nu\cdots}$.

Instead of using this passive formulation in which the coordinates x^μ transform and scalars are invariant, an (equivalent) active formulation can be used in which the coordinates x^μ are inert and the fields ϕ and $h_{(n)}$ transform as in (4.14), (5.2). Then instead of $\delta F(x^\mu, \partial_\mu\phi) \doteq 0$, one has

$$\begin{aligned}\delta F(x^\mu, \partial_\mu\phi) &\equiv F(x^\mu, \partial_\mu\phi + \partial_\mu\delta\phi, h_{(s)} + \delta h_{(s)}) - F(x^\mu, \partial_\mu\phi, h_{(s)}) \\ &= \frac{\partial \Lambda}{\partial y_\mu}\frac{\partial F(x^\mu, \partial_\mu\phi)}{\partial x^\mu}\,. \end{aligned} \tag{5.4}$$

Such an $F(x, y)$ will be referred to as a $\mathscr{W}$-scalar.

To construct invariant actions, one needs scalar densities rather than scalars. It is straightforward to construct densities $D(x, y)$ that can be integrated over the whole of the cotangent bundle (i.e. over both x and y) by introducing a metric G_{MN} on the cotangent bundle and constructing the fundamental density $\sqrt{\det(G_{MN})}$.Then $\int d^dx d^dy \sqrt{G}L$ is invariant under the full group of diffeomorphisms of the cotangent bundle for any scalar L. However, for $\mathscr{W}$-gravity one requires integrals over the base manifold rather than ones over the whole bundle, i.e. integrals of the form

$$S = \int d^d x\tilde{F}(x, y(x))\,, \tag{5.5}$$

where $y_\mu(x)$ is some vector field. In particular, for vector fields of the form $y_\mu(x) = \partial_\mu\phi$ the integral (5.5) becomes a candidate action for $\mathscr{W}$-gravity. Consider, then, the integral (5.5) where the "cometric density function" $\tilde{F}(x, y)$ has the expansion (4.4). If the coefficients $\tilde{h}_{(n)}$ in (4.4) transform as tensor densities under Diff($\mathscr{N}$), then the integral will be invariant (up to a surface term) under diffeomorphisms.

The next step is to attempt to find transformations of the tensor densities $\tilde{h}_{(n)}$ such that the integral is invariant under $\mathscr{W}$-transformations. For active transformations with x^μ inert and y_μ transforming as

$$\delta y_\mu = \frac{\partial \Lambda}{\partial x^\mu} \tag{5.6}$$

one requires transformations of $\tilde{h}_{(n)}$ such that

$$\delta\tilde{F} = \frac{\partial}{\partial x^\mu}[\Omega^\mu(\tilde{F}, \Lambda)] \tag{5.7}$$

for some $\Omega^\mu(\tilde{F}, \Lambda)$ constructed from $\tilde{F}$, Λ and their derivatives, so that the integral (5.5) is invariant. If $\Omega^\mu = \tilde{F}k^\mu$ for some $k^\mu(x, y(x))$, then the surface term arising from the variation of (5.5) can be cancelled by a variation of x^μ, $\delta x^\mu = -k^\mu$. That is, the change in the measure d^dx resulting from the transformation $\delta x^\mu = -k^\mu$ of x^μ would be cancelled by the variation of $\tilde{F}$ under the passive transformations given by (5.6), $\delta x^\mu = -k^\mu$ and $\tilde{F}'(x', y') = \tilde{F}(x, y)J$, where J is the jacobian $J = |\partial x/\partial x'|$. In particular, it would be expected that in this case k^μ would be given by $k^\mu = -\partial\Lambda/\partial y_\mu$, in agreement with (4.6). If $\tilde{F}$ transforms as in (5.7) for some Ω^μ, it will be referred to as a $\mathscr{W}$-density, while in the special case in which $\Omega^\mu = \tilde{F}k^\mu$ for some $k^\mu(x, y(x))$, so that the active viewpoint is equivalent to a passive one, it will be referred to as a proper $\mathscr{W}$-density. Surprisingly, it turns out that it is only possible to contruct $\mathscr{W}$-densities with y-independent transformations of the tensor densities $\tilde{h}_{(n)}$ in dimensions $d = 1, 2$ and that these are not proper densities, as will now be seen. This is in contrast to the case of $\mathcal{W}$-scalars, which can be constructed in any dimension d, and the case of ordinary gravity, where densities are proper.

Consider, then, the variation of the integral (5.5) under the y transformation (5.6). The change in $\tilde{F}$ is given by

$$\delta\tilde{F} = \frac{\partial\tilde{F}}{\partial y_\mu}\frac{\partial}{\partial x^\mu}\Lambda(x, y(x))$$

$$= \sum_{m,n=2}^{\infty} [\tilde{h}_{(n)}^{\nu\mu_1\ldots\mu_{n-1}}\,\partial_\nu\lambda_{(m)}^{\mu_n\mu_{n+1}\ldots\mu_{m+n-2}}\,y_{\mu_1}y_{\mu_2}\ldots y_{\mu_{m+n-2}}$$

$$+ (m-1)\tilde{h}_{(n)}^{\nu\mu_1\ldots\mu_{n-1}}\,\lambda_{(m)}^{\mu_n\mu_{n+1}\ldots\mu_{m+n-3}\rho}\,y_{\mu_1}y_{\mu_2}\ldots y_{\mu_{m+n-3}}\partial_\nu y_\rho]\,. \tag{5.8}$$

The strategy is to attempt to write the term involving $\partial_\nu y_\rho$ in (5.8) as a total derivative term plus a term with no derivatives on any y_μ, as such a term can be cancelled by a suitable variation of the gauge fields $\tilde{h}_{(s)}$. This would leave $\tilde{F}$ with the $\mathcal{W}$-density transformation rule (5.7).

In one dimension, $d = 1$, (5.8) can indeed be rewritten as

$$\delta\tilde{F} = \frac{\partial}{\partial x}\Omega + \sum_{n,m=2}^{\infty}\frac{y^{n+m-2}}{n+m-2}[(n-1)\tilde{h}_{(n)}\,\partial\lambda_{(m)} - (m-1)\lambda_{(m)}\,\partial\tilde{h}_{(n)}]\,, \tag{5.9}$$

where

$$\Omega(x, y) = \sum_{n,m=2}^{\infty}\frac{m-1}{n+m-2}\tilde{h}_{(n)}\,\lambda_{(m)}\,y^{n+m-2} \tag{5.10}$$

and the one-dimensional indices $\mu, \nu, \ldots$ have been suppressed ($h_{(p)} = h_{(p)}^{111\ldots1}$, etc.). If the tensor densities $\tilde{h}_{(p)}$ transform as

$$\delta\tilde{h}_{(p)} = \sum_{m,n}\delta_{m+n,p+2}\,[(m-1)\lambda_{(m)}\,\partial\tilde{h}_{(n)} - (n-1)\tilde{h}_{(n)}\,\partial\lambda_{(m)}]\,, \tag{5.11}$$

then the variation of $\tilde{h}_{(p)}$ cancels the second term on the right-hand side of (5.9), so that

$$\delta\tilde{F} \equiv \tilde{F}(x, \phi + \delta\phi, \tilde{h}_{(p)} + \delta\tilde{h}_{(p)}) - \tilde{F}(x, \phi, \tilde{h}_{(p)}) = \partial_x\Omega\,. \tag{5.12}$$

Then the integral (5.5) is invariant up to a surface term under (5.6), (5.11),

$$\delta S = \int dx\,\partial_x\Omega\,, \tag{5.13}$$

and this will vanish with suitable boundary conditions. Note that Ω can be rewritten as

$$\Omega(x, y) = \mathbf{N}^{-1}\left(\frac{\partial\Lambda}{\partial y}[\mathbf{N}\tilde{F}]\right), \tag{5.14}$$

where $\mathbf{N}$ is the number operator $\mathbf{N} = y\partial/\partial y$ for y, so that $\mathbf{N}y^s = sy^s$ and $\mathbf{N}^{-1}y^s = s^{-1}y^s$. Thus with the transformation (5.11), $\tilde{F}$ transforms as a $\mathcal{W}$-density, although the surprising presence of the number operator in (5.14) implies that it is not a proper $\mathcal{W}$-density, so that the surface term variation (5.7) cannot be completely cancelled by a transformation of x^μ. Note that (5.11) implies that $\tilde{h}_{(s)}$ transforms as a contravariant tensor density of weight s under the one-dimensional diffeomorphisms with parameter $\lambda = \lambda_{(2)}$,

$$\delta \tilde{h}_{(s)} = \lambda \partial \tilde{h}_{(s)} - (s-1)\tilde{h}_{(s)}\, \partial \lambda \,. \tag{5.15}$$

In one dimension, $d = 1$, a $\mathscr{W}$-density can be related to a $\mathscr{W}$-scalar transformation (5.2) as follows. If $\tilde{h}_{(n)}$ transforms as in (5.11), then $h_{(n)} \equiv n\tilde{h}_{(n+1)}$ has precisely the transformation (5.2). This means that the quantity

$$K(x, y, \tilde{h}_{(n)}) \equiv \frac{\partial \tilde{F}}{\partial y} = \sum_n \tilde{h}_{(n+1)}(x) y^n = \sum_n \frac{1}{n} h_{(n)}(x) y^n \tag{5.16}$$

is a $\mathscr{W}$-scalar. In particular, the variation (5.11) leads to the change of K,

$$\delta_h K \equiv K(x, y, \tilde{h}_{(n)} + \delta \tilde{h}_{(n)}) - K(x, y, \tilde{h}_{(n)}) = \partial_y \Lambda \partial_x K - \partial_y K \partial_x \Lambda = -\{\Lambda, K\}\,, \tag{5.17}$$

and the transformation of x and y under the symplectic diffeomorphism (4.6) leads to a change of K given by $\delta K = \{\Lambda, K\}$, which cancels (5.17). This gives the important result that for any $d = 1$ $\mathscr{W}$-density $\tilde{F}$, the derivative $\dfrac{\partial F}{\partial y}$ is a $\mathscr{W}$-scalar.

