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Methods of KAM-Theory for Long-Range Quasi-Periodic Operators on $\mathbb{Z}^\nu$. Pure Point Spectrum

V. A. Chulaevsky[1] **and E. I. Dinaburg**[2]

[1] Department of Mathematics, University of California at Irvine, Irvine, CA 92717, USA. Permanent address: Institute of Mathematical Problems in Biology, Russian Academy of Sciences, Pushchino, Moscow Region 142292, Russia
[2] O. Yu. Schmidt Institute of Earth Physics, Russian Academy of Sciences, B. Gruzinskaya ul., 10, Moscow 123810, Russia

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Abstract. We consider the class of quasi-periodic self-adjoint operators $\hat{H}(x) = \hat{D}(x) + \hat{V}(x)$, $x \in S^1 = \mathbb{R}^1/\mathbb{Z}^1$, on a multi-dimensional lattice $\mathbb{Z}^\nu$, with the matrix elements

$$\hat{D}_{mn}(x) = \delta_{mn} D(x + n\omega)\,, \qquad \hat{V}_{mn}(x) = V(m - n, x + n\omega)\,,$$

where $D(x+1) = D(x)$, $V(n, x+1) = V(n, x)$, $\omega \in \mathbb{R}^\nu$, and $|V(n,x)| \le \varepsilon e^{-r|n|}$, $r > 0$. We prove that, if ε is small enough, $V(n, \cdot)$ and $D(\cdot)$ satisfy some conditions of smoothness, and $D(\cdot)$ is non-degenerate, then for a.e. ω and for a.e. $x \in S^1$ the operator $\hat{H}(x)$ has pure point spectrum. All its eigenfunctions belong to $l^1(\mathbb{Z}^\nu)$.

1. Introduction

In the spectral theory of almost periodic media, two important classes of quantum Hamiltonians have been investigated particularly well: nearest-neighbor Hamiltonians like the "almost-Mathieu" operator on $\mathbb{Z}^1$,

$$(H_\varepsilon(x)\psi)(n) = \varepsilon(\psi(n-1) + \psi(n+1)) + \cos(x + n\omega)\psi(n)\,,$$

which describes a quasi-periodic medium with infinite number of resonances (Sinai [1], Fröhlich, Spencer, and Wittwer [2]), and long-range Hamiltonians like

$$(H_\varepsilon(x)\psi)(n) = \varepsilon \sum_{n \in \mathbb{Z}^\nu} a(n-m)\psi(m) + \tan(x + n\omega)\psi(n)\,,$$

with $|a(n)| \le e^{-r|n|}$, $r > 0$, which describe media with no resonances (see Bellissard, Lima, and Scoppola [3]). The main purpose of the present paper is to extend the perturbation-theoretic analysis of resonances, originally proposed by Sinai [1] and going back to the KAM (Kolmogorov-Arnold-Moser) theory. Many authors mentioned that the methods of the KAM theory appear naturally in localization problems (see in particular [4]). We refer also to a related work by Bellissard [6].

There are two crucial points in our analysis which allow us to apply successfully the KAM-approach to the investigation of multiple resonances:
– sparseness of resonances on the lattice, or, in our terms, sparseness of the moments in our inductive procedure when a given eigenvalue (EV) undergoes a resonant splitting. This phenomenon has been already discovered and used in the papers by Sinai [1], by Fröhlich, Spencer, and Wittwer [2], and by Surace [5].
– uniform boundedness of the multiplicity of "elementary" resonances which we *cannot treat separately*. An accurate formulation of this property can be given in terms of the smoothness of the EV's considered as functions on the phase space S^1 of the underlying dynamical system. Roughly speaking, an "elementary" resonance is related to an intersection of the graphs of the EV's on the phase space, so the multiplicity of an "elementary" resonance cannot be higher than $\dim S^1 + 1 = 2$.

One of our goals in this paper has been to divide the problem of localization for long-range operators on the lattice $\mathbb{Z}^\nu$ related to a finite-dimensional, smooth, ergodic dynamical system into two separate problems:
– a perturbation theoretic analysis, in the spirit of the KAM theory, of resonant phenomena in disordered media;
– a geometrical analysis, following essentially the approach by Sinai [1], of the regularity properties of eigenvalues of self-adjoint operators in question considered as functions on the phase space of the underlying dynamical system.

We have to mention also that, as it follows from our proof (see Sect. 5), the localization holds for a set of frequencies of Lebesgue measure 1.

2. Formulation of the Result

Definition 1. Let φ be a complex-valued function on the lattice $\mathbb{Z}^\nu$. For any $\varrho \geq 0$ we define a norm $\|\varphi\|_\varrho$ as follows:

$$\|\varphi\|_\varrho = \sum_{n\in\mathbb{Z}^\nu} |\varphi(n)| e^{\varrho|n|} .$$

The Banach space of all functions φ with $\|\varphi\|_\varrho < \infty$ is denoted by $\mathscr{H}^\varrho$.

We shall also call the norm $\|\cdot\|_\varrho$ a ϱ-norm. The ϱ-norms and the corresponding Banach spaces $\mathscr{H}^\varrho$ have the following convenient property: for any $\varrho' \geq \varrho''$ the ϱ'-norm dominates ϱ''-norm and, therefore, $\mathscr{H}^{\varrho''} \supseteq \mathscr{H}^{\varrho'}$.

Fix an arbitrary vector $\omega \in \mathbb{R}^\nu$ and define on the unit circle S^1 an action $T = T_\omega$ of the additive group $\mathbb{Z}^\nu$ as follows:

$$T_\omega : \mathbb{Z}^\nu \times S^1 \to S^1$$
$$(n, x) \mapsto x + n_1\omega_1 + \ldots + n_\nu\omega_\nu .$$

Fix a number $\varrho \geq 0$ and consider the Banach space $L^\infty(S^1, \mathscr{H}^\varrho)$ of functions $\varphi : \mathbb{Z}^\nu \times S^1 \to \mathbb{C}$ for which

$$\|\varphi\|^\sim_\varrho = \sum_{n\in\mathbb{Z}^\nu} \operatorname{ess\,sup}_{x\in S^1} |\varphi(n,x)| e^{\varrho|n|} < \infty .$$

We can define in this linear space the structure of C^*-algebra with the multiplication

$$(\varphi\psi)(n,x) = \sum_{m\in\mathbb{Z}^\nu} \varphi(n-m, T^m_\omega x)\psi(m,x) ,$$

with the natural *-conjugation $\varphi^*(n,x) = \overline{\varphi(-n, T_\omega^n x)}$, (where $\bar{z}$ means the complex conjugate of $z \in \mathbb{C}$) and with the norm $\|\cdot\|_\varrho^\sim$. We shall denote this C^*-algebra as $\mathscr{A}^\varrho$ or, in order to emphasize the dependence of $\mathscr{A}^\varrho$ upon the vector $\omega \in \mathbb{R}^\nu$, as $\mathscr{A}^\varrho(\omega)$. Sometimes we shall omit the indication to ω in $\mathscr{A}^\varrho(\omega)$ and in T_ω writing simple $\mathscr{A}^\varrho$ and T.

Now we introduce an important C^*-representation of $\mathscr{A}^\varrho$ in the algebra of operator-valued functions on S^1. Namely, for any $\varphi \in \mathscr{A}^\varrho$ we define a family of operators $\hat{\varphi}(x)$ in $\mathscr{H}^\varrho$ with the following matrix elements:

$$\hat{\varphi}_{nm}(x) = \varphi(n-m, T^m x).$$

For any $\varphi \in \mathscr{A}^\varrho$ introduce its decomposition into the sum of its "diagonal" and "off-diagonal" components [related to the decomposition of the corresponding operators $\hat{\varphi}(x)$]:

$$\varphi(n,x) = D_\varphi(x)\delta_{n0} + V_\varphi(n,x), \quad D_\varphi(x) = \varphi(0,x), \quad V_\varphi(n,x) = (1-\delta_{n0})\varphi(n,x).$$

It follows from our definitions that the matrix of the operator $\hat{\varphi}(x)$ in the standard basis is given by

$$\hat{\varphi}_{nm}(x) = D_\varphi(T^n x)\delta_{nm} + (1-\delta_{nm})V_\varphi(n-m, T^m x).$$

It is not difficult to see that for any $\varrho > 0$ and any $\varphi \in \mathscr{A}^\varrho$ the operators $\hat{\varphi}(x)$, defined on a dense subset $\mathscr{H}^\varrho \subset l^2(\mathbb{Z}^\nu)$, are bounded and, therefore, we can continue them to bounded operators on $l^2(\mathbb{Z}^\nu)$. This follows from the exponential decay of the matrix coefficients $\hat{\varphi}_{nm}(x)$ as $|n-m| \to \infty$. It will be always clear from the context whether an operator $\hat{\varphi}(x)$ is considered in $\mathscr{H}^\varrho$ or in $l^2(\mathbb{Z}^\nu)$.

