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Chern-Simons Invariants of 3-Manifolds Decomposed along Tori and the Circle Bundle Over the Representation Space of T^2

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Abstract. We describe a cut-and-paste method for computing Chern-Simons invariant of flat G-connections on 3-manifolds decomposed along tori, especially for $G = SU(2)$ and $SL(2, C)$. We use this method to make computations of $SU(2)$ Chern-Simons invariants of graph manifolds which generalize Fintushel and Stern's computations for Seifert-fibered spaces. We also use this technique to give a simple derivation of a formula of Yoshida relating the flat $SL(2, C)$ Chern-Simons invariant of the holonomy representation to the volume and the metric Chern-Simons invariant for cusped hyperbolic 3-manifolds.

1. Introduction

This paper is a continuation of [KK2]. In that paper we described a method for computing the Chern-Simons invariants of $SU(2)$ representations of a 3-manifold obtained by surgery on a knot K in a closed manifold M in terms of the image of the restriction $R(M - K) \to R(T)$, where $R(X)$ denotes the space of conjugacy classes of representations of the fundamental group of X in $SU(2)$ and T is the boundary torus of $M - K$. The main purpose of this paper is to show how to compute Chern-Simons invariants of a closed manifold in terms of an arbitrary decomposition of the manifold along tori. Cutting a 3-manifold along tori is a useful procedure in 3-manifold theory. In addition to surgery on knots and links, this includes also decompositions along incompressible tori in the sense of Jaco-Shalen and Johannson [J]. This cuts a 3-manifold into simpler pieces, namely Seifert-fibered 3-manifolds and complete hyperbolic 3-manifolds. The basic idea is to define Chern-Simons invariants for a manifold whose boundary consists entirely of tori. We then show how to use these methods to explicitly compute Chern-Simons invariants of various representations of 3-manifolds with toral boundaries, including many Seifert-fibered and hyperbolic manifolds.

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The outline of the paper is as follows. In Sect. 1 we construct explicitly an S^1 bundle $E(T)$ over $R(T)$ when T is a torus or a union of tori and prove (see Sect. 2 for the precise formulation):

Theorem 2.1. *If X is a 3-manifold with $\partial X = T$, then the Chern-Simons invariant defines a lifting $c_X : R(X) \to E(T)$ of the restriction $R(X) \to R(T)$. Moreover, there is an inner product $\langle\,\rangle$: $E(T) \times E(-T) \to S^1$ so that if Z is a closed manifold decomposed along T, $Z = X \cup_T Y$, then $c_Z(\varrho) = \langle c_X(\varrho), c_Y(\varrho)\rangle$.*

Although the existence of such a bundle was known (see [RSW]), the point of this result is that the construction we give of $E(T)$ is totally explicit and makes computations very easy. This is especially true when combined with Theorem 2.7, which shows how to compute the difference between the Chern-Simons invariants of two representations which lie on a path of representations.

Theorem 2.7. *Let X be an oriented 3-manifold with toral boundary $\partial X = T_1 \cup \cdots \cup T_n$ and let $\varrho(t) : \pi_1 X \to SU(2)$, $t \in [0,1]$ be a path of representations. Let $(\alpha_1(t),\ \beta_1(t), \ldots, \alpha_n(t),\ \beta_n(t))$ be a lift of $\varrho(t)\,|_{\partial X}$ to $\mathbb{R}^{2n}$. Suppose*

$$c_X(\varrho(t)) = [\alpha_1(t),\ \beta_1(t), \ldots, \alpha_n(t),\ \beta_n(t); z(t)]$$

for all $t \in I$. Then

$$z(1)z(0)^{-1} = \exp\left(2\pi i\left(\sum_{a=1}^{n} \int_0^1 \alpha_a \frac{d\beta_a}{dt} - \beta_a \frac{d\alpha_a}{dt}\right)\right).$$

In particular, if $\varrho(1)$ is the trivial representation (so that $z(1) = 1$ by Corollary 2.6) then

$$c_X(\varrho(0)) = \Bigg[\alpha_1(0).\beta_1(0), \ldots, \alpha_n(0),\ \beta_n(0);$$
$$\exp\left(-2\pi i\left(\sum_{a=1}^{n} \int_0^1 \left(\alpha_a \frac{d\beta_a}{dt} - \beta_a \frac{d\alpha_a}{dt}\right)\right)\right)\Bigg].$$

We refer to Sect. 2 for definitions of $[\alpha_1(0),\ \beta_1(0), \ldots, \alpha_n(0),\ \beta_n(0); z]$. (The bundle $E(T)$ is a quotient of the trivial bundle $\mathbb{R}^{2n} \times S^1$ by a discrete group.) Theorem 2.7 implies that the lift c_X is parallel with respect to the connection $\sum \alpha_k d\beta_k - \beta_k d\alpha_k$ on $E(T)$ (Corollary 2.8). Stated this way, this corresponds to the facts in [RSW] that there is a natural connection on $E(T)$ whose curvature is the symplectic form and that the image $R(X) \to R(T)$ is Lagrangian. The explicit formula for this connection makes computations easy.

We then extend the results to $SL(2,\mathbb{C})$ representations. $A/\mathbb{C}^*$ bundle $E_{\mathbb{C}}(T)$ over the character variety of the torus $R_{\mathbb{C}}(T)$ is constructed and the analogues of Theorems 2.1 and 2.7 are proven. The arguments for $SU(2)$ do not carry over immediately to $SL(2,\mathbb{C})$ because of the presence of non-diagonalizable reducible connections, and so the we carry out the necessary analysis to extend the results to the character varieties. We also make some comments about how to extend these results to $SO(3)$ and $PSL(2,\mathbb{C})$ representations.

In Sect. 4 we then carry out explicit computations of the Chern-Simons invariants for several different types of representations of 3-manifolds with toral boundary. Theorem 4.1 computes the Chern-Simons invariants for abelian representations of X

when $H^1(X;\mathbb{Z})$ is free abelian. We then compute the representation spaces and Chern-Simons invariants for $F \times S^1$, where F is a punctured surface. Then we show how the results of Auckly [A] can be used to compute the Chern-Simons invariants of certain binary dihedral representations (the *special representations* of [KK2]) of punctured surface bundles over the circle. We also compute the Chern-Simons invariants of certain Seifert-fibered spaces, namely the complements of regular fibers in Seifert-fibered homology spheres (Proposition 4.4). (The method works with no substantial changes for any Seifert-fibered 3 manifold with boundary but the formulas are messier.) These computations can then be used to understand Chern-Simons invariants of closed 3-manifolds obtained by glueing together manifolds along tori by using the inner product of Theorem 2.1.