For dimensions $d > 1$, the problem is to write the term involving $\partial_\mu y_\nu$ in (5.8) as a surface term plus a term without any derivatives acting on any y that can be cancelled by an appropriate $\tilde{h}_{(s)}$ variation. This is not possible for $d > 2$ or for $d = 2$; this is easily seen in the special case $\tilde{F} = \frac{1}{2}\eta^{\mu\nu} y_\mu y_\nu$, when (5.8) becomes

$$\delta \tilde{F} = \partial_\nu(\Lambda y^\nu) - \Lambda(\partial_\nu y^\nu) \tag{5.18}$$

and there is no way to get rid of the $\partial_\nu y^\nu$ term (where $y^\nu = \eta^{\mu\nu} y_\mu$).

However, for the two-dimensional case, if one further imposes the constraint that

$$\eta_{\mu\nu}\, \lambda_{(m)}^{\mu\nu\rho\ldots} = 0 \tag{5.19}$$

then (using $\partial_\mu y_\nu = \partial_\nu y_\mu$) it follows that

$$\begin{aligned} \Lambda(\partial_\nu y^\nu) = \Bigg(\partial_- \Bigg[\sum_{m=2}^{\infty} \frac{2}{m} \lambda_{(m)}^{++\ldots+} (y_+)^m \Bigg] \\ - \sum_{m=2}^{\infty} \frac{2}{m} (\partial_- \lambda_{(m)}^{++\ldots+})(y_+)^m \Bigg) + (+ \leftrightarrow -)\,, \end{aligned} \tag{5.20}$$

which is of the required form of a total derivative plus a term with no derivatives on any y. Thus for $d = 2$ $\mathscr{W}$-gravity linearised about this flat background, linearised $\mathscr{W}$-densities exist only if the gauge group is restricted to a subgroup of the symplectic diffeomorphisms in which the parameters satisfy a constraint whose linearised form is (5.19). This is in agreement with the discussion of linearised $\mathscr{W}$-gravity of Sect. 2. This suggests that in $d = 2$, a $\mathscr{W}$-density might exist if the gauge group is restricted by some constraint whose linearised form is (5.19), and this is indeed the case. A lengthy calculation (using the fact that in two dimensions any tensor can be written as $T_{\mu\nu\rho..} = T_{(\mu\nu)\rho\ldots} + T_{[\mu\nu]\rho\ldots}$ and the anti-symmetric part is proportional to the two-dimensional alternating tensor[3] $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$: $T_{[\mu\nu]\rho\ldots} = -\frac{1}{2}\varepsilon_{\mu\nu} T_{\rho\ldots}$, where $T_{\rho\ldots} = \varepsilon^{\alpha\beta} T_{\alpha\beta\rho\ldots}$) leads to the result that (5.8) can be

[3] The alternating tensor satisfies $\varepsilon^{\mu\nu}\varepsilon_{\nu\rho} = \delta^\mu_\rho$ and $\varepsilon^{01} = 1$

written as

$$\delta \tilde{F} = \frac{\partial}{\partial x^\mu} \Omega^\mu + X + \sum_{n,m=2}^{\infty} \frac{1}{m+n-2} y_{\mu_1} y_{\mu_2} \ldots y_{\mu_{m+n-2}}$$
$$\times \Bigg[(n+m-2) \tilde{h}_{(n)}^{\nu\mu_1 \ldots \mu_{n-1}} \partial_\nu \lambda_{(m)}^{\mu_n \mu_{n+1} \ldots \mu_{m+n-2}}$$
$$- \frac{(m-1)(n-1)}{m+n-3} \partial_\nu (\tilde{h}_{(n)}^{\mu_1 \ldots \mu_n} \lambda_{(m)}^{\mu_{n+1} \ldots \mu_{m+n-2}\nu})$$
$$- \frac{(m-1)(m-2)}{m+n-3} \partial_\nu (\tilde{h}_{(n)}^{\nu\mu_1 \ldots \mu_{n-1}} \lambda_{(m)}^{\mu_n \mu_{n+1} \ldots \mu_{m+n-2}}) \Bigg], \tag{5.21}$$

where

$$X = \sum_{n=2}^{\infty} \sum_{m=2}^{n-2} a_n [\varepsilon_{\nu\sigma} \varepsilon_{\rho\tau} \tilde{h}_{(n-m)}^{\mu_1 \ldots \mu_{n-m-2}\nu\rho} \lambda_{(m)}^{\mu_{n-m-1} \ldots \mu_{n-4}\sigma\tau}]$$
$$\times \varepsilon^{\alpha\beta} \varepsilon^{\gamma\delta} y_{\mu_1} y_{\mu_2} \ldots y_{\mu_{n-4}} y_\alpha y_\gamma \partial_\beta y_\delta \tag{5.22}$$

for certain coefficients a_n, and

$$\Omega^\nu = \sum_{n,m=2}^{\infty} \frac{m-1}{(m+n-2)(m+n-3)} y_{\mu_1} y_{\mu_2} \ldots y_{\mu_{m+n-2}}$$
$$\times [(n-1)(\tilde{h}_{(n)}^{\mu_1 \ldots \mu_n} \lambda_{(m)}^{\mu_{n+1} \ldots \mu_{m+n-2}\nu})$$
$$+ (m-2)(\tilde{h}_{(n)}^{\nu\mu_1 \ldots \mu_{n-1}} \lambda_{(m)}^{\mu_n \mu_{n+1} \ldots \mu_{m+n-2}})] . \tag{5.23}$$

Then if the tensor densities transform as

$$\delta \tilde{h}_{(p)}^{\mu_1 \mu_2 \ldots \mu_p} = \sum_{m,n} \delta_{m+n,p+2} \Bigg[(m-1) \lambda_{(m)}^{(\mu_1 \mu_2 \ldots} \partial_\nu \tilde{h}_{(n)}^{\ldots \mu_p)\nu} - (n-1) \tilde{h}_{(n)}^{\nu(\mu_1 \mu_2 \ldots} \partial_\nu \lambda_{(m)}^{\ldots \mu_p)}$$
$$+ \frac{(m-1)(n-1)}{p-1} \partial_\nu \{ \lambda_{(m)}^{\nu(\mu_1 \mu_2 \ldots} \tilde{h}_{(n)}^{\ldots \mu_p)} - \tilde{h}_{(n)}^{\nu(\mu_1 \mu_2 \ldots} \lambda_{(m)}^{\ldots \mu_p)} \} \Bigg] \tag{5.24}$$

the integral (5.5) transforms as

$$\delta S = \int d^2x (\partial_\mu \Omega^\mu + X) . \tag{5.25}$$

This means that the action will be invariant up to a surface term under the transformations (5.6), (5.24) for which the parameters $\lambda_{(m)}$ satisfy the constraint $X = 0$. This constraint gives the required non-linear generalisation of (5.19) and will be discussed in the next section. From (5.24), the $\tilde{h}_{(s)}$ transform as tensor densities under the $\lambda_{(2)}$ transformations. Note that on restricting to one dimension, the transformation (5.24) reduces to (5.11).

6. Covariant Actions

Before constructing $\mathcal{W}$-invariant actions, it will be useful to consider the analogous problem of deriving the coupling of a matter system in d dimensions to gravity, without using any knowledge of geometry. Suppose one has a matter current $S_{\mu\nu} = S_{(\mu\nu)}$ which transforms under diffeomorphisms as a tensor, $\delta S_{\mu\nu} = k^\rho \partial_\rho S_{\mu\nu} + 2S_{\rho(\mu} \partial_{\nu)} k^\rho$, e.g., in the sigma-model example, one has the tensor $S_{\mu\nu} = g_{ij} \partial_\mu \phi^i \partial_\nu \phi^j$.