It is easy to see that the condition of self-adjointness of the image $\hat{\varphi}$ of an element $\varphi \in \mathscr{A}^\varrho$, i.e. self-adjointness of the operators $\hat{\varphi}(x)$ acting in $l^2(\mathbb{Z}^\nu)$, reads as

$$\varphi(n,x) = \overline{\varphi(-n, T^n x)}.$$

Note that the algebra $\mathscr{A}^\varrho$ contains the element

$$\Delta(n,x) = \sum_{m\in\mathbb{Z}^\nu : |m|=1} \delta_{m,n},$$

which corresponds to the discrete nearest-neighbor Laplacian in the above mentioned representation. However, it contains also elements related to long-range self-adjoint operators on the lattice, e.g. the element

$$\varphi = e^\Delta = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!}$$

or elements φ with $|\varphi(n,x)| \le \exp(-r|n|)$ for $r > 0$. In other words, in the framework of the algebra $\mathscr{A}^\varrho$ we can consider operators $\hat{A}(x)$ with $|\hat{A}_{nm}(x)| \le \exp(-r|n-m|)$ for $r > 0$.

Now, we can formulate the main result of the present paper.

Theorem. *Let $d(x)$ be a C^2-function on S^1 with exactly two critical points, the maximum and the minimum, both of which are non-degenerate, and let $D \in \mathscr{A}^\varrho$ be the "diagonal" element of the form $D(n,x) = \delta_{n0}d(x)$. Then for any $\varrho > 0$ and for any $\delta > 0$ there exist a set $\Omega_{\varrho,\delta} \subseteq \Omega = [0,1)^\nu$, $\operatorname{mes}\Omega_{\varrho,\delta} > 1-\delta$, and a positive number $\varepsilon(D,\varrho,\delta)$ such that if the off-diagonal component V of the operator belongs* *to* $\mathscr{A}^{\varrho}(\omega)$ *and* $\|V\|_{\varrho} \le \varepsilon(D,\varrho,\delta)$, $\|dV/dx\|_{\varrho} \le \varepsilon(D,\varrho,\delta)$, $\|d^2V/dx^2\|_{\varrho} = \varepsilon(D,\varrho,\delta)$, *then for a.e.* $x \in S^1$ *the operators* $\hat{\varphi}(x)$ *in* $l^2(\mathbb{Z}^\nu)$ *corresponding to the element* $\varphi = D + V \in \mathscr{A}^{\varrho}$ *have pure point spectrum. All the corresponding eigenfunctions belong to* $\mathscr{H}^0 = l^1(\mathbb{Z}^\nu)$. *The multiplicity of any eigenvalue does not exceed* 2.

Remark. There is a visible difference between the regularity conditions we impose on elements $\varphi \in \mathscr{A}^{\varrho}$ and the conditions on $d(x)$, $V(n,x)$ in the formulation of the above theorem. The reason why we require only boundedness (and measurability) of functions $\varphi(n,x)$, $\varphi \in \mathscr{A}^{\varrho}$, is that in the proof of this theorem we shall introduce a sequence of functions $d^{(s)}(x)$, $V^{(s)}(n,x)$ which are only piecewise-smooth in the variable $x \in S^1$, and the number of points of discontinuity of those functions will grow as $s \to \infty$. In order to be able to treat those functions as elements of the same C^*-algebra, we have to relax conditions on their regularity.

3. Proof of the Theorem

During the proof we shall use some general standard notions from the ergodic theory. However, we define all the necessary objects only in the case where the space of the dynamical system is the unit circle S^1 and the system is generated by ν incommensurate irrational rotations of the circle.

Definition 2. An equipped partition of order $s \in \mathbb{N}$ of the unit circle S^1 is a sequence of triples $(\xi^{(t)}, \zeta^{(t)}, \tau^{(t)})$, $0 \le t \le s$, where $\xi^{(t)}$ and $\zeta^{(t)}$ are finite partitions of S^1, $\xi^{(t)} = \{C^{(t)}_{\xi,i}, i = 1,\dots,m(\xi^{(t)})\}$, $\zeta^{(t)} = \{C^{(t)}_{\zeta,i}, i = 1,\dots,m(\zeta^{(t)})\}$, and $\tau^{(t)}: S^1 \to \mathbb{Z}^+$ which is constant on any $C^{(t)}_{\zeta,i}$, such that

(i) $\xi^{(t)} \ge \zeta^{(t)}$, i.e. any element of $\zeta^{(t)}$ is a union of elements of $\xi^{(t)}$; moreover, each element $C^{(t)}_{\zeta,i}$ is a union of at most two elements of $\xi^{(t)}$:

$$C^{(t)}_{\zeta,i} = \bigcup_{j=1}^{k(i)} C^{(t)}_{\xi,l_j}, \qquad 1 \le k(i) \le 2.$$

(ii) For any $t' \le t''$ $\xi^{(t')} \le \xi^{(t'')}$, $\zeta^{(t')} \le \zeta^{(t'')}$, $\tau^{(t')}(x) \le \tau^{(t'')}(x)$; $\tau^{(s)}(x) \le s$.

(iii) Consider a point $x \in S^1$ with $\tau^{(s)}(x) = t$. Let $C^{(t)}_{\zeta}(x)$ be the element $C^{(t)}_{\zeta,i}$ of the partition $\zeta^{(t)}$ containing x, and let

$$\bigcup_{j=1}^{k(x)} C^{(t)}_{\xi,m_j}(x) = C^{(t)}_{\zeta,i}(x).$$

If $k(x) = 2$, then there exists a vector $r(i) \in \mathbb{Z}^\nu$ such that

$$C^{(t)}_{\xi,m_2} = T^{r(i)} C^{(t)}_{\xi,m_1}.$$

(iv) The interior of any element of $\xi^{(s)}$ is an interval in S^1.

Definition 3. An equipped partition $\{(\xi^{(t)}, \zeta^{(t)}, \tau^{(t)}), 1 \le t \le s'\}$ of order s' is an extension of an equipped partition $\{\tilde{\xi}^{(t)}, \tilde{\zeta}^{(t)}, \tilde{\tau}^{(t)}), 1 \le t \le s''\}$ of order $s'' < s'$ if for any $t \le s''$ $\xi^{(t)} = \tilde{\xi}^{(t)}$, $\zeta^{(t)} = \tilde{\zeta}^{(t)}$, $\tau^{(t)} = \tilde{\tau}^{(t)}$.

Let $\varepsilon \le \varrho^{1/\sigma}$. We shall use in the sequel three sequences of positive numbers defined recursively by the following formulas:

$$\varepsilon_{s+1} = \varepsilon_s^{1+\kappa},\ R^{(s+1)} = [\varepsilon_s^{-\sigma}\Gamma \ln \varepsilon_s^{-1}],\ \delta^{(s+1)} = \varepsilon_{s+1}^{\mu},\ \varepsilon_0 = \varepsilon,\ \varrho_{s+1} = \varepsilon_{s+1}^{\sigma},$$

where we can put

$$\kappa = 1/4,\ \ \mu \ge 2(\sigma(B+\nu+1))^{1/2},\ \ B \ge 2,\ \ \Gamma = 100, \tag{1}$$

and μ and σ are small enough; in particular, $\mu + \sigma(B+\nu+1) < 1/2$. The notation $[\cdot]$ is used here for the integer part. We set $V^{(0)}(n,x) = V(n,x)$, $d^{(0)}(x) = d(x)$, $D^{(0)}(n,x) = \delta_{n0}d^{(0)}(x)$, and $H^{(0)}(n,x) = D^{(0)}(n,x) + V^{(0)}(n,x)$. Define the following equipped partition $\{\xi^{(0)}, \zeta^{(0)}, \tau^{(0)})\}$ of order $s = 0$: $\tau^{(0)}(x) \equiv 0$, and the partitions $\xi^{(0)} = \zeta^{(0)}$ for any $\omega \in \Omega$ consist of two $\varepsilon^{\mu/2}$-neighborhoods of critical points of the function $d(x)$ and of two complimentary intervals.

For simplicity of the notations, we shall write $\|\cdot\|_s$ instead of $\|\cdot\|_{\varrho s}$ where it cannot lead to a confusion.

Let be given

(1_s) a Borel set $\Omega^{(s)} = \Omega \setminus \bigcup_{t=1}^{s} \Omega_t$ such that mes $\Omega_t \le \varepsilon^{a(1+\kappa)^t}$, $a > 0$;

(2_s) for any $\omega \in \Omega^{(s)}$ a equipped partition

$$\{(\xi^{(t)}, \zeta^{(t)}, \tau^{(t)}), 0 \le t \le s\} = \{(\xi^{(t)}(\omega), \zeta^{(t)}(\omega), \tau^{(t)}(\omega)), 0 \le t \le s\}$$

and two functions $D^{(s)}(n,x) = \delta_{n0}d^{(s)}(x)$, $V^{(s)}(n,x)$ such that for any $n \in \mathbb{Z}^\nu$ restrictions of $D^{(s)}(\cdot)$ and $V^{(s)}(n,\cdot)$ to the interior of any element of the partition $C^{(s)}_{\xi,\imath}$ belongs to $C^2(C^{(s)}_{\xi,\imath})$, and the boundary values of the functions as well as those of their first and second order derivatives exist and are finite. Moreover, $d^{(s)}$ takes each value at exactly two points, taking into account their multiplicities. On any element of partition $C^{(s)}_{\xi,i}$ the function $d(s)$ either is monotone with the derivative separated from zero, or has two intervals of monotonicity and exactly one critical point (minimum or maximum) separating them; this critical point is non-degenerate.

