Commun. Math. Phys. 155, 93–101 (1993)

Communications in
Mathematical
Physics

The Topological Structure of the Unitary and Automorphism Groups of a Factor*

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Received June 18, 1992; in revised form September 3, 1992

Dedicated to Huzihiro Araki

Abstract. It is proved that a large class of II_1 factors have unitary group which is contractible in the strong operator topology, but whose fundamental group in the norm topology is isomorphic to the additive real numbers as proven by Araki-Smith-Smith [1]. The class includes the approximately finite dimensional factor of type II_1 and the group factor associated with the free group on infinitely many generators. This contractibility is used to prove the contractibility of the automorphism group of the approximately finite dimensional factor of type II_1 and type II_∞. It is further shown that the fundamental group of the automorphism group of the approximately finite dimensional factor of type III_λ, $0 < \lambda < 1$, is isomorphic to the integer group $\mathbb{Z}$.

Introduction

While the basic topological properties of the unitary group $U(n)$ of a finite dimensional Hilbert space, its homotopy groups etc., are well known for quite some time, the first results concerning the quantum theoretical setting of the infinite dimensional Hilbert space appeared only in the 60's and 70's. In 1965 Kuiper proved that in sharp contrast with the finite dimensional case the unitary group of an infinite dimensional Hilbert space, $U(\infty)$, endowed with the operator norm topology, is contractible [11]. This result was later extended to unitary groups $\mathcal{U}(\mathcal{M})$ of properly infinite von Neumann factors $\mathcal{M}$ by Breuer and Singer [4, 5]. However, for factors of type II_1 the situation is quite different, as H. Araki, L. Smith, and M.-S. B. Smith showed in 1971 that the first homotopy group $\pi_1(\mathcal{U}(\mathcal{M}))$ is then isomorphic to $\mathbb{R}$ and thus nontrivial, [1].

In recent years, the merge of topological methods in operator algebras and noncommutative geometry stimulated much interest in studying further topological properties of the unitary groups of more general operator algebras (with their operator norm topology).

In the case of von Neumann algebras though the natural topologies to consider are the weak and strong operator topologies, which for infinite dimensional $\mathcal{M}$ are

* This research is supported in part by NSF Grant DMS-9206984

strictly weaker than the norm topology. As it turns out, in these topologies $U(\infty)$ is again contractible (cf. [8]) by an argument that goes back to Kakutani's proof in 1943 of the contractibility of the unit sphere of the infinite dimensional Hilbert space [10]. Moreover, the same proof shows that $\mathscr{U}(\mathscr{M})$ is contractible for any properly infinite von Neumann algebra $\mathscr{M}$. But it was left as an open problem as to whether or not the same is true for a factor of type II_1 as well. This problem is more difficult than it appears at first glance and bare handed elementary techniques have failed over the years to provide an answer even in the approximate finite dimensional case. Also, the Araki-Smith-Smith result could not be used.

In this paper we will use the Connes-Takesaki theory to show that if the associated factor $\mathscr{M} \overline{\otimes} \mathscr{L}(\mathfrak{H})$ of type II_∞ with a factor $\mathscr{M}$ of type II_1 admits a one parameter group of automorphisms scaling the trace then $\mathscr{U}(\mathscr{M})$ is indeed contractible. The Connes-Takesaki continuous decomposition of the Araki-Woods factor of type III_1 then shows that the AFD factor of type II_1 $\mathscr{R}_0$ and in fact all factors of type II_1 that split $\mathscr{R}_0$ have contractible unitary group. Also, by the recent result of Radulescu [14], the same follows true for factors that split the factor of type II_1 associated with the free group on infinitely many generators.

Moreover, we will use the density of the inner automorphism group in the group Aut($\mathscr{R}_0$) of all automorphisms of $\mathscr{R}_0$, and one of Michael's selection principles to show that Aut($\mathscr{R}_0$) and Aut($\mathscr{R}_0 \overline{\otimes} \mathscr{L}(\mathfrak{H})$) are contractible, when endowed with the usual pointwise norm convergence topology on the predual. Finally we will deduce that $\pi_1(\mathscr{U}(\mathscr{R}))$ is isomorphic to $\mathbb{Z}$ for AFD factors $\mathscr{R}$ of type III_λ.

Finally, by making use of the Fermion algebra presentation of the AFD factor $\mathscr{R}_0$ of type II_1, we show that Aut($\mathscr{R}_0$) can serve as EG, a contractible topological space on which G acts freely, for any separable locally compact group G, thus we obtain a concrete description of the classifying space BG for principal G-bundles.