Note that it would be inappropriate at this stage to subtract a trace to obtain the usual stress tensor, as that would involve introducing a background metric. The fact that the current $S_{\mu\nu}$ transforms linearly implies that the following action

$$S = \int d^d x\, \tilde{h}^{\mu\nu} S_{\mu\nu} \tag{6.1}$$

can be made diffeomorphism invariant by attributing to the field $\tilde{h}^{\mu\nu}(x)$ a suitable transformation law. Indeed, the action is invariant provided $\tilde{h}^{\mu\nu}(x)$ transforms as a tensor density:

$$\delta\tilde{h}^{\mu\nu} = k^\rho \partial_\rho \tilde{h}^{\mu\nu} - 2\tilde{h}^{\rho(\mu}\partial_\rho k^{\nu)} + \tilde{h}^{\mu\nu}\partial_\rho k^\rho \,. \tag{6.2}$$

If $d \neq 2$, one can define $h^{\mu\nu} = \tilde{h}^{\mu\nu}[\det(-\tilde{h}^{\mu\nu})]^{1/(2-d)}$ and show that it transforms as a tensor. The density can be rewritten in terms of the tensor as $\tilde{h}^{\mu\nu} = \sqrt{-h}h^{\mu\nu}$ (where $h = \det(h_{\mu\nu})$ and $h_{\mu\nu}$ is the inverse of $h^{\mu\nu}$) and this can be substituted into the action (6.1) to give the usual coupling to a metric tensor $h^{\mu\nu}$. Both h and $\tilde{h}$ have the same number of components and the two formulations are equivalent (at least for non-degenerate metrics). If $d = 2$, however, the tensor density cannot be written in terms of a tensor in this way. Nevertheless, in two dimensions, $\det(\tilde{h}^{\mu\nu})$ is a scalar, so that one can consistently impose the constraint $\det(\tilde{h}^{\mu\nu}) = -1$ to eliminate one of the three components of $\tilde{h}^{\mu\nu}$. This constraint can then be solved in terms of an unconstrained tensor $h^{\mu\nu}$ by writing $\tilde{h}^{\mu\nu} = \sqrt{-h}h^{\mu\nu}$. This solution is invariant under Weyl scalings of the metric, $h_{\mu\nu} \to \sigma h_{\mu\nu}$, so that $\tilde{h}_{\mu\nu}$ depends on only two of the three components of $h_{\mu\nu}$, as one of the components is pure gauge.

To summarise, the geometric coupling to gravity was recovered by first finding a gauge field $\tilde{h}$ in terms of which the action was linear and then rewriting this in terms of a gauge field with covariant transformation properties in the case $d \neq 2$, or imposing a covariant constraint in the case $d = 2$. We now use a similar approach to seek the coupling of a sigma-model to d-dimensional $\mathscr{W}$-gravity, which in the case $d = 2$ has the linearised form (2.9). Consider the case in which the target space dimension is $D = 1$. We require an action of the form

$$S = \int d^d x \tilde{F}(x^\mu, \partial_\nu \phi) \tag{6.3}$$

for some cometric $\mathscr{W}$-density $\tilde{F}$, with expansion (4.4) in terms of the tensor densities $\tilde{h}_{(n)}^{\mu_1 \dots \mu_n}$ on $\mathscr{N}$, and demand that it have a $\mathscr{W}$-symmetry invariance under which ϕ transforms as in (4.14) and the transformations of the density gauge fields $\tilde{h}_{(n)}^{\mu_1 \dots \mu_n}$ are independent of ϕ.[4] If such an action is found, the next stage is to rewrite in terms of a cometric (4.5) whose components $h_{(n)}$ are tensors if $d \neq 2$, or if $d = 2$, to impose invariant constraints and solve in terms of a cometric function with higher-spin $\mathscr{W}$-Weyl symmetries, so as to recover the linearised results given earlier. Note that the gravitational coupling for any tensor current $S_{\mu\nu}$ is given by (6.1). For $\mathscr{W}$-gravity, we will find the coupling for the matter currents $\partial_{\mu_1}\phi \dots \partial_{\mu_n}\phi$, but the same coupling then immediately works for any set of matter currents $S^{(n)}_{\mu_1 \dots \mu_n}$ which transform into one another under $\mathscr{W}$-diffeomorphisms in the same way as $\partial_{\mu_1}\phi \dots \partial_{\mu_n}\phi$.

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[4] If this requirement were dropped, it would be straightforward to find a $\mathscr{W}$-gravity coupling for all d, but it would not give a universal $\mathscr{W}$-gravity which could be coupled to all matter systems with $\mathscr{W}$-symmetry and would not give the non-linear form of the linearised action (2.9). Note also that an active viewpoint is now adopted, so that the coordinates x do not transform

The results are as follows. Invariance of the action (6.3) requires that $\tilde{F}$ is a $\mathcal{W}$-density transforming as (5.7) for some Ω^μ, and the results of the last section can now be applied to the cotangent vector field $y_\mu = \partial_\mu \phi$. First, if $d > 2$, there are no $\mathcal{W}$-densities for which $\tilde{h}_{(s)}$ has no ϕ dependence in its transformation rules and so there is no such invariant action.

Next, if $d = 1$, so that the sigma-model can be interpreted as a particle action, $\mathcal{W}$-densities indeed exist, so that $\mathcal{W}$-gravity actions can be constructed. Specifically, the action (6.3), (4.4) is invariant (up to a surface term) under the transformations (4.9), (4.14) and (5.11), where the one-dimensional indices $\mu, \nu, \ldots$ have been suppressed. The gauge group is the symplectic diffeomorphism group of the cotangent bundle of the one-dimensional manifold $\mathcal{N}$, $\mathrm{Diff}_0(T^*\mathcal{N})$. This gives the one-dimensional $\mathcal{W}$-gravity theory of [5].

In one dimension, one can in fact go much further and construct the action explicitly from an invariant cometric line element (i.e. a $\mathcal{W}$-scalar) $F(x, y)$. For comparison, the coupling to gravity (as opposed to $\mathcal{W}$-gravity) is given by the truncation of the action (6.3) to $\frac{1}{2}\int dx\, \tilde{h}_{(2)}(\partial\phi)^2$ and in this case the tensor density $\tilde{h}_{(2)}$ can be rewritten in terms of a contravariant inverse metric tensor $h_{(2)}$ by

$$\tilde{h}_{(2)} = \sqrt{h_{(2)}}\,. \tag{6.4}$$

(This is the one-dimensional form of $\tilde{h}^{\mu\nu} = \sqrt{h}h^{\mu\nu}$, with positive definite metric $h_{11} = (h^{11})^{-1} = h = \det[h_{\mu\nu}]$.) In a similar spirit, we will now show that the action (6.3), (4.4) with $d = 1$ can be rewritten in terms of a cometric

$$F(x, \partial\phi) = \sum_{n=2}^{\infty} \frac{1}{n} h_{(n)}\,(\partial\phi)^n \tag{6.5}$$

which transforms as a scalar under $\mathrm{Diff}_0(T^*\mathcal{N})$, i.e. under the (active) transformation in which $h_{(n)}$ transforms as in (5.2) and ϕ as in (4.9), (4.14), $\delta F = [\partial\Lambda/\partial(\partial\phi)]\partial_x F$. (Equivalently, F is invariant under the "passive" transformations in which, in addition to the above transformations, the coordinate x transforms as $\delta x = -[\partial\Lambda/\partial(\partial\phi)]$.)

It was seen in the last section that, given any $\mathcal{W}$-density $\tilde{F}$ the derivative $\dfrac{\partial\tilde{F}}{\partial y}$ is a $\mathcal{W}$-scalar. This suggests identifying $\dfrac{\partial\tilde{F}}{\partial y}$ with the $\mathcal{W}$-scalar (6.5). However, this is not quite correct, since the Taylor expansions of $\tilde{F}$ and F start at order y^2, while that of $\dfrac{\partial\tilde{F}}{\partial y}$ starts at order y. It follows that $\left(\dfrac{\partial\tilde{F}}{\partial y}\right)^2$ is a $\mathcal{W}$-scalar whose expansion starts at order y^2 and so can be identified with (6.5). We then give the cometric density function $\tilde{F}$ in terms of a $\mathcal{W}$-scalar cometric function F as

$$\left[\frac{\partial\tilde{F}(x, y)}{\partial y}\right]^2 = 2F(x, y), \quad \frac{\partial\tilde{F}(x, y)}{\partial y} = \sqrt{2F(x, y)}\,. \tag{6.6}$$

The factor of two is a consequence of our conventions, while the square root in (6.6) was to be expected from comparison with the pure gravity limit; indeed, the term of lowest order in y in the Taylor expansion of (6.6) reproduces (6.4). These relations can be integrated to give $\tilde{F}$ explicitly, using the boundary condition that $\tilde{F}(x, y)$ is a power series in y starting with the y^2 term. In terms of the number operator

$\mathbf{N} = y\dfrac{\partial}{\partial y}$, using the identity $\tilde{F} = \mathbf{N}^{-1}(y\partial\tilde{F}\partial y)$, we have

$$\tilde{F}(x, y) = \mathbf{N}^{-1}[y\sqrt{2F(x, y)}] \tag{6.7}$$

so that

$$\mathbf{N}\tilde{F}(x, y) \equiv \sum_{n=2}^{\infty} \tilde{h}_{(n)} y^n = \left(2 \sum_{n=2}^{\infty} h_{(n)} y^{n+2}\right)^{\frac{1}{2}} \tag{6.8}$$

and expanding in y gives the $\tilde{h}_{(n)}$ in terms of the $h_{(n)}$.

