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# Gravity in Non-Commutative Geometry

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*Dedicated to H. Araki*

**Abstract.** We study general relativity in the framework of non-commutative differential geometry. As a prerequisite we develop the basic notions of non-commutative Riemannian geometry, including analogues of Riemannian metric, curvature and scalar curvature. This enables us to introduce a generalized Einstein-Hilbert action for non-commutative Riemannian spaces. As an example we study a space-time which is the product of a four dimensional manifold by a two-point space, using the tools of non-commutative Riemannian geometry, and derive its generalized Einstein-Hilbert action. In the simplest situation, where the Riemannian metric is taken to be the same on the two copies of the manifold, one obtains a model of a scalar field coupled to Einstein gravity. This field is geometrically interpreted as describing the distance between the two points in the internal space.

## 1. Introduction

The poor understanding we have of physics at very short distances might lead one to expect that our description of space-time at tiny distances is inadequate. No convincing alternative description is known, but different routes to progress have been proposed. One such proposal is to try to formulate physics on some non-commutative space-time. There appear to be too many possibilities to do this, and it is difficult to see what the right choice is. So the strategy is to consider slight variations of commutative geometry, and to see whether reasonable models can be constructed. This is the approach followed by Connes [1], and Connes and Lott [2, 3]. They consider a model of commutative geometry (a Kaluza-Klein theory with an internal space consisting of two points), but use non-commutative geometry to define metric properties. The result is an economical way of deriving the standard model in which, roughly speaking, the Higgs field appears as the component of the gauge field in the internal direction.

In this paper, we show how gravity, in its simplest form, can be introduced in this context. We first propose a generalization of the basic notions of Riemannian

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geometry. This construction is based on the definition of the Riemannian metric as an inner product on cotangent space. Connes has proposed to define metric properties of a non-commutative space corresponding to an involutive unital algebra $A$ in terms of $K$-cycles over $A$, [3]. It will be shown that every $K$-cycle over $A$ yields a notion of "cotangent bundle" associated to $A$ and a Riemannian metric on the cotangent bundle [1]. We then introduce orthonormal bases, "vielbeins", in a space of sections of the cotangent bundle, analogues of the spin connection, torsion and Riemann curvature tensor, and we derive Cartan structure equations. This enables us to define an analogue of scalar curvature. After this, we propose a generalized Einstein-Hilbert action and see how it looks like in the case of a Kaluza-Klein model with a two-point internal space.

The construction illustrates an interesting feature of non-commutative geometry for commutative spaces: the fact that the metric structure is more general allows one to consider a class of metric spaces more general than Riemannian manifolds, in which however differential geometric notions, such as connections and curvature, still make sense.

The physical picture emerging from this is of a gravitational field described by a Riemannian metric on a four-dimensional space-time plus a scalar field which encodes the distance between the two points in the internal space. This field is massless and couples in a minimal way to gravity. Its vacuum expectation value turns out to determine the scale of weak interactions in the formalism of [3].

## 2. Riemannian Geometry

In this section we develop some concepts of Riemannian geometry in the more general context of non-commutative spaces. Let $\Omega^\cdot$ be a $\mathbb{Z}$-graded differential algebra over $\mathbb{R}$ or $\mathbb{C}$. This means that $\Omega^\cdot = \bigoplus_0^\infty \Omega^n$ is a graded complex of vector spaces with differential $d\colon \Omega^n \to \Omega^{n+1}$ and that there is an associative product $m\colon \Omega^n \otimes \Omega^m \to \Omega^{n+m}$. In particular, $A = \Omega^0$ is an algebra, and $\Omega^n$ is a two sided $A$ module. We will always assume that $\Omega^\cdot$ has a unit $1 \in A$. The algebra $A$ is to be thought of as a generalization of the algebra of functions on a manifold, and $\Omega^\cdot$ as a generalization of the space of differential forms. The most important example for us is Connes' algebra of universal forms $\Omega^\cdot(A)$ over an algebra $A$. It is generated by symbols $f$, of degree zero, and $df$, of degree one, $f \in A$, with relations $d(fg) = df\, g + f\, dg$, $f, g \in A$, and $d1 = 0$. The notation is consistent, since $\Omega^0(A) = A$.

In non-commutative geometry, a notion of *vector bundles* over a non-commutative space described by an algebra $A$ is provided by finitely generated, projective left $A$ modules. A *connection* on a left $A$ module $E$ is, by definition, a linear map $\nabla\colon E \to \Omega^1 \bigotimes_A E$ such that, for any $f \in A$ and $s \in E$,

$$\nabla(fs) = df \otimes s + f\nabla s\,. \tag{1}$$

For any left $A$ module $E$, define $\Omega^\cdot E$ to be the graded left $\Omega^\cdot$ module, $\Omega^\cdot E = \Omega^\cdot \bigotimes_A E$, of "$E$-valued differential forms". A connection $\nabla$ on $E$ extends uniquely

[1] We thank A. Connes for help in finding the right construction of a metric

to a linear map of degree one $\nabla : \Omega^{\cdot} E \to \Omega^{\cdot} E$ with the property that, for any homogeneous $\alpha \in \Omega^{\cdot}$, $\phi \in \Omega^{\cdot} E$,

$$\nabla(\alpha\phi) = d\alpha\, \phi + (-1)^{\deg(\alpha)} \alpha \nabla \phi\,. \tag{2}$$

The *curvature* of $\nabla$ is then $R(\nabla) = -\nabla^2 : E \to \Omega^2 \bigotimes_A E$, and obeys $-\nabla^2(fs) = f(-\nabla^2)s$, for any $f \in A$ and $s \in E$.

Suppose now that $\Omega^{\cdot}$ is involutive, i.e. there is an antilinear antiautomorphism $\alpha \mapsto \alpha^*$ with $\alpha^{**} = \alpha$, for all $\alpha \in \Omega^{\cdot}$. Assume that $\deg(\alpha^*) = \deg(\alpha)$ and $(d\alpha)^* = (-1)^{\deg(\alpha)+1} d(\alpha^*)$, for homogeneous $\alpha$. If $A$ is any involutive algebra then the algebra $\Omega^{\cdot}(A)$ of universal differential forms is involutive, with the above properties, if we set $(df)^* = -d(f^*)$, for $f \in A$. In general, elements of $A$ of the form $g = \sum_i f_i^* f_i$, are called non-negative ($g \geq 0$). The module $E$ is called *hermitian* if it has a hermitian inner product $\langle\, , \rangle : E \times E \to A$, which is by definition a sesquilinear form such that

(i) $\langle fs, gt \rangle = f\langle s, t\rangle g^*$, $f, g \in A$, $s, t \in E$.
(ii) $\langle s, s\rangle \geq 0$.
(iii) The map $s \mapsto \langle s, \cdot \rangle$ from $E$ to the left $A$ module $E^* = \{l : E \to A, l(fs + gt) = l(s)f^* + l(t)g^*\}$ is an isomorphism.