(3_s) for a.e. $x \in S^1$ the following inequalities hold:

(a) $\|d^{(s)}\|_s \le \text{const},\ \left\|\frac{\partial}{\partial x} d^{(s)}(x)\right\|_s \le \text{const},\ \|V^{(s)}\|_s \le \varepsilon_s;$

(b) $\left\|\frac{\partial^2}{\partial x^2} d^{(s)}(x)\right\|_s \le 4(s - \tau(C^{(s)}_\xi)))\,(\varepsilon_{\tau(C^{(s)}_\xi(x))+1})^{-1};$

(c) $\left\|\frac{\partial}{\partial x} V^{(s)}(x)\right\|_s \le \varepsilon_s,\ \left\|\frac{\partial^2}{\partial x^2} V^{(s)}(x)\right\|_s \le \varepsilon_s^{1/2};$

(d) for any fixed x either $\left|\frac{\partial^2}{\partial x^2} d^{(s)}(x)\right| \ge C_1(d^{(0)}) > 0$, or $\left|\frac{\partial}{\partial x} d^{(s)}(x)\right| \ge C_2(d^{(0)})\varepsilon_s$; moreover, the latter inequality can be violated only in a neighborhood of the critical point of the function $d^{(t)}(x)$, $t = \tau^{(s)}(x)$, of radius $\varepsilon_{\tau^{(s)}(x)}$, where the uniform lower bound for the second derivative holds.

Remark. All the constants in the conditions (1_s)–(4_s) do not depend upon s, although some of them may depend upon $d^{(0)}, V^{(0)}$.

(4_s) If $n, m \in \mathbb{Z}^\nu$ satisfy the inequality $|n - m| < R^{(s)}$, and if, in addition, $|d^{(s-1)}(T^n x) - d^{(s-1)}(T^m x)| \le \delta_s |n-m|^{-B}$ [where the constant B is the same as in (1)], $\delta_s = \varepsilon_s^\mu$, $|V^{(s-1)}(n-m, T^m x)| \geq \frac{1}{4}\varepsilon_s$, then we say that the condition of s-resonance is fulfilled for $d^{(s-1)}(T^n x)$ and $d^{(s-1)}(T^m x)$. We shall also say that the sites $n \in \mathbb{Z}^\nu$ and $m \in \mathbb{Z}^\nu$ belong to the same resonant group. Any resonant group contains exactly two sites of the lattice.

For any $s \geq 0$, we shall use the following notation:

$$H^{(s)}(n,x) = D^{(s)}(n,x) + V^{(s)}(n,x) = d^{(s)}(x)\delta_{n0} + V^{(s)}(n,x).$$

Inductive Lemma. *If the conditions* (1_s)–(4_s) *are fulfilled for some* $s \geq 0$, *then there exists a set* $\Omega_{s+1} \subset \Omega$ *such that*

(A) $\operatorname{mes} \Omega_{s+1} < \varepsilon^{a(1+\kappa)^{s+1}}$ for some constant $a > 0$ (independent of s);

(B) *for any* $\omega \in \Omega^{(s+1)} = \Omega \backslash \Omega_{s+1}$ there exist two functions $Q_c^{(s+1)}(n,x)$, $Q_u^{(s+1)}(n,x)$, $n \in \mathbb{Z}^\nu$, $x \in S^1$, such that

$$Q_u^{(s+1)} = \prod_{k=1}^{16} \exp\{iW_u^{(s+1,k)}\}$$

and

$$\|W_u^{(s+1,k)}(\cdot, x)\|_{\varrho_{s+1}} \leq \varepsilon_s^{k\kappa+\gamma'}, \qquad \gamma' > 0,$$

(the norm is taken in the space $\mathscr{H}^{\varrho_{s+1}}$ for any fixed x), the operator $\hat{Q}_c^{(s+1)}$ is a product of rotations in two-dimensional subspaces; each subspace in consideration is spanned by two columns of the matrix of the operator

$$\prod_{r=1}^{s} (\hat{Q}_u^{(r)} \hat{Q}_c^{(r)}) \hat{Q}_u^{(s+1)}$$

which correspond to the couple of sites in $(s+1)$-resonance; and the function

$$H^{(s+1)} = (Q_c^{(s+1)})^{-1} (Q_u^{(s+1)})^{-1} H^{(s)} (Q_u^{(s+1)})(Q_c^{(s+1)})$$

can be decomposed into the sum of its "diagonal" and "off-diagonal" components,

$$H^{(s+1)}(n,x) = d^{(s+1)}(x)\delta_{n0} + (1-\delta_{n0})V^{(s+1)}(n,x),$$

where the functions $d^{(s+1)}, V^{(s+1)}$ satisfy conditions (1_{s+1})–(4_{s+1}) with respect to a new equipped partition $(\zeta^{(s+1)}, \xi^{(s+1)}, \tau^{(s+1)})$ which is defined as follows:

(I) if $C_\xi^{(s)}(x)$ does not contain any point of $(s+1)$-resonance, then

$$C_\xi^{(s+1)}(x) = C_\xi^{(s)}(x), \qquad \tau^{(s+1)}(x) = \tau^{(s)}(x);$$

(II) if $C_\zeta^{(s)}(x)$ does not contain any point of $(s+1)$-resonance, then

$$C_\zeta^{(s+1)}(x) = C_\zeta^{(s)}(x);$$

(III) if x satisfies (I), but does not satisfy (II), then

$$C_\zeta^{(s+1)}(x) = C_\xi^{(s+1)}(x);$$

(IV) assume that $C_\xi^{(s)}(x)$ contain some points of $(s+1)$-resonance; then we decompose it into elements of the new partition $\xi^{(s+1)}$ of the following types:

(a) connected components $C_\xi^{(s+1)}(x)$ of the set of $(s+1)$ resonance such that $\inf |d^{(s)}(T^n x) - d^{(s)}(T^m x)| < \varepsilon_{(s+1)}/4$ over $C_\xi^{(s+1)}(x)$, with $\tau^{(s+1)}(x) = s+1$;

(b) connected components of the complement to the latter set with $\tau^{(s+1)}(x) = \tau^{(s)}(x)$.

The partition $\xi^{(s+1)}$ consists of the elements listed above. The partition $\zeta^{(s+1)}$ is defined in the following way:

(1) $\zeta^{(s+1)} \leq \xi^{(s+1)}$;

(2) elements $C^{(s+1)}_{\xi,i}, C^{(s+1)}_{\xi,j}$ belong to the same element $C^{(s+1)}_{\zeta,k}$ if for the functions $d^{(s)}(T^n x)\chi_{C^{(s+1)}_{\xi,i}}(x)$, $d^{(s)}(T^m x)\chi_{C^{(s+1)}_{\xi,j}}(x)$ the condition of a $(s+1)$-resonance is fulfilled for some $n, m \in \mathbb{Z}^\nu$. □

Remark. We say that the Diophantine property is fulfilled for $\omega \in \mathbb{R}^\nu$ if fractional part of $|(n, \omega)|$ is more than $\varepsilon_s^B |n|^{-\nu-1}$ for all $|n| < R^{(s+1)}$. The set Ω_{s+1} consists of all such ω.

The proof of the Inductive Lemma is given in Sect. 4, and now we derive the statement of the Theorem from the Inductive Lemma. This is done in several steps. First of all, it is not difficult to see that we can start the induction.

Lemma 1. *Let the functions $V^{(0)}(x) = V(x)$ and $d^{(0)}(x) = d(x)$ satisfy the conditions of the theorem. Then there exists an equipped partition $(\xi^{(0)}, \zeta^{(0)}, \tau^{(0)})$ such that the conditions (1_0)–(4_0) are also satisfied.*

Lemma 2. *If the Inductive Lemma holds and, therefore, the conditions (1_s)–(4_s) are fulfilled for any $s \geq 1$, then for a.e. $x \in S^1$ there exists an integer $s^* = s^*(x)$ such that for any $s \geq s^*$.*

$$\tau^{(s)}(x) = \tau^{(s^*)}(x) .$$

In other words, a.e. $x \in S^1$ undergoes only a finite number of resonances. □

Proof. This is a direct consequence of (4_s) and of estimates in Proposition 2 (see Sect. 5), combined with Borel-Cantelli lemma. □

Lemma 3. *Consider the sequence $\{\hat{\mathscr{U}}^{(s)}_+, \hat{\mathscr{U}}^{(s)}_-\}$ of operators corresponding to the elements of the algebra $\mathscr{A}$*

$$\mathscr{U}^{(t)}_+ = \prod_{r=1}^{t} (Q^{(r)}_u Q^{(r)}_c) \qquad \mathscr{U}^{(t)}_- = (\mathscr{U}^{(t)}_+)^{-1} .$$

Then for any function φ on $\mathbb{Z}^\nu$ with compact support the sequence $\{\hat{\mathscr{U}}^{(s)}_\pm(x)\varphi\}$ converges in the norm topology in $l^1(\mathbb{Z}^\nu)$ for a.e. $x \in S^1$.

