

Darboux Coordinates and Liouville-Arnold Integration in Loop Algebras[★]

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Abstract. Darboux coordinates are constructed on rational coadjoint orbits of the positive frequency part $\tilde{\mathfrak{g}}^+$ of loop algebras. These are given by the values of the spectral parameters at the divisors corresponding to eigenvector line bundles over the associated spectral curves, defined within a given matrix representation. A Liouville generating function is obtained in completely separated form and shown, through the Liouville-Arnold integration method, to lead to the Abel map linearization of all Hamiltonian flows induced by the spectral invariants. As illustrative examples, the case $\mathfrak{g} = \mathfrak{sl}(2)$, together with its real forms, is shown to reproduce the classical integration methods for finite dimensional systems defined on quadrics, with the Liouville generating function expressed in hyperellipsoidal coordinates. For $\mathfrak{g} = \mathfrak{sl}(3)$, the method is applied to the computation of quasi-periodic solutions of the two component coupled nonlinear Schrödinger equation, a case which requires further symplectic constraints in order to deal with singularities in the spectral data at ∞ .

Introduction

In [AHP, AHH1, AHH2], a unified approach was developed to the representation of both finite dimensional integrable Hamiltonian systems and quasi-periodic solutions of integrable PDE's as isospectral flows in loop algebras. This involves identifying certain reduced symplectic vector spaces through an equivariant moment map with a set of coadjoint orbits whose elements are rational functions of the complexified loop parameter. The flows induced by Hamiltonians from the ring of spectral invariants are determined, through the Adler-Kostant-Symes (AKS) theorem, by isospectral equations of Lax type,

$$\frac{d\mathcal{N}(\lambda)}{dt} = [(d\phi)_+, \mathcal{N}(\lambda)], \quad (0.1)$$

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where $\mathcal{N}(\lambda)$ is an $r \times r$ matrix valued rational function of the complexified loop parameter λ and $(d\phi)_+$ denotes the projection of the differential of the spectral invariant function $\phi(\mathcal{N})$ to the positive half of the loop algebra.

The commuting invariants are generated by the coefficients of the characteristic equation

$$\det(\mathcal{N}(\lambda) - \lambda\zeta I) = 0, \quad (0.2)$$

which determines the invariant spectral curve \mathcal{S} . The associated linear flows of eigenvector line bundles may then be shown to linearize on the Jacobi variety of \mathcal{S} [vMM, AvM] and the solutions to the equations of motion expressed in terms of theta functions [RS, AHH1] using the Dubrovin-Krichever-Novikov technique [K, KN, Du].

The Hamiltonian content of this algebro-geometric integration method approach is not immediately evident. The link between the algebro-geometric method of integration and the Hamiltonian point of view was made within the context of differential algebras by Gel'fand and Dickey [GD, D], the linearization being based upon the Liouville quadrature method. However, within the loop algebra setting, the symplectic content of the algebro-geometric integration method has only been developed in certain specific examples.

In Sect. 1, we use the Lie algebraic and algebro-geometric structures associated with isospectral flows induced by spectral invariants on the dual $\tilde{\mathfrak{g}}^{+*}$ of the positive frequency half of a loop algebra $\tilde{\mathfrak{g}}$ to introduce a set of Darboux (i.e., canonical) coordinates on rational coadjoint orbits: the “spectral divisor coordinates” associated with the eigenvector line bundle over \mathcal{S} (Theorems 4 and 5). These are determined by solving a pair of polynomial equations with coefficients that are polynomials in the matrix elements of $\mathcal{N}(\lambda)$. They are shown to give a completely separated solution of the Hamilton-Jacobi problem for all Hamiltonians within the spectral ring; that is, a Liouville generating function for the canonical transformation to linearizing coordinates for all flows induced by spectral invariants on such orbits. The transformation is expressed in terms of abelian integrals, giving the Abel map linearization (Theorem 6) through the classical Liouville canonical transformation method, with the Liouville-Arnold torus identified with the Jacobi variety of the underlying spectral curve. Though the Jacobi inversion method, combined with reciprocity theorems for abelian integrals, this leads to θ -function formulae for the integrated flow (Corollary 1.7).

Section 2 consists of examples applying the general method both to finite dimensional problems and to an integrable system of PDE's. First, the $\tilde{\mathfrak{sl}}(2)$ case with n simple poles is shown to reproduce the standard linearization results for well-known classical examples of finite dimensional systems (cf. [M]). For this case, the “spectral divisor coordinates” are just hyperellipsoidal coordinates, the spectral curve is hyperelliptic and the reductions corresponding to fixing spectral data at ∞ lead to constraints defined by quadrics. Secondly, the finite gap solutions of the coupled 2-component nonlinear Schrödinger (CNLS) equation are obtained as an illustration of the $\tilde{\mathfrak{sl}}(3)$ case, involving trigonal curves. The structure of the spectrum at infinity leads in this case to further singularities in the curve, resulting in a decrease of the arithmetic genus relative to the generic case, and an incomplete set of Darboux coordinates on the coadjoint orbit. This example is used to indicate how such problems may be dealt with by restricting to an invariant symplectic submanifold on which the spectral curves share the same generic type of singularities. The linearization on the constrained manifold then proceeds in the same way as in the unconstrained case.

1. Darboux Coordinates and Linearization of Flow

1a. Rational Orbits and Spectral Curves

The Hamiltonian systems to be considered here involve isospectral flows of matrices determined by equations of Lax type:

$$\frac{d\mathcal{N}(\lambda)}{dt} = [\mathcal{B}(\lambda), \mathcal{N}(\lambda)], \tag{1.1}$$

where $\mathcal{N}(\lambda), \mathcal{B}(\lambda)$ are $r \times r$ matrices depending on a complex parameter λ . The matrix $\mathcal{N}(\lambda)$ is taken to be of the form

$$\mathcal{N}(\lambda) = \lambda Y + \lambda \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i}, \tag{1.2}$$

where $Y \in \mathfrak{gl}(r)$, and $\{\alpha_i \in \mathbb{C}\}_{i=1, \dots, n}$ are constants. Thus, we are considering rational $\mathcal{N}(\lambda)$ with fixed, simple poles at the finite points $\{\alpha_i\}$ and possibly at ∞ . Rational matrices with higher order poles may be dealt with similarly, but will not be considered here for the sake of notational simplicity.

The particular form (1.2) arises naturally as the translate by λY of the image

$$\mathcal{N}_0(\lambda) = \lambda \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i} \tag{1.3}$$

of a moment map from a symplectic vector space parametrizing rank- r perturbations of a fixed $N \times N$ matrix with eigenvalues $\{\alpha_i\}_{i=1, \dots, n}$ into the dual $(\tilde{\mathfrak{g}}^+)^*$ of a loop algebra, represented by $r \times r$ matrix functions of the complexified loop parameter λ , holomorphic in a suitable domain [AHP, AHH2]. This serves to embed a large class of integrable systems as Lax pair flows in $(\tilde{\mathfrak{g}}^+)^*$. The image space for such maps is a Poisson subspace of $(\tilde{\mathfrak{g}}^+)^*$, with respect to the Lie Poisson structure, the symplectic leaves (coadjoint orbits) consisting of rational functions of λ . Since a specific $r \times r$ matrix representation is involved, we view \mathfrak{g} as a subalgebra of $\mathfrak{gl}(r, \mathbb{C})$ or $\mathfrak{sl}(r, \mathbb{C})$, obtained generally by reductions under involutive automorphisms.

The loop algebra elements $X \in \tilde{\mathfrak{gl}}(r)$ are viewed as smooth maps $X: S^1 \mapsto \mathfrak{gl}(r)$ from a fixed circle S^1 in the complex λ -plane, containing the points $\{\alpha_i\}$ in its interior, and the subalgebra $\tilde{\mathfrak{gl}}(r)^+$ consists of those $X(\lambda)$ that extend as holomorphic functions to the interior of S^1 . The loop group $\tilde{Gl}(r)$ similarly consists of smooth maps $g: S^1 \mapsto Gl(r)$, while the subgroup $\tilde{Gl}(r)^+$ consists again of those $g(\lambda)$ that extend holomorphically inside S^1 . The subspace $\tilde{\mathfrak{gl}}(r)_- \subset \tilde{\mathfrak{gl}}(r)$ of loops extending holomorphically outside S^1 to ∞ is identified with a dense subspace of the dual space $\tilde{\mathfrak{gl}}(r)^{+*}$ through the dual pairing:

$$\langle \mu, X \rangle := \frac{1}{2\pi i} \oint_{S^1} \text{tr}(\mu(\lambda) X(\lambda)) \frac{d\lambda}{\lambda}, \quad \mu \in \tilde{\mathfrak{gl}}(r)_-, \quad X \in \tilde{\mathfrak{gl}}(r)^+. \tag{1.4}$$

The matrix $\mathcal{B}(\lambda)$ has the form:

$$\mathcal{B}(\lambda) = (d\Phi(\mathcal{N}(\lambda)))_+, \tag{1.5}$$

where $\Phi \in I(\widetilde{\mathfrak{gl}}(r)^*)$ is an element of the ring of Ad^* -invariant polynomials on $\widetilde{\mathfrak{gl}}(r)^*$ and the subscript $+$ means projection to the subspace $\widetilde{\mathfrak{gl}}(r)^+$. In general, no notational distinction will be made between $\widetilde{\mathfrak{gl}}(r)^{+*}$ and $\widetilde{\mathfrak{gl}}(r)_-$. The coadjoint action of $\widetilde{Gl}(r)^+$ on rational elements \mathcal{N}'_0 of the form (1.3) is given by:

$$g: \widetilde{\mathfrak{gl}}(r)_- \rightarrow \widetilde{\mathfrak{gl}}(r)_-,$$

$$g: \lambda \sum_{i=1}^n \frac{N_i}{\lambda - \alpha_i} \mapsto \lambda \sum_{i=1}^n \frac{g(\alpha_i) N_i g(\alpha_i)^{-1}}{\lambda - \alpha_i}. \tag{1.6}$$

Equation (1.1) is Hamilton’s equation on the coadjoint orbit $\mathcal{Q}_{\mathcal{N}'_0} \subset \widetilde{\mathfrak{gl}}(r)^{+*}$, with respect to the orbital (Kostant-Kirillov) symplectic form ω_{orb} , corresponding to the Hamiltonian:

$$\phi(\mu) = \Phi(\mu + \lambda Y). \tag{1.7}$$

The Poisson commutative ring of such functions on $\mathcal{Q}_{\mathcal{N}'_0}$ will be denoted \mathcal{F}_Y . According to the “shifted” version [FRS] of the Adler-Kostant-Symes theorem [A, Ko, S], such systems generate commuting Lax pair flows. Moreover, they may be shown to be completely integrable on “generic” coadjoint orbits [RS, AHP, AHH1] in $\widetilde{\mathfrak{gl}}(r)^{+*}$. On such orbits, the (AKS) ring of commuting invariants is generated by the coefficients of the characteristic polynomial of $\mathcal{N}'(\lambda)$.

In analyzing the spectrum, it is convenient to deal with matricial polynomials in λ , so we define

$$\hat{\mathcal{L}} := \frac{a(\lambda)}{\lambda} \mathcal{N}'(\lambda) = Y a(\lambda) + L_0 \lambda^{n-1} + \dots + L_{n-1}, \tag{1.8}$$

where

$$a(\lambda) := \prod_{i=1}^n (\lambda - \alpha_i). \tag{1.9}$$

The matrix

$$L_0 = \lim_{\lambda \rightarrow \infty} \mathcal{N}'_0(\lambda) = \sum_{i=1}^n N_i \tag{1.10}$$

may be viewed as a moment map generating the conjugation action of $Gl(r)$ on $\widetilde{\mathfrak{gl}}(r)^{+*}$:

$$Gl(r) \times \widetilde{\mathfrak{gl}}(r)^{+*} \rightarrow \widetilde{\mathfrak{gl}}(r)^{+*},$$

$$(g, X(\lambda)) \mapsto gX(\lambda)g^{-1}. \tag{1.11}$$

The matrix $\hat{\mathcal{L}}$ satisfies the same Lax equation (1.1) as $\mathcal{N}'(\lambda)$, and the coefficients of its characteristic polynomial:

$$\mathcal{P}(\lambda, z) := \det(\hat{\mathcal{L}}(\lambda) - zI) \tag{1.12}$$

generate the same ring of invariants as that of $\mathcal{N}'(\lambda)$.

Remark. It is also possible to view

$$\lambda^{-n+1} [\hat{\mathcal{L}} - Y a(\lambda)] := \mathcal{L}(\lambda) \tag{1.13}$$

directly as an element of an orbit in $\widetilde{\mathfrak{gl}}(r)^{+*}$ (polynomial in λ^{-1}). Since the ring of invariants is the same, the results are equivalent, with a suitable redefinition of

the Hamiltonians and parametrization of the spectral curve (cf. [AHP]). We retain our present conventions, with $\mathcal{N}_0(\lambda)$ viewed as the point in $\widetilde{\mathfrak{gl}}(r)^{+*}$ undergoing Hamiltonian flow, since these are adapted to examining the particular spectral constraints occurring at the finite values $\{\lambda = \alpha_i\}$ that appear in specific examples (cf. Sect. 2).

The spectral curve $\mathcal{S}_0 \subset \mathbb{C}^2$ defined by the characteristic equation

$$\mathcal{P}(\lambda, z) = 0 \tag{1.14}$$

is invariant under the AKS Hamiltonian flows. Let m be the degree of $\hat{\mathcal{L}}(\lambda)$, ($m = n$ if $Y \neq 0$ or $m = n - 1$ if $Y = 0$) and let $\{k_i\}$ denote the ranks of the matrices $\{N_i\}_{i=1, \dots, n}$ in (1.2) (coadjoint invariants, and hence invariants of any Hamiltonian flow in $\mathcal{Q}_{\mathcal{S}_0}$).