Any hermitian inner product on $E$ extends uniquely to a sesquilinear map $\Omega^{\cdot} E \times \Omega^{\cdot} E \to \Omega^{\cdot}$ such that $\langle \alpha\phi, \beta\psi\rangle = \alpha\langle \phi, \psi\rangle \beta^*$ for all $\alpha, \beta \in \Omega^{\cdot}$, $\phi, \psi \in \Omega^{\cdot} E$. A connection $\nabla$ on a hermitian $A$ module $E$ is *unitary* if, for all $s, t \in E$, $d\langle s, t\rangle = \langle \nabla s, t\rangle - \langle s, \nabla t\rangle$ [the minus sign appears here because we have set $(df)^* = -df^*$]. One has then for homogeneous $\phi, \psi \in \Omega^{\cdot} E$

$$d\langle \phi, \psi\rangle = \langle \nabla\phi, \psi\rangle - (-1)^{\deg(\phi)\deg(\psi)} \langle \phi, \nabla\psi\rangle\,. \tag{3}$$

Next, we attempt to introduce the notion of *non-commutative Riemannian geometry*. The first step consists in introducing a notion of distance on a non-commutative space. Apparently, in non-commutative geometry, a natural notion of distance is provided by $K$-cycles. Recall that a $K$-cycle over an involutive algebra $A$ is a pair $(H, D)$, where $H = H_+ \oplus H_-$ is a $\mathbb{Z}_2$ graded Hilbert space with a $*$-action of $A$ by even bounded operators, and $D$ is a possibly unbounded, odd self-adjoint operator, called Dirac operator, such that $[D, f]$ is bounded, for all $f \in A$, and $(D^2 + 1)^{-1}$ is compact. Then $\pi(f_0 df_1 \ldots df_n) = f_0[D, f_1] \ldots [D, f_n]$ defines an involutive [i.e. with $\pi(\alpha^*) = \pi(\alpha)^*$] representation of the algebra $\Omega^{\cdot}(A)$ of universal forms. One shows then that the graded subcomplex $\mathrm{Ker}(\pi) + d\,\mathrm{Ker}(\pi)$ is a two-sided ideal of $\Omega^{\cdot}(A)$, so that the quotient

$$\Omega^{\cdot}_D(A) = \Omega^{\cdot}(A)/(\mathrm{Ker}(\pi) + d\,\mathrm{Ker}(\pi)) \tag{4}$$

is a graded differential algebra[2].

In *Riemannian geometry*, we choose $A$ and $\Omega^{\cdot}(A)$ to be algebras over the field $\mathbb{R}$. In order to introduce a notion of metric on a non-commutative space, we must try to define an analogue of the tangent- or cotangent bundle over a manifold for non-commutative spaces. One might be tempted to define a "space of sections of the tangent bundle" as the space of derivations of $A$. However, this turns out to be not a very useful notion, (as many interesting algebras $A$ have too few derivations). It

[2] This algebra was introduced in the Cargèse lecture notes of Connes and Lott [3]. It replaces the algebra of universal forms used in [1,2] and allows for a more transparent treatment of "auxiliary fields"

is more promising to introduce an analogue of the cotangent bundle in our context. Given a $K$-cycle $(H, D)$ over $A$, we define a "space of sections of the cotangent bundle" as $\Omega^1_D(A)$. We note that $\Omega^1_D(A)$ is a left $A$ module, hence a vector bundle over $A$.

A *Riemannian metric* is a hermitian inner product – more generally, a non-degenerate inner product – on $\Omega^1_D \equiv \Omega^1_D(A)$ which, in the examples considered below, determines a notion of *distance* coinciding with the one obtained from the Dirac operator, as in [3]. In view of these examples and results in [3], it is natural to ask, whether $\Omega^1_D(A)$ can be equipped with a Riemannian metric that is uniquely determined by the $K$-cycle $(H, D)$ over $A$? In order to answer this question, we have to introduce an analogue for non-commutative spaces of the notions of "volume form" and "integration". Following Connes, see [3], we say that a $K$-cycle $(H, D)$ is $(d, \infty)$-summable if

$$\mathrm{tr}_H(D^2+1)^{-p} < \infty\,, \quad \text{for all } p > \frac{d}{2}\,.$$

Let $\mathrm{Tr}_\omega$ denote the Dixmier trace [3]. The integral of an element $\alpha \in \Omega^{\cdot}_D(A)$ is defined by

$$\int \alpha := \mathrm{Tr}_\omega(\pi(\alpha)|D|^{-d})\,. \tag{5}$$

An alternative definition of $\int \alpha$ which, in the examples considered below, is equivalent to the one just given is the following one:

$$\int \alpha := \lim_{\beta \searrow 0} \frac{\mathrm{tr}_H(\pi(\alpha)e^{-\beta D^2})}{\mathrm{tr}_H(e^{-\beta D^2})}\,,$$

(assuming the limit exists). The advantage of this definition is that it might still be meaningful in examples where $d = \infty$.

When $d < \infty$ and $\int(\cdot)$ is defined by (5) then $\int(\cdot)$ defines a *trace* on the algebra $\Omega^{\cdot}_D(A)$, hence on the subalgebra $A = \Omega^0_D(A) \subset \Omega^{\cdot}_D(A)$, which is invariant under cyclic permutations. It also defines a scalar product on $\Omega^{\cdot}_D(A)$: For $\alpha, \beta \in \Omega^{\cdot}_D(A)$, we define

$$(\alpha, \beta) = \int \alpha \cdot \beta^* = \mathrm{Tr}_\omega(\pi(\alpha \cdot \beta^*)\,|D|^{-d})\,. \tag{6}$$

This scalar product permits us to choose special representatives in the equivalence classes defining the elements of $\Omega^{\cdot}(A)/(\mathrm{Ker}(\pi) + d\,\mathrm{Ker}(\pi))$ as follows: Given $\alpha \in \Omega^{\cdot}_D(A)$, we define $\alpha^\perp$ to be the operator on $H$ corresponding to $\alpha$ and orthogonal to $\mathrm{Ker}(\pi) + d\,\mathrm{Ker}(\pi)$ with respect to the scalar product $(\cdot,\cdot)$. From now on we identify the elements $\alpha \in \Omega^{\cdot}_D(A)$ with the operators $\alpha^\perp$ and omit the symbol $\perp$.