We turn now to the case of two-dimensional $\mathscr{W}$-gravity. Consider the action (6.3), (4.4) and ϕ transformation (4.9), (4.14) with $d = 2$. The tensor densities $\tilde{h}_{(s)}$ will eventually be expressed in terms of tensors $h_{(s)}$ in such a way that in the linearised limit, (2.6), (2.8) and (2.9) will be recovered. However, even in the linearised theory, the action was not invariant under the full $\mathrm{Diff}_0(T^*\mathscr{N})$ group under which ϕ transforms as (4.14), but only under the subgroup in which the parameters satisfied a constraint whose linearised form is (3.10). This is of course borne out by the full non-linear analysis, with the result that the action (4.4), (6.3) is only invariant under the ϕ transformation (4.9), (4.14) together with a transformation of the $\tilde{h}_{(s)}$ which is independent of ϕ if the parameters $\lambda_{(s)}$ satisfy a constraint whose linearised form is (5.19). The transformation (5.21) implies that an invariant action is obtained if the constraint $X = 0$ is imposed, where X is given by (5.22). The condition that $X = 0$ for all $y(x)$ implies that

$$\sum_{m=2}^{n-2} \varepsilon_{\nu\sigma}\varepsilon_{\rho\tau}\, \tilde{h}_{(n-m)}^{\nu\rho(\mu_1\ldots\mu_{n-m-2}}\, \lambda_{(m)}^{\mu_{n-m-1}\ldots\mu_{n-4})\sigma\tau} = 0 \tag{6.9}$$

for each $n \geqq 2$. This constraint can be rewritten in terms of Λ (4.9) and $\tilde{F}$ as

$$\varepsilon^{\mu\rho}\varepsilon^{\nu\sigma} \frac{\partial^2 \Lambda}{\partial y_\mu\, \partial y_\nu} \frac{\partial^2 \tilde{F}}{\partial y_\rho\, \partial y_\sigma} = 0 \tag{6.10}$$

or, equivalently,

$$\left[\frac{\partial^2 \tilde{F}}{\partial y_\mu \partial y_\nu}\right]^{-1} \frac{\partial^2 \Lambda}{\partial y_\mu\, \partial y_\nu} = 0\,. \tag{6.11}$$

Introducing frames $\tilde{e}^a_\mu$ such that $\tilde{h}^{\mu\nu}_{(2)} = \tilde{e}^\mu_a \tilde{e}^\nu_b \eta^{ab}$ and expanding (6.10) in y gives the first few constraints as

$$\eta_{ab}\lambda^{ab}_{(3)} = 0, \quad \eta_{ab}\,\lambda^{abc}_{(4)} = \frac{2}{3}\,\tilde{h}^{abc}_{(3)}\,\lambda_{(3)ab}\,,$$

$$\eta_{ab}\lambda^{abcd}_{(5)} = \tilde{h}^{ab(c}_{(3)}\,\lambda^{d)}_{(4)ab} - \tilde{h}^{a(c}_{(3)a}\,\lambda^{d)b}_{(4)b} + \frac{1}{2}\,\tilde{h}^{abcd}_{(4)}\,\lambda_{(3)ab}\,. \tag{6.12}$$

This generalises (3.10) and the trace of $\lambda_{(s)}$ is set equal to an $\tilde{h}$-dependent expression involving the $\lambda_{(r)}$ for $r < s$, so that these constraints can be solved in terms of the trace-free parts of the parameters, leaving just two parameters for each spin.

The action (6.3), (4.4) is then invariant under the transformations (4.14), (4.9) and (5.24) provided the parameters satisfy the constraint (6.10). As in the case of gravity, the linear coupling to tensor densities is fully gauge-invariant, but is non-minimal. In the case of gravity, the constraint $\det(\tilde{h}^{\mu\nu}) = -1$ can be imposed and solved as $\tilde{h}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ to give the usual Weyl-invariant formulation (3.4). For $\mathcal{W}$-gravity, some generalisation of this constraint is needed that is preserved by $\mathcal{W}$-gravity transformations. From the analysis of Sect. 2, the linearised form of this constraint should imply that the $\tilde{h}^{\mu \cdots \nu}$ are traceless (with respect to the flat metric $\eta_{\mu\nu}$ about which one is expanding in the linearised approximation), but the non-linear constraint should not refer to any fixed background metric. Consider the following constraints on the gauge fields $h_{(2)}, h_{(3)}, h_{(4)}$:

$$\det(\tilde{h}^{\mu\nu}_{(2)}) = -1 , \tag{6.13}$$

$$\tilde{h}_{\mu\nu} \tilde{h}^{\mu\nu\rho}_{(3)} = 0 , \tag{6.14}$$

$$\tilde{h}_{\mu\nu} \tilde{h}^{\mu\nu\rho\sigma}_{(4)} = \frac{2}{3} \tilde{h}_{\mu\alpha} \tilde{h}_{\nu\beta\gamma} \tilde{h}^{\mu\beta\rho}_{(3)} \tilde{h}^{\nu\alpha\sigma}_{(3)} , \tag{6.15}$$

where $\tilde{h}_{\mu\nu}$ is the inverse of $\tilde{h}^{\nu\rho}_{(2)}$, $\tilde{h}_{\mu\nu} \tilde{h}^{\nu\rho}_{(2)} = \delta^{\rho}_{\mu}$. Linearising these constraints implies that, as required, $h_{(3)}$ and $h_{(4)}$ are traceless with respect to $\eta_{\mu\nu}$, to lowest order in the gauge fields. Furthermore, it is straightforward to check that these constraints are preserved by the transformations (5.24), so that they can be consistently imposed on the gauge fields. The full set of constraints are generated by the constraint

$$\det(\tilde{G}^{\mu\nu}(x, y)) = -1 , \tag{6.16}$$

where

$$\tilde{G}^{\mu\nu}(x, y) = \frac{\partial^2 \tilde{F}(x, y)}{\partial y_\mu \partial y_\nu} . \tag{6.17}$$

Expanding the constraint (6.16) in y_μ, one finds the coefficient of y^{n+2} is a non-linear constraint on $h_{(n)}$ which (for $n > 0$) sets the trace $\tilde{h}_{\mu\nu} \tilde{h}^{\mu\nu\rho\sigma}_{(n)}$ equal to a non-linear function of the $h_{(m)}$ for $m < n$, so that the constraint has the correct linearised limit. The first three constraints from the expansion of (6.16) are precisely (6.13), (6.14), (6.15). A lengthy calculation shows that this infinite set of constraints on the density gauge fields $\tilde{h}^{\mu\nu \cdots}_{(s)}$ is preserved by the transformations (5.24), and so can be consistently imposed on the gauge fields without spoiling the invariance of the action. Rather than give the lengthy direct proof of this result, we shall instead present an indirect but simple derivation of this constraint in Sect. 7. Equation (6.16), (6.17) is the real Monge–Ampère equation for a function of the two variables y_μ; this equation is discussed in detail in [29], where the existence of solutions is established (subject to certain conditions).

The constraint (6.16) can be interpreted as follows. Let z_μ be complex coordinates on $\mathbb{R}^4$ with real part y_μ, so that $z_\mu = y_\mu + i u_\mu$ for some u_μ. Thus $(x^\mu, z_\mu, \bar{z}_\mu)$ are coordinates for a bundle $\mathbb{C}\, T^* \mathcal{N}$ which is a complexification of $T^* \mathcal{N}$, whose fibre at x^μ is $\mathbb{C}^2$, the complexification of the cotangent space $T^*_x \mathcal{N} \simeq \mathbb{R}^2$. Then substituting $y_\mu = \frac{1}{2}(z_\mu + \bar{z}_\mu)$ in $\tilde{F}(x, y)$ gives a function

$$K_x(z, \bar{z}) = \tilde{F}(x, z + \bar{z}) \tag{6.18}$$

for each point x on the base space $\mathcal{N}$, which can be interpreted as the Kähler potential for the metric

$$\frac{\partial^2 K_x(z, \bar{z})}{\partial z_\mu \partial \bar{z}_\nu} = \tilde{G}^{\mu\nu}_x(y) \tag{6.19}$$

on the complexified cotangent space at x, $\mathbb{C}T_x^*\mathcal{N} \simeq \mathbb{C}^2$. The fact that K_x is independent of $u_\mu = -\frac{i}{2}(z_\mu - \bar{z}_\mu)$ implies that K_x is the Kähler potential for a Kähler metric of signature (2, 2) on $\mathbb{R}^4$ with two commuting holomorphic Killing vectors, $\partial/\partial u_\mu$. The condition $\det(\tilde{G}_x^{\mu\nu}(y)) = -1$ is then the Plebanski equation [28] (or complex Monge–Ampère equation [29]), which requires that the metric is Ricci-flat and so hyperkähler and this implies that for each x, the corresponding curvature tensor is either self-dual or anti-self-dual. Thus, for each x, $\tilde{F}(x, y) = K_x(z, \bar{z})$ is the Kähler potential for a hyperkähler metric on $\mathbb{R}^4$ with two commuting (tri-) holomorphic Killing vectors and signature (2, 2). (For Euclidean $\mathcal{W}$-gravity, with $h_{\mu\nu}$ has signature (2, 0) and the internal hyperkähler metric $\tilde{G}^{\mu\nu}$ has signature (4, 0)). Thus $K_x(z, \bar{z})$ gives a two-parameter family of metrics labelled by the points $x^\mu \in \mathcal{N}$, so that in this way we obtain a bundle over $\mathcal{N}$ whose fibres are $\mathbb{C}^2$, equipped with a half-flat metric with two Killing vectors.