We then use these computations and Theorem 2.1 to compute the Chern-Simons invariants of certain closed 3-manifolds. First we consider circle bundles over closed surfaces; these manifolds arise as boundaries of the complements of surfaces in 4-manifolds. We prove:

Theorem 4.3. *Let $\varrho : \pi_1 M(n) \to SU(2)$ be a representation of the circle bundle over a closed, oriented surface F with Euler class n. Conjugate ϱ so that the fiber is sent to $e^{2\pi i\beta}$. Then either $\beta = \frac{k}{n}$, in which case $cs(\varrho) = -\frac{k^2}{n}$, or else n is odd and $\beta = \frac{1}{2}$, in which case $cs(\varrho) = -\frac{n}{4}$.*

We then compute the Chern-Simons invariants of certain graph manifolds, namely the manifolds $Z = X \cup Y$, where X and Y are Seifert-fibered but the identification $\partial X \to \partial Y$ is not fiber preserving. This is the application which motivated the investigations of this paper. In an earlier paper [KKR] we showed how to compute the Floer Homology grading ([F]) of a representation of these manifolds in terms of Atiyah, Patodi, and Singer ϱ_α invariants and the Chern-Simons invariants. To finish the computation we needed to compute their Chern-Simons invariants. This is supplied by:

Theorem 4.5. *Let $\varrho : \pi_1 Z_\varphi \to SU(2)$ be a representation whose restriction to X and Y is non-abelian. The Chern-Simons invariant of ϱ is*

$$-\varepsilon_X \frac{e_X^2}{4a} - \varepsilon_Y \frac{e_Y^2}{4c} - \frac{p^2 u}{4v} - \frac{wk^2}{4}(2p+z) \operatorname{Mod} \mathbb{Z}.$$

The reader familiar with Fintushel and Stern's computations for Seifert-fibered homology spheres [FS1, FS2] will recognize the first two terms. They are "internal," depending only on the geometry of X (resp. Y) and the restriction of ϱ to X (resp. Y). The other two terms are "external" in the sense that they depend only on the gluing map $\partial X \to \partial Y$ and the restriction of ϱ to this torus.

We finish Sect. 4 with a computation which shows how to interpret Yoshida's formula [Y] relating the volume and metric Chern-Simons invariants on a hyperbolic 3-manifold obtained by hyperbolic Dehn surgery in terms of Theorem 2.1.

2. Manifolds with Boundary

Chern-Simons invariants of connections on manifolds with boundary are not gauge invariant, and there is a useful formalism described in [RSW] for dealing with them.

In that paper a certain complex line bundle over the space of flat connections on a surface F is constructed. We recall the definition of their bundle: Let F be an oriented closed surface and let $P = F \times SU(2)$ be the trivialized principal $SU(2)$ bundle over F. Let $\mathscr{A}(F)$ be the space of connections on F and $\mathscr{G}(F)$ the group of gauge transformations. The trivialization determines isomorphisms:

$$\mathscr{A}(F) \cong \Omega^1_F \otimes su(2) \quad \text{and} \quad \mathscr{G}(F) \cong \mathrm{Maps}(F, SU(2)).$$

If X is a 3 manifold and $A \in \Omega^1_X \otimes su(2)$ a lie-algebra valued 1-form on X define its Chern-Simons invariant to be:

$$cs_X(A) = \frac{1}{8\pi^2} \int_X \mathrm{Tr}(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A).$$

Now let $A \in \mathscr{A}(F)$ and $g \in \mathscr{G}(F)$. Extend A to $\tilde{A}$ over some 3-manifold and extend g to $\tilde{g}$ over X. Then define:

$$\Theta(A, g) = e^{2\pi i (cs_X(\tilde{g} \cdot \tilde{A}) - cs_X(\tilde{A}))},$$

where $\tilde{g}$ acts on $\tilde{A}$ in the usual way; so as 1-forms $\tilde{g} \cdot \tilde{A} = \tilde{g}\tilde{A}\tilde{g}^{-1} - d\tilde{g}\tilde{g}^{-1}$. Then $\Theta(A, g)$ is well defined; this follows from the fact that on a closed manifold the Chern-Simons invariant is well defined $\mathrm{mod}\, \mathbb{Z}$.

Let $\mathscr{G}(F)$ act on the trivial circle bundle over $\mathscr{A}(F)$ by:

$$g \cdot (A, z) = (g \cdot A,\ \Theta(A, g)z).$$

To see that this defines a (topological) quotient bundle $E(F)$ over $\mathscr{B}(F) = \mathscr{A}(F)/\mathscr{G}(F)$ one checks that if $g \cdot A = A$, then $\Theta(A, g) = 1$, i.e. that the fiber over fixed points is itself fixed.

Let $\mathscr{A}(X)$ denote the connections on a 3-manifold X and let $\mathscr{B}(X)$ denote the orbit space of $\mathscr{A}(X)$ under action of the gauge transformations. If $\partial X = F$, then it is a tautology that the map $A \mapsto e^{2\pi i cs(A)}$ defines a lifting of the restriction map $\mathscr{B}(X) \to \mathscr{B}(F)$ to the total space of $E(F)$.

We will construct this bundle explicitly over the space of flat connections modulo gauge transformations on a torus T by considering a 2-dimensional subspace of $\mathscr{A}(F)$ which maps onto the flat connections modulo gauge equivalence, and which is invariant under a certain discrete subgroup of $\mathscr{G}$. The bundle we construct will be explicitly defined, and so after we define it we will have to prove that it is indeed $E(T)$.

Let T be an oriented torus, and let $R(T)$ denote the space of conjugacy classes of $SU(2)$ representations of $\pi_1 T$ into $SU(2)$. As is well known, the holonomy defines a homeomorphism (in fact an analytic isomorphism) from the space of gauge equivalence classes of flat $SU(2)$ connections on T to $R(T)$. As a space $R(T)$ is homeomorphic to S^2; as a variety it has 4 singular points.

Let $V(T)$ be the two dimensional vector space

$$V(T) = \mathrm{Hom}(\pi_1(T), \mathbb{R}).$$

Then the map $V(T) \to R(T)$ defined by

$$v \mapsto (\gamma \mapsto e^{2\pi i v(\gamma)})$$

is a branched cover. (Here we are identifying $SU(2)$ with the unit quaternions. In what follows we will always use this notation, as well as identifying the lie algebra of $SU(2)$ with the pure quartenions $\mathbb{R}i \oplus \mathbb{C}j$.)