Let $\overline{\Omega^{\cdot}_D(A)} \equiv L^2(\Omega^{\cdot}_D(A))$ and $\bar{A} \equiv L^2(A)$ denote the completions of $\Omega^{\cdot}_D(A), A$, respectively, in the norm defined by the scalar product (6). It is easy to show that $\overline{\Omega^{\cdot}_D(A)}$ is a left- and right $\Omega^{\cdot}_D(A)$ module, and $\bar{A}$ is a left- and right $A$ module. We define $P_n$ to be the projection onto the subspace $L^2\left(\bigoplus_{m=0}^{n} \Omega^m_D(A)\right)$ that is orthogonal in the scalar product (6). For $\alpha \in \Omega^{\cdot}_D(A)$, we abbreviate $P_n\alpha$ by $\alpha_n$. [Note that in the classical case, where $A = C^\infty(M)$, $M$ is an even-dimensional, compact Riemannian manifold with spin structure, and $D$ is the usual Dirac operator, an $n$-form corresponds to the operator $\alpha - \alpha_{n-1}$, for some $\alpha \in \Omega^n_D(A)$.]

Given $\alpha$ and $\beta$ in $\Omega^1_D(A)$, we define an element, $\langle\alpha,\beta\rangle_0 \in \bar{A}$ by the equation

$$(c, \langle\alpha,\beta\rangle_0) \equiv \int c\langle\alpha,\beta\rangle_0^* := \int c\beta\alpha^* , \tag{7}$$

for all $c \in A$.

By the Cauchy-Schwarz inequality for $(\cdot,\cdot)$,

$$\begin{aligned} |(c, \langle\alpha,\beta\rangle_0)| &= \left|\int c\beta\alpha^*\right| \leq \sqrt{\int cc^*}\sqrt{\int \alpha\beta^*\beta\alpha^*} \\ &= \text{const}\,\sqrt{(c,c)} , \end{aligned}$$

which shows that $\langle\alpha,\beta\rangle_0$ indeed belongs to $\bar{A}$. The element $\langle\alpha,\beta\rangle_0$ defines an operator on the dense subspace $A \subset \bar{A}$ by the equation

$$(\langle\alpha,\beta\rangle_0 c, d) := \int \alpha\beta^* cd^* .$$

Since

$$\begin{aligned} |(\langle\alpha,\beta\rangle_0, c, d)| &= \left|\int \alpha\beta^* cd^*\right| \\ &\leq \text{const}\sqrt{\int cc^*}\sqrt{\int dd^*} \\ &= \text{const}\,\sqrt{(c,c)}\sqrt{(d,d)} , \end{aligned}$$

$\langle\alpha,\beta\rangle_0$ actually is an operator in $\bar{A}$ which is a bounded operator on the Hilbert space $L^2(A) \equiv \bar{A}$.

It is straightforward to check that the definition of $\langle\cdot,\cdot\rangle_0$ extends to $\overline{\Omega^1_D(A)} \times \overline{\Omega^1_D(A)}$: For $\alpha,\beta$ in $\overline{\Omega^1_D(A)}$, $\langle\alpha,\beta\rangle_0$ defines a unique linear functional on the dense subspace $A \subset \bar{A}$. Next, we note that, for $\alpha,\beta$ in $\overline{\Omega^1_D(A)}$,

$$\begin{aligned} \int c\langle a\alpha, b\beta\rangle_0^* &= \int cb\beta\alpha^* a^* \\ &= \int a^* cb\beta\alpha^* \\ &= \int a^* cb\langle\alpha,\beta\rangle_0^* \\ &= \int cb\langle\alpha,\beta\rangle_0^* a^* \\ &= \int c(a\langle\alpha,\beta\rangle_0 b^*)^* , \end{aligned}$$

for arbitrary $a$, $b$ and $c$ in $A$. Hence

$$\langle a\alpha, b\beta\rangle_0 = a\langle\alpha,\beta\rangle_0 b^* . \tag{8}$$

Furthermore, for every $\alpha \in \overline{\Omega^1_D(A)}$, $\langle\alpha,\alpha\rangle_0$ defines a positive-semidefinite quadratic form on the dense subspace $A \subset \bar{A}$. Finally if for some $\alpha \in \overline{\Omega^1_D(A)}$,

$$\langle\alpha,\beta\rangle_0 = 0 , \quad \text{for all } \beta \in \overline{\Omega^1_D(A)} ,$$

then $(1, \langle\alpha, \alpha\rangle_0 1) = \int \alpha\alpha^* = 0$, and hence

$$|(\alpha, \gamma)| = \left|\int \alpha\gamma^*\right| \leq \sqrt{\int \alpha\alpha^*}\sqrt{\int \gamma\gamma^*} = 0\,,$$

for all $\gamma \in \Omega_D^{\cdot}(A)$. Thus $\alpha = 0$, as an element of $\overline{\Omega_D^1(A)}$.

We conclude that $\langle\cdot, \cdot\rangle_0$ defines a hermitian inner product on the left $A$ module $\overline{\Omega_D^1(A)}$ satisfying (i)–(iii), above; i.e., $\langle\cdot, \cdot\rangle_0$ defines a Riemannian metric on $\overline{\Omega_D^1(A)}$. Since $\bar{A}$ contains $A$, it may happen that $\langle\alpha, \beta\rangle_0$ belongs to $A$, for arbitrary $\alpha$ and $\beta$ in $\Omega_D^1(A)$. In this case, $\langle\cdot, \cdot\rangle_0$ defines a Riemannian metric on $\Omega_D^1(A)$. In general, we say that $\langle\cdot, \cdot\rangle_0$ defines a generalized Riemannian metric on $\Omega_D^1(A)$. We wish to thank A. Connes for having suggested to us this construction of the metric $\langle\cdot, \cdot\rangle_0$.