We mention that our proofs give a first example of the use of the Connes-Takesaki continuous decomposition of factors of type III_1 to solve a problem for factors of type II_1 which could not be solved by genuine type II_1 techniques. It would be worth noting here that the existence of a one parameter automorphism group of an approximately finite dimensional factor of type II_1 was proved first through the structure analysis of factors of type III_1 as shown in [15, Sect. 11].

Since our method of contracting $\mathscr{U}(\mathscr{M})$ depends on the deformation of the (trace on the) algebra $\mathscr{M}$, it doesn't apply to Connes' rigid factors with property T. In this case the problem may in fact have a negative answer. Or, if it has a positive one, a new technique should be found.

The increasing interest in quantization-deformation of groups and algebras in recent years give strong motivation towards settling this problem, especially for technical reasons: if some deformation can be made for each value of a parameter, then to do it continuously one needs to apply continuous selection principles. These in turn depend on the contractibility of the unitary or automorphism group of the algebras deformed.

Factors of Type II_1

We begin by proving that factors of type II_1 for which the associated factors of type II_∞ admit a one parameter group of trace scaling automorphisms have contractible unitary group in a very "strong" sense:

Theorem 1. *Let $\mathscr{R}$ be a separable factor of type II_1. If the tensor product $\mathscr{R} \overline{\otimes} \mathscr{B}$ of $\mathscr{R}$ and a factor $\mathscr{B}$ of type I_∞ admits a one parameter automorphism group $\{\theta_s : s \in \mathbb{R}\}$*

scaling the trace of $\mathcal{R}\bar{\otimes}\mathcal{B}$, i.e. $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$, *with* τ *a faithful semi-finite normal trace on* $\mathcal{R}\bar{\otimes}\mathcal{B}$, *then the unitary group* $\mathcal{U}(\mathcal{R})$ *of* $\mathcal{R}$ *is contractible with respect to the* σ*-strong operator topology in such a way that there exists a continuous map* α:

$$(t,u) \in [0,\infty) \times \mathcal{U}(\mathcal{R}) \mapsto \alpha_t(u) \in \mathcal{U}(\mathcal{R})$$

with the properties:

i) $\alpha_0(u) = u$, $\lim\limits_{t\to\infty} \alpha_t(u) = 1$;

ii) *each* α_s, $s \geq 0$, *is an injective endomorphism of* $\mathcal{U}(\mathcal{R})$ *into itself;*

iii) $\alpha_s \circ \alpha_t = \alpha_{s+t}$, $s, t \geq 0$.

iv) $\|\alpha_s(u) - \alpha_s(v)\|_2 = e^{-s}\|u - v\|_2$, $u, v \in \mathcal{U}(\mathcal{R})$.

Proof. Let $\bar{\mathcal{R}} = \mathcal{R}\bar{\otimes}\mathcal{B}$ and $\mathcal{M} = \bar{\mathcal{R}} \rtimes_\theta \mathbb{R}$ be the crossed product of $\bar{\mathcal{R}}$ by $\{\theta_s : s \in \mathbb{R}\}$. By the structure theorem for factors of type III [7, 15], $\mathcal{M}$ is a separable factor of type III_1. Let φ be a faithful normal state on $\mathcal{M}$ and $\{\sigma_t^\varphi\}$ be the modular automorphism group of $\mathcal{M}$. Then the Connes-Takesaki duality theorem [7, 15], implies that $\{\bar{\mathcal{R}}, \mathbb{R}, \theta\}$ is conjugate to $\{\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}, \mathbb{R}, \hat{\sigma}^\varphi\}$, where $\hat{\sigma}^\varphi$ means of course the action of $\mathbb{R}$ dual to σ^φ. Furthermore, the modular automorphism group $\{\sigma_t^{\hat{\varphi}}\}$ of the weight $\hat{\varphi}$ on $\hat{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ dual to φ is given by the one parameter unitary group $\{u(t)\}$ associated with the crossed product. Let h be the generator of $\{u(t)\}$ so that $u(t) = e^{ith}$. Then the von Neumann algebra $\mathcal{A}$ generated by $\{u(t)\}$ is isomorphic to $L^\infty(\mathbb{R})$ and h corresponds to the function: $\lambda \in \mathbb{R} \mapsto \lambda \in \mathbb{R}$. The dual action $\{\hat{\sigma}_s^\varphi\}$ corresponds to the translation of $\mathbb{R}$ by $t \in \mathbb{R}$. Since $\hat{\varphi}$ is invariant under $\{\hat{\sigma}_s^\varphi\}$ and semi-finite on $\mathcal{A}$, it is precisely given by the Lebesgue integral on $\mathbb{R}$. The faithful semi-finite normal trace τ on $\hat{\mathcal{M}}(= \bar{\mathcal{R}})$ is then given by $\tau = \hat{\varphi}(e^{-h}\cdot)$, [7, 15], hence it is semi-finite on $\mathcal{A}$. Now, let e_t be the spectral projection of h corresponding to the half line $[t, +\infty)$. Then we have