The covering group is isomorphic to a semi-direct product of $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}/2$. To see this, fix an oriented basis μ, λ for $\pi_1 T$. Give G the presentation:

$$G = \langle x, y, b \mid [x,y] = bxbx = byby = b^2 = 1 \rangle.$$

Then via the isomorphism $V(T) \to \mathbb{R}^2$ defined by $v \mapsto (v(\mu), v(\lambda))$ the action of G on $V(T) = \mathbb{R}^2$ is

$$x(\alpha, \beta) = (\alpha + 1, \beta), \quad y(\alpha, \beta) = (\alpha, \beta + 1), \quad b(\alpha, \beta) = (-\alpha, -\beta).$$

Let G act on the trivial S^1 bundle over $V(T)$ in the following way, still using μ and λ to identify $V(T)$ with $\mathbb{R}^2$ as above. Then let G act by:

$$\begin{aligned} x(\alpha, \beta; z) &= (\alpha + 1, \beta; ze^{2\pi i \beta}), \\ y(\alpha, \beta; z) &= (\alpha, \beta + 1; ze^{-2\pi i \alpha}), \\ b(\alpha, \beta; z) &= (-\alpha, -\beta; z). \end{aligned}$$

It is easy to see that this defines an action extending the action of G on $\mathbb{R}^2$. For example, $[x, y](\alpha, \beta; z) = (\alpha, \beta; ze^{4\pi i}) = (\alpha, \beta; z)$.

If $g \in G$ fixes $(\alpha, \beta) \in \mathbb{R}^2$ then $g = x^r y^s b$ for some integers r, s and $(\alpha, \beta) = \left(\frac{r}{2}, \frac{s}{2}\right)$. Thus

$$\begin{aligned} g \cdot (\alpha, \beta; z) &= (-\alpha + r, -\beta + s; ze^{2\pi i (s\alpha - r\beta + rs)}) \\ &= (\alpha, \beta; z). \end{aligned}$$

So g also fixes the fiber over (α, β).

Thus the quotient circle bundle $E(T)$ over $R(T)$ is defined:

$$E(T) = V(T) \times S^1 / G.$$

Although we have written down the action of G on $V(T) \times S^1$ in coordinates determined by the choice of μ and λ, the bundle $E(T)$ depends only on the orientation of T. To see this, suppose $m = p\mu + q\lambda$ and $l = r\mu + s\lambda$ is another choice of basis for $\pi_1 T$, where p, q, r and s are integers satisfying $ps - qr = 1$. Then $x^a y^b (v(m), v(l)) = (v(m) + pa + qb,\ v(l) + ra + sb)$. One then checks that

$$e^{2\pi i (v(l)(pa+qb) - v(m)(ra+sb))} = e^{2\pi i (v(\lambda)a - v(\mu)b)}.$$

We will fix a basis μ, λ for $\pi_1 T$ for most of this section and just write elements of $E(T)$ as $[\alpha, \beta; z]$, where the square brackets indicate the orbit of G.

Notice that if the orientation of T is reversed, then $E(T)$ is replaced by the inverse line bundle. As a smooth object this should be thought of as an orbifold bundle, double covered by an "honest" bundle over the torus $V(T)/\mathbb{Z} \oplus \mathbb{Z}$.

We can define a natural bundle map from $E(T) \times E(-T)$ to $\{pt\} \times S^1$, the trivial bundle over a point, (which we view as a bundle over $R(\phi)$ given by taking the pair $([\alpha, \beta; z], [\alpha, \beta; w])$ to $zw \in S^1$.

The definition generalizes easily to the case of a union $T_1 \cup \cdots \cup T_n$ of tori. Then $R(T_1 \cup \cdots \cup T_n) = R(T_1) \times \cdots \times R(T_n)$ and we take the bundle $E(T_1 \cup \cdots \cup T_n)$ to be the "tensor product" of the $E(T_i)$. More precisely the product action of G^n on $(\mathbb{R}^2)^n$ extends to an action of G^n on $(\mathbb{R}^2)^n \times S^1$ using the same formula as above, so that for example

$$y_k \cdot (\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n; z) = (\alpha_1 . \beta_1, \cdots, a_k, \beta_k + 1, \ldots, \alpha_n, \beta_n; ze^{-2\pi i \alpha_k}).$$

We denote the quotient bundle by $E(T_1 \cup \cdots \cup T_n)$. Again we have a natural "partial" inner product map

$$\langle , \rangle : E(T_1 \cup \cdots \cup T_n) \times E(-T_1 \cup \cdots \cup -T_m) \to E(T_{m+1} \cup \cdots \cup T_n).$$

Given a 3-manifold X with ∂X a union of tori, let $\mathscr{A}_F(X)$ denote the $SU(2)$-connections on X which are flat near the boundary. The gauge group acts on these connections and we let $\mathscr{B}_F(X)$ denote the orbit space. Notice that there is a restriction map

$$\mathscr{B}_F(X) \to R(\partial X)$$

given by taking the holonomy of the flat connection on the boundary. Although the Chern-Simons invariant is not well-defined on $\mathscr{B}_F(X)$ as an element of $\mathbb{R}/\mathbb{Z}$, the following result shows that we can define it as a section of this circle bundle. It is convenient to introduce the notation:

$$c_X(A) = e^{2\pi i cs_X(A)}$$

for $A \in \Omega^1_X \otimes su(2)$. So $c_X(A)$ is just a different way to express the Chern-Simons invariant.

2.1. Theorem. 1. *The map $A \mapsto c_X(A)$ defines a lifting of the restriction map $\mathscr{B}_F(X) \to R(\partial X)$ to $E(\partial X)$:*

$$\begin{tikzcd} & E(\partial X) \arrow[d] \\ \mathscr{B}_F(X) \arrow[r] \arrow[ur] & R(\partial X). \end{tikzcd}$$

2. *Let X, Y be oriented 3-manifolds with toral boundaries, and let $C = T_1 \cup \cdots \cup T_n$ be a collection of tori. Suppose that we are given a diffeomorphism h_X of C with a part of the boundary of X and a diffeomorphism h_Y of $-C$ with part of the boundary of Y. Let $Z = X \cup_{h_Y \circ h_X^{-1}} Y$. If A is a connection on Z which is flat near C, then*

$$c_Z(A) = \langle c_X(A_{|X}),\ c_Y(A_{|Y}) \rangle.$$

3. *The bundle $E(T) \to R(T)$ has Euler class equal to -1. More generally if $C = T_1 \cup \cdots \cup T_n$ is a union of tori, then $R(C) = \times_k R(T_k)$, so that $H^2(R(C)) = \oplus_k H^2(R(T_k))$. In this case the Euler class is $(-1, -1, \ldots, -1)$*

We will prove this later in this section. We first develop some of the ideas we will need.