If $\tilde{F}$ satisfies the constraint (6.16), the constraint (6.10) on the infinitesimal parameters Λ can be rewritten, to lowest order in Λ, as

$$\det\left(\frac{\partial^2[\tilde{F} + \Lambda](x, y)}{\partial y_\mu\, \partial y_\nu}\right) = -1 \tag{6.20}$$

which implies that $\tilde{F} + \Lambda$ also corresponds to a Kähler potential for a hyperkähler metric with two killing vectors, so that for each x, Λ represents a deformation of the hyperkähler geometry.

The field equation obtained by varying ϕ in (6.3) is

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y_\mu}\tilde{F} = 0 \tag{6.21}$$

which can be rewritten as

$$\sum_n \partial_\mu[\tilde{h}_{(n)}^{\mu\nu_1 \ldots \nu_{n-1}}\, \partial_{\nu_1}\phi \ldots \partial_{\nu_{n-1}}\phi] = 0\,. \tag{6.22}$$

7. Twistor Transform Solution of Monte–Ampère–Plebanski Constraints

The general solution of the Monge–Ampère equation (6.16) can be given implicitly by a Penrose transform construction. For solutions with one (triholomorphic) Killing vector, the Penrose transform reduces to a Legendre transform solution [35] which was first found in the context of supersymmetric non-linear sigma-models [34]. This will now be used to solve (6.16); see [35] for a discussion of the twistor space interpretation. It will be convenient to introduce the notation $y_0 = \zeta$, $y_1 = \xi$. The first step is to write $\tilde{F}(x, \zeta, \xi)$ as the Legendre transform with respect to ζ of some H, so that

$$\tilde{F}(x, \zeta, \xi) = \pi\zeta - H(x, \pi, \xi)\,, \tag{7.1}$$

where the equation

$$\frac{\partial H}{\partial \pi} = \zeta \tag{7.2}$$

gives π implicitly as a function of x, ζ, ξ, so that $\pi = \pi(x, \zeta, \xi)$. Then it is straightforward to show that

$$\frac{\partial \tilde{F}}{\partial \zeta} = \pi, \quad \frac{\partial \tilde{F}}{\partial \xi} = -\frac{\partial H}{\partial \xi} \tag{7.3}$$

and

$$\frac{\partial \pi}{\partial \zeta} = \left(\frac{\partial^2 H}{\partial \pi^2}\right)^{-1} \quad \frac{\partial \pi}{\partial \xi} = -\left(\frac{\partial^2 H}{\partial \pi^2}\right)^{-1} \frac{\partial^2 H}{\partial \pi\, \partial \xi} \tag{7.4}$$

and to use these to obtain

$$\begin{aligned} \frac{\partial^2 \tilde{F}}{\partial \zeta^2} &= \left(\frac{\partial^2 H}{\partial \pi^2}\right)^{-1}, \\ \frac{\partial^2 \tilde{F}}{\partial \zeta\, \partial \xi} &= -\left(\frac{\partial^2 H}{\partial \pi^2}\right)^{-1} \frac{\partial^2 H}{\partial \pi\, \partial \xi}, \\ \frac{\partial^2 \tilde{F}}{\partial \xi^2} &= -\frac{\partial^2 H}{\partial \xi^2} + \left(\frac{\partial^2 H}{\partial \pi^2}\right)^{-1} \left(\frac{\partial^2 H}{\partial \pi\, \partial \xi}\right)^2 . \end{aligned} \tag{7.5}$$

It then follows that

$$\det\left(\frac{\partial^2 \tilde{F}}{\partial y_\mu\, \partial y_\nu}\right) = -\left(\frac{\partial^2 H}{\partial \pi^2}\right)^{-1} \frac{\partial^2 H}{\partial \xi^2} . \tag{7.6}$$

Then the Monge–Ampère equation (6.16) will be satisfied if and only if H satisfies

$$\frac{\partial^2 H}{\partial \pi^2} - \frac{\partial^2 H}{\partial \xi^2} = 0 \tag{7.7}$$

and the general solution of this is

$$H = H_1(x, \pi + \xi) + H_2(x, \pi - \xi) \tag{7.8}$$

for arbitrary functions H_1, H_2. Then the general solution of (6.16) is given by the Legendre transform (7.1) of (7.8) and the action (6.3) can be given in the first order form

$$S = \int d^2x \tilde{F}(x, y) = \int d^2x (\pi \dot{\phi} - H_1(x^\mu, \pi + \phi') - H_2(x^\mu, \pi - \phi')) , \tag{7.9}$$

where $y_0 = \dot{\phi}, y_1 = \phi'$. This is essentially the canonical formulation of $\mathcal{W}$-gravity of [9]. The field equation for the auxiliary field π is (7.2) and this can be used in principle to eliminate π from the action. However, it will not be possible to solve Eq. (7.2) explicitly in general.

The close relation between the forms of the action (7.9) and (3.6), (3.7) suggests that there may be a covariant Legendre-type transform technique that leads to the form of the action (3.6), (3.7). Indeed, $\tilde{F}$ can be written as a transform of a function H as follows:

$$\tilde{F}(x^\mu, y_\nu) = 2\pi^\mu y_\mu - \frac{1}{2} \eta^{\mu\nu} y_\mu y_\nu - 2H(x, \pi) , \tag{7.10}$$

where the equation

$$y_\mu = \frac{\partial H}{\partial \pi^\mu} \tag{7.11}$$

implicitly determines $\pi_\mu = \pi_\mu(x^\mu, y_\rho)$. H is not quite a Legendre transform of $\tilde{F}$ with respect to y_0 and y_1 because of the y^2 term in (7.10). Then

$$\frac{\partial \tilde{F}}{\partial y_\mu} = 2\pi^\mu - \eta^{\mu\nu} y_\nu \tag{7.12}$$

and the transform (7.10) can be inverted to give

$$H(x, \pi) = -\frac{1}{2}\tilde{F}(x^\mu, y_\nu) + \pi^\mu y_\mu - \frac{1}{4}\eta^{\mu\nu} y_\mu y_\nu , \tag{7.13}$$

where (7.12) implicitly gives $y_\mu = y_\mu(x, \pi)$. As the transform is invertible, any $\tilde{F}$ can be written as the transform of some H and vice versa. Using

$$\frac{\partial^2 \tilde{F}}{\partial y_\mu\, \partial y_\nu} = -\eta^{\mu\nu} + 2\left(\frac{\partial^2 H}{\partial \pi^\mu\, \partial \pi^\nu}\right)^{-1} , \tag{7.14}$$

it follows that

$$\det\left(\frac{\partial^2 \tilde{F}}{\partial y_\mu\, \partial y_\nu}\right) = -1 + 2\Delta^{-1}\left(\eta^{\mu\nu}\frac{\partial^2 H}{\partial \pi^\mu\, \partial \pi^\nu} - 2\right), \tag{7.15}$$

where

$$\Delta = \det\left(\frac{\partial^2 H}{\partial \pi^\mu\, \partial \pi_\nu}\right) . \tag{7.16}$$

Then $\tilde{F}$ will satisfy (6.16) if and only if its transform H satisfies

$$\frac{1}{2}\eta^{\mu\nu}\frac{\partial^2 H}{\partial \pi^\mu\, \partial \pi^\nu} = \frac{\partial^2 H}{\partial \pi^+\, \partial \pi^-} = 1 . \tag{7.17}$$

The general solution of this is

$$H = \pi^+ \pi^- + L(x, \pi^+) + \tilde{L}(x, \pi^-) \tag{7.18}$$

which can be used to give the action

$$\begin{aligned} S &= \int d^2x\left(2\pi^\mu y_\mu - 2H(x, \pi) - \frac{1}{2}\eta^{\mu\nu} y_\mu y_\nu\right) \\ &= \int d^2x\left(2\pi^\mu y_\mu - \eta_{\mu\nu}\pi^\mu\pi^\nu - \frac{1}{2}\eta^{\mu\nu} y_\mu y_\nu - 2L(x, \pi^+) - 2\tilde{L}(x, \pi^-)\right) . \end{aligned} \tag{7.19}$$

The field equation for π^μ is (7.11), and using this to substitute for π gives the action (6.3) subject to the constraint (6.16). Alternatively, expanding the functions L, $\tilde{L}$ as

$$\begin{aligned} L(x, \pi^+) &= \sum_{s=2}^{\infty} \frac{1}{s} h^{(+s)}(\pi^+)^s , \\ \tilde{L}(x, \pi^-) &= \sum_{s=2}^{\infty} \frac{1}{s} h^{(-s)}(\pi^-)^s , \end{aligned} \tag{7.20}$$

reproduces the action (3.6). In this way, the auxiliary fields π^μ of the approach of [2] have a natural twistor interpretation, and we learn that the fact that the actions (7.9), (3.6) are linear in the gauge fields reflects the fact that the twistor transform converts the self-duality equation into a linear twistor-space problem.