$$\tau(e_0) = \int_0^\infty e^{-\lambda} d\lambda = 1 < +\infty\,,$$

and

$$\hat{\sigma}_s^\varphi(e_0) = e_s\,, \qquad s \in \mathbb{R}\,.$$

Therefore, we obtain

$$\hat{\sigma}_s^\varphi(e_0) = e_s \leq e_0\,, \qquad s \geq 0\,.$$

As $\hat{\mathcal{M}}_{e_0} \cong \mathcal{R}$, we conclude that $\mathcal{R}$ admits a one parameter semi-group $\{\theta_s : s \geq 0\}$ of endomorphisms such that $\tau(\theta_s(1)) = e^{-s}$, $s \geq 0$, where τ is the canonical trace on $\mathcal{R}$. We now simply set

$$\alpha_s(u) = 1 - \theta_s(1) + \theta_s(u)\,, \qquad u \in \mathcal{U}(\mathcal{R})\,.$$

As $\lim\limits_{s\to+\infty} \theta_s(1) = 0$ σ-strongly, we complete the proof. Q.E.D.

If $\mathcal{R}_0$ is an AFD factor of type II_1, then $\mathcal{R}_0\bar{\otimes}\mathcal{B}$ appears as the core of an AFD factor of type III_1, which is in turn isomorphic to the factor of type III_1 constructed by Araki and Woods [2], so that the next result immediately follows from the arguments in [15, Sect. 11]:

Corollary 2. *The unitary group* $\mathcal{U}(\mathcal{R})$ *of a factor* $\mathcal{R}$ *of type* II_1 *admits a contraction path* $\{\alpha_s\}$ *satisfying* (i)–(iv) *of Theorem if the factor* $\mathcal{R}$ *is either:* a) *approximately* *finite dimensional;* b) *strongly stable in the sense that $\mathscr{R}$ is isomorphic to the tensor product of itself with the* ADF *factor $\mathscr{R}_0$ of type II_1;* c) *isomorphic to the factor $\mathscr{R}(F_\infty)$ associated with the free group F_∞ of infinite generators;* d) *or isomorphic to the tensor product of $\mathscr{R}(F_\infty)$ and any other factor $\mathscr{R}_1$.*

The strong stability of a factor in (b) is also called the McDuff property of a factor, cf. [12]. The cases (c) and (d) follow from the recent result of Radulescu [14].

The following result is a straightforward adaptation of an old result of Michael [13]:

Lemma 3. *Let $\mathscr{R} \subset \mathscr{P}$ be a pair of factors. If $\mathscr{R}$ has the property described in Theorem 1, then the quotient homogeneous space $\mathscr{U}(\mathscr{P})/\mathscr{U}(\mathscr{R})$ admits a continuous cross-section.* [1]

Proof. The homogeneous space $X = \mathscr{U}(\mathscr{P})/\mathscr{U}(\mathscr{R})$ is a complete metric space relative to the metric inherited from the L^2-metric on the unitary group of $\mathscr{U}(\mathscr{P})$. In view of the theory of continuous cross-sections due to Michael, [13], we have only to check that the fibre $\mathscr{U}(\mathscr{R})$ admits a geodesic structure. With $\{\alpha_t\}$ as in Theorem 1, we set

$$M = \{(u,v) \in \mathscr{U}(P) \times \mathscr{U}(P) : uv^* \in \mathscr{U}(\mathscr{R})\}\,;$$
$$k(u,v,t) = \alpha_{-\log(1-t)}(uv^*)v\,, \qquad (u,v) \in M\,.$$

It is straightforward to check that $\{M, k\}$ satisfies the requirements in [13, Definition 5.1]. Thus, our assertion follows:

Theorem 4. *The automorphism group* $\mathrm{Aut}(\mathscr{R}_0)$ *of an* AFD *factor $\mathscr{R}_0$ of type II_1 is contractible.*