Lemma 1.1. *The spectral polynomial $\mathcal{P}(\lambda, z)$ has the form:*

$$\mathcal{P}(\lambda, z) = (-z)^r + z^{r-1} \mathcal{P}_1(\lambda) + \sum_{j=2}^r A_j(\lambda) \mathcal{P}_j(\lambda) z^{r-j}, \tag{1.15}$$

where

$$A_j(\lambda) := \prod_{i=1}^n (\lambda - \alpha_i)^{\max(0, j - k_i)} \tag{1.16}$$

and

$$\deg \mathcal{P}_j(\lambda) = \sum_{i=1}^n \min(j, k_i) - j(n - m) =: \kappa_j. \tag{1.17}$$

Remark. This means that $\mathcal{P}(\lambda, z)$ and all its partial derivatives in λ or z up to order $r - k_i - 1$ vanish at $(\alpha_i, 0)$.

Proof. This follows immediately by expanding $\det(\hat{\mathcal{L}}(\lambda) - zI)$ and using the fact that $\hat{\mathcal{L}}(\alpha_i) = N_i \prod_{j=1, j \neq i}^n (\alpha_i - \alpha_j)$ has rank k_i . \square

The structure of $\mathcal{P}(\lambda, z)$ implies that on \mathcal{S}_0 , $z \sim O(\lambda^m)$ as $\lambda \rightarrow \infty$. This suggests assigning z a homogeneity degree m , thereby giving $\mathcal{P}(\lambda, z)$ an overall degree rm . We may then compactify \mathcal{S}_0 , regarding it as the affine part of an r -sheeted branched cover of \mathbf{P}^1 , by embedding it in the total space \mathcal{F} of $\mathcal{O}(m)$, the m^{th} power of the hyperplane section bundle over \mathbf{P}^1 , whose sections are homogeneous functions of degree m (cf. [AHH1]). The pair (λ, z) is viewed as the base and fibre coordinates over the affine neighborhood $U_0 := \pi^{-1}(\mathbf{P}^1 - \{\infty\})$. Over $U_1 := \pi^{-1}(\mathbf{P}^1 - \{0\})$, we have coordinates $(\tilde{\lambda}, \tilde{z})$ related to (λ, z) on $U_0 \cap U_1$ by:

$$\tilde{\lambda} = \frac{1}{\lambda}, \quad \tilde{z} = \frac{z}{\lambda^m}. \tag{1.18}$$

Re-expressing (1.14) as a polynomial equation in $(\tilde{\lambda}, \tilde{z})$ extends \mathcal{S}_0 to U_1 , thereby defining its compactification $\mathcal{S} \subset \mathcal{F}$.

Let us assume that \mathcal{S} has no multiple components. Let (λ_0, z_0) belong to \mathcal{S} , and suppose that the multiplicity of the eigenvalue z_0 of $\hat{\mathcal{L}}(\lambda_0)$ is $k > 1$. It follows from the constructions of [AHH1] that there is a partial desingularisation $\tilde{\mathcal{S}}$ of \mathcal{S} such

that, generically, the number of points (with multiplicity) in $\tilde{\mathcal{S}}$ over (λ_0, z_0) equals the number of Jordan blocks of $\hat{\mathcal{L}}(\lambda_0)$ with eigenvalue z_0 , and $\tilde{\mathcal{S}}$ is smooth over (λ_0, z_0) . If, for example, $\hat{\mathcal{L}}(\lambda_0)$ has only one Jordan block of size k with eigenvalue z_0 , then $\tilde{\mathcal{S}} = \mathcal{S}$ and, generically, \mathcal{S} has a smooth k -fold branch point over \mathbf{P}^1 . In the opposite extreme, if $\hat{\mathcal{L}}(\lambda_0)$ has k independent eigenvectors with eigenvalue z_0 then generically there are k points (with multiplicity) over (λ_0, z_0) and \mathcal{S} has a k -fold node.

These remarks are of particular importance when $\lambda_0 = \alpha_i$, since the Jordan form of $\hat{\mathcal{L}}(\alpha_i)$ is an invariant of the coadjoint orbit. Thus, if $\hat{\mathcal{L}}(\alpha_i)$ is diagonalisable with multiple eigenvalues, the generic spectral curve for the orbit will be singular.

Genericity Conditions. In what follows, we only consider the singularities that follow from the specific structure (1.2), (1.8) assumed for $\hat{\mathcal{L}}(\lambda_0)$. We shall make the simplifying assumption that the N_i (and hence $\hat{\mathcal{L}}(\alpha_i)$) are diagonalizable, with the only multiple eigenvalue being $z = 0$, with multiplicity $r - k_i$. This property is, of course, “generic” for orbits with $\text{rank}(N_i) = k_i$, but is only assumed in order to simplify the exposition. If other Jordan forms are allowed for the N_i ’s, the only effect is to change the specific form (1.22) for the spectral polynomial $\mathcal{P}(\lambda, z)$, (1.27) for the genus formula determining the dimension of $\mathcal{Q}_{\mathcal{N}_0}$ and the explicit expressions (1.82a), (1.82b) for the abelian differentials. All these can easily be modified to hold for other cases. The main results, contained in Theorems 1.3–1.6, Corollary 1.7 and the subsequent sections, remain valid *mutatis mutandis*.

We also assume that one of the following two conditions holds:

Case (i): $Y = 0$ and L_0 has a simple spectrum ($m = n - 1$).

Case (ii): $Y \neq 0$ and has a simple spectrum ($m = n$).

Again, these conditions are generic and invariant on coadjoint orbits, but in Sect. 2 it will be indicated how they may be relaxed.

Finally, we make a further spectral genericity assumption regarding the singularities of the curve \mathcal{S} ; namely, that the only singularities occur at the points $(\alpha_i, 0)$, where there is an $r - k_i$ -fold node with $r - k_i$ distinct branches intersecting transversally. This implies that the eigenspaces of $\hat{\mathcal{L}}(\lambda)$ all be 1-dimensional except at $\lambda = \alpha_i$, where, by the structure of $\mathcal{N}_0(\lambda)$, the eigenvalue $z = 0$ has an eigenspace of dimension $r - k_i$. The desingularization $\tilde{\mathcal{S}}$ is then smooth and is isomorphic to \mathcal{S} away from $(\alpha_i, 0)$. This condition is generic in the space of \mathcal{N}_0 ’s of the form (1.3) and, if satisfied at any point of $\mathcal{Q}_{\mathcal{N}_0}$, it is also valid in a neighborhood of the isospectral manifold through that point. (In particular, it is invariant under the AKS flows.)

The coefficients of the polynomials $\mathcal{P}_j(\lambda)$ generate the AKS ring on each coadjoint orbit $\mathcal{Q}_{\mathcal{N}_0} \subset \tilde{\mathfrak{gl}}(r)^{+*}$ and should be viewed as functions on the Poisson submanifold consisting of rational elements of the form (1.3) (with $\text{rank}(N_i) = k_i$). Note that

$$\mathcal{P}_1(\lambda) = \text{tr } \hat{\mathcal{L}}(\lambda), \tag{1.19}$$

and hence its coefficients are Casimir invariants (i.e. constants on all coadjoint orbits). The nonzero eigenvalues $\{z_{i\kappa}\}_{i=1, \dots, n, \kappa=1, \dots, k_i}$ over the points $\{\lambda = \alpha_i\}_{i=1, \dots, n}$ are also Casimir invariants, since they are determined as the nonzero roots of the

characteristic equation:

$$\det \left[N_i \prod_{j=1, j \neq i}^n (\alpha_i - \alpha_j) - zI \right] = 0, \tag{1.20}$$

which is invariant under the coadjoint action (1.6). The $N := \sum_{i=1}^n k_i$ trivial invariants $\{z_{i\kappa}\}$ determine, in particular, the coefficients of $\mathcal{P}_1(\lambda)$, since

$$\sum_{\kappa=1}^{k_i} z_{i\kappa} = \text{tr } \hat{\mathcal{L}}(\alpha_i) = \mathcal{P}_1(\alpha_i), \quad i = 1, \dots, n. \tag{1.21}$$

(For case (i), this is sufficient to determine the degree $n - 1$ polynomial $\mathcal{P}_1(\lambda)$; for case (ii), the degree n coefficient is just $\text{tr } Y$.) This may all be summarized by noting that the spectral curves $\tilde{\mathcal{S}}$ on the orbit $\mathcal{Q}_{\mathcal{M}_0}$ are constrained to pass through the $N + n$ points $\{(\alpha_i, z_{i\kappa}), (\alpha_i, 0)\}$, with $r - k_i$ branches intersecting at the singular points $\{(\alpha_i, 0)\}$, the values $\{z_{i\kappa}\}$ being fixed. It should also be noted that for $Y \neq 0$ the leading ($\text{deg } \kappa_i$) terms in the polynomials $\mathcal{P}_j(\lambda)$ are constants, determined entirely by the symmetric invariants of Y . For $Y = 0$, the leading terms are not constants, but they are determined as symmetric invariants of L_0 , and hence are constant on its level sets.

A way to express $\mathcal{P}(\lambda, z)$ in terms of independent, non-Casimir invariants is to choose a reference point $\mathcal{N}_R \in \mathcal{Q}_{\mathcal{M}_0}$ on the orbit and parametrize the difference between $\mathcal{P}(\lambda, z)$ and its value $\mathcal{P}_R(\lambda, z)$ at \mathcal{N}_R .

Proposition 1.2. *In a neighborhood of the point $\mathcal{N}_R \in \mathcal{Q}_{\mathcal{M}_0}$, the characteristic polynomial has the form:*

$$\mathcal{P}(\lambda, z) \equiv \mathcal{P}_R(\lambda, z) + a(\lambda) \sum_{j=2}^r a_j(\lambda) p_j(\lambda) z^{r-j}, \tag{1.22}$$

where

$$a_j(\lambda) = \prod_{i=1}^n (\lambda - \alpha_i)^{\max(0, j - k_i - 1)}, \tag{1.23}$$

$$p_j(\lambda) =: \sum_{a=0}^{\delta_j} P_{ja} \lambda^a, \tag{1.24}$$

and $\{p_j(\lambda)\}_{j=1, \dots, r}$ are polynomials of degree:

$$\delta_j \equiv \text{deg } p_j(\lambda) = \begin{cases} d_j - j & \text{if } Y = 0 \\ d_j & \text{if } Y \neq 0 \end{cases} \tag{1.25a}$$

$$d_j \equiv \sum_{i=1}^n \min(j - 1, k_i). \tag{1.25b}$$

For $Y = 0$, the leading coefficients $P_{j\delta_j}$ are constant translates of the elementary symmetric invariants of L_0 , while for $Y \neq 0$, the leading coefficients $P_{j\delta_j}$ are all constants; namely, the elementary symmetric invariants of Y (translated by the

corresponding leading terms in $\mathcal{P}_R(\lambda, z)$). The number of spectral parameters $\{P_{ja}\}$, ($a = 0, \dots, \delta_j + n - m - 1, j = 2, \dots, r$) defining the polynomials $p_j(\lambda)$ on generic orbits is thus:

$$\begin{aligned} d &\equiv \sum_{j=2}^r (d_j - (n - m)(j - 1)) \\ &= \tilde{g} + r - 1, \end{aligned} \tag{1.26}$$

where

$$\tilde{g} = \frac{1}{2} (r - 1)(mr - 2) - \frac{1}{2} \sum_{i=1}^n (r - k_i)(r - k_i - 1). \tag{1.27}$$

In a neighborhood of any generic point on $\mathcal{Q}_{\mathcal{M}_0}$, these spectral invariants are all independent.

Proof. The structure of $\mathcal{P}(\lambda)$ follows Lemma 1.1, plus the fact that $\mathcal{P}(\lambda, z) - \mathcal{P}_R(\lambda, z)$ vanishes at each $\lambda = \alpha_i$, while z vanishes at least linearly in $\lambda - \alpha_i$ along each branch through $(\alpha_i, 0)$. From formula (1.6) and the above genericity conditions regarding the residues N_i , the dimension of the coadjoint orbit $\mathcal{Q}_{\mathcal{M}_0}$ is easily computed to be $2d$. From the proof of complete integrability of the AKS flows on such orbits given in [AHH1], it follows that the isospectral foliation is Lagrangian, and hence the d spectral parameters $\{P_{ja}\}$ are independent. The expression of $P_{j\delta_j}$ in terms of the elementary symmetric invariants of L_0 or Y follows directly from the fact that the leading term in $\hat{\mathcal{L}}(\lambda)$ in Eq. (1.12) is either $L_0\lambda^{n-1}$ or $Y\lambda^n$. \square

It follows from the adjunction formula applied to the curve $\tilde{\mathcal{S}}$ obtained by blowing up \mathcal{S} once at each point $(\alpha_i, 0)$ (cf. [AHH1, GH]) that \tilde{g} in Eq. (1.27) is also equal to the (arithmetic) genus of $\tilde{\mathcal{S}}$. If we reduce such an orbit under the $Gl(r, \mathbb{C})$ action (1.9) for case (i), or the action of the stabilizer $G_Y \subset Gl(r, \mathbb{C})$ of Y for case (ii), the dimension of the reduced space is precisely $2\tilde{g}$, and the projected spectral invariants again define completely integrable Hamiltonian systems [AHH1]. These facts suggest exploiting the orbital symplectic structure further so as to explicitly integrate the isospectral flows via Hamiltonian methods. This will be the content of the following subsections.