Conversely, we know from [2, 3] that the action (3.6) is invariant under $\mathcal{W}_\infty$ transformations and that this action can be rewritten as (7.19) provided that H satisfies the constraint (7.17). However, $H(x, \pi)$ can be expressed in terms of a function $\tilde{F}(x, y)$ using the inverse transform (7.13), and using this the action becomes simply $\int d^2x\, \tilde{F}(x, y)$ while from (7.15) it follows that the constraint (7.17) becomes precisely (6.16). Thus the fact that (7.19) subject to (7.17) is an invariant action implies that the action (6.3) subject to the constraint (6.16) is also invariant This establishes the result that the constraint (6.16) is consistent with the $\mathcal{W}$-transformations, as claimed in the last section.

8. Covariant Formulation and $\mathcal{W}$-Weyl Invariance

The constraints on the gauge fields $\tilde{h}_{(n)}^{\mu\nu\cdots}$ generated by (6.16) can be solved in terms of unconstrained gauge fields in a number of ways. We shall first review the solution of [14] which led to gauge fields which transformed naturally under $\mathcal{W}$-Weyl symmetry and then discuss a solution which it is conjectured will lead to an expression of the gauge fields $\tilde{h}_{(n)}^{\mu\nu\cdots}$ occurring in the expansion of a $\mathcal{W}$-density $\tilde{F}(x, y)$ in terms of the gauge fields $h_{(n)}^{\mu\nu\cdots}$ in the expansion of a $\mathcal{W}$-scalar.

The constraint (6.13) can be solved in terms of an unconstrained metric tensor $g_{\mu\nu} = g_{(2)\mu\nu}$ as

$$\tilde{h}_{(2)}^{\mu\nu} = \sqrt{-g}\, g_{(2)}^{\mu\nu}\,. \tag{8.1}$$

Similarly, the constraint (6.14) can be solved in terms of an unconstrained third rank tensor $g_{(3)}^{\mu\nu\rho}$:

$$\tilde{h}_{(3)}^{\mu\nu\rho} = \sqrt{-g}\left[\, g_{(3)}^{\mu\nu\rho} - \frac{3}{4} g^{(\mu\nu}\, g_{(3)}^{\rho)\alpha\beta}\, g_{\alpha\beta} \right] \tag{8.2}$$

and (6.15) can be solved in terms of an unconstrained fourth rank tensor $\hat{g}_{(4)}^{\mu\nu\rho\sigma}$:

$$\tilde{h}_{(4)}^{\mu\nu\rho\sigma} = \sqrt{-g}\left[\, \hat{g}_{(4)}^{\mu\nu\rho\sigma} + g^{(\mu\nu}\, Q^{\rho\sigma)} - \frac{1}{8} g^{(\mu\nu}\, g^{\rho\sigma)}\, Q^{\alpha\beta}\, g_{\alpha\beta} \right], \tag{8.3}$$

where

$$Q^{\rho\sigma} = \frac{2}{3} h^{\alpha\gamma(\rho}\, h^{\sigma)\beta\delta}\, g_{\alpha\beta} g_{\gamma\delta} - \hat{g}_{(4)}^{\rho\sigma\alpha\beta}\, g_{\alpha\beta}\,,$$

$$h^{\mu\nu\rho} = g_{(3)}^{\mu\nu\rho} - \frac{3}{4} g^{(\mu\nu} g_{(3)}^{\rho)\alpha\beta}\, g_{\alpha\beta}\,. \tag{8.4}$$

This can be repeated for all spins, giving the constrained tensor densities $\tilde{h}_{(n)}^{\mu_1\mu_2\ldots\mu_n}$ in terms of unconstrained tensors $g_{(n)}^{\mu_1\mu_2\ldots\mu_n}$, which can be assembled into a function

$$f(x, y) = \sum_{n=2}^{\infty} \frac{1}{n}\, g_{(n)}^{\mu_1\mu_2\ldots\mu_n}(x)\, y_{\mu_1}\, y_{\mu_2} \ldots y_{\mu_n}\,. \tag{8.5}$$

The generating function $\tilde{F}$ for the tensor densities $\tilde{h}_{(n)}$ can then be written as

$$\tilde{F}(x, y) = \Omega(x, y) f(x, y)\,, \tag{8.6}$$

where Ω is determined in terms of f by requiring (8.6) to satisfy (6.16). The function Ω has an expansion of the form

$$\Omega(x, y) = \sum_{n=0}^{\infty} \Omega_{(n+2)}^{\mu_1 \mu_2 \dots \mu_n}(x) y_{\mu_1} y_{\mu_2} \dots y_{\mu_n} \tag{8.7}$$

and substituting (8.6) in (6.16) gives a set of equations which can be solved to give the tensors $\Omega_{(n)}$ in terms of the unconstrained tensors $g_{(n)}$ in (8.5). This gives

$$\Omega_{(2)} = \sqrt{-g} \,,$$

$$\Omega_{(3)}^{\mu} = -\frac{1}{2}\sqrt{-g}\, g_{(3)}^{\mu\nu\rho} g_{\nu\rho} \,,$$

$$\Omega_{(4)}^{\mu\nu} = \sqrt{-g}\left[\frac{1}{2} Q^{\mu\nu} - \frac{1}{16} g^{\mu\nu} Q^{\rho\sigma} g^{\rho\sigma}\right], \tag{8.8}$$

where $Q^{\mu\nu}$ is given by (8.4) and $g_{(4)}^{\mu\nu\rho\sigma}$ is related to the tensor $\hat{g}_{(4)}^{\mu\nu\rho\sigma}$ in (8.3) by the field redefinition

$$g_{(4)}^{\mu\nu\rho\sigma} = \hat{g}_{(4)}^{\mu\nu\rho\sigma} - \frac{4}{3} g_{(3)}^{(\mu\nu\rho} \Omega_{(3)}^{\sigma)} \,. \tag{8.9}$$

The solution (8.1) to the constraint (6.13) is invariant under the Weyl transformation $g_{\mu\nu} \to \sigma(x) g_{\mu\nu}$, and this suggests that (8.6) should be invariant under higher spin generalisations of this. Indeed, writing $\tilde{F}$ in terms of f gives an action which is invariant under the $\mathscr{W}$-Weyl transformations

$$\delta f(x, y) = \sigma(x, y) f(x, y) \,. \tag{8.10}$$

Expanding

$$\sigma(x, y) = \sigma_{(2)}(x) + \sigma_{(3)}^{\mu}(x) y_{\mu} + \sigma_{(4)}^{\mu\nu}(x) y_{\mu} y_{\nu} + \dots \tag{8.11}$$

these can be written as

$$\delta g_{(n)}^{\mu_1 \dots \mu_n} = n \sum_{r=2}^{n} \frac{1}{r} g_{(r)}^{(\mu_1 \dots \mu_r} \sigma_{(n-r+2)}^{\mu_{r+1} \dots \mu_n)} \,. \tag{8.12}$$

These transformations can be used to remove all traces from the gauge fields, leaving only traceless gauge fields. These $\mathscr{W}$-Weyl transformations are similar to those given in (3.8) and have the same linearised limit, but have the advantage that they do not give a privileged position to the spin-two gauge field.