Proof. First, let M be a type I subfactor of $\mathscr{R}_0$, and let $\mathrm{Iso}(M, \mathscr{R}_0)$ be the set of all injections of M into $\mathscr{R}_0$ equipped with the pointwise σ-strong convergence topology. It follows that $\mathrm{Iso}(M, \mathscr{R}_0)$ is naturally identified with the homogeneous space $\mathscr{U}(\mathscr{R}_0)/\mathscr{U}(M^c)$ where $M^c = M' \cap \mathscr{R}_0$. Since $M^c \cong \mathscr{R}_0$, $\mathrm{Iso}(M, \mathscr{R}_0)$ admits a cross-section, say $u(\alpha) \in \mathscr{U}(\mathscr{R}_0)$ for $\alpha \in \mathrm{Iso}(M, \mathscr{R}_0)$. Now, each $\sigma \in \mathrm{Aut}(\mathscr{R}_0)$ gives rise to an element $\sigma|_M \in \mathrm{Iso}(M, \mathscr{R}_0)$. We write $u_M(\sigma)$ for $u(\sigma|_M)$, so that

$$\sigma(x) = u_M(\sigma)xu_M(\sigma)^*\,, \qquad x \in M\,.$$

Let $\{M_n\}$ be an increasing sequence of type I subfactors whose union is σ-weakly dense in $\mathscr{R}_0$. For each $n \in \mathbb{N}$ and $\sigma \in \mathrm{Aut}(\mathscr{R}_0)$ set

$$u_n(\sigma) = u_{M_n}(\sigma)\,,$$

so that we have

$$\sigma(x) = u_n(\sigma)xu_n(\sigma)^*\,, \qquad x \in M_n\,, \qquad \sigma \in \mathrm{Aut}(\mathscr{R}_0)\,.$$

For each fixed $n \in \mathbb{N}$, the map: $\sigma \in \mathrm{Aut}(\mathscr{R}_0) \mapsto u_n(\sigma) \in \mathscr{U}(\mathscr{R}_0)$ is continuous. Thus, we obtain a sequence $\{u_n(\cdot)\}$ of $\mathscr{U}(\mathscr{R}_0)$-valued continuous functions on $\mathrm{Aut}(\mathscr{R}_0)$ such that

$$\sigma(x) = \lim_{n\to\infty} u_n(\sigma)xu_n(\sigma)^*\,, \qquad x \in \bigcup_{n=1}^{\infty} M_n\,,$$

[1] Katayama pointed out that the authors' original preprint misquoted this lemma in its application

in fact for each $x \in \bigcup_{n=1}^{\infty} M_n$, we have $\sigma(x) = \mathrm{Ad}(u_n(\sigma))(x)$ for a large n. Since $\bigcup_{n=1}^{\infty} M_n$ is σ-weakly dense in $\mathscr{R}_0$, $\{\mathrm{Ad}(u_n(\sigma))\}$ converges to σ relative to the strong operator topology on $L^2(\mathscr{R}_0, \tau)$, which agrees with the topology of $\mathrm{Aut}(\mathscr{R}_0)$.

For each $n \in \mathbb{N}$, let $\{\alpha_t^{(n)} : 0 \le t \le 1\}$ be a contraction path in $\mathrm{End}(\mathscr{U}(M_n^c))$ such that

$$\alpha_0^{(n)}(u) = 1\,, \qquad \alpha_1^{(n)}(u) = u\,, \qquad u \in \mathscr{U}(M_n^c))\,.$$

For each $n \in \mathscr{Z}_+$ and $0 \le t < 1$, we set

$$\gamma_{\sigma,n+t} = \mathrm{Ad}(\alpha_t^{(n)}(u_{n+1}(\sigma)^* u_n(\sigma)) \circ (\mathrm{Ad}(u_n(\sigma)^*)) \circ \sigma\,,$$

where $u_0(\sigma) = 1$. Observe that $u_{n+1}(\sigma)^* u_n(\sigma)$ belongs to M_n^c, so that

$$\alpha_t^{(n)}(u_{n+1}(\sigma)^* u_n(\sigma)) \in \mathscr{U}(M_n^c)\,.$$

Hence $\{\alpha_t^{(n)}(u_{n+1}(\sigma)^* u_n(\sigma)) : n \in \mathbb{N}\}$ is a central sequence, so that

$$\lim_{n\to\infty} \mathrm{Ad}(\alpha_t^{(n)}(u_{n+1}(\sigma)^* u_n(\sigma)) = \mathrm{id}\,.$$

As $\lim_{t\to 1-} \gamma_{\sigma,n+t} = \gamma_{\sigma,n+1}$, for each $\sigma \in \mathrm{Aut}(\mathscr{R}_0)$, the path: $t \in [0,\infty) \mapsto \gamma_{\sigma,t} \in \mathrm{Aut}(\mathscr{R}_0)$ is continuous and

$$\lim_{t\to\infty} \gamma_{\sigma,t} = \mathrm{id}\,, \qquad \sigma \in \mathrm{Aut}(\mathscr{R}_0)\,.$$

By construction, γ is jointly continuous $[n, n+1) \times \mathrm{Aut}(\mathscr{R}_0)$. Since $\lim_{\substack{t\to 1\\ \sigma\to\theta}} \gamma_{\sigma,n+t} = \gamma_{\theta n+1}$, we conclude that γ is jointly continuous on $\mathbb{R}_+ \times \mathrm{Aut}(\mathscr{R}_0)$ and that

$$\gamma_{\sigma,0} = \sigma \quad \text{and} \quad \lim_{t\to+\infty} \gamma_{\sigma,t} = \mathrm{id}\,.$$

This means that $\mathrm{Aut}(\mathscr{R}_0)$ is contractible. Q.E.D.