Proof of Lemma 3. It suffices to prove convergence for the functions $\varphi(m) = \delta_{nm}$, since any function with compact support is finite linear combination of those functions φ. In other words, we prove the $l^1(\mathbb{Z}^\nu)$-convergence of the columns of the matrices of $\hat{\mathscr{U}}^{(s)}_\pm(x)$ for a.e. $x \in S^1$. Consider the function $s^*(x)$ defined in Lemma 2 and suppose $s^*(x) < \infty$. Lemma 2 claims that $\mathrm{mes}\{x : s^*(x) < \infty\} = 1$. Let $\varphi^* = \hat{\mathscr{U}}^{(s^*(x))}_+(x)$. Then, by definition of the integer $s^*(x)$, for any $t > s^*(x)$,

$$\begin{aligned} \hat{\mathscr{U}}^{(t)}_+(x)\varphi &= \left(\prod_{r=s^*(x)+1}^{t} \hat{Q}^{(r)}_u \right) \hat{\mathscr{U}}^{(s^*(x))}_+(x)\varphi \\ &= \left(\prod_{r=s^*(x)+1}^{t} Q^{(r)}_u \right) \varphi^* . \end{aligned}$$

The inductive estimates on the non-resonant perturbations complete the proof. □

Lemma 4. *The operators* $\hat{\mathscr{U}}_{\pm}^{(s)}$ *converge in the strong operator topology on the space of bounded operators acting in the Banach space* $l^1(\mathbb{Z}^\nu)$.

Proof of Lemma 4. Any function $\psi \in l^1(\mathbb{Z}^\nu)$ can be approximated in the norm topology by functions $\psi^{(k)}$ with finite support. Since operators $\hat{\mathscr{U}}_{\pm}^{(s)}$ are uniformly bounded, the sequence $\{\hat{\mathscr{U}}_{\pm}^{(s)}\}$ is a Cauchy sequence in the space $l^1(\mathbb{Z}^\nu)$. Indeed, for any $\delta > 0$ there exist a number $k(\delta)$ such that $\|\psi - \psi^{(k)}\| < \delta$, and, therefore,

$$\begin{aligned}
\|(\hat{\mathscr{U}}_{\pm}^{(s)} - \hat{\mathscr{U}}_{\pm}^{(s-1)})\psi\| &= \|(\hat{\mathscr{U}}_{\pm}^{(s)} - \hat{\mathscr{U}}_{\pm}^{(s+1)})(\psi^{(k)} + \psi - \psi^{(k)})\| \\
&\le \|(\hat{\mathscr{U}}_{\pm}^{(s)} - \hat{\mathscr{U}}_{\pm}^{(s+1)})\psi^{(k)}\| + (\|\hat{\mathscr{U}}_{\pm}^{(s)}\| + \|\hat{\mathscr{U}}_{\pm}^{(s+1)}\|)\,\|\psi - \psi^{(k)}\| \\
&\le \|(\hat{\mathscr{U}}_{\pm}^{(s)} - \hat{\mathscr{U}}_{\pm}^{(s+1)})\psi^{(k)}\| + \delta \sup_s \|\hat{\mathscr{U}}_{\pm}^{(s)}\|\,.
\end{aligned}$$

Since $\psi^{(k)}$ has finite support, the above quantity does not exceed 2δ for sufficiently large s. Since $l^1(\mathbb{Z}^\nu)$ is complete, any Cauchy sequence converges in the norm topology. □

Lemma 5. (i) *The limits*

$$\hat{\mathscr{U}}_{\pm} = \operatorname*{s\text{-}lim}_{t\to\infty} \hat{\mathscr{U}}_{\pm}^{(t)},$$

where $\hat{\mathscr{U}}_{\pm}^{(t)}$ *are considered as operators on* $l^2(\mathbb{Z}^\nu)$, *also exist and are partial isometries on* $l^2(\mathbb{Z}^\nu)$.

(ii) $\hat{\mathscr{U}}_{+}\hat{\mathscr{U}}_{-} = 1$ *and, therefore, both* $\hat{\mathscr{U}}_{+}$ *and* $\hat{\mathscr{U}}_{-}$ *are unitary operators on* $l^2(\mathbb{Z}^\nu)$.

(iii) *The operators* $\hat{\mathscr{U}}_{-}\hat{H}(x)\hat{\mathscr{U}}_{+} = \text{s-lim}\,\hat{D}^{(t)}$ *are diagonal in the standard basis for* $l^2(\mathbb{Z}^\nu)$.

Proof of Lemma 5. (i) The proof of this statement is quite similar to that of Lemma 4. Actually, we mention the l^1-convergence of the operators $\hat{\mathscr{U}}_{\pm}^{(s)}$ just to show that all the EF of the operators $\hat{H}(x)$ are summable on $\mathbb{Z}^\nu$. The operators $\hat{\mathscr{U}}_{\pm}^{(s)}$ are unitary in $l^2(\mathbb{Z}^\nu)$, so their strong limits must be partial isometries.

(ii) For any $s \ge 1$, $\hat{\mathscr{U}}_{+}^{(s)}\hat{\mathscr{U}}_{-}^{(s)} = 1$, so by strong convergence, the limits $\hat{\mathscr{U}}_{\pm}^{(s)}$ satisfy $\hat{\mathscr{U}}_{+}\hat{\mathscr{U}}_{-} = 1$, and, therefore, the inverses $(\hat{\mathscr{U}}_{+})^{-1} = \hat{\mathscr{U}}_{-}$, $(\hat{\mathscr{U}}_{-})^{-1} = \hat{\mathscr{U}}_{+}$ exist. Since $\hat{\mathscr{U}}_{+}$ and $\hat{\mathscr{U}}_{-}$ are shown to be partial isometries, they are actually unitary operators.

(iii) The off-diagonal components of the operators $\hat{\mathscr{U}}_{-}^{(t)}(x)\hat{H}(x)\hat{\mathscr{U}}_{\pm}^{(t)}(x)$ decay as $t \to \infty$, so the off-diagonal component of the limit is zero, provided that the limit exists. As we have already shown, the limit in question exists for a.e. $x \in S^1$. □

The statement (iii) of the Lemma 5 combined with (ii) imply that for a.e. x the operator $\hat{H}(x)$ has pure point spectrum, and Lemma 4 implies that any EF of $\hat{H}(x)$ belongs to $l^1(\mathbb{Z}^\nu)$. □

Remark. The analysis of the eigenvalues given in the proof of the Inductive Lemma shows that the multiplicity of any EV cannot be more than 2.

4. Proof of the Inductive Lemma

First of all, we introduce the following decomposition of the element $H^{(s)} \in \mathscr{A}^{\varrho_s}$:

$$H^{(s)} = D^{(s)} + C^{(s)} + U^{(s)} = D^{(s)} + C^{(s)} + U_1^{(s)} + U_2^{(s)},$$

where
(1) $D^{(s)}$ is the diagonal component of $H^{(s)}$, i.e. $D^{(s)}(n,x) = \delta_{n0} d^{(s)}(x)$;
(2) $C^{(s)}$ represents the "resonant" component, namely,

$$C^{(s)}(n,x) = \begin{cases} V^{(s)}(n,x) & \text{if } |d^{(s)}(T^n x) - d^{(s)}(x)| \le \delta_{s+1}|n|^{-B}, \\ 0 & \text{otherwise}; \end{cases}$$

(3) $H^{(s)} = D^{(s)} + C^{(s)} + U^{(s)}$;
(4) the elements $U_1^{(s)}$ and $U_2^{(s)}$ are defined as follows:

$$U_1^{(s)}(n,x) = U^{(s)}(n,x)\chi_{\{m\in\mathbb{Z}^\nu : |m| \le R^{(s+1)}\}}(n),$$
$$U_2^{(s)}(n,x) = U^{(s)}(n,x)\chi_{\{m\in\mathbb{Z}^\nu : |m| > R^{(s+1)}\}}(n).$$

Now we define explicitly the function $W_u^{(s+1,p)}(x)$. We emphasize that it is non-zero only for non-resonant (n,x). For those (n,x) we apply the standard formula of the KAM-theory (see, e.g. [3] where it was used in a similar situation):

$$W_u^{(s+1,p)}(n,x) = \frac{iV^{(s,p)}(n,x)}{d^{(s)}(T^n x) - d^{(s)}(x)}.$$

Here $V^{(s,0)}(n,x) = V^{(s)}(n,x)$; the functions $V^{(s,p)}(n,x)$ for $p > 0$ will be defined later in this argument: we just need to introduce some notations for their definition.

Besides the $\varrho^{(s)}$-norms, we will use also $\varrho^{(s,p)}$-norms with $\varrho^{(s,p)} = \varrho^{(s)}\left(1 - \dfrac{p+1}{l+2}\right)$, $0 \le p \le l = 16$. For the sake of simplicity of notations, we will write $\|\cdot\|_{s,p}$ instead of $\|\cdot\|_{\varrho^{(s,p)}}$.