It is straightforward to see that the arguments described above can be generalized to construct Riemannian metrics, $\langle\cdot, \cdot\rangle_0$, on the spaces $\overline{\Omega_D^n(A)}$, for all $n = 1, 2, 3, \ldots$, which are uniquely determined by the $K$-cycle $(H, D)$ over $A$.

Next, we consider some connection $\nabla$ on $\Omega_D^1(A)$. We say that $\nabla$ is *unitary* with respect to the generalized Riemannian metric $\langle\cdot, \cdot\rangle_0$ on $\Omega_D^1(A)$ iff, for arbitrary $\alpha$, $\beta$ and $\gamma$ in $\Omega_D^1(A)$,

$$\begin{aligned}(\gamma, d\langle\alpha, \beta\rangle_0)\colon &= ([D, \gamma], \langle\alpha, \beta\rangle_0)\\ &= (P_0[D, \gamma], \langle\alpha, \beta\rangle_0)\\ &= (\gamma, \langle\nabla\alpha, \beta\rangle_0) - (\gamma, \langle\alpha, \nabla\beta\rangle_0)\,,\end{aligned} \tag{9}$$

where

$$(\gamma, \langle\nabla\alpha, \beta\rangle_0)\colon = \int \gamma\beta(\nabla\alpha)^*\,, \quad \text{etc.}$$

We define the *torsion* of a connection $\nabla$ on $\Omega_D^1(A)$ by

$$T(\nabla) = d - m \circ \nabla\,. \tag{10}$$

It is an $A$ linear operator from $\Omega_D^1(A)$ to $\Omega_D^2(A)$. The connections of interest in Riemannian geometry are those with vanishing torsion. Among such connections we should like to find ones that can be interpreted as natural generalizations of *Levi-Civita connections*. A connection $\nabla$ on $\Omega_D^1(A)$ is a Levi-Civita connection iff $T(\nabla) = 0$ and $\nabla$ is unitary with respect to the Riemannian metric defined on $\Omega_D^1(A)$.

It is straightforward to derive *Cartan structure equations* in this context. Suppose that $\Omega_D^1$ is a trivial vector bundle, i.e. a free, finitely generated $A$ module, with Riemannian metric. (The following analysis could be generalized to situations where $\Omega_D^1$ is a non-trivial vector bundle by introducing a suitable family of subspaces of $H$ invariant under $\pi(\Omega_D^1)$ with the property that the restriction of $\pi(\Omega_D^1)$ to every subspace in this family is trivial.) Let $E^A$, $A = 1, \ldots, N$, be a basis of $\Omega_D^1$ which is orthonormal in the metric on $\Omega_D^1$. We define $\Omega^A{}_B \in \Omega_D^1$ by

$$\nabla E^A = -\Omega^A{}_B \otimes E^B\,. \tag{11}$$

The components of torsion and curvature are defined by

$$\begin{aligned}T(\nabla)E^A &= T^A\,,\\ R(\nabla)E^A &= R^A{}_B \otimes E^B\,.\end{aligned} \tag{12}$$

The Cartan structure equations follow by inserting the definitions of $T(\nabla)$ and $R(\nabla)$:

$$\begin{aligned} T^A &= dE^A + \Omega^A{}_B E^B , \\ R^A{}_B &= d\Omega^A{}_B + \Omega^A{}_C \Omega^C{}_B . \end{aligned} \tag{13}$$

Next, we determine the transformation properties of the components, $\Omega^A{}_B, T^A$ and $R^A{}_B$, of a connection $\nabla$, its torsion $T(\nabla)$ and its curvature $R(\nabla)$, respectively, under a change of the orthonormal basis of $\Omega^1_D(A)$. We recall that $\Omega^1_D(A)$ is a left $A$ module of some dimension $N$, and $\{E^B\}_{B=1}^N$ is a basis of $\Omega^1_D(A)$ which is orthonormal with respect to a (generalized) Riemannian metric $\langle \cdot, \cdot \rangle$ on $\Omega^1_D(A)$. We introduce a new basis, $\{\tilde{E}^B\}_{B=1}^N$, of $\Omega^1_D(A)$ by setting

$$\tilde{E}^B = M^B{}_C E^C , \tag{14}$$

where $M = (M^B{}_C)$ is an $N \times N$ matrix with matrix elements $M^B{}_C \in A$. Requiring that $\{\tilde{E}^B\}_{B=1}^N$ be again orthonormal with respect to $\langle \cdot, \cdot \rangle$ implies that

$$\begin{aligned} \delta^{BC} = \langle \tilde{E}^B, \tilde{E}^C \rangle &= M^B{}_D \langle E^D, E^K \rangle \, (M^C{}_K)^* \\ &= M^B{}_D \delta^{DK} (M^C{}_K)^* = M^B{}_D (M^{CD})^* , \end{aligned} \tag{15}$$

where we have used property (i). [Note that, since $\delta^{BC} = \delta^B_C = \delta_{BC} = 1$ if $B = C$, $= 0$ if $B \neq C$, $M^{BC} = M^B{}_C = M_B{}^C$.] It follows from (15) that $M$ is a unitary $N \times N$ matrix with matrix elements in $A$, i.e., $M \in U_N(A)$. In real Riemannian geometry we assume that $A$ is an algebra over $\mathbb{R}$. Then $a^*$ is the same as the transposed of $a \in A$, and it would be more natural to call $N \times N$ matrices satisfying (15) "orthogonal" and replace $U_N(A)$ by $O_N(A)$.