AFD Factors of Type II_∞

Let $\mathscr{R}_1$ be an AFD factor of type II_∞. We then have a split exact sequence:

$$1 \to \overline{\mathrm{Int}}(\mathscr{R}_1) \to \mathrm{Aut}(\mathscr{R}_1) \xrightarrow{\mathrm{mod}} \mathbb{R}_+^* \to 1\,.$$

Thus $\mathrm{Aut}(\mathscr{R}_1)$ is precisely the semi-direct product of $\overline{\mathrm{Int}}(\mathscr{R}_1)$ and $\mathbb{R}_+^*$, so that its algebraic and topological structures are completely determined by those of $\overline{\mathrm{Int}}(\mathscr{R}_1)$. We will prove the following

Theorem 1. *In the above context,* $\overline{\mathrm{Int}}(\mathscr{R}_1)$ *is contractible and therefore* $\mathrm{Aut}(\mathscr{R}_1)$ *is contractible.*

An element $\alpha \in \mathrm{Aut}(\mathscr{R}_1)$ is approximately inner precisely when $\mathrm{mod}(\alpha) = 1$. Let $\mathscr{R}_1 = \mathscr{R}_0 \overline{\otimes} \mathscr{B}$ be a tensor product decomposition of $\mathscr{R}_1$ with $\mathscr{R}_0$ the AFD factor of type II_1 and $\mathscr{B}$ a factor of type I_∞. We know from the previous section that $\mathscr{U}(\mathscr{R}_1)/\mathscr{U}(\mathscr{R}_0)$ admits a continuous cross-section.

Lemma 2. *If $\alpha \in \overline{\mathrm{Int}}(\mathscr{R}_1)$, then there exists $u \in \mathscr{U}(\mathscr{R}_1)$ such that*

$$\alpha(x) = uxu^*, \qquad x \in \mathscr{B},$$

Hence there exists a continuous map $u : \alpha \in \overline{\mathrm{Int}}(\mathscr{R}_1) \mapsto u(\alpha) \in \mathscr{U}(\mathscr{R}_1)$ such that

$$\alpha(x) = u(\alpha)xu(\alpha)^*, \qquad x \in \mathscr{B}, \qquad \alpha \in \overline{\mathrm{Int}}(\mathscr{R}_1).$$

Proof. The second assertion follows from the first and the lifting lemma for $\mathscr{U}(\mathscr{R}_1)/\mathscr{U}(\mathscr{R}_0)$.

Let $\{e_{ij} : i, j \in \mathbb{N}\}$ be a matrix unit of $\mathscr{B}$. Since $\mathrm{mod}(\alpha) = 1$, $\alpha \in \overline{\mathrm{Int}}(\mathscr{R}_1)$, we have $\alpha(e_{11}) \sim e_{11}$. Choose $v \in \mathscr{R}_1$ with $v^*v = e_{11}$ and $vv^* = \alpha(e_{11})$, and set

$$u = \sum_{j \in \mathbb{N}} \alpha(e_{j1})ve_{1j} \in \mathscr{R}_1.$$

It is straightforward to check that $u \in \mathscr{U}(\mathscr{R}_1)$ and

$$\alpha(x) = uxu^*, \qquad x \in \mathscr{B}. \qquad \text{Q.E.D.}$$

Proof of Theorem 1. Let $u : \alpha \in \overline{\mathrm{Int}}(\mathscr{R}_1) \mapsto u(\alpha) \in \mathscr{U}(\mathscr{R}_1)$ be a continuous map such that

$$\alpha(x) = u(\alpha)xu(\alpha)^*, \qquad x \in \mathscr{B}.$$

Then we have $\mathrm{Ad}(u(\alpha)^*) \circ \alpha = \alpha_0 \otimes \mathrm{id}$ with respect to the decomposition $\mathscr{R}_1 = \mathscr{R}_0 \overline{\otimes} \mathscr{B}$. As $\mathscr{U}(R_1)$ and $\mathrm{Aut}(\mathscr{R}_0)$ both are contractible, $\overline{\mathrm{Int}}(\mathscr{R}_1)$ is also contractible. Q.E.D.