Let X be a compact, oriented 3-manifold with toral boundary $\partial X = T_1 \cup \cdots \cup T_n$. For each k, choose an identification of T_k with $S^1 \times S^1$. This identifies $\mathbb{R}^2$ with the universal cover of T_k via the covering map given by $(x, y) \mapsto (e^{ix}, e^{iy})$. It also determines a symplectic basis $\mu_k, \lambda_k \in \pi_1(T_k)$ by letting μ be the image of a horizontal line and λ the image of a vertical line in $\mathbb{R}^2$ under the covering map. Also the forms dx and dy in $\mathbb{R}^2$ factor through to give us corresponding forms in $\Omega^1(T_k)$, which we will still denote by dx and dy. Let $T_k \times [0, 1] \subset X$ be a collar with T_k identified with $T_k \times 1$. This allows us to define 1-forms $\{dx, dy, dr\}$ on X near T_k. We assume the orientation of T_k as the boundary of X agrees with the orientation inherited from the cover $\mathbb{R}^2 \to T_k$, so that $\{dx, dy, dr\}$ is an oriented basis of 1-forms on X. Thus we orient ∂X with "outward normal last" convention. Stokes' theorem says $\int_X d\omega = \int_{\partial X} \omega$ on a 3-manifold X with this convention.

With respect to the trivialization of the $SU(2)$ bundle over X any connection A can be written as

$$A = \alpha\, dx + \beta\, dy + \gamma\, dr$$

in a neighborhood of T_k, where

$$\alpha, \beta, \gamma : T_k \times I \to su(2).$$

We will focus on the connections for which α and β are constants and γ is zero:

2.2. Definition. We say A is a *normal form* if for each k there exist $\alpha_k, \beta_k \in \mathbb{R}$ so that in a neighborhood of T_k,

$$A = i\alpha_k\, dx + i\beta_k\, dy.$$

Similarly, if $A(t)$ is a path of connections on X we say that the path is in normal form if each A_t is; in other words there exist functions $\alpha_k, \beta_k : I \to \mathbb{R}$ so that

$$A(t) = i\alpha_k(t)\, dx + i\beta_k(t)\, dy.$$

Using the local definition of the curvature $F^A = dA + A \wedge A$ it is easy to see that if $A = i\alpha_k\, dx + i\beta_k\, dy$ near T_a, then A is flat near T_k. Furthermore, the holonomy representation of $\pi_1 T_k$ (with respect to a base point near T_k) is given by

$$\mu_k \mapsto e^{2\pi i \alpha_k}, \quad \lambda_k \mapsto e^{2\pi i \beta_k}.$$

This can be computed directly from the definition of holonomy. Alternatively one uses developing maps as explained in [KK2]. The map $D : \mathbb{R}^2 \to SU(2)$ given by $(x, y) \mapsto e^{-2\pi i(\alpha x + \beta y)}$ is a developing map for this holonomy and $A = -dDD^{-1}$.

Notice that the representations

$$\mu_a \mapsto e^{2\pi i \alpha}, \quad \lambda \mapsto e^{2\pi i \beta}$$

and

$$\mu_a \mapsto e^{2\pi i \tilde{\alpha}}, \quad \lambda \mapsto e^{2\pi i \tilde{\beta}}$$

are conjugate in $SU(2)$ if and only if there exist integers m, n and $\varepsilon \in \{\pm 1\}$ such that $\tilde{\alpha} = \varepsilon i(\alpha + m)$ and $\tilde{\beta} = \varepsilon i(\beta + n)$. Since gauge-equivalent flat connections have conjugate holonomy it follows that if $i\alpha_k\, dx + i\beta_k\, dy$ and $i\tilde{\alpha}_k\, dx + i\widetilde{\beta_k}\, dy$ are gauge equivalent then a, $\tilde{\alpha}$ and β, $\tilde{\beta}$ are related as above.

2.3 Proposition. 1. *Let A be a connection on X which is flat in a neighborhood of ∂X. Then there exists a gauge transformation g so that $g \cdot A$ is in normal form.*
2. *Let $A(t)$ be a path of connections on X so that $A(t)$ is flat near ∂X for each t. Then there exists a path of gauge transformations $g(t)$ so that $g(t) \cdot A(t)$ is in normal form for each t. Furthermore, if $A(0)$ is already in normal form we can choose $g(0) = Id$.*
3. *Let A be any connection on X which is flat near ∂X. Then for any choice of $\alpha_k, \beta_k \in \mathbb{R}$, $k = 1, \ldots, n$ such that (up to conjugation)*

$$(\varrho_A(\mu_k), \varrho_A(\lambda_k)) = (e^{2\pi i \alpha_k}, e^{2\pi i \beta_k})$$

(where ϱ_A denotes the holonomy representation near ∂X) there exists a gauge transformation g supported near ∂X so that

$$g \cdot A = i\alpha_k \, dx + i\beta_k \, dy$$

near T_k.

Proof. 1. First observe that this is a local question since if we can find a gauge transformation g near ∂X the obstructions to extending g over X lie in $H^i(X, \partial X; \pi_{i-1}(SU(2)))$ and these groups are all zero. So it suffices to show that any flat connection on $T^2 \times I$ can be gauge transformed to this form. But this follows from the fact that the holonomy gives a homeomorphism between the set of flat connections modulo gauge equivalence and the space of conjugacy classes of representations of the fundamental group.

2. If $g(t)$ is a path of gauge transformations defined in a neighborhood of ∂X, then $g(t)$ can be extended to a path on X since $H^i(X \times I, \partial X \times I; \pi_{i-1} SU(2)) = 0$. If $A(0)$ is already in normal form, we can take $g(0)$ to be the identity since the obstructions then lie in $H^i(X \times I, \partial X \times I \cup X \times 0; \pi_{i-1} SU(2)) = 0$.