1b. Divisor Coordinates on Reduced Orbits

Define

$$\mathcal{H}(\lambda, z) := \hat{\mathcal{L}}(\lambda) - zI, \tag{1.28}$$

and let $\tilde{\mathcal{H}}(\lambda, z)$ denote its classical adjoint (matrix of cofactors). Let $V_0 \in \mathbb{C}^r$ be an eigenvector of L_0 in case (i), or of Y in case (ii). From the results of [AHH1], it follows that the set of polynomial equations:

$$\tilde{\mathcal{H}}(\lambda, z) V_0 = 0 \tag{1.29}$$

have, away from $(\alpha_i, 0)$, precisely \tilde{g} generically distinct finite solutions $\{(\lambda_\mu, z_\mu)\}_{\mu=1, \dots, \tilde{g}}$ that may be viewed as functions on the coadjoint orbit $\mathcal{Q}_{\mathcal{M}_0}$. (Changing to the coordinates $(\tilde{\lambda}, \tilde{z})$, there are also $r - 1$ further solutions with $\tilde{\lambda} = 0$, i.e., $\lambda = \infty$.)

If V_0 is not chosen as an eigenvector of L_0 or Y , the remaining $r - 1$ solutions will generically also be at finite values of (λ, z) .

The significance of these functions in terms of the algebraic geometry of the spectral curves $\tilde{\mathcal{S}}$ may be summarized as follows (cf. [AHH1] and [AHH6] for the detailed construction). To each matricial polynomial $\hat{\mathcal{L}}(\lambda)$ is associated a degree $\tilde{g} + r - 1$ line bundle $\tilde{E} \rightarrow \tilde{\mathcal{S}}$ over the partly desingularized spectral curve $\tilde{\mathcal{S}}$. Away from the degenerate eigenvalues this coincides with the dual of the bundle of eigenvectors of $\hat{\mathcal{L}}^T(\lambda)$ over \mathcal{S} . At a smooth point (λ, z) of \mathcal{S} , the fibre of \tilde{E} is the cokernel of the map $\mathcal{H}(\lambda, z)$:

$$0 \rightarrow \mathbb{C}^r \xrightarrow{\mathcal{H}(\lambda, z)} \mathbb{C}^r \rightarrow \tilde{E} \rightarrow 0. \tag{1.30}$$

More generally, this exact sequence defines the direct image E over \mathcal{S} of \tilde{E} over $\tilde{\mathcal{S}}$. Vectors V_0 in \mathbb{C}^r then give sections of E by projection. These sections vanish precisely at the points where V_0 is in the image of $\mathcal{H}(\lambda, z)$. Since $\mathcal{H}(\lambda, z)\tilde{\mathcal{H}}(\lambda, z) = \mathcal{P}(\lambda, z)I$, this is equivalent to (1.29), at least over the open set of points in \mathcal{S} corresponding to nondegenerate eigenvalues, for which the corank of $\mathcal{H}(\lambda, z)$ is one. From [AHH1], the degree of \tilde{E} is $\tilde{g} + r - 1$, so sections of \tilde{E} have $\tilde{g} + r - 1$ zeroes. The choice of V_0 as an eigenvector of the leading term in $\hat{\mathcal{L}}(\lambda)$ implies that $r - 1$ of these are over $\lambda = \infty$, and the coordinates of the remaining \tilde{g} points are the finite solutions $\{(\lambda_\mu, z_\mu)\}_{\mu=1, \dots, \tilde{g}}$.

In evaluating Poisson brackets, it is preferable to introduce another normalization, corresponding to the eigenvalues of $\frac{\mathcal{N}(\lambda)}{\lambda}$ rather than $\hat{\mathcal{L}}(\lambda)$, by defining:

$$\zeta := \frac{z}{a(\lambda)} \tag{1.31}$$

and

$$\mathcal{M}(\lambda, \zeta) := \frac{\mathcal{N}(\lambda)}{\lambda} - \zeta I, \tag{1.32}$$

with classical adjoint $\tilde{\mathcal{M}}(\lambda, \zeta)$. Then

$$\tilde{\mathcal{H}}(\lambda, z) = [a(\lambda)]^{r-1} \tilde{\mathcal{M}}(\lambda, \zeta) \tag{1.33}$$

and Eq. (1.29) is equivalent to:

$$\tilde{\mathcal{M}}(\lambda, \zeta)V_0 = 0. \tag{1.34}$$

The \tilde{g} solutions $\{(\lambda_\mu, z_\mu)\}_{\mu=1, \dots, \tilde{g}}$ are thus related to the solutions $\{(\lambda_\mu, \zeta_\mu)\}_{\mu=1, \dots, \tilde{g}}$ of (1.34), by:

$$\zeta_\mu = \frac{z_\mu}{a(\lambda_\mu)}. \tag{1.35}$$

Viewing $\{(\lambda_\mu, \zeta_\mu)\}_{\mu=1, \dots, \tilde{g}}$ as functions on \mathcal{Q}_{orb} , we may evaluate their Poisson brackets with respect to the orbital (Kostant-Kirillov) symplectic structure ω_{orb} .

Theorem 1.3. *The Poisson brackets of the functions $(\lambda_\mu, \zeta_\mu)_{\mu=1, \dots, \tilde{g}}$ are:*

$$\{\lambda_\mu, \lambda_\nu\} = 0, \quad \{\zeta_\mu, \zeta_\nu\} = 0, \quad \{\lambda_\mu, \zeta_\nu\} = \delta_{\mu\nu}. \tag{1.36}$$

Proof. Choose a basis in which the leading term in $\mathcal{L}(\lambda)$ (i.e. L_0 for case (i) and Y for case (ii)) is diagonal, and let $V_0 = (1, 0 \dots 0)^T$. Let $\tilde{\mathcal{M}}_{ij}(\lambda, \zeta)$ denote the ij^{th} component of $\tilde{\mathcal{M}}(\lambda, \zeta)$. The points (λ_ν, ζ_ν) are then determined by the conditions

$$\tilde{\mathcal{M}}_{k1}(\lambda_\nu, \zeta_\nu) = 0 \tag{1.37}$$

for all k . Generically, these points are cut out by only two of these equations, say

$$\tilde{\mathcal{M}}_{11} = \tilde{\mathcal{M}}_{21} = 0. \tag{1.37a}$$

That is, generically the matrix

$$F_\nu := \begin{pmatrix} \frac{\partial \tilde{\mathcal{M}}_{11}}{\partial \lambda} & \frac{\partial \tilde{\mathcal{M}}_{11}}{\partial \zeta} \\ \frac{\partial \tilde{\mathcal{M}}_{21}}{\partial \lambda} & \frac{\partial \tilde{\mathcal{M}}_{21}}{\partial \zeta} \end{pmatrix} (\lambda_\nu, \zeta_\nu) \tag{1.38}$$

is invertible. By implicit differentiation, the Poisson brackets of the functions (λ_ν, ζ_ν) are then:

$$\begin{pmatrix} \{\lambda_\nu, \lambda_\mu\} & \{\lambda_\nu, \zeta_\mu\} \\ \{\zeta_\nu, \lambda_\mu\} & \{\zeta_\nu, \zeta_\mu\} \end{pmatrix} = (F_\nu)^{-1} \\ \times \begin{pmatrix} \{\tilde{\mathcal{M}}_{11}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{11}(\lambda_\mu, \zeta_\mu)\} & \{\tilde{\mathcal{M}}_{11}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{21}(\lambda_\mu, \zeta_\mu)\} \\ \{\tilde{\mathcal{M}}_{21}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{11}(\lambda_\mu, \zeta_\mu)\} & \{\tilde{\mathcal{M}}_{21}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{21}(\lambda_\mu, \zeta_\mu)\} \end{pmatrix} ((F_\mu)^T)^{-1}. \tag{1.39}$$

To determine the brackets in the matrix on the right-hand side of Eq. (1.39) we first recall that if f and g are functions on the orbit $\mathcal{Q}_{\mathcal{N}_0}$, their Poisson bracket at a point $\mu \in \mathcal{Q}_{\mathcal{N}_0} \subset \tilde{\mathfrak{gl}}(r)_-$ is given by

$$\{F, G\} = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle, \tag{1.40}$$

where $\frac{\delta f}{\delta \mu}$ is the differential of f at μ , considered as an element of $\tilde{\mathfrak{gl}}(r)^+$, and the pairing \langle, \rangle is defined by Eq. (1.4). The ij^{th} coefficient of $\mathcal{M}(\lambda, \zeta)$ evaluated at the point (λ_0, ζ_0) , viewed as a function of $\mu \in \tilde{\mathfrak{gl}}(r)_-$, may be written

$$\mathcal{M}_{ij}(\lambda_0, \zeta_0) = - \left\langle \mu, \frac{e_{ji}}{\lambda - \lambda_0} \right\rangle - \zeta_0 \delta_{ij}, \tag{1.41}$$

where e_{ji} is the matrix with a 1 in the ji^{th} place and zeroes elsewhere. It follows that

$$\frac{\delta \mathcal{M}_{ij}(\lambda_0, \zeta_0)}{\delta \mu} = - \frac{e_{ji}}{\lambda - \lambda_0}, \tag{1.42}$$

and hence, dropping the 0 subscripts,

$$\{\mathcal{M}_{ij}(\lambda, \zeta), \mathcal{M}_{kl}(\sigma, \eta)\} \\ = \frac{1}{\lambda - \sigma} [(\mathcal{M}_{il}(\lambda, \zeta) - \mathcal{M}_{il}(\sigma, \eta))\delta_{jk} - (\mathcal{M}_{kj}(\lambda, \zeta) - \mathcal{M}_{kj}(\sigma, \eta))\delta_{il}]. \tag{1.43}$$

Since $\tilde{\mathcal{M}}(\lambda, \zeta)$ is the classical adjoint of $\mathcal{M}(\lambda, \zeta)$ we have

$$\tilde{\mathcal{M}}(\lambda, \zeta) \mathcal{M}(\lambda, \zeta) = \det(\mathcal{M}(\lambda, \zeta)) I. \tag{1.44}$$

Differentiating with respect to a parameter t yields

$$\frac{d\tilde{\mathcal{M}}(\lambda, \zeta)}{dt} = \frac{\tilde{\mathcal{M}}(\lambda, \zeta) \operatorname{tr} \left(\left(\frac{d}{dt} \tilde{\mathcal{M}}(\lambda, \zeta) \right) \tilde{\mathcal{M}}(\lambda, \zeta) \right) - \tilde{\mathcal{M}}(\lambda, \zeta) \left(\frac{d}{dt} \mathcal{M}(\lambda, \zeta) \right) \tilde{\mathcal{M}}(\lambda, \zeta)}{\det(\mathcal{M}(\lambda, \zeta))} \tag{1.45}$$

away from points (λ, ζ) where $\det(\mathcal{M}(\lambda, \zeta)) = 0$, i.e., points on the spectral curve. Thus, away from the spectral curve,

$$\frac{\partial \tilde{\mathcal{M}}_{ij}(\lambda, \zeta)}{\partial \mathcal{M}_{pq}(\lambda, \zeta)} = \frac{\tilde{\mathcal{M}}_{qp}(\lambda, \zeta) \tilde{\mathcal{M}}_{ij}(\lambda, \zeta) - \tilde{\mathcal{M}}_{ip}(\lambda, \zeta) \tilde{\mathcal{M}}_{qj}(\lambda, \zeta)}{\det \mathcal{M}(\lambda, \zeta)}. \tag{1.46}$$

The derivation property of the bracket

$$\begin{aligned} & \{ \tilde{\mathcal{M}}_{ij}(\lambda, \zeta), \tilde{\mathcal{M}}_{kl}(\sigma, \eta) \} \\ &= \sum_{pqrs} \frac{\partial \tilde{\mathcal{M}}_{ij}(\lambda, \zeta)}{\partial \mathcal{M}_{pq}(\lambda, \zeta)} \frac{\partial \tilde{\mathcal{M}}_{kl}(\sigma, \eta)}{\partial \mathcal{M}_{rs}(\sigma, \eta)} \{ \mathcal{M}_{pq}(\lambda, \zeta), \mathcal{M}_{rs}(\sigma, \eta) \} \end{aligned} \tag{1.47}$$

then gives

$$\begin{aligned} \{ \tilde{\mathcal{M}}_{i1}(\lambda, \zeta), \tilde{\mathcal{M}}_{k1}(\sigma, \eta) \} &= \left(\frac{1}{\lambda - \sigma} \right) \left[\frac{1}{\det \mathcal{M}(\sigma, \eta)} [(\tilde{\mathcal{M}}(\sigma, \eta) \tilde{\mathcal{M}}(\lambda, \zeta))_{k1} \tilde{\mathcal{M}}_{i1}(\sigma, \eta) \right. \\ &\quad \left. - (\tilde{\mathcal{M}}(\sigma, \eta) \tilde{\mathcal{M}}(\lambda, \zeta))_{i1} \tilde{\mathcal{M}}_{k1}(\sigma, \eta) \right] \\ &\quad + \frac{1}{\det \mathcal{M}(\lambda, \zeta)} [(\tilde{\mathcal{M}}(\lambda, \zeta) \tilde{\mathcal{M}}(\sigma, \eta))_{i1} \tilde{\mathcal{M}}_{k1}(\lambda, \zeta) \\ &\quad \left. - (\tilde{\mathcal{M}}(\lambda, \zeta) \tilde{\mathcal{M}}(\sigma, \eta))_{k1} \tilde{\mathcal{M}}_{i1}(\lambda, \zeta) \right]. \end{aligned} \tag{1.48}$$

By Eq. (1.37), $\tilde{\mathcal{M}}_{k1}(\lambda_\nu, \zeta_\nu)$ vanishes for all k, ν . Taking the limits $(\lambda, \zeta) \rightarrow (\lambda_\mu, \zeta_\mu)$, $(\sigma, \eta) \rightarrow (\lambda_\nu, \zeta_\nu)$ along any path transversal to the curve \mathcal{S} , the right-hand side of Eq. (1.48) has limit zero for $\nu \neq \mu$ (the simple zero in $\det \mathcal{M}$ is cancelled by a double zero in the numerator), implying

$$\begin{aligned} \{ \tilde{\mathcal{M}}_{11}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{11}(\lambda_\mu, \zeta_\mu) \} &= \{ \tilde{\mathcal{M}}_{11}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{21}(\lambda_\mu, \zeta_\mu) \} \\ &= \{ \tilde{\mathcal{M}}_{21}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{21}(\lambda_\mu, \zeta_\mu) \} = 0 \end{aligned} \tag{1.49}$$

when $\mu \neq \nu$. Hence $\{\lambda_\nu, \lambda_\mu\}$, $\{\zeta_\nu, \zeta_\mu\}$ and $\{\lambda_\nu, \zeta_\mu\}$ all vanish when $\nu \neq \mu$.