The relation (8.6) implies that a $\mathscr{W}$-Weyl transformation can be used to set $\tilde{F}(x, y) = f(x, y)$, so that $\tilde{h}_{(n)}^{\mu_1 \dots \mu_n} = g_{(n)}^{\mu_1 \dots \mu_n}$ in this $\mathscr{W}$-Weyl gauge. This means that in general the transformations of $g_{(n)}^{\mu_1 \dots \mu_n}$ can be taken to be equal to those of $\tilde{h}_{(n)}^{\mu_1 \dots \mu_n}$, or to be related to these by a possible $\mathscr{W}$-Weyl transformation. For example, this gives the transformation of $g_{(2)}^{\mu\nu}$ to be that of (6.2), up to a Weyl transformation

$$\delta g^{\mu\nu} = k^{\rho} \partial_{\rho} g^{\mu\nu} - 2 g^{\rho(\mu} \partial_{\rho} k^{\nu)} + g^{\mu\nu}(\sigma + \partial_{\rho} k^{\rho}) \,. \tag{8.13}$$

Then shifting $\sigma \to \sigma' = \sigma + \partial_{\rho} k^{\rho}$ absorbs the $\partial_{\rho} k^{\rho}$ terms into the Weyl transformation and the transformation becomes the standard one for an inverse metric:

$$\delta g^{\mu\nu} = k^{\rho} \partial_{\rho} g^{\mu\nu} - 2 g^{\rho(\mu} \partial_{\rho} k^{\nu)} + g^{\mu\nu} \sigma \,. \tag{8.14}$$

Similarly, the term proportional to $[\partial_\nu \lambda_{(p)}^{\nu(\mu_1\mu_2\cdots}]g_{(2)}^{\cdots\mu_p)}$ in the variation $\delta g_{(p)}^{\mu_1\mu_2\cdots\mu_p}$ given by replacing $\tilde{h}$ by g in (5.24) can be absorbed into a $\mathcal{W}$-Weyl transformation, but the resulting transformation for $g_{(n)}$ is not that corresponding to that of a $\mathcal{W}$-scalar and does not seem to have any obvious geometric interpretation.

The constraint (6.20) on the parameters $\lambda_{(n)}$ can be solved in a similar fashion in terms of unconstrained parameters $k_{(n)}^{\mu_1\cdots\mu_{n-1}}$ and the transformations of the unconstrained gauge fields can be defined to take the form $\delta g_{(n)}^{\mu_1\mu_2\cdots\mu_n} = \partial^{(\mu_1}k_{(n)}^{\mu_2\cdots\mu_n)} + \ldots$. The $g_{(n)}$ might be thought of as gauge fields for the whole of the symplectic diffeomorphisms of $T^*\mathcal{N}$ (with parameters $k_{(n)}$), and appear in the action only through the combinations $\tilde{h}_{(n)}$. The transformations of $\tilde{h}_{(n)}$ and ϕ then only depend on the parameters $k_{(n)}$ in the form $\lambda_{(n)}$.

In gravity theory, it is sufficient to have a metric $h_{\mu\nu}$ in order to construct actions, as densities can be constructed using $\sqrt{-\det[h_{\mu\nu}]}$. In $\mathcal{W}$-gravity, it is natural to ask whether a cometric function F which transforms as a $\mathcal{W}$-scalar can be used to construct actions, and in particular whether a $\mathcal{W}$-density $\tilde{F}$ can be constructed from a $\mathcal{W}$-scalar F. If so, this would lend weight to the idea that the cometric F might play a fundamental role in $\mathcal{W}$-geometry in the same way that the line element does in Riemannian geometry. This would be particularly attractive, as a $\mathcal{W}$-scalar transforms naturally under the whole of the symplectic diffeomorphisms of the cotangent bundle, $\mathrm{Diff}_0(T^*\mathcal{N})$, while a $\mathcal{W}$-density only transforms under the subgroup of this defined by the constraint (6.10). Thus, as in the previous paragraph, we would have gauge fields $h_{(n)}$ for the whole of $\mathrm{Diff}_0(T^*\mathcal{N})$ with the $\mathcal{W}$-scalar transformation law (5.24),

$$\delta h_{(n)}^{\mu_1\mu_2\cdots\mu_n} = \partial^{(\mu_1}k_{(n)}^{\mu_2\cdots\mu_n)} + \ldots \tag{8.15}$$

with $\tilde{h}_{(n)}^{\mu_1\cdots\mu_n} = [h_{(n)}^{\mu_1\cdots\mu_n} - (\text{traces})] + \ldots$ plus non-linear terms, and $\delta\tilde{h}_{(n)}^{\mu_1\cdots\mu_n} = \partial^{(\mu_1}\lambda_{(n)}^{\mu_2\cdots\mu_n)} + \ldots$, where $\lambda_{(n)}^{\mu_1\cdots\mu_{n-1})} = [k_{(n)}^{\mu_1\cdots\mu_{n-1})} - (\text{traces})] + \ldots$ plus non-linear terms.

It is straightforward to show that the first few density gauge fields $\tilde{h}_{(n)}$, subject to the constraints generated by (6.16), can be written in terms of the first few tensor gauge fields $h_{(n)}$ as follows:

$$\tilde{h}_{(2)}^{\mu\nu} = \sqrt{-h}\,h_{(2)}^{\mu\nu}\,, \tag{8.16}$$

$$\tilde{h}_{(3)}^{\mu\nu\rho} = \frac{2}{3}\sqrt{-h}\left[h_{(3)}^{\mu\nu\rho} - \frac{3}{4}h^{(\mu\nu}h_{(3)}^{\rho)\alpha\beta}h_{\alpha\beta}\right], \tag{8.17}$$

$$\begin{aligned}\tilde{h}_{(4)}^{\mu\nu\rho\sigma} = {}& \sqrt{-h}\left[K^{\mu\nu\rho\sigma} - h^{(\mu\nu}K^{\rho\sigma)\alpha}{}_{\alpha} + \frac{1}{8}h^{(\mu\nu}h^{\rho\sigma)}K^{\alpha\beta}{}_{\alpha\beta}\right]\\ &+ \frac{2}{3\sqrt{-h}}\left[h^{(\mu\nu}\tilde{h}_{(3)\alpha\beta}^{\rho}\tilde{h}_{(3)}^{\sigma)\alpha\beta} - \frac{1}{8}h^{(\mu\nu}h^{\rho\sigma)}\tilde{h}_{(3)}^{\alpha\beta\gamma}\tilde{h}_{(3)\alpha\beta\gamma}\right],\end{aligned} \tag{8.18}$$

where indices are raised and lowered with $h^{\mu\nu} = h_{(2)}^{\mu\nu}$ and its inverse $h_{\mu\nu}$, $h = \det[h_{\mu\nu}]$ and

$$K^{\mu\nu\rho\sigma} = \frac{1}{2}h^{\mu\nu\rho\sigma} - \frac{1}{3}h_{(3)}^{\alpha(\mu\nu}h_{(3)\alpha}^{\rho\sigma)}\,. \tag{8.19}$$

This means that given a set of gauge fields $h_{(n)}$ transforming under $\mathrm{Diff}_0(T^*\mathcal{N})$ as in (5.2), then the gauge fields $\tilde{h}_{(n)}$ defined by these equations transform as in (5.24).

I conjecture that all the $\tilde{h}_{(n)}$ can be written in terms of $h_{(n)}$ gauge fields in this way, although I have as yet no general proof; this is currently under investigation.

9. Summary and Discussion

We have seen that symplectic diffeomorphisms of the cotangent bundle of the space-time (or world-sheet) $\mathcal{N}$ play a fundamental role in $\mathcal{W}$-gravity, generalising the role played by the diffeomorphisms of $\mathcal{N}$ in ordinary gravity theories. For any dimension d of $\mathcal{N}$, we found an infinite set of symmetric tensor gauge fields $h_{(n)}^{\mu_1 \ldots \mu_n}$, $n = 2, 3, \ldots$, transforming under the action of a gauge group isomorphic to $\mathrm{Diff}_0(T^*, \mathcal{N})$ as

$$\delta h_{(p)}^{\mu_1 \ldots \mu_p} = p \sum_{m,n} \delta_{n+m,p+2} \left[\frac{m-1}{n} k_{(m)}^{\nu(\mu_1 \ldots} \partial_\nu h_{(n)}^{\ldots \mu_p)} - h_{(n)}^{\nu(\mu_1 \ldots} \partial_\nu k_{(m)}^{\ldots \mu_p)} \right], \quad (9.1)$$

where $k_{(m)}^{\mu_1 \ldots \mu_{m-1}}(x)$ are unconstrained infinitesimal symmetric tensor parameters. These transformations had a geometric interpretation: they were precisely the transformations needed for the generating function

$$F(x^\mu, y_\mu) = \sum_n \frac{1}{n} h_{(n)}^{\mu_1 \ldots \mu_n}(x) y_{\mu_1} \ldots y_{\mu_n} \quad (9.2)$$

to transform as a $\mathcal{W}$-scalar, i.e. to be invariant under the action of the gauge group $\mathrm{Diff}_0(T^* \mathcal{N})$ (as described in Sect. 5, with $y = \partial \phi$). This suggested regarding F as the natural generalisation of the invariant line element of Riemannian geometry.