Notice that we cannot in general gauge transform a path into normal form leaving *both endpoints* $A(0)$ and $A(1)$ fixed. The obstruction to doing this is usually non-zero.

3. Using the first part of this proposition we may put A in normal form by a gauge transformation. Suppose that $A = i\alpha \, dx + i\beta \, dy$ near T_1. We will show that there are gauge transformations g_x, g_y and g_b equal to the identity outside a neighborhood of T_1 so that

$$g_x \cdot A = i(\alpha + 1) \, dx + i\beta dy,$$
$$g_y \cdot A = i\alpha \, dx + i(\beta + 1) dy,$$

and

$$g_b \cdot A = -i\alpha \, dx - i\beta \, dy.$$

By composing these gauge transformations we can get any normal form on T_1, and similarly near the entire collection of tori ∂X.

Let $h : S^1 \to S^1 \subset SU(2)$ be the map $e^{ix} \mapsto e^{-ix}$. As a map into $SU(2)$, h is nullhomotopic. Let h_t, $t \in [0,1]$ be a nullhomotopy which is constant for t near 0 or 1, with h_0 constant at $1 \in SU(2)$ and $h_1 = h$. Use h_t to define a map $f : S^1 \times [0,1] \to SU(2)$ which equals h near $S^1 \times 1$ and is equal to 1 near $S^1 \times 0$. The map g_x is then defined in a collar of T_1 to be the composition of the projection $T_1 = S^1 \times S^1 \to S^1$ onto the first factor with the map f. Then extend g_x to be 1 outside this collar. Notice that $dg_x g_x^{-1} = -i \, dx$ near $T_1 \times 1$.

If we let $B = g_x \cdot A$, then near T_1:

$$B = gA_g^{-1} - dgg^{-1} = A + i \, dx.$$

We construct g_y in the same way, taking the projection $T_1 \to S^1$ onto the second factor.

To construct g_b, repeat the construction starting with the map $h : T_1 \to SU(2)$ which is the constant map at j. It will be convenient in what follows to assume that the extension to $T_1 \times I$ factors through the projection $T_1 \times I \to I$.

Since $jij^{-1} = -i$, this has the effect of reversing the sign of α and β. □

Remark. We will use the gauge transformations g_x, g_y and g_b later. We note here that

$$\frac{dg_x}{dy} = 0, \quad \frac{dg_y}{dx} = 0, \quad \frac{dg_b}{dx} = 0 \quad \frac{dg_b}{dy} = 0.$$

The following result shows that gauge equivalent connections in normal form have the same Chern-Simons invariants if they agree near the boundary.

2.4. Theorem. *Suppose that X is a 3-manifold whose boundary components are tori: $\partial X = T_1 \cup \ldots \cup T_n$. Let A and B be connections on X in normal form with respect to ∂X and suppose that*
1. *A and B are equal near the boundary.*
2. *A and B are gauge equivalent. Then $cs(A)$ and $cs(B)$ coincide* Mod $\mathbb{Z}$.

Proof. Let g be a gauge transformation such that $g \cdot A = B$. We first assume that g is trivial near ∂X. In this case, the connections $A \cup A$ and $A \cup B$ on $(-X) \cup X$ are gauge equivalent by the gauge transformation $1 \cup g$. Since cs is additive over unions, we may cancel $cs_{-X}(A)$ to conclude that $cs_X(A) = cs_X(B)$.

Now drop the assumption that g is trivial near ∂X. Since $A = B$ near ∂X, we know that

$$i\alpha_k\, dx + i\beta_k\, dy = g(i\alpha_k\, dx + i\beta_k\, dy)g^{-1} - dgg^{-1}$$

near each T_k. We conclude that $\dfrac{\partial g}{\partial r} = 0$. This implies that g is constant in the r direction, and fixes A as a connection on T_k. The group of those gauge transformations on T_k which fix $A_{|T_k}$ is isomorphic to the centralizer in $SU(2)$ of the image of the holonomy of $A_{|T_k}$. Because $\pi_1(T_k)$ is abelian, this centralizer is either $S^1 \subset SU(2)$ or all of $SU(2)$. In any case, it is connected. Hence for each k we may choose a path $g_k(t)$ of gauge transformations on T_k from $g_k(0) = 1$ to $g_k(1) = g_{|T_k}$ such that for all t, $g_k(t)$ fixes $A_{|T_k}$. (Assume each path is constant near each end.) View each g_k as a gauge transformation on $T_k \times I$, where $T_k \times I$ is a collar of T_k on which A and B are in normal form. Define a gauge transformation $\tilde{g}$ on X by setting $\tilde{g} = g$ outside of the collars $T_k \times I$ and $\tilde{g} = g_k(t)$ on $T_k \times \{t\}$, for $t \in I$. Let $C = \tilde{g} \cdot A$. By the first paragraph of this proof, we know that $cs(C) = cs(A)$. We will now show that $cs(C) = cs(B)$.

Note that on the complement of the collars the two connections are identical, so we need only consider their Chern-Simons integrals over the collars. For each $s \in I$, define a gauge transformation h_s on $T_k \times I \times \{s\}$ by setting $h_s = g_k(1 - s + st)$ on $T_k \times \{t\} \times \{s\}$. Consider the flat connection $\tilde{A}_s = h_s \cdot A$ on $T_k \times I \times \{s\}$. Define a connection $\mathbb{A}$ on $T_k \times I \times I$ to be the union of the $\tilde{A}_s$. Since $\mathbb{A}$ is actually a path of flat connections on $T_k \times I$, it follows that

$$0 = \int_{T_k \times I \times I} \mathrm{Tr}(F^{\mathbb{A}} \wedge F^{\mathbb{A}}).$$

By Stokes' theorem, however, this integral is equal to the sum of a Chern-Simons integral over $T_k \times \partial I \times I$ and one over $T_k \times I \times \partial I$. The former integral vanishes (because it is a constant path of connections over T_k), while the latter is equal to

$$\int_{T_k \times I} \mathrm{Tr}\left(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A\right) - \int_{T_k \times I} \mathrm{Tr}\left(dC \wedge C + \tfrac{2}{3} C \wedge C \wedge C\right)$$

which proves these latter two integrals are equal and completes the proof of the theorem. $\square$

The next result shows how the Chern-Simons invariant changes when a flat connection is altered near the boundary. A consequence of this theorem is that the bundle $E(T)$ is the same as the bundle constructed in [RWS].