In order to determine the transformation properties of the components $\Omega^A{}_B$ of $\nabla$ under the change of basis given in (14), we use their definition, Eq. (11): Then

$$\begin{aligned} \nabla \tilde{E}^B &= \nabla (M^B{}_C E^C) = dM^B{}_C E^C + M^B{}_C \nabla E^C \\ &= dM^B{}_C E^C - M^B{}_C \Omega^C{}_K E^K . \end{aligned} \tag{16}$$

But we also have that

$$\nabla \tilde{E}^B = -\tilde{\Omega}^B{}_C \tilde{E}^C = -\tilde{\Omega}^B{}_C M^C{}_K E^K . \tag{17}$$

Comparing (16) and (17), for arbitrary $B = 1, \ldots, N$, and using (15), it follows that

$$\tilde{\Omega}^B{}_C = M^B{}_D \Omega^D{}_K (M_C{}^K)^* - dM^B{}_K (M_C{}^K)^* . \tag{18}$$

It then follows from the first equation in (12) that

$$\tilde{T}^B = M^B{}_C T^C . \tag{19}$$

Using the second equation in (12), Eqs. (18) and (15) and the identity

$$dM^B{}_K (M_C{}^K)^* = -M^B{}_K d(M_C{}^K)^*$$

which follows from (15), one verifies by a direct calculation that the components of curvature transform as follows:

$$\tilde{R}^B{}_C = M^B{}_D R^D{}_K (M_C{}^K)^* . \tag{20}$$

These transformation properties enable us to define an analogue, $r(\nabla)$, of *scalar curvature*:

$$r(\nabla) := P_0(E^*_B R^B{}_C E^C) . \tag{21}$$

It follows from Eqs. (14), (20), and (15) that $r(\nabla)$ is independent of our choice of the orthonormal basis $\{E^B\}_{B=1}^N$ of $\Omega_D^1(A)$. [At this point, our assumption that $\Omega_D^1(A)$ be a trivial bundle is likely to become superfluous.] Note that, by the definition of $P_0, r(\nabla)$ belongs to $\bar{A}$.

If the metric on $\Omega_D^1(A)$ is the generalized Riemannian metric $\langle \cdot, \cdot \rangle_0$ determined by the $K$-cycle $(H, D)$ over $A$ we may define an analogue of the *Einstein-Hilbert action* by setting

$$\begin{aligned} I(\nabla) := \int r(\nabla) &= \int E_B^* R^B{}_C E^C \\ &= (R^B{}_C E^C, E_B) . \end{aligned} \tag{22}$$

If $\nabla$ is chosen to be a *Levi-Civita connection* associated with $(H, D)$, i.e., $T(\nabla) = 0$, and $\nabla$ is unitary with respect to $\langle \cdot, \cdot \rangle_0$, then the action functional $I = I(\nabla)$ defines a functional on a "space of $K$-cycles over $A$."

## 3. The Example of a Two-Sheeted Space

We now introduce a class of algebras and of $K$-cycles for which the Riemannian geometry concepts introduced above are well defined.

Let $X$ be a compact even dimensional $C^\infty$ spin manifold, with a reference Riemannian metric $g_0$ and fixed spin structure, $A$ the algebra of smooth real functions on $X$ and Cliff$(T^*X)$ the Clifford bundle over $X$, whose fiber at $x$ is the (real) Clifford algebra of the cotangent space Cliff$(T_x^*X)$ associated to $g_0(x)$. Let $S$ be the spinor bundle. Thus $S$ is a $\mathbb{Z}_2$ graded complex vector bundle over $X$, with a representation of the Clifford algebra of the cotangent space on each fiber $S_x$, such that $\mathrm{End}_{\mathbb{C}}(S_x) \simeq \mathrm{Cliff}(T_x^*X) \otimes \mathbb{C}$. A section of $\mathrm{End}(S) \simeq \mathrm{Cliff}(T^*X) \otimes \mathbb{C}$ is called real if it takes values in the real Clifford algebra. We consider $K$-cycles $(H, D)$ where:
(a) $D$ is an odd first order elliptic differential operator on the space $C^\infty(S)$ of smooth sections of $S$.
(b) For each $f \in A$, $[D, f]$ is a real section of End$(S)$.
(c) $H = L^2(S, \varrho d^d x)$ is the space of square integrable sections of $S$, where $\varrho(y)$ is a density for which $D$ is self-adjoint.

We will also need the following variant with group action: Let $X, A$, Cliff$(T^*X)$ and $S$ be as above, and suppose that $X$ is a finite smooth covering of a manifold $Y$. That is, $p: X \to Y$ is a principal $G$ bundle with base space $Y = X/G$, and $G$ is a finite group. The reference metric will be chosen to be preserved by the group action, and we assume that the group action lifts to $S$. Denote by $p_*S$ the vector bundle over $Y$ whose fiber over $y$ is the direct sum $\bigoplus_{p(x)=y} S_x$. Both $A$ and the group $G$ act on the sections of $p_*S$. A linear operator on the space of smooth sections of $p_*S$ is called equivariant if it commutes with the action of $G$. The vector space $\mathrm{End}_{\mathbb{C}}(p_*S_y)$ is the space of matrices indexed by $p^{-1}(y)$ with entries in $\mathrm{Cliff}(T_y^*Y) \otimes \mathbb{C}$. A vector in $\mathrm{End}_{\mathbb{C}}(p_*S_y)$ is called real if its matrix entries belong to the real Clifford algebra, and a section of End$(p_*S)$ is called real if it takes real values.

In this setting, we consider $K$-cycles $(H, D)$ where:
(a′) $D$ is an odd equivariant first order elliptic differential operator on the space $C^\infty(p_*S)$ of smooth sections of $p_*S$.
(b′) For each $f \in A$, $[D, f]$ is multiplication by a real section of End$(p_*S)$.

(c′) $H = L^2(p_*S, \varrho d^d y)$ is the space of square integrable sections of $p_*S$, where $\varrho(y)$ is a density for which $D$ is selfadjoint.

These data define a Riemannian geometry on the graded differential algebra $\Omega_D^{\cdot}(A)$. The Riemannian metric is defined to be

$$G(\alpha, \beta) = \mathrm{tr}(\pi(\alpha)\pi(\beta^*)), \qquad \alpha, \beta \in \Omega_D^1(A). \tag{23}$$

This is independent of the choice of repesentatives $\alpha, \beta$ since $d\,\mathrm{Ker}(\pi) \cap \Omega^1(A) = 0$ and therefore $\Omega_D^1(A)$ is isomorphic to $\pi(\Omega^1(A))$. The trace over the Clifford algebra is defined fiberwise. We normalize it in such a way that the trace of the identity is one. It is not hard to check that the metric $G(\cdot, \cdot)$ coincides with the metric $\langle \cdot, \cdot \rangle_0$ determined by the $K$-cycle $(H, D)$, as constructed above.