We set $U_2^{(s,0)} = U_2^{(s)}$, $U_1^{(s,0)} = U_1^{(s)}$. By definition of the $\varrho^{(s,p)}$-norm, we have

$$\|W_u^{(s+1,0)}\|_{s,1} \le \|W_u^{(s+1,0)}\|_{s,0} = \sum_n{}' \frac{|U_1^{(s,0)}(n,x)| e^{\varrho^{(s,0)}|n|}}{|d^{(s)}(T^n x) - d^{(s)}(x)|},$$

where summation in $\sum'$ is taken over all $(s+1)$-nonresonant pairs (n,x), i.e. those with $U_1^{(s,0)}(n,x) \ne 0$. Then

$$\begin{aligned}\|W_u^{(s+1,0)}(n,x)\|_{s,1} &\le \sum_n |V^{(s,0)}(n,x)| e^{\varrho^{(s,0)}|n|} |n|^B \\ &\le \delta_{s+1}^{-1} \Gamma^B \varepsilon_s^{-\sigma B} (\ln \varepsilon_s^{-1})^B \|V^{(s,0)}(n,x)\|_{s,0} \\ &\le \varepsilon^{1-[(1+\kappa)\mu + \sigma(B+1)]}\end{aligned}$$

if ε is sufficiently small. We have used in the above argument the condition

$$|n| \le R^{(s+1)} = \Gamma \varepsilon_s^{-\sigma} \ln \varepsilon_s^{-1}.$$

Let us estimate now the $\varrho^{(s,1)}$-norm of $U_2^{(s,0)}$:

$$\begin{aligned}
\|U_2^{(s,0)}\|_{s,1} &= \sum_{n > R^{(s+1)}} |U_2^{(s,0)}(n,x)| \exp(\varrho^{(s,1)}|n|) \\
&= \sum_{|n| > R^{(s+1)}} |U_2^{(s,0)}(n,x)| \exp(\varrho^{(s,1)}|n|) \\
&\quad \times \exp((\varrho^{(s,1)} - \varrho^{(s,0)})\,|n|) \\
&\leq \max_{|n| > R^{(s+1)}} \exp((\varrho^{(s,1)} - \varrho^{(s,0)})\,|n|)\, \|U_2^{(s,0)}\|_{s,0} \\
&\leq \varepsilon_s \exp((l+2)^{-1} \varrho^{(s)} R^{(s+1)}) \\
&= \varepsilon_s \varepsilon_s^{(l+2)^{-1}\Gamma} \varepsilon_{s+1}^{((1+\kappa)^{-1}(1+\Gamma/(l+2)))}\,.
\end{aligned}$$

Define the function

$$H_u^{(s+1,0)} = \exp(-iW_u^{(s+1,0)}) H^{(s)} \exp(iW_u^{(s+1,0)})\,.$$

The decay of the off-diagonal component of $H_u^{(s+1,0)}$ can be estimated in the following way. First of all, we can write that

$$\begin{aligned}
H_u^{(s+1,0)} &= D^{(s)} + C^{(s)} + U_1^{(s)} + U_2^{(s)} + i[D^{(s)}, W_u^{(s+1,0)}] \\
&\quad + \sum_{k=2}^{\infty} \frac{1}{k!} (\mathrm{ad}_{iW_u^{(s+1,0)}})^k D^{(s)} + \sum_{k=1}^{\infty} \frac{1}{k!} (\mathrm{ad}_{iW_u^{(s+1,0)}})^k U_1^{(s)} \\
&\quad + (\exp(-iW_u^{(s+1,0)}) C^{(s)} \exp(iW_u^{(s+1,0)}) - C^{(s)}) \\
&\quad + (\exp(-iW_u^{(s+1,0)}) U_2^{(s)} \exp(iW_u^{(s+1,0)}) - U_2^{(s)})\,.
\end{aligned}$$

It follows from the definition of the element $W_u^{(s+1,0)}$ that

$$U_1^{(s)} + i[D^{(s)}, W_u^{(s+1,0)}] = 0\,.$$

Besides,

$$\begin{aligned}
&\|\exp(-iW_u^{(s+1,0)}) U_2^{(s)} \exp(iW_u^{(s+1,0)}) - U_2^{(s)}\|_{s+1,0} \\
&\quad \leq 4\|W_u^{(s+1,0)}\|_{s+1,0} \|U_2^{(s)}\|_{s+1,0}
\end{aligned}$$

and

$$\begin{aligned}
&\|\exp(-iW_u^{(s+1,0)}) C^{(s)} \exp(iW_u^{(s+1,0)}) - C^{(s)}\|_{s+1,0} \\
&\quad \leq 4\|W_u^{(s+1,0)}\|_{s+1,0} \|C^{(s)}\|_{s+1,0}\,.
\end{aligned}$$

Furthermore, if we denote

$$S = \sum_{k=2}^{\infty} \frac{1}{k!} (\mathrm{ad}_{iW_u^{(s+1,0)}})^k D^{(s)} + \sum_{k=1}^{\infty} \frac{1}{k!} (\mathrm{ad}_{iW_u^{(s+1,0)}})^k U_1^{(s)}\,,$$

then

$$S = i[U_1^{(s)}, W_u^{(s+1,0)}] + \sum_{k=2}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) (\mathrm{ad}_{iW_u^{(s+1,0)}})^k U_1^{(s)}$$

and

$$\|S\|_{s+1,1} \leq 2\|W_u^{(s+1,0)}\|_{s,1} \|U_1^{(s)}\|_{s,1}\,.$$

Now we expand $H_u^{(s+1,0)}$ into the sum

$$H_u^{(s+1,0)} = D^{(s,0)} + C^{(s,0)} + U_1^{(s,0)} + U_2^{(s,0)}$$

in the same way as we represented $H_u^{(s)}$. We emphasize that the non-resonant pairs (n, x) for $H_u^{(s+1,0)}$ and for $H^{(s)}$ are the same. Note also that the above norm estimates imply that

$$\|D^{(s,0)} - D^{(s)}\|_{s,1} < 4\|W_u^{(s+1,0)}\|_{s,0}(\|C^{(s)}\|_{s,1} + \|U_1^{(s)}\|_{s,1} + \|U_2^{(s)}\|_{s,1})$$

and

$$\|C^{(s,0)} - C^{(s)}\|_{s,1} < 4\|W_u^{(s+1,0)}\|_{s,0}(\|C^{(s)}\|_{s,1} + \|U_1^{(s)}\|_{s,1} + \|U_2^{(s)}\|_{s,1}).$$

It is worth to mention that the same upper bound holds for $U_1^{(s,0)}$:

$$\|U_1^{(s,0)}\|_{s,1} < 4\|W_u^{(s+1,0)}\|_{s,0}(\|C^{(s)}\|_{s,1} + \|U_1^{(s)}\|_{s,1} + \|U_2^{(s)}\|_{s,1}).$$

By iterating this procedure k times we get the element $H_u^{(s+1,k)}$ with

$$\begin{aligned} H_u^{(s+1,k)} &= \exp(iW_u^{(s+1,k)})H_u^{(s+1,k-1)} \exp(iW_u^{(s+1,k)}) \\ &= D^{(s,k)} + C^{(s,k)} + U_1^{(s,k)} + U_2^{(s,k)}, \end{aligned}$$

where the elements of the decomposition satisfy the following inequalities:

$$\begin{aligned} \|D^{(s,k)} - D^{(s)}\|_{s,1} &< 4\left(\sum_{t=0}^{k} \|W_u^{(s+1,t)}\|_{s,t+1}\right) \\ &\quad \times (\|C^{(s)}\|_{s,k+1} + \|U_1^{(s)}\|_{s,k+1} + \|U_2^{(s)}\|_{k+1}), \\ \|C^{(s,k)} - C^{(s)}\|_{s,k+1} &< 4\left(\sum_{t=0}^{k} \|W_u^{(s+1,t)}\|_{s,t+1}\right) \\ &\quad \times (\|C^{(s)}\|_{s,k+1} + \|U_1^{(s)}\|_{s,k+1} + \|U_2^{(s)}\|_{s,k+1}), \\ \|U_1^{(s,k)}\|_{s,k+1} &< 4\|W_u^{(s+1,k)}\|_{s,k} \\ &\quad \times (\|C^{(s,k)}\|_{s,k} + \|U_1^{(s,k-1)}\|_{s,k} + \|U_2^{(s,k-1)}\|_{s,k}). \end{aligned}$$

Therefore, we see that

$$\|U_1^{(s,k)}\|_{s,k+1} \le 8\varepsilon_s \varepsilon_s^{k(1-\mu(1+\kappa)-\sigma(B+1))}.$$

If μ and $\sigma \le \sigma(B)$ satisfy

$$1 - \mu(1+\kappa) - \sigma(B+1) > \kappa + \gamma$$

with $\gamma > 0$, then for $k = l = 16$ we have

$$\|U_1^{(s,1)}\|_{s,l+1} \le 8\varepsilon_s^{1+l\kappa+l\gamma'} \le (\varepsilon_{s+1})^{\frac{1+l\kappa}{1+\kappa}+\frac{l\gamma'}{2}} \le \varepsilon_{s+1}^{4+\gamma''}$$

with $\gamma'' > 0$ (if $l = 16$), and

$$\|U_2^{(s,l)}\|_{s,l+1} \le \varepsilon_{s+1}^{4+\gamma''}.$$

Note that

$$|D^{(s,l)}(x) - D^{(s)}(x)| \le \varepsilon_{s+1}^{1+\gamma''}$$

and

$$|C^{(s,l)}(n,x) - C^{(s)}(n,x)| < \varepsilon_{s+1}(\varepsilon_{s+1})^{\frac{\kappa}{1+\kappa}}$$

if (n,x) is a $(s+1)$-resonant pair, while for a nonresonant pair $C^{(s)}(n,x) = C^{(s)}(n,x) = 0$. Furthermore,

$$\|C^{(s,l)}\|(n,x)\|_{s,l+1} \le \varepsilon_{s+1}\,.$$

Thus, if $|D^{(s,l)}(x) - D^{(s)}(y)| > \varepsilon^{\mu}_{s+1}$, then

$$|D^{(s,l)}(x) - D^{(s,l)}(y)| > \varepsilon^{\mu}_{s+1} - 2\varepsilon^{1+\gamma''}_{s+1}\,,$$

and if $|D^{(s)}(x) - D^{(s)}(y)| < \varepsilon^{\mu}_{s+1}$, then

$$|D^{(s,l)}(x) - D^{(s,l)}(y)| < \varepsilon^{\mu}_{s+1} + 2\varepsilon^{1+\gamma''}_{s+1}\,.$$