To compute the bracket for $\nu = \mu$ we first note that the brackets on the diagonal of the matrix on the right-hand side of Eq. (1.39) are zero in this case. Thus, to show that $\{\lambda_\nu, \zeta_\nu\} = 1$ it suffices to show that

$$\{ \tilde{\mathcal{M}}_{11}(\lambda_\nu, \zeta_\nu), \tilde{\mathcal{M}}_{21}(\lambda_\nu, \zeta_\nu) \} = \det(F_\nu). \tag{1.50}$$

To compute the left-hand side of (1.50) we first take the limit $(\lambda, \zeta) \rightarrow (\sigma, \eta)$ in (1.43) using the derivation property (1.47) of the bracket to show

$$\begin{aligned} & \{ \tilde{\mathcal{M}}_{11}(\lambda, \zeta), \tilde{\mathcal{M}}_{21}(\lambda, \zeta) \} \\ &= \sum_{prs} \left(\frac{\partial \tilde{\mathcal{M}}_{11}}{\partial \mathcal{M}_{pr}} \frac{\partial \tilde{\mathcal{M}}_{21}}{\partial \mathcal{M}_{rs}} - \frac{\partial \tilde{\mathcal{M}}_{11}}{\partial \mathcal{M}_{rs}} \frac{\partial \tilde{\mathcal{M}}_{21}}{\partial \mathcal{M}_{pr}} \right) \frac{d \mathcal{M}_{ps}}{d\lambda}. \end{aligned} \tag{1.51}$$

On the other hand

$$\det \begin{pmatrix} \frac{\partial \tilde{M}_{11}}{\partial \lambda} & \frac{\partial \tilde{M}_{11}}{\partial \zeta} \\ \frac{\partial \tilde{M}_{21}}{\partial \lambda} & \frac{\partial \tilde{M}_{21}}{\partial \zeta} \end{pmatrix} (\lambda, \zeta) = \sum_{prq} \left(\frac{\partial \tilde{\mathcal{M}}_{11}}{\partial \mathcal{M}_{pr}} \frac{\partial \tilde{\mathcal{M}}_{21}}{\partial \mathcal{M}_{qq}} - \frac{\partial \tilde{\mathcal{M}}_{11}}{\partial \mathcal{M}_{qq}} \frac{\partial \tilde{\mathcal{M}}_{21}}{\partial \mathcal{M}_{pr}} \right) \frac{d \mathcal{M}_{pr}}{d \lambda}. \tag{1.52}$$

Equation (1.50) now follows by substituting Eq. (1.46) into Eqs. (1.51) and (1.52) and using the fact that $\tilde{\mathcal{M}}(\lambda_\nu, \zeta_\nu)$ has rank 1. \square

The implication of Theorem 1.3 is that the functions $\{(\lambda_\mu, \zeta_\mu)\}_{\mu=1, \dots, \tilde{g}}$ *nearly* provide a Darboux coordinate system on the coadjoint orbit $\mathcal{Q}_{\mathcal{J}_0}$. However, the dimensions are not quite right. For case (i), we have

$$\dim \mathcal{Q}_{\mathcal{J}_0} = 2\tilde{g} + (r + 2)(r - 1) \tag{1.53a}$$

for generic orbits, while for case (ii),

$$\dim \mathcal{Q}_{\mathcal{J}_0} = 2(\tilde{g} + r - 1). \tag{1.53b}$$

(Note that in these formulae, it is the value of \tilde{g} that is different, according to Eq. (1.27), *not* the dimension of $\mathcal{Q}_{\mathcal{J}_0}$ which, of course, is the same.)

On the other hand, for case (i), the Marsden-Weinstein reduced coadjoint orbit \mathcal{Q}_{red} , obtained by fixing the value of the $Gl(r)$ moment map L_0 and quotienting by its stabilizer $G_{L_0} \subset Gl(r)$, is of dimension $2\tilde{g}$. Similarly, for case (ii) we may reduce by the stabilizer $G_Y \subset Gl(r)$ of Y , since the shifted AKS Hamiltonians of the form (1.7) are invariant under this subgroup and the restriction of L_0 to the corresponding subalgebra \mathfrak{g}_Y is concerned under the flows. The reduced orbit under this action, also denoted \mathcal{Q}_{red} , is again of dimension $2\tilde{g}$. (Note again that the value of \tilde{g} for the latter case is, by Eq. (1.27), $\frac{1}{2}r(r - 1)$ greater than for the former.) Thus, if the coordinates (λ_μ, ζ_μ) could be shown to be projectable to the reduced spaces, and if the reduced Poisson brackets remain the same as in Eq. (1.36), we would have Darboux coordinates on \mathcal{Q}_{red} .

For case (ii) this may be seen immediately. Since V_0 was assumed to be an eigenvector of Y , with no degeneracy allowed, the defining equation (1.34) is invariant under the stabilizer $G_Y \subset Gl(r)$ (an $r - 1$ dimensional abelian group under our hypotheses). Thus $(\lambda_\mu, \zeta_\mu)_{\mu=1, \dots, \tilde{g}}$ are all invariant under the Hamiltonian G_Y -action, and the Poisson brackets of their projection to \mathcal{Q}_{red} are the same as on $\mathcal{Q}_{\mathcal{J}_0}$.

For case (i), we cannot quite apply Hamiltonian symmetry reduction under $Gl(r)$, since the functions (λ_μ, ζ_μ) are only invariant under the stabilizer subgroup G_{L_0} . However, we may still compute the Poisson brackets on the reduced space by the procedure used for constrained Hamiltonian systems. Let us first choose the reduction condition given by the level set:

$$L_0 = \text{diag}\{l_i\}, \tag{1.54}$$

where the eigenvalues $\{l_i\}$ are, by our genericity assumption, distinct. The diagonal terms in Eq. (1.54) are the first class constraints, which generate the Hamiltonian G_{L_0} -action, and the terms with $i > 1$ may be chosen as the independent generators.

Applying the standard procedure of modifying the Hamiltonian by adding a linear combination of the remaining, second class constraints, we see that the following modified functions generate flows that are tangential to the constrained submanifold:

$$\hat{\lambda}_\mu = \lambda_\mu - \sum_{i,j=1,i \neq j}^r \frac{\{\lambda_\mu, (L_0)_{ij}\}}{l_i - l_j} (L_0)_{ji}, \tag{1.55a}$$

$$\hat{\zeta}_\mu = \zeta_\mu - \sum_{i,j=1,i \neq j}^r \frac{\{\zeta_\mu, (L_0)_{ij}\}}{l_i - l_j} (L_0)_{ji}. \tag{1.55b}$$

Evaluating their Poisson brackets, we find, again:

$$\{\hat{\lambda}_\mu, \hat{\lambda}_\nu\} = 0, \quad \{\hat{\zeta}_\mu, \hat{\zeta}_\nu\} = 0, \quad \{\hat{\lambda}_\mu, \hat{\zeta}_\nu\} = \delta_{\mu\nu} \tag{1.56}$$

since, by implicit differentiation of the defining equations (1.37), the second factor in (1.55a, b) involves terms of the form $\{\mathcal{N}_{k1}(\lambda, \zeta), (L_0)_{ij}\}$ which, applying the chain rule and Eq. (1.46), vanish unless $i = 1$. The cross terms in the Poisson brackets (1.56) therefore all contain terms proportional to $\{\mathcal{N}_{k1}(\lambda, \zeta), (L_0)_{i1}\}$, $i \neq 1$, which vanish at $(\lambda, \zeta) = (\lambda_\mu, \zeta_\mu)$. Since the functions $(\hat{\lambda}_\mu, \hat{\zeta}_\mu)$ coincide with (λ_μ, ζ_μ) on the constrained manifold and generate tangential flow, it follows that the projections of (λ_μ, ζ_μ) to \mathcal{Q}_{red} (the quotient of the constrained manifold by G_{L_0}) satisfy the same Poisson bracket relations as (1.56). Finally, for other values of L_0 than (1.54), we just repeat the same argument with respect to a diagonalizing basis of eigenvectors.

Combining these results we obtain, for both cases (i) and (ii):

Theorem 1.4. *The projections of $(\lambda_\mu, \zeta_\mu)_{\mu=1, \dots, \tilde{g}}$ to the reduced orbit \mathcal{Q}_{red} , in both case (i) ($Y = 0$) and case (ii) ($Y \neq 0$, with distinct eigenvalues), are Darboux coordinates; that is, the reduced symplectic form is:*

$$\omega_{\text{red}} = \sum_{\mu=1}^{\tilde{g}} d\lambda_\mu \wedge d\zeta_\mu. \tag{1.57}$$

*Remark.*¹ The proof of Theorem 1.3 did not depend on the fact that there are \tilde{g} finite points in the spectral divisor. If the vector V_0 is not chosen as an eigenvector of Y , the number of such finite points, and corresponding coordinate pairs (λ_μ, ζ_μ) , may be between \tilde{g} and $\tilde{g} + r - 1$. The number of points over $\lambda = \infty$ equals the number of eigenvalues \tilde{z} of the asymptotic form of $\hat{\mathcal{S}}(\lambda)$ (i.e., Y for case (ii) and L_0 for case (i)), for which V_0 is in the image of $Y - \tilde{z}I$ for case (ii) (resp. $L_0 - \tilde{z}I$ for case (i)). This is zero for generically chosen (non-diagonal) Y (or L_0) or, equivalently, if Y is taken as a diagonal matrix, and V_0 chosen as a vector with no vanishing components (e.g. $V_0 = (1, 1, \dots, 1)^T$). In this case, the number of finite spectral divisor coordinate pairs (λ_μ, ζ_μ) will actually be $\tilde{g} + r - 1$, sufficient to provide a Darboux coordinate system for the full orbit in case (ii) and an $r(r - 1)$ codimensional symplectic submanifold in case (i) (cf. Sect. 1c). However, for the examples involving integrable systems that will be of interest to us (cf. Sect. 3), it is not this type of spectral Darboux system that is needed for directly determining solutions, but those derived in the following subsection. The problem lies with the invertibility of the Abel map (cf. Sect. 1d), which requires a degree \tilde{g} divisor. The remaining $r - 1$ points of the spectral divisor are related to the singular differentials having pole singularities over $\lambda = \infty$.

¹ Thanks are due to B. Dubrovin for raising the point discussed in this remark

There remains then the question of the nonreduced orbits $\mathcal{Q}_{\mathcal{A}_0^0}$. Can the functions $(\lambda_\mu, \zeta_\mu)_{\mu=1, \dots, \tilde{g}}$ somehow be completed to provide a Darboux coordinate system on $\mathcal{Q}_{\mathcal{A}_0^0}$? The answer is: yes, for case (ii), and partially for case (i). The construction is given in the following subsection.

1c. Darboux Coordinates on Unreduced Orbits

In case (i) we shall obtain Darboux coordinates, not on the complete coadjoint orbit $\mathcal{Q}_{\mathcal{A}_0}$, but on a constrained submanifold $\mathcal{Q}_{\mathcal{A}_0^0} \subset \mathcal{Q}_{\mathcal{A}_0}$ consisting of elements for which the off-diagonal elements of L_0 vanish:

$$(L_0)_{ij} = 0 \quad \text{if } i \neq j. \tag{1.58}$$

By our earlier genericity assumptions, the diagonal elements $(L_0)_{ii}$ are hence distinct, and it is easily verified that $\mathcal{Q}_{\mathcal{A}_0^0} \subset \mathcal{Q}_{\mathcal{A}_0}$ is a symplectic submanifold of dimension

$$\dim \mathcal{Q}_{\mathcal{A}_0^0} = 2(\tilde{g} + r - 1). \tag{1.59}$$

(Note that $m = n - 1$ in the genus formula (1.27) and we are dealing with case (1.53a), not (1.53b).) For case (ii), we choose a basis in which Y is diagonal:

$$Y = \text{diag}\{Y_i\}. \tag{1.60}$$

Thus in both cases, the leading term of $\hat{\mathcal{L}}(\lambda)$ is diagonal. As in the proof of Theorem 1.3, we also choose the eigenvector V_0 in (1.29) to be $V_0 = (1, 0, 0, \dots, 0)^T$. In both cases, let

$$P_i := (L_0)_{ii}, \quad i = 1, \dots, r. \tag{1.61}$$

These generates the action of the group D of diagonal matrices, which equals G_{L_0} and G_Y , respectively, for cases (i) and (ii). The generator P_1 is not independent of the others, since the sum:

$$\sum_{i=1}^r P_i = \text{tr } L_0 \tag{1.62}$$

is a Casimir. These generators Poisson commute amongst themselves and also with the D -invariant functions $(\lambda_\mu, \zeta_\mu)_{\mu=01, \dots, \tilde{g}}$, since Eq. (1.37), which determines them, is D -invariant. In case (i), let

$$q_i := \ln(L_1)_{i1} + \frac{1}{2} \sum_{j \neq i, j > 1}^r \ln(P_i - P_j), \tag{1.63}$$

while for case (ii), let

$$q_i := \ln(L_0)_{i1}. \tag{1.64}$$

With these definitions, we have:

Theorem 1.5. *The coordinate functions $(\lambda_\mu, \zeta_\mu, q_i, P_i)_{\mu=1, \dots, \tilde{g}; i=2, \dots, r}$ form a Darboux system on $\mathcal{Q}_{\mathcal{A}_0^0}$ in case (i), and $\mathcal{Q}_{\mathcal{A}_0}$ in case (ii); that is, the only nonvanishing Poisson brackets between them are given by:*

$$\{\lambda_\mu, \zeta_\nu\} = \delta_{\mu\nu}, \quad \{q_i, P_j\} = \delta_{ij}. \tag{1.65}$$

Equivalently,

$$\omega_{\text{orb}} = \sum_{\mu=1}^{\tilde{g}} d\lambda_{\mu} \wedge d\zeta_{\mu} + \sum_{i=2}^r dq_i \wedge dP_i, \tag{1.66}$$

where the equality refers to the full orbit $\mathcal{Q}_{\mathcal{A}_0^0}$ in case (ii), and the restriction of ω_{orb} to $\mathcal{Q}_{\mathcal{A}_0^0}^0$ in case (i).