AFD Factors of Type III_λ, $0 < \lambda < 1$

Let $\mathscr{R}$ be an AFD factor of type III_λ, $0 < \lambda < 1$. Then we have a split short exact sequence:

$$1 \to \overline{\mathrm{Int}}(\mathscr{R}) \to \mathrm{Aut}(\mathscr{R}) \xrightarrow{\mathrm{mod}} \mathbb{R}_+^*/\lambda^{\mathbb{Z}} \to 1.$$

As in the case of type II_∞, the topological and the algebraic structures of $\overline{\mathrm{Int}}(\mathscr{R})$ determine that of $\mathrm{Aut}(\mathscr{R})$. We will prove the following:

Theorem 1. *In the above context, $\overline{\mathrm{Int}}(\mathscr{R})$ is contractible. Therefore, we have*

$$\pi_1(\mathrm{Aut}(\mathscr{R})) \cong \mathbb{Z}.$$

To prove the theorem, we identify $\mathscr{R}$ with the infinite tensor product:

$$\{\mathscr{R}, \omega\} = \prod_{n=1}^{\infty} \otimes \{M_n, \omega_n\},$$

where M_n is the 2×2-matrix algebra, $M_2(\mathbb{C})$ and

$$\omega_n \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} = \frac{1}{1+\lambda} x_{00} + \frac{\lambda}{1+\lambda} x_{11}.$$

As $\overline{\mathrm{Int}}(\mathscr{R}) = \mathrm{Ker(mod)}$, for each $\alpha \in \overline{\mathrm{Int}}(\mathscr{R})$ there exists $u \in \mathscr{U}(\mathscr{R})$ such that $\omega \circ \alpha = \omega \circ \mathrm{Ad}(u)$. Hence the $\overline{\mathrm{Int}}(\mathscr{R})$ orbit of ω is precisely the $\mathrm{Int}(\mathscr{R})$ orbit of ω. Since the centralizer $\mathscr{R}_\omega$ of ω is isomorphic to $\mathscr{R}_0$, the AFD-factor of type II_1, we have a continuous cross-section: $\alpha \in \overline{\mathrm{Int}}(\mathscr{R}) \mapsto u(\alpha) \in \mathscr{U}(\mathscr{R})$ such that

$$\omega \circ \alpha \circ \mathrm{Ad}(u(\alpha)) = \omega .$$

Since $\mathscr{U}(\mathscr{R})$ is contractible, it suffices to prove that the subgroup $\{\alpha \in \overline{\mathrm{Int}}(\mathscr{R}) : \omega \circ \alpha = \omega\}$, say G, is contractible.

Let $\{e_{ij}(n) : 0 \leq i,\, j \leq 1\}$ be the standard matrix unit of M_n. We then have

$$\sigma_t^\omega(e_{j,k}(n)) = \lambda^{(j-k)it} e_{j,k}(n) , \qquad 0 \leq j , \qquad k \leq 1 .$$

As each $\alpha \in G$ commutes with $\{\sigma_t^\omega\}$,

$$\sigma_t^\omega(\alpha(e_{j,k}(n))) = \lambda^{(j-k)it} \alpha(e_{j,k}) .$$

Lemma 2. *For each $n \in \mathbb{N}$, there exists a continuous map: $\alpha \in G \mapsto u_n(\alpha) \in \mathscr{U}(\mathscr{R}_\omega)$ such that*

$$\alpha(x) = u_n(\alpha) x u_n(\alpha)^* , \qquad x \in M_1 \vee \ldots \vee M_n .$$

Proof. Consider the $2^n \times 2^n$-matrix unit of $M_1 \vee \ldots \vee M_n$:

$$e_{i,j}(n) = e_{i_1,j_1}(1) e_{i_2,j_2}(2) \ldots e_{i_n,j_n}(n) ,$$

where $i = (i_1, i_2, \ldots, i_n)$, $j = (j_1, j_2, \ldots, j_n) \in \{0,1\}^n$. With $0 = (0,0,\ldots,0)$, $e_{0,0}(n)$ belongs to $\mathscr{R}_\omega$, and $\alpha(e_{0,0}(n))$ and $e_{0,0}(n)$ are equivalent in $\mathscr{R}_\omega$ for each $\alpha \in G$. Thus choose $v \in \mathscr{R}_\omega$ such that $v^*v = e_{0,0}(n)$ and $vv^* = \alpha(e_{0,0}(n))$, then set

$$u = \sum_j \alpha(e_{j,0}) v e_{0,j}$$

to obtain a unitary $u \in \mathscr{R}_\omega$ such that $\alpha(x) = uxu^*$, $x \in M_1 \vee \ldots \vee M_n$. Since $(M_1 \vee \ldots \vee M_n)' \cap \mathscr{R}_\omega \cong \mathscr{R}_0$, the homogeneous space $\mathscr{U}(\mathscr{R}_\omega)/\mathscr{U}(M_1' \cap \ldots \cap M_n' \cap \mathscr{R}_\omega)$ admits a continuous cross section, which in turn yields a continuous map $u : \alpha \in G \mapsto u_n(\alpha) \in \mathscr{U}(\mathscr{R}_\omega)$ such that

$$\alpha(x) = u_n(\alpha) x u_n(\alpha)^* , \qquad x \in M_1 \vee \ldots \vee M_n . \qquad \text{Q.E.D.}$$