Set

$$H^{(s+1,l)}_u = H^{(s+1)}_u = D^{(s+1)}_u + C^{(s+1)}_u + M^{(s+1)}_u\,,$$

where $D^{(s+1)}_u = D^{(s,l)}$, $C^{(s+1)}_u = C^{(s,l)}$, $M^{(s+1)}_u = U^{(s,l)}_1 + U^{(s,l)}_2$. The norm of the element $M^{(s+1)}_u$ can be estimated as follows:

$$\|M^{(s+1)}_u\|_{s,l+1} \le \varepsilon^{4+\gamma'''}_{s+1}\,, \qquad \gamma''' > 0\,.$$

Let us estimate the derivatives of the elements $W^{(s+1,k)}_u$:

$$\begin{aligned}\left\|\frac{dW^{(s+1,k)}_u}{dx}\right\|_{s,l+1} &\le \left\|\frac{dW^{(s+1,k)}_u}{dx}\right\|_{s,k+1}\\ &\le \text{const}\,\|U^{(s,k)}_1\|_{s,k+1}\varepsilon^{-2\mu-2\sigma(B+1)}_{s+1}\\ &\le \text{const}\,\varepsilon^{1+(k+1)\kappa-2\mu-2\sigma(B+1)}_{s}\\ &< (\varepsilon_{s+1})^{\frac{1+(k-1/2)\kappa}{1+\kappa}}\end{aligned}$$

and similarly

$$\left\|\frac{d^2W^{(s+1,k)}_u}{dx^2}\right\|_{s,l+1} < (\varepsilon_{s+1})^{\frac{1+(k-1/2)\kappa}{1+\kappa}-1}\,.$$

This yields

$$\left\|\frac{dM^{(s+1)}_u}{dx}\right\|_{s,l+1} \le \varepsilon^{3+\gamma'''}_{s+1}\,, \qquad \left\|\frac{d^2M^{(s+1)}_u}{dx^2}\right\|_{s,l+1} \le \varepsilon^{3+\gamma'''}_{s+1}\,.$$

It is easy to verify also that

$$\left\|\frac{dC^{(s+1)}_u}{dx}\right\|_{s,l+1} < \varepsilon^{1+\gamma'''}_{s+1}\,, \qquad \left\|\frac{d^2C^{(s+1)}_u}{dx^2}\right\|_{s,l+1} < \varepsilon^{1/2}_{s+1}\,.$$

Since $\varrho^{(s+1)} < \varrho^{(s,l+1)}$, we can replace in all the estimates the $\varrho^{(s,l+1)}$-norm by the $\varrho^{(s+1)}$-norm. Thus, we come to the following

Proposition 1. *There exists a positive constant γ''' such that*

$$\left\|\frac{dM^{(s+1)}_u}{dx}\right\|_{s+1} \le \varepsilon^{3+\gamma'''}_{s+1}\,, \qquad \left\|\frac{d^2M^{(s+1)}_u}{dx^2}\right\|_{s+1} \le \varepsilon^{3+\gamma'''}_{s+1}$$

and

$$\left\| \frac{dC_u^{(s+1)}}{dx} \right\|_{s+1} < \varepsilon_{s+1}^{1+\gamma'''}, \qquad \left\| \frac{d^2C_u^{(s+1)}}{dx^2} \right\|_{s+1} < \varepsilon_{s+1}^{1/2}.$$

Consider the function $d^{(s)}(x)$ obtained at the step s of the inductive construction. Recall that it takes any value at precisely two points (taking into account their multiplicities). Consider all intersections of the graphs of $\{d^{(s)}(T_\omega^n x), |n| \le R^{(s+1)}\}$.

Proposition 2. *There exists a measurable subset* $\Omega_s \subset \Omega$ *such that* $\operatorname{mes}\Omega_s \le 1/R^{(s+1)}$ *and for any* $\omega \in \Omega \backslash \Omega_s$
(i) *all the triple intersections*

$$Z_{ij}^{(3)} = \{x \in S^1 : d^{(s)}(x) = d^{(s)}(T_\omega^i x) = d^{(s)}(T_\omega^j x)\}, \qquad i, j \in \mathbb{Z}^\nu, |i|, |j| \le R^{(s+1)},$$

are empty. Moreover, there is no triple "almost intersection" satisfying the following condition:

$$|d^{(s)}(x) - d^{(s)}(T^i x)| \le \varepsilon_s^{2\sigma(B+\nu+1)}, \qquad |d^{(s)}(x) - d^{(s)}(T^j x)| \le \varepsilon_s^{2\sigma(B+\nu+1)}$$

with $|i|, |j| \le R^{(s+1)}$.
(ii) *Let* $y \in Z_i = \{x \in S^1 : d^{(s)}(x) = d^{(s)}(T^i x)\}$, $|i| \le R^{(s)}$. *Then the angle between the graphs intersecting at the point* y *is not less than* $\varepsilon_s^{\sigma(B+\nu+1)}$.

The proof of the Proposition 2 is given in Sect. 5.

Now we show how the statement of the Inductive Lemma can be proven with the help of Proposition 2. The general scheme is the following:
(1) Consider a 2×2 matrix $K^{(s+1)}$ with the following entries:

$$K_{ii}^{(s+1)}(x) = d^{(s)}(T^{n_i} x), \quad K_{ij}^{(s+1)}(x) = V^{(s+1)}(n_i - n_j, T^{n_j} x), \quad i, j = 1, 2, \ i \ne j,$$

the resonance condition (4_s) is fulfilled. Putting $y = T^{n_1} x$ we find out that $(0, n) = (0, n_2 - n_1)$ is such a group of indices. So we can restrict ourselves by the consideration of only maximal connected groups with $n_1 = 0$.

A connected component of the region on S^1 in which the resonance condition $|d^{(s)}(x) - d^{(s)}(T^n x)| < \varepsilon_{s+1}^{\mu}$ takes place will be called zones of resonances of order s. If the condition (4_s) is satisfied for a point x in some resonance zone then $|V^{(s+1)}(n, x)| > \varepsilon_{s+1}/8$ for all points of the zone. It follows from the inductive estimates of derivatives of $V^{(s+1)}(x)$ and from the estimate of size of any resonance zone given in Sect. 5. Any resonance zone is a small neighborhood of intersection of several translates of the graph of $d^{(s)}(x)$. Therefore we arrive at the auxiliary spectral problem for the matrix $K^{(s+1)}(x)$.
(2) If the condition (4_s) is fulfilled at a point of some resonance zone we perform a conjugation $Y^{(s+1)}(x)$ which diagonalize $K^{(s+1)}(x)$ for every point x in the zone and define a new function $d^{(s+1)}(x)$ in the zone as one of EV $\lambda_i(x)$ ($i = 1, 2$) of $K^{(s+1)}(x)$; the precise meaning of this statement will be clear from the rigorous proof given below. It is worth to mention that $|\lambda_1(x) - \lambda_2(x)| > \varepsilon_{s+1}/4$.
(3) Assume that the point x is contained in some resonance zone of order s. Then this point can be contained in some resonance zone of order $s + k$ for the same connected resonance group during the successive inductive steps ($k = 1, 2, \ldots$). The number of those steps is bounded by const $|\ln \mu|$.

If (4_s) is fullfilled on one of those steps we have to perform the corresponding conjugation. However it is not difficult to see that in this case the conjugation is close to the identity in ϱ_{s+k} norm and the needed estimates of derivatives are valid. Indeed, this case can be considered as the nonresonant one. Moreover $\tau^{(s+1)}(x) = \tau^{(s)}(x)$.

(4) We also have to treat the "potential" zones of resonance, if any appeared at previous steps of induction as it is explained in the previous paragraph. Inside those zones, which were not considered as resonant zones at previous steps of induction, we proceed as follows:
– define the corresponding matrix $K(x)$ again;
– if the off-diagonal component still has the norm which is too small (less than $\varepsilon_{s+2}, \varepsilon_{s+3}, \dots$, at the steps $s+2$, $s+3, \dots$), we consider again the corresponding sites as non-resonant, until this process eventually stops (if it does).
(5) If for a group of diagonal elements $\hat{D}^{(s)}_{n_1 n_1}, \hat{D}^{(s)}_{n_2 n_2}$ the above mentioned events happen infinitely many times, then they converge to a common limit as $s \to \infty$. Therefore, in this case the operator $\hat{D}(x)$ has an EV of multiplicity $k > 1$. However, this multiplicity cannot be bigger than 2 (this is related to the absence of triple intersections of graphs of $d^{(s)}(x)$ and their translates). And besides, all the EF corresponding to that degenerate EV are localized, by virtue of our estimates of the ϱ-norms of the off-diagonal components $V^{(s)}$.