## 4. The Gravity Action for a Two-Sheeted Space

Let us apply the formalism introduced in the previous section to an example. As in [1–3], we take $X$ to be two copies of a compact, say four-dimensional, spin manifold $Y$:

$$X = Y \times \mathbb{Z}_2,$$

and we have the trivial $\mathbb{Z}_2$ bundle $p: X \to Y$. The algebra $A$ is then $C^\infty_{\mathbb{R}}(Y) \oplus C^\infty_{\mathbb{R}}(Y)$. It is convenient to think of $A$ as a subalgebra of diagonal matrices in the algebra $M_2(\mathbb{C}) \otimes C^\infty(\mathrm{Cliff}(T^*Y))$ of two by two matrices whose entries are smooth sections of the Clifford bundle. The chirality operator $\gamma^5$ belongs to the real Clifford algebra and defines a $\mathbb{Z}_2$ grading of the spinor bundle $S$. The operator

$$\Gamma = \begin{pmatrix} \gamma^5 & 0 \\ 0 & -\gamma^5 \end{pmatrix}$$

defines a $\mathbb{Z}_2$ grading of $C^\infty(p_*S) = C^\infty(S) \oplus C^\infty(S)$ (the minus sign is a matter of convention).

We work in local coordinates. Let us introduce gamma matrices $\gamma^a$ with $(\gamma^a)^* = -\gamma^a$, $a = 1, \ldots, 4$, obeying the relations $\gamma^a\gamma^b + \gamma^b\gamma^a = -2\delta^{ab}$. Then $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ is self-adjoint and has square one. We set $\gamma^{ab} = \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a) = -(\gamma^{ba})^*$.

The Dirac operator can then be represented as a two by two matrix $(D_{ij})$, $i, j \in \{+, -\}$, whose entries are first order differential operators acting on spinors of $Y$. What are the restrictions on these entries imposed by (a′)–(c′)? First of all, $\mathbb{Z}_2$ equivariance implies that $D_{+-} = D_{-+}$ and $D_{++} = D_{--}$, and the fact that $[D, f]$ is a multiplication operator implies that $D_{+-}$ should be a multiplication operator. The most general form of $D$, compatible with self-adjointness, reality and oddness is then

$$D = \begin{pmatrix} \gamma^a \varepsilon_a^\mu \partial_\mu + \ldots & \psi + \gamma^5\phi \\ \psi + \gamma^5\phi & \gamma^a \varepsilon_a^\mu \partial_\mu + \ldots \end{pmatrix},$$

where $\varepsilon_a^\mu$, $\psi$, and $\phi$ are real functions. Since $D$ is elliptic, $\varepsilon_a^\mu \partial_\mu$ is a basis of the tangent space, and we can define a Riemannian metric $g$ on $Y$ by $g(\varepsilon_a, \varepsilon_b) = \delta_{ab}$. The dots in the definition of $D$ indicate zero order contributions which do not contribute to $\pi$.

The representation $\pi$ on one-forms can now be computed. Let $\alpha = \Sigma_i a_i db_i \in \Omega^1(A)$ be a representative of a one-form in $\Omega_D^1(A)$. Then $\pi(\alpha)$ is parametrized by two classical one-forms $\alpha_{1\mu}, \alpha_{2\mu}$, and two functions $\alpha_5, \tilde{\alpha}_5$, on $Y$:

$$\pi(\alpha) = \begin{pmatrix} \gamma^\mu \alpha_{1\mu} & \bar{\gamma}\alpha_5 \\ -\bar{\gamma}\tilde{\alpha}_5 & \gamma^\mu \alpha_{2\mu} \end{pmatrix}.$$

We use the notation $\gamma^\mu = \gamma^a \varepsilon_a^\mu$, $\bar{\gamma} = \psi + \gamma^5 \phi$. In terms of the variables $a_i = a_{i1} \oplus a_{i2}$ and $b_i = b_{i1} \oplus b_{i2}$, we have

$$\begin{aligned} \alpha_{1\mu} &= \sum_i a_{i1} \partial_\mu b_{i1}\,, \\ \alpha_{2\mu} &= \sum_i a_{i2} \partial_\mu b_{i2}\,, \\ \alpha_5 &= \sum_i a_{i1}(b_{i2} - b_{i1})\,, \\ \tilde{\alpha}_5 &= \sum_i a_{i2}(b_{i2} - b_{i1})\,. \end{aligned}$$

The Riemannian metric $G: \Omega_D^1(A) \otimes \Omega_D^1(A) \to A$ can be expressed, using the isomorphism $\Omega_D^1(A) = \pi(\Omega^1(A))$, in terms of components:

$$G(\alpha, \beta) = (g^{\mu\nu}\alpha_{1\mu}\beta_{1\nu} + g^{55}\tilde{\alpha}_5\tilde{\beta}_5) \oplus (g^{\mu\nu}\alpha_{2\mu}\beta_{2\nu} + g^{55}\alpha_5\beta_5)\,,$$

where $g^{\mu\nu} = -\operatorname{tr}(\gamma^\mu\gamma^\nu) = \varepsilon_a^\mu \varepsilon_a^\nu$ and $g^{55} = \operatorname{tr}\bar{\gamma}^2 = \psi^2 + \phi^2$.

To compute torsion and curvature, we must understand two-forms, $\Omega_D^2(A)$. This space is isomorphic to the quotient of $\pi(\Omega^2(A))$ by the space of "auxiliary fields" $\pi(d\operatorname{Ker}(\pi|_{\Omega^1(A)}))$. We proceed to compute the general form of auxiliary fields. If $\alpha = \Sigma_i a_i db_i \in \operatorname{Ker}(\pi)$, we obtain for $\pi(d\alpha) = \Sigma_i [D, a_i]\,[D, b_i]$,

$$\pi(d\alpha) = \begin{pmatrix} -g^{\mu\nu}\partial_\mu a_{i1}\partial_\nu b_{i1} & -2\psi\gamma^\mu a_{i1}\partial_\mu b_{i2} \\ -2\psi\gamma^\mu a_{i2}\partial_\mu b_{i1} & -g^{\mu\nu}\partial_\mu a_{i2}\partial_\nu b_{i2} \end{pmatrix},$$

and it is not difficult to see that, for a suitable choice of $a_i, b_i$ subject to the constraint $\pi(\alpha) = 0$, any expression of the form

$$\begin{pmatrix} X_1 & \psi\gamma^\mu Y_\mu \\ \psi\gamma^\mu \tilde{Y}_\mu & X_2 \end{pmatrix}$$

can be obtained.