As well as considering $\mathcal{W}$-scalars, we also considered $\mathcal{W}$-densities $\tilde{F}$, which we found could only exist in dimensions $d = 1, 2$. The $\mathcal{W}$-density $\tilde{F}$ generated an infinite set of gauge fields $\tilde{h}_{(n)}^{\mu_1 \ldots \mu_n}$. In the case $d = 1$, these gauge fields transformed under local $\mathrm{Diff}_0(T^* \mathcal{N}) \sim \mathcal{W}_\infty$ transformations as

$$\delta \tilde{h}_{(p)} = \sum_{m,n} \delta_{m+n,p+2} \left[(m-1) \lambda_{(m)} \partial \tilde{h}_{(n)} - (n-1) \tilde{h}_{(n)} \partial \lambda_{(m)} \right]. \quad (9.3)$$

For $d = 2$, we considered gauge fields with the transformation

$$\begin{aligned} \delta \tilde{h}_{(p)}^{\mu_1 \mu_2 \ldots \mu_p} = \sum_{m,n} \delta_{m+n,p+2} \Bigg[& (m-1) \lambda_{(m)}^{(\mu_1 \mu_2 \ldots} \partial_\nu \tilde{h}_{(n)}^{\ldots \mu_p)\nu} - (n-1) \tilde{h}_{(n)}^{\nu(\mu_1 \mu_2 \ldots} \partial_\nu \lambda_{(m)}^{\ldots \mu_p)} \\ & + \frac{(m-1)(m-1)}{p-1} \partial_\nu \{ \lambda_{(m)}^{\nu(\mu_1 \mu_2 \ldots} \tilde{h}_{(n)}^{\ldots \mu_p)\nu} - \tilde{h}_{(n)}^{\nu(\mu_1 \mu_2 \ldots} \lambda_{(m)}^{\ldots \mu_p)} \} \Bigg]. \end{aligned} \quad (9.4)$$

Note that we could consider this transformation for any dimension d; in particular, it reduces to (9.3) if $d = 1$. However, for $d > 2$, the corresponding generating function $\tilde{F}$ is never a $\mathcal{W}$-density, while for $d = 2$ $\tilde{F}$ is not a $\mathcal{W}$-density for the full group $\mathrm{Diff}_0(T^* \mathcal{N})$ but only for the subgroup defined by the constraint (6.9), or equivalently, (6.10). This formulation is redundant, in the sense that there are more gauge fields than are needed, and it was shown that the following constraint could be consistently imposed on the gauge fields:

$$\det \left(\frac{\partial^2 \tilde{F}(x, y)}{\partial y_\mu \partial y_\mu} \right) = -1 . \quad (9.5)$$

This constraint is preserved by the gauge transformations (9.4), and implies that the linearised gauge fields are traceless.

For $d = 1$, we showed that gauge fields $\tilde{h}_{(n)}$ transforming as in (9.3) could be explicitly constructed from gauge fields $h_{(n)}$ transforming as in (9.1), so that given any $d = 1$ $\mathcal{W}$-scalar $F(x, y)$, we obtain a $\mathcal{W}$-density by writing

$$\tilde{F}(x, y) = \mathbf{N}^{-1}[y\sqrt{2F(x, y)}] . \tag{9.6}$$

For $d = 2$, we showed that the first few gauge fields $\tilde{h}_{(n)}$, $n = 2, 3, 4$, could be expressed in terms of $\mathcal{W}$-scalar gauge fields $h_{(n)}$ (provided that the gauge fields satisfied the constraints generated by (9.5)) and conjectured that there was such a construction for all n. The reformulation in terms of the $h_{(n)}$ involved many redundant gauge fields (in the linearised theory, these are the traces of the $h_{(n)}$) which could be gauged away using $\mathcal{W}$-Weyl transformations.

The action for a single scalar field ϕ coupled to $\mathcal{W}$-gravity in either one or two dimensions ($d = 1, 2$) is then given by the integral of the $\mathcal{W}$-density $\tilde{F}(x, \partial\phi)$ over $\mathcal{N}$,

$$S = \int d^d x \sum_n \tilde{h}_{(n)}^{\mu_1 \ldots \mu_n}(x) S^{(n)}_{\mu_1 \ldots \mu_n} , \tag{9.7}$$

where the currents $S^{(n)}_{\mu_1 \ldots \mu_n}$ are defined by

$$S^{(n)}_{\mu_1 \ldots \mu_n} = \frac{1}{n} \partial_{\mu_1}\phi \ldots \partial_{\mu_n}\phi . \tag{9.8}$$

If $d = 2$, this remains invariant if the constraints generated by (9.5) are imposed on the gauge fields $\tilde{h}_{(n)}$, and it seems that the action can then be reformulated in terms of $\mathcal{W}$-scalar gauge fields $h_{(n)}$.

So far we have restricted ourselves to the rather trivial case of a single boson. However, for any matter current $S_{\mu\nu}$ that transforms under diffeomorphisms in the same way as the free boson current $\frac{1}{2} g_{ij} \partial_\mu \phi^i \partial_\nu \phi^j$, i.e. which transforms as a tensor, the action $S = \int d^d x\, \tilde{h}^{\mu\nu} S_{\mu\nu}$ is invariant provided that $\tilde{h}^{\mu\nu}$ transforms as a tensor density. In the same way, given any matter system which can be used to construct a set of currents $S^{(n)}_{\mu_1 \ldots \mu_n}$ which transform in the same way under $\mathcal{W}$-gravity transformations as the single-boson currents (9.8), then the action (9.7) involving these new currents will be $\mathcal{W}$-invariant, provided that the gauge fields $\tilde{h}$ transform as in (9.3) or (9.4). This immediately gives actions for a large set of matter systems; this will be discussed further elsewhere.

Another important issue is the generalisation of these results to other $\mathcal{W}$-algebras. As will be shown in [37], the gauge fields $\tilde{h}$ for $\mathcal{W}_N$ gravity are generated by a $\mathcal{W}$-density $\tilde{F}$ which, in addition to the constraint (9.5), satisfies a non-linear $(N + 1)^{\text{th}}$ order differential constraint, which implies that only the gauge fields $\tilde{h}_{(2)}, \tilde{h}_{(3)}, \ldots \tilde{h}_{(N)}$ are independent. Whereas the constraint (9.5) is related to self-dual geometry, the new $(N + 1)^{\text{th}}$ order differential constraint is similar to the type of constraint that arises in the study of special geometry [38]. The truncation to the $\mathcal{W}$-gravity theory corresponding to the algebra $\mathcal{W}_{\infty/2}$ (i.e. the subalgebra of $\mathcal{W}_\infty$ generated by currents of even spin) is more straightforward: it corresponds to setting to zero all of the gauge fields of odd spin, $h_{(2n+1)}$.

One motivation for the study of $\mathcal{W}$-geometry is to try to understand finite $\mathcal{W}$-transformations (as opposed to those with infinitesimal parameters) and the moduli space for $\mathscr{W}$-gravity. The infinitesimal transformations for the scalar field ϕ were derived from studying infinitesimal symplectic diffeomorphisms and it follows that the large W-transformations of ϕ are given by the action of large $\mathrm{Diff}_0\, T^*\mathcal{N}$ transformations on $y_\mu = \partial_\mu \phi$. The finite transformations of the gauge fields $h_{(n)}$ are given by requiring the invariance of the generating function $F(x, y)$, while the finite transformations for the $\tilde{h}_{(n)}$ follow from requiring the invariance of $\int \tilde{F}$, or from the construction of $\tilde{F}$ in terms of F. It seems natural to conjecture that the transformations of the gauge fields can be defined to give invariance under the full group of symplectic diffeomorphisms, as opposed to invariance under the subgroup generated by exponentiating infinitesimal ones, but this remains to be proved.

The gauge-fixing of $\mathscr{W}$-gravity and the generalisation of the Liouville theory that emerges in $\mathscr{W}$-conformal gauge were discussed in [11]. Consider now the moduli space M_n for gauge fields $\tilde{h}_{(n)}$ subject to the constraints generated by (9.5) [11]. Linearising about a Euclidean background $\tilde{F} = \frac{1}{2}\tilde{h}^{\mu\nu}_{(2)} y_{\mu\nu}$ and choosing complex coordinates $z, \bar{z}$ on the Riemann surface $\tilde{\mathcal{N}}$ such that the background is $\tilde{F} = y_z y_{\bar{z}}$, and using the linearised transformations $\delta \tilde{h}^{zz\cdots z}_{(n)} = \partial_{\bar{z}} \lambda^{zz\cdots z}_{(n)}$, it follows by standard arguments that the tangent space to the moduli space M_n at a point corresponding to the background configuration is the space of holomorphic n-differentials, i.e. the n^{th} rank symmetric tensors $\mu_{zz\ldots z}$ with n lower z indices satisfying $\partial_{\bar{z}} \mu_{zz\ldots z} = 0$ [11]. It follows from the Riemann–Roch theorem that the dimension of this space on a genus-g Riemann surface (the number of anti-ghost zero-modes) is $\dim(M_n) = (2n-1)(g-1) + k(n, g)$, where $k(n, g)$ is the number of solutions $\kappa^{zz\cdots z}$ (with $n-1$ "z" indices) to $\partial_{\bar{z}} \kappa^{zz\cdots z} = 0$ (the number of ghost zero-modes). It would be of great interest to use information about the global structure of the symplectic diffeomorphism group to learn more about the structure of these moduli spaces.

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