2.5 Theorem. *Let X be a manifold whose boundary is a union of tori $\partial X = T_1 \cup \cdots \cup T_n$ and let A and B be gauge-equivalent connections in normal form. Suppose that near T_k,*

$$A = i\alpha_k\, dx + i\beta_k\, dy$$

and

$$B = \varepsilon_k(i(\alpha_k + m_k)\, dx + i(\beta_k + n_k)dy)$$

for some collection of integers m_k, n_k, $k = 1, \dots, n$ and signs $\varepsilon_k \in \{\pm 1\}$. Then

$$cs(B) - cs(A) = \sum_{k=1}^{n} m_k\beta_k - n_k\alpha_k \operatorname{Mod} \mathbb{Z}.$$

Proof. Suppose g is a gauge transformation so that $g \cdot A$ agrees with B near the boundary. Then we have seen that $cs(g \cdot A) = cs(B)$.

The proof therefore reduces to showing that $cs(g_x \cdot A) = cs(A) + \beta_1$, $cs(g_y \cdot A) = cs(A) - \alpha_1$, and $cs(g_b \cdot A) = cs(A)$, where g_x, g_y, g_b are the gauge transformations constructed in the proof of Proposition 2.3.

The difference $cs(g_x \cdot A) - cs(A)$ is the integral of a function supported in $S^1 \times S^1 \times I$, since g_x is the identity outside a collar. Moreover, the Chern-Simons integrand vanishes for A on $S^1 \times S^1 \times I$ since $dA = 0$ and $A \wedge A \wedge A = 0$. Thus we must show that if $C = g_x \cdot A$ on $S^1 \times S^1 \times I$, then

$$\frac{1}{8\pi^2} \int_{S^1 \times S^1 \times I} \operatorname{Tr}\left(dC \wedge C + \frac{2}{3} C \wedge C \wedge C\right) = \beta_1.$$

We do this computation directly. This is similar to the argument in the proof of Theorem 4.2 of [KK]. Write $g = g_x$. First notice that $g \cdot A$ is flat on $S^1 \times S^1 \times I$ so that

$$\operatorname{Tr}\left(dC \wedge C + \tfrac{2}{3} C \wedge C \wedge C\right) = -\tfrac{1}{3} \operatorname{Tr}(C \wedge C \wedge C).$$

Using the remark after the proof of Proposition 2.3,

$$\begin{aligned} C = g \cdot A &= gAg^{-1} - dgg^{-1} \\ &= \left(gi\alpha_1 g^{-1} - \frac{\partial g}{\partial x} g^{-1}\right) dx + gi\beta_1 g^{-1} dy - \frac{\partial g}{\partial r} g^{-1} dr. \end{aligned}$$

A bit of manipulation yields:

$$-\frac{1}{3} \operatorname{Tr}(C \wedge C \wedge C) = \beta_1 \operatorname{Tr}\left(\left[g^{-1}\frac{\partial g}{\partial x},\ g^{-1}\frac{\partial g}{\partial r}\right] i\right) dx\, dy\, dr. \qquad (*)$$

We can now integrate out the y variable since the form is constant in the y direction and since we are integrating over a product manifold. So

$$\frac{1}{8\pi^2} \int_{S^1 \times S^1 \times I} \operatorname{Tr}\left(dC \wedge C + \frac{2}{3} C \wedge C \wedge C\right) = \frac{\beta_1}{4\pi} \int_{S^1 \times I} \operatorname{Tr}\left(\left[g^{-1}\frac{\partial g}{\partial x},\ g^{-1}\frac{\partial g}{\partial r}\right] i\right) dx\, dr.$$

Let $\omega = \mathrm{Tr}(ig^{-1}dg)$. Then

$$d\omega = \mathrm{Tr}\left(\left[g^{-1}\frac{\partial g}{\partial x},\, g^{-1}\frac{\partial g}{\partial r}\right]i\right) dr\, dx$$

on $S^1 \times I$.

So we need to compute $\int_{S^1\times I} d\omega$, which by Stokes' theorem equals $\int_{S^1\times 1}\omega - \int_{S^1\times 0}\omega$. (A bit of care must be taken to see that the signs are correct.) Since g is the constant map at $1 \in SU(2)$ near $S^1 \times 0$, $\int_{S^1\times 0}\omega$ is zero. Near $S^1 \times 1$, $g(e^{ix}) = e^{-ix}$ and so

$$\mathrm{Tr}(ig^{-1}dg) = 2\,\mathrm{Re}(ie^{ix}(-i)e^{-ix}dx) = 2\,dx.$$

(For $v, w \in su(2)$, $\mathrm{Tr}(vw) = 2\,\mathrm{Re}(vw)$ when viewed as unit quarternions.) Therefore, the integral of ω over $S^1 \times 1$ is 4π. Hence:

$$\mathrm{Tr}\left(dC \wedge C + \tfrac{2}{3}C \wedge C \wedge C\right) = \beta_1.$$

A similar argument applies to g_y. The opposite sign arises from orientation considerations. The argument involving g_b is simpler. Since $\frac{\partial g_b}{\partial x}$ vanishes, equation $(*)$ shows that the Chern-Simons integral vanishes on the collar, completing the proof. □

We have the following corollary:

2.6 Corollary. *Let X be a compact, oriented 3-manifold whose boundary is a union of tori. Suppose A is a flat connection in normal form with respect to ∂X so that the holonomy representation of A is trivial. Then*

$$cs_X(A) \in \mathbb{Z}.$$

Proof. The connection A is gauge-equivalent to the trivial connection, whose Chern-Simons invariant is 0. Thus

$$cs(A) \equiv \sum_k m_k\beta_k - n_k\alpha_k \,\mathrm{Mod}\,\mathbb{Z}.$$

but the α_k and β_k are integers, since the holonomy representation is trivial. □

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Given $[A] \in \mathscr{B}_F(X)$, choose a representative A in normal form. If $A = i\alpha_k\, dx + i\beta_k\, dy$ near T_k let

$$c_X([A]) = [\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n; e^{2\pi i cs_X(A)}].$$

Theorem 2.5 shows that $c_X([A])$ transforms properly.

The second assertion of the theorem is now obvious.

To prove the third assertion, observe that the region $[0,\pi]\times[0,2\pi]$ is a fundamental region for the action of G on $\mathbb{R}^2$. The identifications of the boundary are indicated in the next figure. Split the region into two pieces: $A = [0,\pi] \times [0,\pi]$ and $B = [0,\pi] \times [\pi, 2\pi]$. The space $R(T)$ is homeomorphic to S^2 and A and B map to hemispheres. The trivial S^1 bundle over $\mathbb{R}^2$ restricts to trivializations over A and B.