Remark. We have to stress that we do *not* claim the existence of degenerate EV of the operators $\hat{H}(x)$ in question. However, the way we prove the theorem in the present paper does not allow us to claim absence of degenerate EV, either.
(6) Even if there exist zones of resonance for which the off-diagonal component of the matrix $K(x)$ is less than ε_{s+1}, as described above, the multiplicities of the corresponding groups of lattice sites cannot exceed the maximal possible value $\dim S^1 + 1 = 2$. This follows from the fact that in this case we do not subject the diagonal elements to a resonant splitting and, therefore, they undergo only a C^2-small perturbation at each step. So, the graphs of these diagonal elements, as functions of x, can get closer to each other in smaller and smaller intervals on the circle S^1. This process is controlled by the inductive estimates of non-degeneracy of the derivatives.

Now, we give the rigorous proof.
1. On the set of points $x \in S^1$ where the condition of a new resonance is fulfilled we can apply the inductive estimates of the derivatives of $d^{(s)}(x)$ and of $V^{(s)}(\cdot, x)$ given by (3_s). On any element of the partition $\xi^{(s)}$, by virtue of ($3b_s$), the following upper bound of the second derivative holds:

$$\left\| \frac{\partial^2}{\partial x^2} d^{(s)}(x) \right\| \leq 4s\varepsilon^{-1}_{\tau^{(s)}(C^{(s)}_\xi)(x)} .$$

Furthermore, we have an upper bound on the size of the new zone of resonance; namely its length is less than $\delta_s^{\mu/2} \leq \varepsilon_s^{(1-\kappa)\mu/2}$. The EV's $\lambda_+ \geq \lambda_-$ of the $K^{(s+1)}(x)$ are given by the following explicit formula:

$$\begin{aligned} \lambda_\pm(x) &= \tfrac{1}{2}(K^{(s+1)}_{11}(x) + K^{(s+1)}_{22}(x)) \\ &\quad \pm [\tfrac{1}{4}(K^{(s+1)}_{11}(x) - K^{(s+1)}_{22}(x))^2 + (K^{(s+1)}_{12}(x))^2]^{1/2} . \end{aligned} \tag{3}$$

2. The difference $|(d^{(s)})'(x) - (d^{(s)})'(T^n x)|$ is estimated in Proposition 2. The second derivative can be estimated with the help of the explicit formula (3) for the EV $\lambda_\pm(x)$ of the matrix $K^{(s+1)}(x)$:

$$|\lambda''_\pm(x)| \leq \text{const}\, |K^{(s+1)}_{12}(x)|^{-1} .$$

Moreover, a similar lower bound for the $\lambda''_\pm(x)$ holds in a neighborhood of the critical point of an EV of $K^{(s+1)}(x)$ of radius at least Const $|K^{(s+1)}_{12}(x)|$.

Remark. In a similar way the upper bounds for the derivatives of the element $Y^{(s+1)}$ can be obtained:

$$\left\| \frac{dY^{(s+1)}_{(x)}}{dx} \right\|_{s+1} < \mathrm{const}\, |K^{(s+1)}_{12}(x)|^{-1} ,$$

$$\left\| \frac{d^2Y^{(s+1)}_{(x)}}{dx^2} \right\|_{s+1} < \mathrm{const}\, |K^{(s+1)}_{12}(x)|^{-2} .$$

Now we apply the solution of the auxiliary two-dimensional spectral problem for the matrix $K^{(s+1)}(x)$ to the construction of the element $Q^{(s+1)}(x) \in \mathscr{A}^{\varrho^{(s+1)}}$. Namely, let $Y^{(s+1)}(x)$ be a unitary transformation which diagonalizes $K^{(s+1)}(x)$; certainly, such a transformation is not unique, but we fix one of those transformations. Let $n_1, n_2 \in \mathbb{Z}^\nu$ be the sites which are in resonance. Then we define the following element $Q^{(s+1)}_c \in \mathscr{A}^{\varrho^{(s+1)}}$:

$$Q^{(s+1)}_c(n_i - n_j, x) = Y^{(s+1)}_{ij}(T^{-n_j}x) , \qquad i,j = 1,2 ,$$

and $Q^{(s+1)}_c(n,x) = \delta_{n0}$ for any $n \notin \{n_i - n_j, i,j = 1,2\}$. In terms of the matrix entries of the operator $\hat{Q}^{(s+1)}_c(x)$ corresponding to $Q^{(s+1)}_c$, this means that

$$(\hat{Q}^{(s+1)}_c(x))_{n_i,n_j} = Q^{(s+1)}_c(n_i - n_j, T^{n_j}x) = Y^{(s+1)}_{ij}(x) , \qquad i,j = 1,2 ,$$

while the restriction of $\hat{Q}^{(s+1)}_c(x)$ on the subspace orthogonal to the n_1^{th} and n_2^{th} columns of $H^{(s)}(x)$ is a diagonal operator.

Remark. As it is mentioned above, the operator $Y^{(s+1)}$ is defined not uniquely. Even if its EV's are distinct, it is defined up to a reflection which interchanges the eigenvectors of $K^{(s+1)}(x)$. This is a direct consequence of the fact that the condition of $(s+1)$-resonance is fulfilled for the ordered pair of diagonal elements $\hat{H}^{(s)}_{nn}(x), \hat{H}^{(s)}_{mm}(x)$ (or, in other words, for the ordered pair (n,m) of lattice sites), then the same is true for the reversed pair $\hat{H}^{(s)}_{nn}(x), \hat{H}^{(s)}_{mm}(x)$ [resp., for (m,n)]. Therefore, we have to introduce a self-consistent way to determine when $\hat{H}^{(s)}_{nn}(x) = \lambda_+$ or $\hat{H}^{(s)}_{nn}(x) = \lambda_-$. In order to do so, we introduce the natural lexicographical order $n \succ m$ in $\mathbb{Z}^\nu$, and put $\hat{H}^{(s)}_{nn}(x) = \lambda_+$ if $n \succ m$ and $\hat{H}^{(s)}_{nn}(x) = \lambda_-$, otherwise. The points n, m should not coincide, so this convention is correct.

Since $Y^{(s+1)}(x)$ diagonalizes $K^{(s+1)}(x)$, the conjugation by $\hat{Q}^{(s+1)}_c(x)$ results in vanishing all off-diagonal entries in the considered resonant cells. In the cells which are non-resonant because of smallness of the off-diagonal entries of $K^{(s+1)}(x)$ we do not carry out any conjugation, so the magnitude of those off-diagonal entries does not change at all, hence, it does not increase.

Now we shall estimate how the conjugation by $Q^{(s+1)}_c(x)$ perturbs the norm of $M^{(s+1)}_u$.

Proposition 3. *Let* $L^{(s+1)}_c = (Q^{(s+1)}_c)^{-1} M^{(s+1)}_u Q^{(s+1)}_c$. *Then*
(i) $\|L^{(s+1)}_c\|_{\varrho^{(s+1)}} \le \mathrm{const}\, \|M^{(s+1)}_u\|_{\varrho^{(s+1)}} \le \varepsilon^4_{s+1}$;
(ii) $\|\partial/\partial x L^{(s+1)}_c\|_{\varrho^{(s+1)}} \le \varepsilon^{1+\gamma/2}_{s+1}$;
(iii) $\|\partial^2/\partial x^2 L^{(s+1)}_c\|_{\varrho^{(s+1)}} \le \varepsilon^{1/2+\gamma/2}_{s+1}$.

Proof of Proposition 3. (i) Note that the restriction of the operators $(Q^{(s+1)}_c)^{\pm 1}$ on any 2-dimensional resonant subspace is a rotation by some angle ϑ, so it is equal to $\cos(\vartheta)1 + \sin(\vartheta)\hat{S}^n$, where $\hat{S}^n$ is an operator of translation by the vector n, $|n| \le R^{(s+1)}$. Therefore,

$$\|\hat{S}^n\|_{s+1} \le \exp(\varrho^{(s+1)}R^{(s+1)}) \le 1 + \varepsilon_s^{\kappa/2}$$

yielding

$$\|\hat{Q}_c^{(s+1)}\|_{s+1} < \sqrt{2}(1 + \varepsilon_s^{\kappa/2}).$$

The statements (ii), (iii) can be proven with the help of the estimates of the derivatives of $Q_c^{(s+1)}$. □

Now we consider the element

$$J_u^{(s+1)} = (Q_c^{(s+1)})^{-1} C_u^{(s+1)} Q_c^{(s+1)}$$

and notice that the function $J_u^{(s+1)}(n, x)$ vanishes for all resonant pairs (n, x), where we carry out the resonant splitting, and coincide with $C_u^{(s+1)}(n, x)$ otherwise. Therefore, the conjugation by $Q_c^{(s+1)}$ does not affect the derivatives of $C_u^{(s+1)}(n, x)$ for those values of n and x. This allows to prove the inductive estimates of the derivatives ($3b_s$) and ($3c_s$) in the Inductive Lemma. □

Remark. In order to get the function $d^{(s+1)}(x)$ taking each value no more than twice it is sufficient to include in our inductive procedure some smoothing operator.

The details can be found in [7] where the proof of the exponential decay of EF's is given.

5. Geometry of Splitting

For the sake of simplicity of notations, we shall not write the superscript (s) in $d^{(s)}(x)$. Note that the graph of the function $d(x)$ which has exactly two critical points and two intervals of monotonicity, can intersect its own translate $d(T^n x)$ only at a point x_*, where $\operatorname{sgn} d'(x_*) = -\operatorname{sgn} d'(x_*)$. Therefore, both of those derivatives can be small simultaneously only if x is close to one of the two critical points x^* of $d(x)$ and, hence, $T^n x_*$ is equally close to the critical point $T^n x_*$ of $d(T^n x_*)$. In other words, $T^n x^*$ must be close to x^*. This simple geometrical property preserves even after resonant splitting. This follows from the analysis of resonances and from the explicit formula for the EV of the axiliary two-dimensional problem for the matrix $K(x)$ given in Sect. 4.