Proof of Theorem 1. Since $u_k(\alpha) \in \mathscr{U}(\mathscr{R}_\omega)$ and

$$\alpha(x) = \lim_{k\to\infty} u_k(\alpha) x u_k(\alpha)^* , \qquad x \in M_1 \vee \ldots \vee M_n ,$$

we conclude

$$\alpha = \lim_{n\to\infty} \mathrm{Ad}(u_n(\alpha)) .$$

As $u_{n+1}(\alpha)^* u_n(\alpha) \in M_1' \cap \ldots \cap M_n' \cap \mathscr{R}_\omega \cong \mathscr{R}_\omega$, the arguments for the contractibility for $\mathrm{Aut}(\mathscr{R}_0)$ applies to α and the sequence $\{\mathrm{Ad}(u_n(\alpha))\}$ to conclude the contractibility of G. Q.E.D.

BG and EG

Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space and A be the Fermion C^*-algebra over $\mathscr{H}$, i.e. the C^*-algebra generated by the canonical anti-commutation relations, CAR, over $\mathscr{H}$:

$$f \in \mathscr{H} \mapsto a(f) \in A$$

such that

$$\{a(f), a(g)\} = a(f)a(g) + a(g)a(f) = 0\,;$$
$$\{a(g)^*, a(f)\} = a(g)^*a(f) + a(f)a(g)^* = (f \mid g)\,.$$

It is well-known that A is a UHF algebra of type 2^∞ and has a unique tracial state τ, which gives rise, via the GNS-representation, to an AFD factor $\mathscr{R}_0$ of type II_1. It is also known that each unitary $u \in \mathscr{U}(\mathscr{H})$, the unitary group on $\mathscr{H}$, gives rise to an automorphism $\alpha_u \in \mathrm{Aut}(\mathscr{R}_0)$ such that

$$\alpha_u(a(f)) = a(uf)\,, \qquad f \in \mathscr{H}\,,$$

and that the map: $u \in \mathscr{U}(\mathscr{H}) \mapsto \alpha_u \in \mathrm{Aut}(\mathscr{R}_0)$ is an injective continuous homomorphism. In his classical work [2], Blattner proved that α_u is inner if and only if $u - 1$ is of the Hilbert-Schmidt class on $\mathscr{H}$.

Proposition 1. *The homomorphism: $u \in \mathscr{U}(\mathscr{H}) \mapsto \alpha_u \in \mathrm{Aut}(\mathscr{R}_0)$ is an open map and the image $G = \{\alpha_u : u \in \mathscr{U}(\mathscr{H})\}$ is a closed subgroup of* $\mathrm{Aut}(\mathscr{R}_0)$.

Proof. Let $\|\ldots\|_2$ denote the L^2-norm given by the trace τ on $\mathscr{R}_0$. It then follows that

$$\|f\| = \sqrt{2}\,\|a(f)\|_2 = \|a(f)\|\,, \qquad f \in \mathscr{H}\,.$$

Hence $\mathscr{K} = a(\mathscr{H})$ is a Hilbert space as a subspace of $\mathscr{R}_0$, which means that the unit ball of $\mathscr{K}$ is compact relative to the weak topology which is then the relative topology of the σ-weak operator topology of $\mathscr{R}_0$. Therefore, the Banach theorem implies the σ-weak closedness of $\mathscr{K}$. The group G is then characterized as the group of all those automorphisms $\alpha \in \mathrm{Aut}(\mathscr{R}_0)$ such that $\alpha(\mathscr{K}) = \mathscr{K}$. Therefore, G is a closed subgroup of $\mathrm{Aut}(\mathscr{R}_0)$. Thus, the map: $u \in \mathscr{U}(\mathscr{H}) \mapsto \alpha_u \in G$ is a bijective continuous homomorphism between two Polish groups. By the open mapping theorem for Polish groups, it is a homeomorphism. Q.E.D.