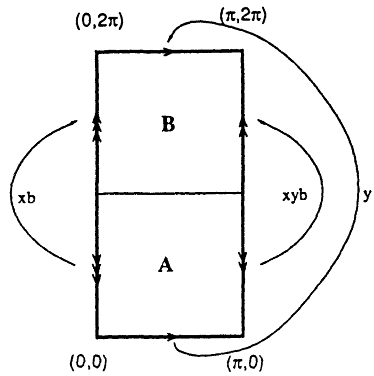

Fig. 1

Thus we need to compute the degree of the clutching function $\partial A \to S^1$. Using the figure one can see that this degree is -1. □

The invariant $c_X(\varrho)$ is an invariant of rel boundary flat cobordism. More precisely, if M is a 4-manifold with boundary $X_o \cup \partial X_0 \times I \cup -X_1$ and $\varrho : \pi_1 M \to SU(2)$ is a representation then $c_{X_0}(\varrho) = c_{X_1}(\varrho)$. This can be seen as follows. Choose a flat connection $\mathbb{A}$ on M with holonomy ϱ and in normal form near $\partial X_0 \times I$. Then the identity

$$\mathrm{Tr}(F^{\mathbb{A}} \wedge F^{\mathbb{A}}) = d\,\mathrm{Tr}\left(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A\right).$$

Stokes' theorem, and the observation that $\mathrm{Tr}\left(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A\right)$ vanishes when $A = i\alpha\, dx + i\beta\, dy$ imply that

$$0 = \frac{1}{8\pi^2} \int\limits_{X \times I} \mathrm{Tr}(F^{\mathbb{A}} \wedge F^{\mathbb{A}}) = cs_{X_0}(A) - cs_{X_1}(A).$$

We turn now to our main tool for computing the Chern-Simons invariants. This result generalizes the main theorem of [KK2]. It should also be viewed as an extension of the observation of the previous paragraph. Rather than considering a flat connection on $X \times I$, we consider a path of flat connections on X which we view as a connection on $X \times I$ which is "flat except in the t direction."

2.7 Theorem. *Let X be an oriented 3-manifold with toral boundary $\partial X = T_1 \cup \cdots \cup T_n$ and let $\varrho(t) : \pi_1 X \to SU(2)$, $t \in [0,1]$ be a path of representations. Let $(\alpha_1(t), \beta_1(t), \ldots, \alpha_n(t), \beta_n(t))$ be a lift of $\varrho(t)$ to $\mathbb{R}^{2n}$. Suppose*

$$c(\varrho(t)) = [\alpha_1(t), \beta_1(t), \ldots, \alpha_n(t), \beta_n(t); z(t)]$$

for all $t \in I$. Then

$$z(1)z(0)^{-1} = \exp\left(2\pi i \left(\sum_{a=1}^{n} \int_0^1 \alpha_a \frac{d\beta_a}{dt} - \beta_a \frac{d\alpha_a}{dt}\right)\right).$$

In particular, if $\varrho(1)$ is the trivial representation (so that $z(1) = 1$ by Corollary 2.6) then

$$c(\varrho(0)) = \Bigg[\alpha_1(0),\ \beta_1(0), \ldots, \alpha_n(0),\ \beta_n(0);$$

$$\exp\left(-2\pi i\left(\sum_{a=1}^{n}\int_0^1\left(\alpha_a\frac{d\beta_a}{dt} - \beta_a\frac{d\alpha_a}{dt}\right)\right)\right)\Bigg].$$

Proof. Choose a path $A(t)$ of flat connections on X in normal form so that near T_a,

$$A(t) = i\alpha_a(t)\,dx + i\beta_a(t)\,dy$$

such that the holonomy of $A(t)$ is $\varrho(t)$. Orient $X \times I$ near $\partial X \times I$ by taking $\{dt,\ dx,\ dy,\ dr\}$ to be an oriented basis. The path $A(t)$ determines a connection $\mathbb{A}$ over $X \times I$ which satisfies $\mathrm{Tr}(F^{\mathbb{A}} \wedge F^{\mathbb{A}}) = 0$, since $F^{\mathbb{A}}$ has only a dt component.

As explained in the paragraph preceding this theorem:

$$0 = cs(A(1)) - cs(A(0)) - \frac{1}{8\pi^2}\int_{\partial X\times I}\mathrm{Tr}\left(dC\wedge C + \frac{2}{3}C\wedge C\wedge C\right),$$

where C is the 1-form over $\partial X \times I$ given by $C(x, y, t) = A(t)$.

On $\partial X \times I$, $C \wedge C \wedge C = 0$ and

$$\mathrm{Tr}(dC \wedge C) = 2\left(\sum_k \alpha_k\frac{d\beta_k}{dt} - \beta_k\frac{d\alpha_k}{dt}\right)dx\,dy\,dt.$$

Therefore:

$$\begin{aligned} cs(A(1)) - cs(A(0)) &= \frac{1}{8\pi^2}\int_{\partial X\times I}\mathrm{Tr}\left(dC\wedge C + \frac{2}{3}C\wedge C\wedge C\right) \\ &= \sum_{k=1}^{n}\int_0^1 \alpha_k\frac{d\beta_k}{dt} - \beta_k\frac{d\alpha_k}{dt}. \end{aligned}$$

Since $e^{2\pi i cs(A_1)}e^{-2\pi i cs(A_0)} = z(1)z(0)^{-1}$, the theorem follows. □

We next give a more abstract restatement of the previous theorem, together with some additional differential geometric information about $E(T)$.

2.8 Corollary. 1. *The connection 1-form*

$$A = -2\pi i\sum_k \alpha_k d\beta_k - \beta_k d\alpha_k$$

on the principal $U(1)$ bundle $\mathbb{R}^{2n} \times U(1)$ descends to give an orbifold connection on $E(T) = \mathbb{R}^{2n} \times U(1)/G_n$. Given a 3-manifold X with boundary a union of tori, the lift $c_X : R(X) \to E(T)$ of the restriction map is parallel with respect to the connection A.