We begin with the proof of the statement (b) of Proposition 2.

(b) Consider a point of the intersection set Z_n introduced in the formulation of Proposition 2. The entire intersection set Z_n consists of two isolated points. At each intersection point x_*, we have $\operatorname{sgn} d'(x_*) = -\operatorname{sgn} d'(T^n x_*)$, so if these derivatives are close to each other, both of them are close to zero. Thus, we can write that

$$|d'(x) - d'(T^n x)| \ge 2\min\{|d'(x)|, |d'(T^n x)|\}.$$

Let $\tau^{(s)}(x_*) = t$. Then, by definition of $\tau^{(s)}(x)$, the point x_* did not belong to any other resonant zone since the moment $t + 1$ up to s. Therefore, we can make use of the property ($3d_s$) and consider two alternative cases:

(1) $|d'(x_*)| \ge C_2\varepsilon_t$, $C_2 = C_2(d^{(0)}) > 0$;

(2) $|d''(x_*)| \ge C_1 = C_1(d^{(0)}) > 0$.

In the case (2), by virtue of the Diophantine properties of the frequencies ω, the critical point x^* appeared in the considered resonant zone of order $t = \tau^{(s)}(x_*)$ is not too close to its translate $T^n x^*$. Namely,

$$\mathrm{dist}(T^n x^*, x^*) \geq (R^{(s+1)})^{-B-\nu-1} \geq \varepsilon_s^{\sigma(B+\nu+2)} .$$

Therefore, we can apply the inequality $|d''(x)| \geq C_1$ on the interval (x^*, x_*) and arrive at the following lower bound for the first derivative:

$$|d'(x_*)| \geq C_1 \varepsilon_s^{\sigma(B+\nu+2)} .$$

Let us set $\beta_s = \min\{C_1 \varepsilon_s^{\sigma(B+\nu+2)}, C_2 \varepsilon_t\}$. The following statement displays the relation between ε_s and ε_t.

Lemma 1. *If σ/μ is small enough, then $\varepsilon_t \geq \varepsilon_s^{2\sigma(B+\nu+1)/\mu}$.*

Proof. We start with the equality which follows directly from the definition of the sequence $\{\varepsilon_s\}$:

$$\varepsilon_t = \varepsilon^{(1+\kappa)^{-s+t}}, \qquad s > t . \tag{4}$$

Now we estimate how large the difference $s - t$ between two successive moments of resonance for any given point $x \in \mathbb{T}^1$ should be. First, we note that the condition of the t-resonance implies that $|D^{(t)}(x) - D^{(t)}(T^m x)| \leq \delta_t = \varepsilon_t^{\mu}$. Furthermore, using the inductive estimate for the first and second derivative we can write that the size of the resonant zone is less than

$$\min\{\varepsilon_t^{\mu/2}, \varepsilon_t^{\mu-\sigma(B+\nu+2)}\} \geq \varepsilon_t^{\mu/2} ,$$

if σ/μ is small enough. The distance between a given point $x \in S^1$ and its translate $T^m x$ at the step $t + k$ is greater than

$$(R^{t+k})^{-(B+\nu+1)} \geq \varepsilon_{t+k}^{\sigma(B+\nu+1)} = \varepsilon_t^{\sigma(B+\nu+1)(1+\kappa)^k} .$$

Therefore, if $\sigma(B + \nu + 1)(1 + \kappa)^k < \mu/2$, then the translates of x cannot get in the considered neighborhood of diameter $\varepsilon_t^{\mu/2}$ of the point x. Resolving the latter inequality and substituting k into (4), we complete the proof. □

Corollary. $|d'(x_*) - d'(T^n x_*)| > |d'(x_*)| \geq \varepsilon_2^{\sigma(B+\nu+2)}$. □

Inside the new resonant zone we can neglect the second order terms in the Taylor expansion of $d(x)$ and $d(T^n x)$, by virtue of the inducitve estimates (3b, c_s). Now we substitute the lower bound for the new resonant EV into the explicit formula for the new resonant EV and obtain the alternative estimates ($3d_s$) for their second derivatives [inside the neighborhood of radius $\varepsilon_s^{1-\sigma(B+\nu+2)}$ or for the first derivatives (outside that neighborhood)].

It is worth to mention that, in principle, it might happen that the resonant condition holds at a point x_* which is not close to an intersection point. We can prove, however, that this possibility is excluded by Diophantine properties of the frequencies ω. Indeed, recall that any element of the partition $\xi^{(s)}$ either is an interval of monotonicity of the function $d(x)$ or consists of two adjacent intervals of monotonicity separated by a critical point, local maximum or local minimum. Moreover, for any value $E \in \mathbb{R}$, there exists at most two elements $C_{\xi,i}^{(s)}$ such that $E \in d(C_{\xi,i}^{(s)})$. Therefore, two intervals $C_{\xi,i}^{(s)}, C_{\xi,j}^{(s)}$ either do not intersect or coincide. In the latter case $d(x)$ is monotonic on either of these intervals and its derivatives have opposite signs on them. In the former case, the intervals are separated by at least one forbidden zone which appeared in a corresponding resonance and having, therefore, the width not less than ε_t, where t is the moment of the resonance. Now we shall show that the difference $s - t$ is large enough so that the spacing ε_t is of lower order of smallness than the quantity $\delta_{s+1} = \varepsilon^{\mu}_{s+1}$. The latter quantity determines, by definition, the resonant condition at the step $s + 1$. Indeed, the Diophantine property of ω implies that any point $x \in s^1$ (hence, any critical point) cannot return to its neighborhood of radius $C(\omega)\,|n|^{-B-\nu}$ under T^n. Therefore, if $|n| \le R^{(s+1)}$ and $t = s + 1 - k$, $k < 0$, then the considered above "resonance without intersection" means that the critical point returned under T^n, $|n| \le R^{(s+1)}$, into its own neighborhood of radius $\delta_t \varepsilon_t^{-\sigma(B+\nu+1)} \le \varepsilon_2^{\mu/2}$:

$$\varepsilon^{\sigma(B+\nu+1)(1+\kappa)^s} = \varepsilon_s^{\sigma(B+\nu+1)} \le (R^{(s+1)})^{-B-\nu} \le \delta_{s+1-k} = \varepsilon_s^{\mu(1+\kappa)^{s+1-k}/2} .$$

Therefore,

$$(1+\kappa)^k \ge \frac{\mu}{2\sigma(B+\nu+1)} .$$

In order to prove that for such k the spacing due to a resonance at the moment $s + 1 - k$ should not be less than the "resonance threshold" δ_{s+1}, we have to show that the following inequality follows from the previous one:

$$\varepsilon^{(1+\kappa)^{s+1-k}} = \varepsilon_{s+1-k} \ge \varepsilon_s^{\mu/2} = \varepsilon^{(1+\kappa)^s \mu/2} ,$$

which is equivalent to $(1+\kappa)^{-k+1} \le \mu/2$. Thus, we have to prove that the inequality

$$(1+\kappa)^{k-1} \ge \frac{\mu}{2\sigma(B+\nu+1)}$$

implies

$$(1+\kappa)^{k-1} \ge 2/\mu .$$

We assume from now on that $\mu \ge 2(\sigma(B+\nu+1))^{1/2}$. Then the above inequalities become equivalent and the needed implication becomes obviously true. Note that the only relation between σ and μ we need in our proofs is that the ratio $\sigma(B+\nu+1)/\mu$ is sufficiently small, which holds if σ is sufficiently small.

(a) Assume that the condition of $(s + 1)$-resonance is fulfilled at a point x for three functions: $d(x)$, $d(T^n x)$, and $d(T^m x)$. Note that the property of double resonances proven above implies that the graphs of all the three functions appeared on the same resonant zone, so there can be no "resonance without intersection" between them. At least two functions should have the same sign of the derivative at the intersection point. So, there exists a point x_* and an integer t, $|t| \le R^{(s+1)}$, such that

$$|d(x_*) - d(T^t x_*)| \le 2\delta_{s+1} , \qquad \operatorname{sgn} d'(x_*) = \operatorname{sgn} d'(T^t x_*) .$$

Recall that inside any element of partition $\xi^{(s)}$ the function $d^{(s)}$ either is monotonic or has one critical point which separates two intervals of its monotonicity. Thus, the previous condition implies that both x_* and $T^t x_*$ lie inside subintervals where either both functions are increasing or both are decreasing. Therefore the difference between a monotonic functgion (monotonic component of $d^{(s)}$) and its translate by T^t is less than $2\delta_{s+1}$. It is not difficult to see that in this case

$$\operatorname{dist}(x_*, T^t x_*) < \delta_{s+1}^{1/4} = \varepsilon_{s+1}^{\mu/4} < \varepsilon_{s+1}^{\sigma(B+\nu+1)} ,$$

provided that $\sigma(B + \nu + 1)/\mu < 1/4$. On the other hand, the above inequality is impossible in view of the Diophantine condition, since $t < 2R^{(s+1)}$. This contradiction completes the proof. □

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