Next, we express $\pi(d\alpha)$ modulo auxiliary fields, for any one-form $\alpha$, in terms of its components:

$$\pi(d\alpha) = \begin{pmatrix} \gamma^{\mu\nu}\partial_\mu\alpha_{1\nu} + 2\phi\psi\gamma^5(\alpha_5 - \tilde{\alpha}_5) & \phi\gamma^\mu\gamma^5(\partial_\mu\alpha_5 + \alpha_{1\mu} - \alpha_{2\mu}) \\ -\phi\gamma^\mu\gamma^5(\partial_\mu\tilde{\alpha}_5 + \alpha_{1\mu} - \alpha_{2\mu}) & \gamma^{\mu\nu}\partial_\mu\alpha_{2\nu} + 2\phi\psi\gamma^5(\alpha_5 - \tilde{\alpha}_5) \end{pmatrix}.$$

This choice of representative in the class of $\pi(d\alpha)$ in $\pi(\Omega^2(A))/\pi(d\operatorname{Ker}(\pi|_{\Omega^1(A)}))$ is uniquely determined by the property to be *orthogonal* to all auxiliary fields, with respect to the inner product on $\Omega^2(A)$ defined by the Dixmier trace:

$$(\alpha, \beta) = \int \alpha\beta^* = \operatorname{Tr}_\omega(\pi(\alpha)\pi(\beta)^*|D|^{-4})\,.$$

For explicit calculations it is convenient to introduce local orthonormal bases $\{E^A\}$ of $\Omega^1_D(A)$. We use the following convention for indices: capital letters $A, B, \ldots$ denote indices taking the values 1 to 5, and lower case letters $a, b, \ldots$ take values from 1 to 4. Introduce a local orthonormal frame of one-forms $e^a_\mu dy^\mu$ on $Y$. The basis is

$$E^a = \begin{pmatrix} \gamma^a & 0 \\ 0 & \gamma^a \end{pmatrix} = \begin{pmatrix} \gamma^\mu e^a_\mu & 0 \\ 0 & \gamma^\mu e^a_\mu \end{pmatrix},$$
$$E^5 = \begin{pmatrix} 0 & \bar{\gamma}\lambda \\ -\bar{\gamma}\lambda & 0 \end{pmatrix}, \qquad \lambda = (\phi^2 + \psi^2)^{-1/2}.$$

Suppose now that the connection $\nabla$ is unitary with respect to the given $K$-cycle. The components of the one-form corresponding to $\pi(\nabla)$ are denoted by

$$\Omega^{AB} = \begin{pmatrix} \gamma^\mu \omega^{AB}_{1\mu} & \bar{\gamma} l^{AB} \\ -\bar{\gamma}\tilde{l}^{AB} & \gamma^\mu \omega^{AB}_{2\mu} \end{pmatrix}.$$

The unitarity condition $(\Omega^{AB})^* = \Omega^{BA}$ implies the component relations

$$\omega^{AB}_{1\mu} = -\omega^{BA}_{1\mu},$$
$$\omega^{AB}_{2\mu} = -\omega^{BA}_{2\mu},$$
$$\tilde{l}^{AB} = -\tilde{l}^{BA}.$$

The components of torsion and curvature are readily computed. As above, we give the representative in $\Omega^2_D(A)$ orthogonal to auxiliary fields. For the torsion we find

$$T^a = \begin{pmatrix} \gamma^{\mu\nu}(\partial_\mu e^a_\nu + \omega^{ab}_{1\mu} e^b_\nu) - 2\phi\psi\lambda\gamma^5 l^{a5} & -\phi\gamma^\mu\gamma^5(l^{ab}e^b_\mu - \lambda\omega^{a5}_{1\mu}) \\ \phi\gamma^\mu\gamma^5(\tilde{l}^{ab}e^b_\mu - \lambda\omega^{a5}_{2\mu}) & \gamma^{\mu\nu}(\partial_\mu e^a_\nu + \omega^{ab}_{2\mu}e^b_\nu) - 2\phi\psi\lambda\gamma^5\tilde{l}^{a5} \end{pmatrix},$$

$$T^5 = \begin{pmatrix} \gamma^{\mu\nu}\omega^{5b}_{1\mu}e^b_\nu - 2\phi\psi\lambda\gamma^5 l^{55} & \phi\gamma^\mu\gamma^5(\partial_\mu\lambda - l^{5b}e^b_\mu) \\ -\phi\gamma^\mu\gamma^5(\partial_\mu\lambda - \tilde{l}^{5b}e^b_\mu) & \gamma^{\mu\nu}\omega^{5b}_{2\mu}e^b_\nu + 2\phi\psi\lambda\gamma^5 l^{55} \end{pmatrix}.$$

The expression for the curvature is

$$R^{AB} = \begin{pmatrix} \gamma^{\mu\nu}R^{AB}_{1\mu\nu} + 2\phi\psi\gamma^5 P^{AB}_1 & \phi\gamma^\mu\gamma^5 Q^{AB}_\mu \\ -\phi\gamma^\mu\gamma^5\tilde{Q}^{AB}_\mu & \gamma^{\mu\nu}R^{AB}_{2\mu\nu} + 2\phi\psi\gamma^5 P^{AB}_2 \end{pmatrix},$$

where

$$R^{AB}_{i\mu\nu} = \partial_\mu\omega^{AB}_{i\nu} - \partial_\nu\omega^{AB}_{i\mu} + \omega^{AC}_{i\mu}\omega^{CB}_{i\nu} - \omega^{AC}_{i\nu}\omega^{CB}_{i\mu}, \qquad i = 1, 2,$$
$$Q^{AB}_\mu = \partial_\mu l^{AB} + \omega^{AB}_{1\mu} - \omega^{AB}_{2\mu} + \omega^{AC}_{1\mu}l^{CB} - \omega^{CB}_{2\mu}l^{AC},$$
$$\tilde{Q}^{AB}_\mu = -\partial_\mu l^{BA} + \omega^{AB}_{1\mu} - \omega^{AB}_{2\mu} + \omega^{CB}_{1\mu}l^{CA} - \omega^{AC}_{2\mu}l^{BC},$$
$$P^{AB}_1 = l^{AB} + l^{BA} + l^{AC}l^{BC},$$
$$P^{AB}_2 = l^{AB} + l^{BA} + l^{CA}l^{CB}.$$

As a gravity action we choose the generalized Einstein-Hilbert action, given in terms of the inner product $(\alpha, \beta) = \mathrm{Tr}_\omega(\pi(\alpha)\pi(\beta)^*|D|^{-4})$ on two-forms defined through the identification of $\Omega^2_D(A)$ with $\pi(\Omega^2(A)) \cap \pi(d\,\mathrm{Ker}(\pi|_{\Omega^1(A)})^\perp)$:

$$I = (R^B{}_C E^C, E_B);$$

see Eq. (22).