Proof. The proof proceeds in two steps. First, as in Theorem 1.3, the Poisson brackets are computed on the full coadjoint orbits. In case (i), we then reduce this to the constrained submanifold, which is symplectic. From the Poisson brackets (1.43) used in the proof of Theorem 1.3 follows:

$$\{(L_0)_{ij}, (L_s)_{kl}\} = (L_s)_{kj} \delta_{il} - (L_s)_{il} \delta_{jk}, \tag{1.67a}$$

$$\{\mathcal{M}_{i,j}(\lambda, \zeta), (L_0)_{kl}\} = (Y - \mathcal{M}(\lambda, \zeta))_{il} \delta_{jk} - (Y - \mathcal{M}(\lambda, \zeta))_{kj} \delta_{il}, \tag{1.67b}$$

$$\begin{aligned} \{\mathcal{M}_{i,j}(\lambda, \zeta), (L_1)_{kl}\} &= \left(\lambda - \sum_m \alpha_m \right) [(Y - \mathcal{M}(\lambda, \zeta))_{il} \delta_{jk} - (Y - \mathcal{M}(\lambda, \zeta))_{kj} \delta_{il}] \\ &\quad + [(L_0)_{il} \delta_{kj} - (L_0)_{kj} \delta_{il}]. \end{aligned} \tag{1.67c}$$

This implies, in addition to the relations (1.36), the brackets:

$$\{\lambda_{\mu}, q_i\} = \{\zeta_{\mu}, q_i\} = 0, \tag{1.68a}$$

$$\{\lambda_{\mu}, P_i\} = \{\zeta_{\mu}, P_i\} = \{P_i, P_j\} = 0, \{q_i, P_j\} = \delta_{ij}, \tag{1.68b}$$

$$\{q_i, q_j\} = \begin{cases} (P_i - P_j)^{-1} & \text{for case (i)} \\ 0 & \text{for case (ii)}, \end{cases} \tag{1.68c}$$

where (1.68a) holds only on the constrained manifold $\mathcal{Q}_{\mathcal{A}_0^0}^0$ for case (i). As in the proof of Theorem 1.3, we must use the fact that $\tilde{\mathcal{M}}_{i1}$ is zero at $(\lambda_{\mu}, \zeta_{\mu})$. In case (ii) this completes the proof. For case (i), the constraints must be taken into account. As in the proof of case (i) of Theorem 1.4, we shift the functions $(\lambda_{\mu}, \zeta_{\nu}, q_i, P_j)$ by terms proportional to the second class constraints $(L_0)_{ij} = 0, i \neq j$ to get functions $(\hat{\lambda}_{\mu}, \hat{\zeta}_{\nu}, \hat{q}_i, \hat{P}_j)$ which agree with $(\lambda_{\mu}, \zeta_{\nu}, q_i, P_j)$ on $\mathcal{Q}_{\mathcal{A}_0^0}^0$ and which generate flows in $\mathcal{Q}_{\mathcal{A}_0^0}$ that are tangential to $\mathcal{Q}_{\mathcal{A}_0^0}^0$. Since

$$\{(L_0)_{ij}, (L_0)_{kl}\} = (L_0)_{kj} \delta_{il} - (L_0)_{il} \delta_{jk}, \tag{1.69}$$

it suffices, for a general function f on $\mathcal{Q}_{\mathcal{A}_0^0}$, to take

$$\hat{f} = f - \sum_{i,j=1, i \neq j}^r \frac{\{f, (L_0)_{ij}\}}{P_i - P_j} (L_0)_{ji}. \tag{1.70}$$

As in the proof of Theorem 1.4, the Poisson brackets (1.36) remain unchanged on the constrained manifold. From Eq. (1.69), it follows (as in Theorem 1.4), that the P_j 's already generate tangential flows and hence the Poisson brackets (1.68b) remain unchanged. Equation (1.68a) also is unchanged since, by the same arguments as in the proof of Theorem 1.4, the additional cross terms obtained after constraining are all proportional to terms of the form $\{q_i, (L_0)_{j1}\}$, which vanish on the constrained manifold. Using Eq. (1.67a), we see that the remaining Poisson bracket (1.68c) gets shifted to zero. \square

Remarks. i) The submanifold $\mathcal{Q}_{\mathcal{J}_0}^0 \subset \mathcal{Q}_{\mathcal{J}_0}$ is, in fact, the relevant phase space for many interesting examples of integrable systems, such as the finite gap solutions of the cubically nonlinear Schrödinger equation (cf. [AHP] and Sect. 2).
 ii) If, in formula (1.2), (1.3), we choose $n = 1$, $\alpha_1 = 0$ and $Y \neq 0$, then $\mathcal{Q}_{\mathcal{J}_0}$ is really a coadjoint orbit in $\mathfrak{gl}(r)^*$ or $\mathfrak{sl}(r)^*$ and Theorem 1.5, together with the Ad^* invariants (Casimirs), provides Darboux coordinate systems for these finite dimensional Lie algebras.

Id. Liouville-Arnold Integration and the Abel Map

We now turn to the integration of the Hamiltonian systems (1.1) generated either by elements of the Poisson commutative ring \mathcal{F}_Y of functions of the form (1.7), with Φ in the ring $I(\widetilde{\mathfrak{gl}}(r)^*)$ of Ad^* -invariants on $\widetilde{\mathfrak{gl}}(r)^*$, or its extension $\mathcal{F}_Y(\mathbf{P})$ by the generators $\{P_i\}_{i=2, \dots, r}$. Thus, our Hamiltonians are all expressible as functions of the invariants $\{P_{ia}, P_i\}$. The notational conventions of the preceding sections allow us to treat cases (i) and (ii) simultaneously, although it should be remembered that the spectral curves and ring of invariants $\mathcal{F}_Y(\mathbf{P})$ depend on the choice of Y , and the relevant symplectic manifold is $\mathcal{Q}_{\mathcal{J}_0}^0$ for case (i) and the entire orbit $\mathcal{Q}_{\mathcal{J}_0}$ for case (ii). The reduced spaces, though both denoted \mathcal{Q}_{red} , are also different, their dimensions $2\tilde{g}$ being given by the genus formula (1.27) with $m = n - 1$ for case (i) and $m = n$ for case (ii). For case (i), \mathcal{Q}_{red} signifies the generic $Gl(r)$ -reduction of $\mathcal{Q}_{\mathcal{J}_0}$ or, equivalently, the reduction of $\mathcal{Q}_{\mathcal{J}_0}^0$ by the abelian $r - 1$ -dimensional group action generated by $\{P_i\}_{i=2, \dots, r}$. For case (ii), \mathcal{Q}_{red} is the reduction of the full orbit $\mathcal{Q}_{\mathcal{J}_0}$ by the latter action.

The $\tilde{g} + r - 1$ independent spectral invariants for case (i) may be chosen to be $(P_{ia_i}, P_j)_{i,j=2, \dots, r; a_i=0, \dots, \delta_i-1}$, since the coefficients $P_{j\delta_j}$ occurring in Proposition 1.2 may be expressed as translates of the elementary symmetric invariants of $L_0 = \text{diag}\{P_i\}$,

$$P_{j\delta_j} = (-1)^{r-j} \sum_{1 \leq i_1 < \dots < i_j} P_{i_1} \dots P_{i_j} + m_j, \tag{1.71}$$

where the constants $\{m_j\}$ depend on the reference polynomial $\mathcal{P}_R(\lambda, z)$. For case (ii), the leading coefficients $\{P_{j\delta_j}\}$ are constants (translates of the elementary symmetric invariants of Y) and the next to leading coefficients are translates of linear combinations of the P_i 's:

$$P_{j,\delta_j-1} = (-1)^{r-j} \sum_{i=1}^r P_i \sum_{1 \leq i_1 < \dots < i_{j-1} \neq i} Y_{i_1} \dots Y_{i_{j-1}} + n_j, \tag{1.72}$$

where again, the constants n_j depend on $\mathcal{P}_R(\lambda, z)$ and the constants Y_i . Thus, the $\tilde{g} + r - 1$ independent invariants may be chosen to be $\{P_{ia_i}, P_j\}_{i,j=2, \dots, r; a_i=1, \dots, \delta_i-2}$. The Hamiltonians may be viewed in the two cases as functions of the independent invariants:

$$h = h(P_{ia_i}, P_j) \quad i, j = 2, \dots, r, \quad a_i = 1, \dots, \delta_i - 1 - \varepsilon. \tag{1.73}$$

with $\varepsilon = 0$ for case (i) and $\varepsilon = 1$ for case (ii). We can use the Darboux coordinates of Theorem 1.5 to express the symplectic forms ω_{orb} or $\omega_{\text{orb}}|_{\mathcal{L}^0_{i_0}}$ as (minus) the exterior derivative of a 1-form:

$$\theta := \sum_{\mu=1}^{\tilde{g}} \zeta_{\mu} d\lambda_{\mu} + \sum_{i=2}^r P_i dq_i. \tag{1.74}$$

Restricting to the invariant Lagrangian manifolds \mathcal{L} obtained by fixing the level sets of $\{P_{i_a}, P_j\}$, there exists (within a suitable neighbourhood of such \mathcal{L} 's) a Liouville generating function $S(\lambda_{\mu}, q_i, P_{i_a}, P_i)$ such that:

$$\theta|_{\mathcal{L}} = dS. \tag{1.75}$$

Integrating from an arbitrary initial point thus gives

$$S(\lambda_{\mu}, q_i, P_{i_a}, P_i) = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_{\mu}} \frac{z(\lambda, P_{i_a}, P_j)}{a(\lambda)} d\lambda + \sum_{i=2}^r q_i P_i, \tag{1.76}$$

where the λ integrals are evaluated within a chosen polygonization of the spectral curve $\tilde{\mathcal{S}}$ and the function

$$z = z(\lambda, P_{i_a}, P_j) \tag{1.77}$$

is determined implicitly along \mathcal{L} by the spectral equation:

$$\mathcal{P}(\lambda, z(\lambda, P_{i_a}, P_j)) = 0. \tag{1.78}$$

Applying the standard canonical transformation procedure, the coordinates (Q_{i_a}, Q_j) canonically conjugate to the invariants (P_{i_a}, P_j) are then

$$Q_{i_a} = \frac{\partial S}{\partial P_{i_a}} = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_{\mu}} \frac{1}{a(\lambda)} \frac{\partial z}{\partial P_{i_a}} d\lambda, \tag{1.79a}$$

$$Q_i = \frac{\partial S}{\partial P_i} = \sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_{\mu}} \frac{1}{a(\lambda)} \frac{\partial z}{\partial P_i} d\lambda + q_i. \tag{1.79b}$$

Evaluating the integrands by implicit differentiation of Eq. (1.78) with respect to the invariants $\{P_{i_a}, P_i\}$, and using Eqs. (1.71), (1.72), we have

$$\frac{\partial z}{\partial P_{i_a}} = -a(\lambda) \frac{a_i(\lambda) z^{r-i} \lambda^a}{\mathcal{P}_z(\lambda, z)}, \quad i = 2, \dots, r, \quad a = 1, \dots, \delta_i - 1 - \varepsilon, \tag{1.80a}$$

$$\frac{\partial z}{\partial P_i} = -a(\lambda) \sum_{j=2}^r \frac{R_{ij} a_j(\lambda) (-z)^{r-j} \lambda^{\delta_j - \varepsilon}}{\mathcal{P}_z(\lambda, z)}, \quad i = 2, \dots, r, \tag{1.80b}$$

where

$$R_{i_j} = \begin{cases} (P_1 - P_i) \sum_{2 \leq i_1 < i_2 \dots < i_{j-2} \neq i} P_{i_1} \dots P_{i_{j-2}} & \text{and } \varepsilon = 0 \quad \text{for case (i)} \\ (Y_1 - Y_i) \sum_{2 \leq i_1 < i_2 \dots < i_{j-2} \neq i} Y_{i_1} \dots Y_{i_{j-2}} & \text{and } \varepsilon = 1 \quad \text{for case (ii)} \end{cases} \tag{1.81}$$

The flow is then given in implicit form by the linear equations:

$$\sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda^\mu} \frac{a_i(\lambda) z^{r-i} \lambda^a}{\mathcal{P}_z(\lambda, z)} d\lambda = C_{ia} - \frac{\partial h}{\partial P_{ia}} t, \tag{1.82a}$$

$$\sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda^\mu} \sum_{j=2}^r \frac{R_{ij} a_j(\lambda) (-z)^{r-j} \lambda^{\delta_j - \varepsilon}}{\mathcal{P}_z(\lambda, z)} d\lambda = q_i + c_i - \frac{\partial h}{\partial P_i}, \tag{1.82b}$$

where $\{C_{ia}, c_i\}_{i=2, \dots, r; a=1, \dots, \delta_i-1-\varepsilon}$ are integration constants and a fixed base point λ_0 has been used in the integration.