Proposition 2. *If G is a locally compact group, then the left regular representation λ of G is a bicontinuous isomorphism of G onto the image $\lambda(G)$, where one considers the relative topology on $\lambda(G)$ induced by the weak operator topology on the unitary group of the Hilbert space $L^2(G)$ of all square integrable functions over G relative to the left Haar measure.*

This is an immediate consequence of the Tannaka-Eymard-Tatsuuma duality theorem, cf. [9]. Namely, the topology of the group G is given by the Fourier algebra $A(G)$ over G, which is in turn the relative topology of the weak operator topology when one views G as the group $\lambda(G)$ acting on $L^2(G)$, i.e.

$$f * \bar{g}^\vee(s) = (\lambda(s)f \mid g)\,, \qquad f, g \in L^2(G)\,, \qquad s \in G\,,$$

where $g^\vee(s) = g(s^{-1})$, $s \in G$.

Combining the above two propositions, we now conclude the following:

Proposition 3. *For any separable locally compact group G, there exists a bicontinuous isomorphism: $s \in G \mapsto \alpha_s \in \mathrm{Aut}(\mathscr{R}_0)$ from G onto $\alpha(G)$ such that $\alpha(G) \cap \mathrm{Int}(\mathscr{R}_0) = \{\mathrm{id}\}$.*

Proof. In view of Propositions 1 and 2, it is sufficient to prove that the map: $u \in \mathscr{U}(L^2(G)) \mapsto \bar{u} = u \otimes 1_{\mathscr{H}_0} \in \mathscr{U}(\mathscr{H})$, with $\mathscr{H} = L^2(G) \otimes \mathscr{H}_0$, is a bicontinuous isomorphism from the unitary group $\mathscr{U}(L^2(G))$ of $L^2(G)$ onto the unitary group $\mathscr{U}(\mathscr{L}(L^2(G)) \otimes \mathbb{C}_{\mathscr{H}_0})$ such that $1 - \bar{u} = (1 - u) \otimes 1_{\mathscr{H}_0}$ is not of the Hilbert-Schmidt class except for $u = 1$, where $\mathscr{H}_0$ is a fixed separable infinite dimensional Hilbert space. But this is a triviality. Q.E.D.

Therefore, the automorphism group $\mathrm{Aut}(\mathscr{R}_0)$ of the AFD factor $\mathscr{R}_0$ of type II_1 contains a replica $\tilde{G}$ of every separable locally compact group G as a closed subgroup which meets with the group $\mathrm{Int}(\mathscr{R}_0)$ of inner automorphisms only at the identity. Thus, Aut $(\mathscr{R}_0)$ can serve as an EG and the quotient homogeneous space $\mathrm{Aut}(\mathscr{R}_0)/\tilde{G}$ as a classifying space BG of principal fibre bundles of G.

References

1. Araki, H., Smith, M.-S.B., Smith, L.: On the homotopical significance of the type of von Neumann algebra factors. Commun. Math. Phys. **22**, 71–88 (1971)
2. Araki, H., Woods, E.J.: A classification of factors. Publ. Research Institute Math. Sci., Kyoto Univ. **4**, 51–130 (1968)
3. Blattner, R.: Automorphic group representations. Pacific J. Math. **8**, 665–677 (1958)
4. Breuer, M.: A generalization of Kuiper's theorem to factgors of type II_∞. J. Math. Mech. **16**, 917–925 (1967)
5. Bruning, J., Willgerodt, W.: Eine Verallgemeinerung eines Satzes von N. Kuiper. Math. Ann. **220**, 47–58 (1976)
6. Connes, A.: Une classification des facteurs de type III. Ann. Sci., Ecole Norm. Sup. **6**, 133–252 (1973)
7. Connes, A., Takesaki, M.: The flow of weights on factors of type III. Tohoku Math. J. **29**, 473–576 (1977)
8. Dixmier, J.: Les C^*-algebres et leurs representations. 2nd ed., Paris: Gauthier-Villars 1969
9. Eymard, P.: L'algebre de Fourier d'un groupe localement compact. Bull. Soc. Math. France **92**, 181–236 (1964)
10. Kakutani, S.: Topological properties of the unit sphere of a Hilbert space. 1943
11. Kuiper, N.H.: The homotopy type of the unitary group of Hilbert space. Topology **3**, 19–30 (1965)
12. McDuff, D.: Central sequences and the hyperfine factor. Proc. Lond. Math. Soc. **21**, 443–461 (1970)
13. Michael, E.: Convex structures and continuous selections. Canadian J. Math. **11**, 556–575 (1959)
14. Radulescu, F.: A one real parameter group of automorphisms of $L(F_\infty)$ scaling the trace. C.R. Acad. Sci. Paris 1992 (to appear)
15. Takesaki, M.: Duality for crossed product and the structure of von Neumann algebras of type III. Acta Math. **131**, 249–310 (1973)

Communicated by A. Jaffe