This action reduces to (and could be alternatively defined as) the integral over $Y$

$$I = \int_Y \mathrm{tr}((E_A)^* R^A{}_B E^B) .$$

[Here the trace is over End($p_* S_y$)]. Inserting the above expressions for $E^A$ and $R^A{}_B$ yields the action as a function of the component fields. Set $U^a_\mu = Q^{a5}_\mu + \tilde{Q}^{a5}_\mu - Q^{5a}_\mu - \tilde{Q}^{5a}_\mu$. The result is

$$I = \int_Y [\varepsilon^\mu_a \varepsilon^\nu_b (R^{ab}_{1\mu\nu} + R^{ab}_{2\mu\nu}) + \lambda\phi^2 \varepsilon^\mu_a U^a_\mu - 4\phi^2\psi^2\lambda^2(P^{55}_1 + P^{55}_2)]\sqrt{g} d^4y .$$

At this point two possibilities are open. One can either take the action $I$ as a starting point, with all fields independent, and eliminate non-dynamical fields by their equations of motion. Or one can impose the torsion constraint, and derive an action for Levi-Civita connections. We will follow the second approach.

It turns out that, in general, one gets an uninteresting model, describing just two decoupled universes. A more interesting example is obtained by imposing the additional condition $\psi = 0$. In other words, we consider only Dirac operators of the form

$$D = \begin{pmatrix} \gamma^a e^\mu_a \partial_\mu + \dots & \gamma^5 \phi(x) \\ \gamma^5 \phi(x) & \gamma^a e^\mu_a \partial_\mu + \dots \end{pmatrix},$$

which is in fact closer to the form of Dirac operators used in particle models [1–6].

The zero torsion condition has the following consequences for the components:
1. $\omega^{ab}_\mu \equiv \omega^{ab}_{1\mu} = \omega^{ab}_{2\mu}$ is the one-form corresponding to the classical Levi-Civita connection of the metric $g_{\mu\nu} = e^a_\mu e^a_\nu$. It is the unique solution of $de^a + \omega^{ab} \wedge e^b = 0$, $\omega^{ab} = -\omega^{ba}$.
2. $l^{ab} = l^{ba}$, $l^{5a} = -l^{a5}$.
3. $\omega^{a5}_{1\mu} = -\omega^{a5}_{2\mu} = \lambda^{-1} l^{ab} e^b_\mu$.
4. $\partial_\mu \lambda = e^a_\mu l^{5a}$.

It is interesting to notice that the zero torsion constraint selects $\mathbb{Z}_2$-equivariant connections. In other words, let $\theta: A \to A$ be the involution $\theta(a_1 \oplus a_2) = a_2 \oplus a_1$. Extend it, using the equivariance of $D$, to the unique involutive automorphism of $\Omega^\cdot_D(A)$ such that $d\theta = \theta d$. Then Levi-Civita connections have the property $\theta \otimes \theta \nabla = \nabla \theta$.

The resulting gravity action is then

$$I = \int_Y [2R - \lambda^{-1} 4 \nabla_\mu \partial^\mu \lambda + 4\lambda^{-2} l^{aa} l^{55} + \lambda^{-2}(l^{aa} l^{bb} - l^{ab} l^{ab})]\sqrt{g} d^4y .$$

The fields $l^{ab}, l^{55}$ decouple, and with the substitution $\lambda = \exp(\sigma)$, we finally obtain the action of a massless scalar coupled to the gravitational field:

$$I = 2 \int_Y [R - 2\partial_\mu \sigma \partial^\mu \sigma]\sqrt{g} d^4x .$$

To understand the role of the field $\sigma$ we can study the coupling of gravity to the Yang-Mills sector. In particular, in the example of the standard model in [3] we see that $g^{\mu\nu}$ is the metric of the Riemannian manifold while $\phi = e^{-\sigma}$ replaces the electroweak scale $\mu$. In other words, the vacuum expectation value of the field $\phi$ determines the electroweak scale, thus forming a connection between gravity and the standard model. From the form of the gravity action, it is clear that the field $\sigma$ has no potential. The only other term we could have added is a cosmological constant

$$\Lambda \int 1,$$

and this is $\sigma$-independent. This implies that at the classical level the vacuum expectation value of $\phi$ is undetermined. It is conceivable that the gravity action acquires a Coleman-Weinberg potential through quantum effects. However, at present this is beyond our capabilities, since the problem of quantization in non-commutative geometry has not as yet been dealt with.

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## References

1. Connes, A: In: The interface of mathematics and particle physics. Quillen, D., Segal, G., Tsou, S. (eds.). Oxford: Clarendon 1990
2. Connes, A., Lott, J.: Nucl. Phys. B Proc. Supp. **18**B, 29 (1990), North-Holland, Amsterdam
3. Connes, A., Lott, J.: The metric aspect of noncommutative geometry, to appear in Proceedings of the 1991 Cargèse summer school. See also Connes, A.: Non-Commutative Geometry. Expanded English translation (to appear) of Géométrie Non Commutative. Paris: Inter Editions 1990
4. Kastler, D.: Marseille preprints
5. Coquereaux, R., Esposito-Farèse, G., Vaillant, G.: Nucl. Phys. B **353**, 689 (1991); Dubois-Violette, M., Kerner, R., Madore, J.: J. Math. Phys. **31**, 316 (1990); Balakrishna, B., Gürsey, F., Wali, K.C.: Phys. Lett. **254**B, 430 (1991)
6. Chamseddine, A.H., Felder, G., Fröhlich, J.: Grand unification in non-commutative geometry. Zürich, preprint 1992

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**Note added in proof.** Since the time this paper was accepted for publication, we have found general, manifestly "coordinate-independent" definitions of analogues of Ricci curvature and scalar curvature in non-commutative geometry, *not* assuming that $\Omega^1_D(A)$ is a free left $A$ module. Furthermore, we have found that the Yang-Mills and fermionic sectors induce a Coleman-Weinberg potential for the field $\sigma$, as conjectured above. Finally, we have analyzed torsion-free connections in more detail and found an alternative definition of the generalized Einstein-Hilbert action. These results will appear in forthcoming publications.