Remark. On any given level set of the P_i 's the Hamiltonians $h(P_{ia}, P_j)$ project to the reduced space \mathcal{Q}_{red} and Eq. (1.82a) alone gives the corresponding linearization of the reduced flow.

We note that the linearizing map defined by Eqs. (1.82a, b) involves $\tilde{g} + r - 1$ abelian integrals on $\tilde{\mathcal{S}}$.

Theorem 1.6. *The \tilde{g} differentials $\{\omega_{ia}\}_{i=1, \dots, \tilde{g}}$ appearing as integrands in Eq. (1.82a) are independent, and form a basis for the space $H^0(\tilde{\mathcal{S}}, K_{\tilde{\mathcal{S}}})$ of abelian differentials of the first kind (where $K_{\tilde{\mathcal{S}}}$ denotes the canonical bundle). Changing over to normalised basis of differentials, the linear flow equation (1.82a) may therefore be expressed as:*

$$\mathbf{A}(\mathcal{D}) = \mathbf{B} + \mathbf{U}t, \tag{1.83}$$

where \mathbf{A} is the Abel map $\mathbf{A}: S^{\tilde{g}}(\tilde{\mathcal{S}}) \rightarrow \text{Jac}(\tilde{\mathcal{S}})$ and $\mathbf{B}, \mathbf{U} \in \mathbb{C}^{\tilde{g}}$ are constants.

The $r - 1$ differentials $\{\omega_i\}_{i=2, \dots, r}$ appearing as integrands in Eq. (1.82b) are abelian differentials of the third kind with simple poles at ∞_i and ∞_1 , where

$$\infty_i \Leftrightarrow \begin{cases} (\tilde{\lambda} = 0, \tilde{z} = P_i) & \text{for case (i)} \\ (\tilde{\lambda} = 0, \tilde{z} = Y_i) & \text{for case (ii)}. \end{cases} \tag{1.84}$$

and residues $+1$ and -1 , respectively. After a suitable translation by elements of $H^0(\tilde{\mathcal{S}}, K_{\tilde{\mathcal{S}}})$ to obtain the standard normalization with respect to a canonical homology basis $\{a_\mu, b_\mu \in H_1(\tilde{\mathcal{S}}, \mathbb{Z})\}_{\mu=1, \dots, \tilde{g}}$, these provide a basis for the $r - 1$ dimensional space of normalized differentials with simple poles over $\lambda = \infty$.

Remark. Combining these results with the remark following Eq. (1.82a, b), we see that for Hamiltonian flows on the reduced orbit (or equivalently for Hamiltonians that are independent of the P_i 's), the linearization map only involves abelian differentials of the first kind. For flows on the unreduced orbit it is necessary to introduce the differentials of the third kind in order to determine the time dependence of the additional coordinates $\{q_i\}$ (vis. Corollary 1.7).

Proof. Every holomorphic 1-form on $\tilde{\mathcal{S}}$ can be obtained by evaluating the Poincaré residue of a meromorphic 2-form on \mathcal{S} with pole divisor at $\tilde{\mathcal{S}}$. Over the affine coordinate neighborhood U_0 such a residue has the form

$$\omega = \frac{f(\lambda, z) d\lambda}{\mathcal{P}_z(\lambda, z)}, \tag{1.85}$$

where, for holomorphicity at $\lambda = \infty$, the total weighted degree of the polynomial $f(\lambda, z)$ must not exceed $m(r - 1) - 2$, and for holomorphicity at the points $\{(\alpha_i, 0)\}$,

the function $f(\lambda, z)$ must vanish to sufficiently high order so as to cancel the zeroes of $\mathcal{P}_z(\lambda, z)$. Since at these points $\mathcal{P}_z(\lambda, z)$ vanishes like $(\lambda - \alpha_i)^{r-k_i}$, while z vanishes along each intersecting branch like $\lambda - \alpha_i$, $f(\lambda, z)$ must be a sum of terms of the form

$$z^{r-i} \prod_{j=1}^n (\lambda - \alpha_j)^{\max(0, i-k_j-1)} \lambda^a \quad j = 2, \dots, r,$$

where, in order to have total degree at most $m(r-1) - 2$, $0 \leq a \leq \delta_j - 1 - \varepsilon$. But these are precisely the 1-forms ω_{ia} of Eq. (1.82a), which therefore span the entire \tilde{g} -dimensional space of holomorphic 1-forms $H^0(\tilde{\mathcal{S}}, K_{\tilde{\mathcal{S}}})$.

Turning to the remaining $r - 1$ differentials $\{\omega_i\}$ of Eq. (1.82b), these have the same structure near the points $(\alpha_i, 0)$ as the ω_{ia} 's, and hence are holomorphic there, but since the numerator polynomial is of degree $m(r-1) - 1$, they have simple poles over $\lambda = \infty$. To obtain the exact location of these poles and their residues, one simply converts to the coordinates $\tilde{z}, \tilde{\lambda}$ and takes a limit. \square

It follows, since (1.82a) is essentially the Abel map, that any function on the Lagrangian manifold \mathcal{L} that is symmetric in the coordinates (λ_μ) may be expressed along the flow lines in terms of quotients of theta functions on the curve $\tilde{\mathcal{S}}$. In particular, for the coordinates $\{q_i(t)\}$ themselves, we have

Corollary 1.7. *For a suitable choice of constants $\{e_i, f_i\}_{i=2, \dots, r}$, the coordinate functions $\{q_i(t)\}$ satisfying Eq. (1.82b) are given by:*

$$q_i(t) = \ln \left[\frac{\theta(\mathbf{B} + t\mathbf{U} - \mathbf{A}(\infty_i) - \mathbf{K})}{\theta(\mathbf{B} + t\mathbf{U} - \mathbf{A}(\infty_1) - \mathbf{K})} \right] + e_i t + f_i, \tag{1.86}$$

where $\mathbf{K} \in \mathbb{C}^{\tilde{g}}$ is the Riemann constant.

Proof. We use the standard method underlying the reciprocity theorems relating different types of abelian differentials (cf. [GH]). Namely, on the polygonization of $\tilde{\mathcal{S}}$ obtained by cutting along a canonical basis $\{a_\mu, b_\mu\}$ of cycles, we define the meromorphic differential

$$d\psi(p) := d(\ln \theta(\mathbf{A}(\mathcal{D}) - \mathbf{A}(p) - \mathbf{K})), \tag{1.87}$$

where \mathcal{D} is the divisor $\sum_{\mu=1}^{\tilde{g}} p_\mu$ formed from the \tilde{g} points $(p_\mu)_{\mu=1, \dots, \tilde{g}}$ with coordinates

(λ_μ, z_μ) and p denotes the point of evaluation on $\tilde{\mathcal{S}}$. Since $d\psi$ has simple poles with residues 1 at the p_μ 's, we may express the abelian sum appearing in Eq. (1.82b) as an integral

$$\sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \omega_i = \oint_{\mathcal{E}} \left[\int_{\lambda^0}^p \omega_i \right] d\psi \tag{1.88}$$

around a contour \mathcal{E} enclosing only these singularities of the integrand, and not the ones at $p = \{\infty_i\}$, which are logarithmic branch points. Deforming the contour to the boundary \mathcal{B} of the polygonization of $\tilde{\mathcal{S}}$ and integrating by parts gives

$$\sum_{\mu=1}^{\tilde{g}} \int_{\lambda_0}^{\lambda_\mu} \omega_i = \oint_{\mathcal{L}} \ln \theta(\mathbf{A}(\mathcal{D}) - \mathbf{A}(p) - \mathbf{K}) \omega_i - \oint_{\mathcal{B}} \left[\int_{\lambda^0}^p \omega_i \right] d\psi. \tag{1.89}$$

where \mathcal{L} is a small loop enclosing the poles at ∞_i, ∞_1 and no other singularities. If the differentials ω_i were normalized, the contributions to the boundary integral from the pairs $\pm a_\mu$ in \mathcal{B} would just be constants (the discontinuity given by the theta multiplier over the b_μ cycle), and the contributions from the b_μ terms would vanish. However, our differentials $\{\omega_i\}$ differ from the normalized ones by linear combinations of the holomorphic differentials $\{\omega_{ia}\}$ in Eq. (1.82). It follows from Eq. (1.83) that these differences contribute linear terms in t with constant coefficients. The remaining terms in Eq. (1.89) may be evaluated by taking residues at ∞_i, ∞_1 , using the results of Theorem 1.6 to yield the logarithmic theta function term in Eq. (1.86). The constants and linear terms in (1.86) are then obtained by summing those from the normalizing shift with those already present in Eq. (1.82b).

Remark. The LHS of (1.82a, b) may be interpreted as an extended Abel map from $S^{\tilde{g}}\tilde{\mathcal{F}}$ to a generalized Jacobi variety $\mathcal{J}(\hat{\mathcal{F}})$ associated to the singularized curve $\hat{\mathcal{F}}$ obtained by identifying the points $\{\infty_i\}$, where $\mathcal{J}(\hat{\mathcal{F}})$ is a $(C^*)^{r-1}$ extension of $\mathcal{J}(\tilde{\mathcal{F}})$. The extended theta function for $\hat{\mathcal{F}}$ is obtained by multiplying the ordinary theta function for the nonsingular curve by exponential factors in the extended directions [C]. This may be viewed as the source of the additional linear terms in Eq. (1.86)

2. Examples

In the following, we examine two applications of the above analysis: finite dimensional integrable systems involving isospectral flows in $\tilde{\mathfrak{sl}}(2)^{+*}$, and the coupled 2-component cubically nonlinear Schrödinger system (CNLS). Details on how these systems arise through moment map embeddings from a space or rank 2 or 3 perturbations of $N \times N$ matrices may be found in [AHP].

2a. Finite Dimensional Systems and Isospectral Flows in $\tilde{\mathfrak{sl}}(2)^{+*}$

The moment map embedding of finite dimensional integrable systems as isospectral flows in loop algebras developed in [AHP] leads, in the case $\tilde{\mathfrak{sl}}(2)^{+*}$, to the following parametrization. In Eqs. (1.1–1.3), let

$$Y = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \tag{2.1}$$

and $\text{rank}(N_i) = k_i = 1$. Then $\mathcal{N}_0(\lambda) \in \tilde{\mathfrak{gl}}(2)^{+*}$ may be taken of the form:

$$\mathcal{N}_0(\lambda) = \lambda \sum_{i=1}^n \frac{G_i^T F_i}{\alpha_i - \lambda} = \lambda G^T (A - \lambda I)^{-1} F, \tag{2.2}$$

where $(F_i, G_i)_{i=1, \dots, n}$ are the rows of a pair $F, G \in M^{n \times 2}$ of $n \times 2$ complex matrices and $A = \text{diag}(\alpha_i) \in M^{n \times n}$ is a diagonal matrix with distinct eigenvalues $(\alpha_i)_{i=1, \dots, n}$. Imposing the trace-free conditions $\text{tr}(G_i^T F_i) = 0$ and using the freedom of replacing

$$F_i \mapsto d_i F_i, \quad G_i \mapsto d_i^{-1} G_i, \quad d_i \in \mathbb{C} - 0, \tag{2.3}$$

we may take (F, G) , without loss of generality, to be of the form:

$$F = \frac{1}{\sqrt{2}}(\mathbf{x}, \mathbf{y}), \quad G = \frac{1}{\sqrt{2}}(\mathbf{y}, -\mathbf{x}), \tag{2.4}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ are viewed as column vectors. (This amounts to a symplectic reduction with respect to the center of $\widetilde{\mathfrak{gl}}(2, \mathbb{C})^{+*}$, giving flows in $\widetilde{\mathfrak{sl}}(2, \mathbb{C})^{+*}$.) The reduced orbital symplectic form is then just

$$\omega = d\mathbf{x}^T \wedge d\mathbf{y}, \tag{2.5}$$

and $\mathcal{N}(\lambda)$ has the form

$$\mathcal{N}(\lambda) = \lambda \begin{pmatrix} a & b \\ c & -a \end{pmatrix} + \frac{\lambda}{2} \begin{pmatrix} -\sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} & -\sum_{i=1}^n \frac{y_i^2}{\lambda - \alpha_i} \\ \sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} & \sum_{i=1}^n \frac{x_i y_i}{\lambda - \alpha_i} \end{pmatrix}, \tag{2.6}$$

where $(x_i, y_i)_{i=1, \dots, n}$ are the components of (\mathbf{x}, \mathbf{y}) . We now also impose the reality conditions

$$\mathbf{x} = \bar{\mathbf{x}}, \quad \mathbf{y} = \bar{\mathbf{y}}, \quad Y = \bar{Y}, \tag{2.7}$$

to obtain flows in $\widetilde{\mathfrak{sl}}(2, \mathbb{R})^{+*}$. The Hamiltonian systems obtained by pulling back the AKS ring $I(\widetilde{\mathfrak{sl}}(2)^*)$ through the map

$$\begin{aligned} \tilde{J}_Y : \mathbb{R}^n \times \mathbb{R}^n &\rightarrow \widetilde{\mathfrak{sl}}(2)^* \\ \tilde{J}_Y : (\mathbf{x}, \mathbf{y}) &\mapsto \lambda Y + \lambda G^T (A - \lambda I)^{-1} F = \mathcal{N}(\lambda) \end{aligned} \tag{2.8}$$

are then Poisson commutative and, with the possible addition of certain quadratic constraints, coincide with those studied by Moser in [M] (cf. also [AHP, AHH1]). (The fibres of this map are generated by the finite group of reflections $(x_i, y_i) \mapsto (-x_i, -y_i)$ of the coordinate axes. Since the points with $(x_i = 0, y_i = 0)$ are excluded from the inverse image of the orbit $\mathcal{Q}_{\mathcal{N}}$ by the condition $\text{rank}(N_i) = 1$, the resulting ambiguity is resolved along the flows by continuity.)

As an illustrative example, consider the Neumann system [N]. This has been amply studied by a variety of methods in the literature [AvM, F, Kn, M, Sch, Ra]. We include it here to show how the general approach reduces to familiar results for this case, giving a complete separation of variables in hyperellipsoidal coordinates and linearization via a hyperelliptic Abel map. To obtain this system, we choose the matrix Y to be

$$Y = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}, \tag{2.9}$$

and the Hamiltonian ϕ to be:

$$\phi(\mathbf{x}, \mathbf{y}) = -\text{tr}(\mathcal{N}(\lambda)^2)_0 = \frac{1}{2} [(\mathbf{x}^T \mathbf{x})(\mathbf{y}^T \mathbf{y}) + \mathbf{x}^T A \mathbf{x} - (\mathbf{x}^T \mathbf{y})^2], \tag{2.10}$$

where the subscript $()_0$ signifies the λ^0 term in the Laurent expansion around $\lambda = 0$ for large λ . To obtain the appropriate phase space, we must also add the symplectic constraints:

$$\mathbf{x}^T \mathbf{x} = 1, \quad \mathbf{y}^T \mathbf{x} = 0, \tag{2.11}$$

defining the cotangent bundle $T^*S^{n-1} \sim TS^{n-1} \subset \mathbb{R}^{2n}$. The Neumann oscillator Hamiltonian is

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2} [\mathbf{y}^T \mathbf{y} + \mathbf{x}^T A \mathbf{x}], \tag{2.12}$$

which coincides with $\phi(\mathbf{x}, \mathbf{y})$ on the constrained manifold. The constraints (2.11) may be viewed as a Marsden-Weinstein reduction under the stabilizer $\text{Stab}(Y) \subset \mathfrak{sl}(2, \mathbb{R})$ (cf. Sect. 1b), in which $\mathbf{x}^T \mathbf{x} = 1$ defines a level set of the moment map generating the flow

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y} + t\mathbf{x}) \tag{2.13}$$

induced by the one-parameter subgroup $\text{Stab}(Y)$, while $\mathbf{x}^T \mathbf{x} = 0$ defines a section over the quotient of the level set by this flow (i.e., of the null foliation it generates). It follows that the H -flow of the constrained system is obtained from the ϕ -flow of the free system simply by orthogonal projection of the momentum \mathbf{y} relative to \mathbf{x} :

$$(\mathbf{x}(t), \mathbf{y}(t))_{\text{free}} \mapsto (\hat{\mathbf{x}}(t), \hat{\mathbf{y}}(t))_{\text{constr.}} := \left(\mathbf{x}(t), \mathbf{y}(t) - \left(\frac{\mathbf{x}^T(t) \mathbf{y}(t)}{\mathbf{x}^T(t) \mathbf{x}(t)} \right) \mathbf{x}(t) \right) \tag{2.14}$$

from the invariant manifold defined by $\mathbf{x}^T \mathbf{x} = 1$. The equations of motion for the unconstrained system are

$$\frac{d\mathbf{x}}{dt} = (\mathbf{x}^T \mathbf{x}) \mathbf{y} - (\mathbf{x}^T \mathbf{y}) \mathbf{x}, \quad \frac{d\mathbf{y}}{dt} = -(\mathbf{y}^T \mathbf{y}) \mathbf{x} - A\mathbf{x} + (\mathbf{x}^T \mathbf{y}) \mathbf{y}. \tag{2.15}$$

These are equivalent (within a quotient by the finite group of reflections in the coordinate axes) to the Lax equation

$$\frac{d\mathcal{N}}{dt} = [\mathcal{B}, \mathcal{N}], \tag{2.16a}$$

where

$$\mathcal{B} = d\phi(\mathcal{N})_+ = \begin{pmatrix} \mathbf{x}^T \mathbf{y} & \lambda + \mathbf{y}^T \mathbf{y} \\ -\mathbf{x}^T \mathbf{x} & -\mathbf{x}^T \mathbf{y} \end{pmatrix}, \tag{2.16b}$$

The invariant spectral curve is thus given by the characteristic equation

$$\det \left(\frac{\mathcal{N}(\lambda)}{\lambda} - \zeta I \right) = 0, \tag{2.17}$$

which, defining as in Sect. 1,

$$z := a(\lambda) \zeta, \quad a(\lambda) := \prod_{i=1}^n (\lambda - \alpha_i), \tag{2.18}$$

determines a genus $g = n - 1$ hyperelliptic curve defined by (cf. Eqs. (1.15), (1.22)):

$$z^2 - a(\lambda) \mathcal{P}(\lambda) = 0, \tag{2.19}$$

where

$$\begin{aligned} \mathcal{P}(\lambda) &:= -\mathcal{P}_2(\lambda) = -\frac{a(\lambda)}{4} \sum_{i=1}^n \frac{I_i}{\lambda - \alpha_i} = P_{n-1} \lambda^{n-1} + P_{n-2} \lambda^{n-2} + \dots + P_0, \\ P_{n-1} &= -\frac{1}{4} \sum_{i=1}^n I_i = -\frac{1}{4} \mathbf{x}^T \mathbf{x}, \quad P_{n-2} = -\frac{1}{4} \sum_{i=1}^n \alpha_i I_i = \frac{1}{2} \phi, \end{aligned} \tag{2.20}$$

and

$$I_i := \sum_{j=1, j \neq i}^n \frac{(x_i y_j - x_j y_i)^2}{\alpha_i - \alpha_j} + x_i^2 \tag{2.21}$$

are the Devaney-Uhlenbeck invariants (cf. [M]).

Applying the prescription of Sect. 1b, with $V_0 = (1, 0)^T$, the reduction with respect to the stabilizer of Y gives rise to the constraints (2.11) as discussed above, and the solutions to Eq. (1.34) give us the Darboux coordinates $(\lambda_\mu, \zeta_\mu)_{\mu=1, \dots, n-1}$ defined by

$$\sum_{i=1}^n \frac{x_i^2}{\lambda - \alpha_i} = \frac{\prod_{\mu=1}^{n-1} (\lambda - \lambda_\mu)}{a(\lambda)}, \tag{2.22a}$$

$$\zeta_\mu = \frac{1}{2} \sum_{i=1}^n \frac{x_i y_i}{\lambda_\mu - \alpha_i} = \sqrt{\frac{\mathcal{P}(\lambda_\mu)}{a(\lambda_\mu)}}. \tag{2.22b}$$

Thus, the spectral divisor coordinates here are just the usual hyperellipsoidal coordinates (λ_μ) , together with their conjugate momenta (ζ_μ) . The Liouville generating function on the isospectral foliation thus becomes

$$S = \sum_{\mu=1}^{n-1} \int \zeta_\mu d\lambda_\mu |_{P_i = \text{cst.}} = \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \sqrt{\frac{\mathcal{P}(\lambda)}{a(\lambda)}} d\lambda, \tag{2.23}$$

and the canonically conjugate coordinates undergoing linear flow are

$$Q_j := \frac{\partial S}{\partial P_j} = \frac{1}{2} \sum_{\mu=1}^{n-1} \int_0^{\lambda_\mu} \frac{\lambda^j d\lambda}{\sqrt{a(\lambda)\mathcal{P}(\lambda)}} = b_j t, \quad j = 0, \dots, n-2, \tag{2.24}$$

where, for our Hamiltonian $\phi = 2P_{n-2}$,

$$b_{n-2} = 2, \quad b_j = 0, \quad j < n-2. \tag{2.25}$$

This reproduces the familiar linearization via the hyperelliptic Abel map obtained through the classical methods of Jacobi (cf. [M]).

The other classical systems treated in [M] as isospectral flows of rank 2 perturbations of a fixed matrix A , such as geodesic flow on hyperellipsoids or the Rosochatius system, follow identically (cf. also [GHHW, AHP]). In all these cases, the spectral divisor Darboux coordinates (λ_μ, ζ_μ) will coincide with the usual hyperellipsoidal coordinates, or some complexification thereof, the curves will be hyperelliptic and the spectral invariants will be an analogue of the Devaney-Uhlenbeck invariants (2.21) encountered in this case.

2b. The CNLS System and Flows in $\tilde{\mathfrak{su}}(1, 2)^{+*}$

The real form of the CNLS system we consider involves two complex functions $u(x, t), v(x, t)$ satisfying the coupled system of equations:

$$iu_t + u_{xx} = 2u(|u|^2 + |v|^2), \tag{2.26a}$$

$$iv_t + v_{xx} = 2v(|u|^2 + |v|^2). \tag{2.26b}$$

These can be obtained as the compatibility conditions for a pair of Lax equations of the type (1.1) with $\mathcal{N}(\lambda) \in \widetilde{\mathfrak{su}}(1, 2)^{+*}$ of the form given in Eq. (1.2) and $Y = 0$, corresponding, as above, to the Hamiltonians

$$H_x(\mathcal{N}) = \frac{1}{2} \left[\frac{a(\lambda)}{\lambda^{n-1}} \operatorname{tr}(\mathcal{N}(\lambda)^2) \right]_0, \tag{2.27a}$$

$$H_t(\mathcal{N}) = \frac{1}{2} \left[\frac{a(\lambda)}{\lambda^{n-2}} \operatorname{tr}(\mathcal{N}(\lambda)^2) \right]_0, \tag{2.27b}$$

If, as above, we set:

$$\hat{\mathcal{L}}(\lambda) = \frac{a(\lambda)}{\lambda} \mathcal{N}(\lambda) = L_0 \lambda^{n-1} + L_1 \lambda^{n-2} + L_2 \lambda^{n-3} + \dots + L_{n-1}, \tag{2.28}$$

then Hamiltonian's equations again take the Lax form

$$\frac{d}{dx} \hat{\mathcal{L}}(\lambda) = [\lambda L_0 + L_1, \hat{\mathcal{L}}(\lambda)], \tag{2.28a}$$

$$\frac{d}{dx} \hat{\mathcal{L}}(\lambda) = [\lambda^2 L_0 + \lambda L_1 + L_2, \hat{\mathcal{L}}(\lambda)]. \tag{2.28b}$$

and the flows commute. If the following invariant constraints are imposed:

$$L_0 = \frac{i}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & \bar{u} & \bar{u} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \tag{2.29}$$

$$L_2 = i \begin{pmatrix} |u|^2 + |v|^2 & -\bar{u}_x & -\bar{v}_x \\ u_x & -|u|^2 & -\bar{v}u \\ v_x & -\bar{u}v & -|v|^2 \end{pmatrix},$$

the CNLS equations are obtained as the compatibility conditions for Eqs. (2.28a, b).

It is possible to obtain an intrinsic characterization of the orbit corresponding to residue matrices N_i of rank $k_i = 1$ as an open, dense subset of \mathbb{C}^{2n} , viewed as a real symplectic space, (cf. [AHP, AHH1]). However, the approach developed in Sect. 1 allows us to treat all orbits of the type (1.3) on the same footing, regardless of the rank $k_i = 1, 2, 3$. Only the explicit formulae (1.27) for the genus \tilde{g} of the spectral curve and (1.25a) for the degrees of the invariant polynomials will change.

By Lemma 1.1 and Proposition 1.2, the invariant spectral curve is given by a polynomial equation of the general form

$$\det(\hat{\mathcal{L}}(\lambda) - zI) = \mathcal{P}(\lambda, z) = \mathcal{P}_R(\lambda, z) + p(\lambda, z) = 0, \tag{2.30a}$$

where

$$\mathcal{P}_R(\lambda, z) := -z^3 + z\mathcal{A}_2(\lambda)\mathcal{P}_{R2}(\lambda) + \mathcal{A}_3(\lambda)\mathcal{P}_{R3}(\lambda) \tag{2.30b}$$

defines a reference curve \mathcal{S}_R , determined by the initial data for a particular solution of the CNLS system, and

$$p(\lambda, z) := a(\lambda) \left(za_2(\lambda) \sum_{a=0}^{\varrho} P_{2a} \lambda^a + a_3(\lambda) \sum_{a=0}^{\sigma} P_{3a} \lambda^a \right) \tag{2.30c}$$

is of the form given by Proposition 1.2 for neighbouring curves, with $\varrho = \delta_2$, $\sigma = \delta_3$, given by formulae (1.25a) and $(P_{2a}, P_{3b})_{a=0, \dots, \varrho, b=0, \dots, \sigma}$ are the Poisson commuting

spectral invariants. It will be convenient to reparametrize the space of polynomials $p(\lambda, z)$ as follows. Set

$$a_2(\lambda) \sum_{a=0}^{\varrho} P_{2a} \lambda^a = \hat{P}_{2,\varrho} \lambda^{n-2} + \hat{P}_{2,\varrho-1} \lambda^{n-3} + \dots + \hat{P}_{2,0} \lambda^{n-2-\varrho} + (\text{lower order}), \tag{2.30d}$$

$$a_3(\lambda) \sum_{a=0}^{\sigma} P_{3a} \lambda^a = \hat{P}_{3,\sigma} \lambda^{2n-3} + \hat{P}_{3,\sigma-1} \lambda^{2n-4} + \dots + \hat{P}_{3,0} \lambda^{2n-3-\sigma} + (\text{lower order}). \tag{2.30e}$$

Note that the lower order terms are completely determined by the terms $\hat{P}_{2,a}, \hat{P}_{3,b}$ explicitly appearing.

There can be singularities in general over the points with $\lambda = \alpha_i$, which are assumed to be resolved as indicated in Sect. 1. Due to the normalizations (2.29a-c) the spectral curve of $\hat{\mathcal{L}}(\lambda)$ also has singularities at $\lambda = \infty$. In terms of the coordinates $\tilde{\lambda} = 1/\lambda$, the eigenvalues of $\lambda^{-m} \hat{\mathcal{L}}(\lambda)$ have the expansion around $\tilde{\lambda} = 0$,

$$\begin{aligned} \tilde{z}_1(\tilde{\lambda}) &= \frac{2i}{3} - (m_2 + m_3) \tilde{\lambda}^3 + \dots, \\ \tilde{z}_2(\tilde{\lambda}) &= -\frac{i}{3} + m_2 \tilde{\lambda}^3 + \dots, \\ \tilde{z}_3(\tilde{\lambda}) &= -\frac{i}{3} + m_3 \tilde{\lambda}^3 + \dots, \end{aligned} \tag{2.31}$$

where for generic $\hat{\mathcal{L}}, m_2 \neq m_3$. This gives the curve \mathcal{S} a triple singularity (tacnode of order 3) at $\lambda = \infty$, and the desingularisation $\tilde{\mathcal{S}}$ is 3-sheeted over $\lambda = \infty$. Furthermore, $\hat{\mathcal{L}}(\lambda)$ is diagonalisable in a neighbourhood of $\lambda = \infty$.

The genus of the curve $\tilde{\mathcal{S}}$ is $\tilde{g}' = \varrho + \sigma - 3$, three less than the genus $\tilde{g} = \varrho + \sigma$ given in formulae (1.26), (1.27) for the generic spectral curve in the coadjoint orbit. Similarly, the degree of \tilde{E} is $\varrho + \sigma - 1$, three less than the generic case. Accordingly, we have six fewer divisor coordinates than in the generic case, and hence an insufficient number to provide a Darboux system on the orbit $Q_{\mathcal{S}}$.

The solution to this problem is to impose six extra constraints on elements of the orbit, so that the dimensions of the constrained submanifold coincides with the number of coordinates. This must be done in an invariant way. Note that, under the Lax equations (2.28a, b), the matrix $\hat{\mathcal{L}}(\lambda), \lambda$ fixed, evolves by conjugation. In $Gl(n, \mathbb{C})$, the spectrum is not the only invariant under conjugation; when there are multiple eigenvalues, we also have the different Jordan canonical forms. For example, among matrices with one double eigenvalue, the generic coadjoint orbits have nondiagonal canonical form, and there are orbits that are two dimensions smaller, consisting of diagonalisable matrices. With this model in mind, we impose the following further invariant constraints, defining a $2(\varrho + \sigma - 1)$ -dimensional symplectic submanifold $Q_s \subset Q_{\mathcal{S}}^0$:

(i) The spectral curve has genus three less than the generic curve in the orbit, and so has three extra singularities, counted with multiplicity. (2.32a)

(ii) At these extra singular points, $\hat{\mathcal{L}}(\lambda)$ is diagonalisable. (2.32b)

The restriction to \mathcal{Q}_s of the form ω_{orb} is given, over a suitable dense set, by

$$\omega_{\text{orb}} = \sum_{\mu=1}^{g'} d\lambda_{\mu} \wedge d\zeta_{\mu} + \sum_{i=2}^3 dq_i \wedge dP_i. \tag{2.33}$$

The constraints (2.32a, b) and the relations between P_2, P_3 and the coefficients of the polynomial $p(\lambda, z)$ imply that the coefficients $P_{2,\varrho}, P_{3,\sigma}, P_{3,\sigma-1}, P_{3,\sigma-2}, P_{3,\sigma-3}$ in Eq. (2.30c) can be expressed in terms of the lower coefficients $(P_{2a}, P_{3b})_{a=0, \dots, \varrho-1, b=0, \dots, \sigma-4}$ and P_2, P_3 . To apply the Liouville method, we only need to know the constraints to order 1 at the CNLS curve \mathcal{S}_R corresponding to the spectral polynomial $\mathcal{P}_R(\lambda, z)$. The constraint (2.32a), requiring the neighboring curve to have the same degree of singularity as \mathcal{S}_R , is equivalent to first order to requiring the induced section of the normal bundle along \mathcal{S}_R :

$$\frac{p(\lambda, z)}{\partial \mathcal{P}_R / \partial t} \cdot \frac{\partial}{\partial z} \tag{2.34}$$

to remain finite at the singular points of \mathcal{S}_R .

At the CNLS curve this means that, passing to the coordinates $\tilde{z}, \tilde{\lambda}$, the three first terms in the Taylor expansion at $\tilde{\lambda} = 0$ of the expression

$$\tilde{\lambda}^{2n-3} (\tilde{z} \tilde{\lambda}^{-n+1} a_2 (\tilde{\lambda}^{-1}) p_2 (\tilde{\lambda}^{-1}) + a_3 (\tilde{\lambda}^{-1}) p_3 (\tilde{\lambda}^{-1})) =: \tilde{z} f_2(\tilde{\lambda}) + f_3(\tilde{\lambda}) \tag{2.35}$$

must vanish when one substitutes $\tilde{z} = -\frac{i}{3}$. This yields the linearized constraints:

$$\hat{P}_{3,\sigma} = \frac{i}{3} \hat{P}_{2,\varrho}, \quad \hat{P}_{3,\sigma-1} = \frac{i}{3} \hat{P}_{2,\varrho-1}, \quad \hat{P}_{3,\sigma-2} = \frac{i}{3} \hat{P}_{2,\varrho-2}. \tag{2.36}$$

Setting

$$P_j =: -\frac{i}{3} + \hat{P}_j, \quad j = 2, 3, \tag{2.37}$$

the linear variation of the (\hat{P}_j) at the CNLS curve can be computed by evaluating the limits of the normal vector field (2.34) along the two branches of the curve as one approaches the singular point. To first order:

$$\hat{P}_j = \lim_{\tilde{\lambda} \rightarrow 0} \frac{\tilde{z}_j(\tilde{\lambda}) f_2(\tilde{\lambda}) + f_3(\tilde{\lambda})}{(-\tilde{z}_j(\tilde{\lambda}) + \tilde{z}_k(\tilde{\lambda}))(-2\tilde{z}_j(\tilde{\lambda}) - \tilde{z}_k(\tilde{\lambda}))}, \tag{2.38}$$

where $(j, k) = (2, 3)$ or $(3, 2)$, and $\tilde{z}_j(\tilde{\lambda})$ is as in (2.31). This gives

$$\hat{P}_2 + \hat{P}_3 = i f_2(0) = i P_{2,\varrho} \tag{2.39}$$

and, if m_2, m_3 are as in (2.31),

$$m_3 \hat{P}_2 + m_2 \hat{P}_3 = \lim_{\tilde{\lambda} \rightarrow 0} -\frac{i}{\tilde{\lambda}^3} \left(-\frac{i}{3} f_2 + f_3 \right) = -i \left[-\frac{i}{3} (\hat{P}_{2,\varrho-3}) + \hat{P}_{3,\sigma-3} \right], \tag{2.40}$$

where c_i, d_i are constants. Isolating $\hat{P}_{3,\sigma-3}$ above, and using (2.39–2.40) to rewrite (2.36) we obtain

$$\begin{aligned} \hat{P}_{2,\varrho} &= -i(\hat{P}_2 + \hat{P}_3), & \hat{P}_{3,\sigma} &= \frac{1}{3}(\hat{P}_2 + \hat{P}_3), & \hat{P}_{3,\sigma-1} &= \frac{i}{3} \hat{P}_{2,\varrho-1}, \\ \hat{P}_{3,\sigma-2} &= \frac{i}{3} \hat{P}_{2,\varrho-2}, & \hat{P}_{3,\sigma-3} &= \frac{i}{3} \hat{P}_{2,\varrho-3} + i(m_3 \hat{P}_2 + m_2 \hat{P}_3). \end{aligned} \tag{2.41}$$

This expresses the terms on the left in terms of the independent complete set of integrals of motion $\hat{P}_{2,0}, \dots, \hat{P}_{2,\ell-1}, \hat{P}_{3,0}, \dots, \hat{P}_{3,\sigma-4}, \hat{P}_2, \hat{P}_3$ (at least to first order around the reference curve, which is all we need to apply the Liouville method). Since $(\lambda_\nu, z_\nu)_{\nu=1, \dots, \tilde{g}}(q_i, P_i)_{i=2,3}$ form a Darboux coordinate system “up to constants of motion,” if we define our generating function $S(\lambda_\nu, q_2, q_3, P_{2a}, P_{3b}, P_2, P_3)$ as in (1.76), then the canonically conjugate coordinates

$$\left\{ \hat{Q}_{2a} = \frac{\partial S}{\partial \hat{P}_{2a}} \right\}_{a=0, \dots, \ell-1}, \quad \left\{ \hat{Q}_{3b} = \frac{\partial S}{\partial \hat{P}_{3b}} \right\}_{b=0, \dots, \sigma-4}, \quad (2.42)$$

$$\left\{ \hat{Q}_i = \frac{\partial S}{\partial \hat{P}_i} \right\}_{i=2,3}$$

undergo linear flow:

$$\hat{Q}_{2a} = c_{2a} - \delta_{a,\ell-1}x - \delta_{a,\ell-2}t, \quad \hat{Q}_{3b} = c_{3b}, \quad \hat{Q}_i = c_i, \quad (2.43)$$

where c_{2a}, c_{3b}, c_i are constants. (Up to additive constants, $H_x = -\hat{P}_{2,\ell-1}, H_t = -\hat{P}_{2,\ell-2}$.) Evaluating the derivatives (2.42), taking (2.41) into account, we obtain:

$$\hat{Q}_{2a} = \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda_\nu} \frac{z \lambda^{n-2-\ell+a} + \frac{i}{3} (\delta_{a,\ell-3} \lambda^{2n-6} + \delta_{a,\ell-2} \lambda^{2n-5} + \delta_{a,\ell-1} \lambda^{2n-4})}{-3z^2 + \mathcal{A}_2(\lambda) \mathcal{P}_{R2}(\lambda)} d\lambda, \quad (2.44a)$$

$$\hat{Q}_{3b} = \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda_\nu} \frac{\lambda^{2n-3-\sigma+b}}{-3z^2 + \mathcal{A}_2(\lambda) \mathcal{P}_{R2}(\lambda)} d\lambda, \quad (2.44b)$$

$$\hat{Q}_2 = \ln u + I + im_3 J, \quad (2.44c)$$

$$\hat{Q}_3 = \ln v + I + im_2 J, \quad (2.44d)$$

where I, J are defined by

$$I = \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda_\nu} \frac{-iz \lambda^{n-2} + \frac{1}{3} \lambda^{2n-3}}{-3z^2 + \mathcal{A}_2(\lambda) \mathcal{P}_{R2}(\lambda)} d\lambda, \quad (2.45a)$$

$$J = \sum_{\nu=1}^{\tilde{g}} \int_0^{\lambda_\nu} \frac{\lambda^{2n-6}}{-3z^2 + \mathcal{A}_2(\lambda) \mathcal{P}_{R2}(\lambda)} d\lambda. \quad (2.45b)$$

We can check explicitly that the integrands of (2.44a, b) form a basis for the holomorphic differentials on the curve \tilde{S} which has been desingularized over $\lambda = \alpha_i$ and over $\lambda = \infty$. If $\infty_1, \infty_2, \infty_3$ are the three points of \tilde{S} over $\lambda = \infty$ corresponding to $\tilde{z} = \frac{2i}{3}, \frac{-i}{3}, \frac{-i}{3}$ respectively, the integrands of $I + im_3 J, I + im_2 J$ have only simple poles over $\lambda = \infty$ with residues $(-1, 1, 0)$ and $(-1, 0, 1)$ respectively at $(\infty_1, \infty_2, \infty_3)$. Proceeding as in Corollary 1.7, we obtain the θ -function formulae

$$u(x, t) = \exp(q_2) = \tilde{K}_2 \exp(e_2 x + d_2 t) \frac{\theta(\mathbf{A}(\infty_2, p) + t\mathbf{U} + x\mathbf{V} - \mathbf{K})}{\theta(\mathbf{A}(\infty_1, p) + t\mathbf{U} + x\mathbf{V} - \mathbf{K})}, \quad (2.46a)$$

$$v(x, t) = \exp(q_2) = \tilde{K}_3 \exp(e_3 x + d_3 t) \frac{\theta(\mathbf{A}(\infty_3, p) + t\mathbf{U} + x\mathbf{V} - \mathbf{K})}{\theta(\mathbf{A}(\infty_1, p) + t\mathbf{U} + x\mathbf{V} - \mathbf{K})}, \quad (2.46b)$$

with $\mathbf{U}, \mathbf{V} \in \mathbb{C}^{\mathfrak{g}'}$ determined from the Hamiltonians $h = H_x, H_t$ as in Theorem 1.6, $(e_i, d_i)_{i=2,3}$ as in Corollary 1.7, and the remaining integration constants determined to satisfy the appropriate initial conditions.

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