2. *The curvature of A is $F^A = -4\pi i\sum_k d\alpha_k \wedge d\beta_k$. This is a real multiple of the symplectic form on the symplectic orbifold $R(T)$.* □

Observe that by Chern-Weil theory the Euler class is $-2\sum_k d\alpha_k d\beta_k$. Restricting to each factor $R(T_k) \subset R(\partial X)$ we get $-2d\alpha_k d\beta_k$. Finally integrating over the fundamental domain for the action $\left[0, \frac{1}{2}\right] \times [0, 1]$ we see that the Euler number is -1. So this gives an alternative argument for the 3rd part of Theorem 2.1.

Consider the following situation. We are given a closed oriented 3-manifold Z and a collection or tori: $T_1 \cup \cdots \cup T_n \subset Z$. Let $\varrho : \pi_1 Z \to SU(2)$ be a representation so that if $X_1 \cup \cdots \cup X_m$ denotes the components of Z cut along C, the restrictions $\varrho_\imath = \varrho_{|X_\imath}$ lie on a piecewise smooth path of representations $\varrho_\imath(t)$ to the trivial representation. (Notice that each torus appears twice in the boundary of $\cup_i X_i$.) Then, making use of Theorems 2.1 and 2.7 we can compute the Chern-Simons invariant of ϱ. Theorem 2.7 is used to compute the Chern-Simons invariants of the pieces, and Theorem 2.1 to compute the effect of gluing the pieces together. In Sect. 4 we will describe how to compute the representations and Chern-Simons invariant for a number of 3 manifolds.

One last remark is needed for computations: For a given representation of a 3-manifold X there is usually a convenient choice of basis of $\pi_1 \partial X$ determined by the topology of X and the representation. (For example, if ϱ is an abelian representation and $\partial X = T^2$ it is convenient to take λ to be the generator of $\operatorname{Ker} H_1 T \to H_1 X$ since then λ is sent to 1 along any path of abelian representations. For this choice, then, $\beta(t)$ can be taken to be the constant path at 0.) However, in glueing X to Y along T we must choose the same normal forms for $A_{|X}$ and $A_{|Y}$ to compute $\langle c(A_{|X}), c(A_{|Y})\rangle$. Any linear change of coordinates of T takes connections in normal form to connections in normal form. More precisely, if

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z})$$

then M determines a change of basis of the 1-forms

$$dX = p\,dx + r\,dy, \quad dY = q\,dx + s\,dy,$$

and so if A is in normal form with respect to dx, dy it remains in normal form with respect to dX, dY. Of course, the Chern-Simons invariant does not depend on the choice of local coordinates. The point is that the vector space $V(T)$ is intrinsic, but to do explicit computations requires fixing a choice of basis for $\pi_1 T$.

We can now outline the "algorithm" we will use to compute Chern-Simons invariants in the simplest case of a decomposition $Z = X \cup Y$ along one torus. If $\varrho : \pi_1 Z \to SU(2)$ is a representation then let ϱ_X and ϱ_Y denote the restrictions to X and Y. Choose convenient bases $\{\mu_X, \lambda_X\}$ for $\pi_1(\partial X)$ (resp. $\{\mu_Y, \lambda_Y\}$ for $\pi_1(\partial Y)$). Suppose there exist paths $\varrho_X(t)$ and $\varrho_Y(t)$ of the restrictions to the trivial representation. Then with respect to the bases we obtain paths $\alpha_X(t)$, $\beta_X(t)$ and $\alpha_Y(t)$, $\beta_Y(t)$. We now compute the expression of Theorem 2.7 to get

$$c(A_{|X}) = [\alpha_X(0),\ \beta_X(0); c_X]$$

and

$$c(A_{|Y}) = [\alpha_Y(0),\ \beta_Y(0); c_Y].$$

Next we express $\{\mu_Y, \lambda_Y\}$ in terms of $\{\mu_X, \lambda_X\}$ using a matrix

$$M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Since the representations ϱ_X and ϱ_Y agree on the boundary, it follows that there are integers m, n such that

$$\begin{pmatrix} \alpha_X(0) \\ \beta_X(0) \end{pmatrix} = \pm \left(M \begin{pmatrix} \alpha_Y(0) \\ \beta_Y(0) \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} \right).$$

Now Theorem 2.1 can be used to adjust $c(A_{|Y})$ to get

$$c(A_{|Y}) = [\alpha_Y(0), \beta_Y(0), c_Y] = [\alpha_X(0), \beta_X(0); c_Y e^{2\pi i(m\beta_X(0) - n\alpha_X(0))}]$$

so that

$$c_Z(\varrho) = c_X c_Y e^{2\pi i(m\beta_X(0) - n\alpha_X(0))}.$$

3. $SL(2, C)$-Representations

The results for $SU(2)$ generalize to the larger group $SL(2, \mathbb{C})$. Even for some $SU(2)$ computations it is convenient to consider $SL(2, \mathbb{C})$ representations since $\mathrm{Hom}(\pi, SL(2, \mathbb{C}))$ is an algebraic variety containing the real subvariety $\mathrm{Hom}(\pi, SU(2))$. In particular, a given $SU(2)$ representation may not lie on a path of $SU(2)$ representations to the trivial representation, but there may nevertheless be a path to the trivial representation in the space of $SL(2, \mathbb{C})$ representations. There are some subtleties which arise in going to the $SL(2, \mathbb{C})$ representations which we describe in this section. The essential point is that all the results of the previous section extend to $SL(2, \mathbb{C})$ representations provided we work with the character varieties instead of the representation spaces.

Consider a (trivial) principal $SL(2, \mathbb{C})$ bundle $P = X \times SL(2, \mathbb{C})$ over a 3 manifold X. As before the trivialization allows us to identify the space of all connections in this bundle with the $sl(2, \mathbb{C})$-valued 1 forms on X. The Chern-Simons invariant of $A \in \Omega_X \otimes sl(2, \mathbb{C})$ is defined as before, but notice that this time the expression $\mathrm{Tr}\left(dA \wedge A + \frac{2}{3} A \wedge A \wedge A\right)$ is a complex-valued 3-form on X. Hence

$$cs_X(A) = \frac{1}{8\pi^2} \int_X \mathrm{Tr}\left(dA \wedge A + \frac{2}{3} A \wedge A \wedge A\right)$$

is in general a complex number. Changing A by a gauge transformation preserves $cs_X(A) \bmod \mathbb{Z}$ when X is closed. We will construct a $\mathbb{C}^*$ bundle over the space of $SL(2, \mathbb{C})$ representations of a surface mod conjugacy in the same way as the $SU(2)$ case.

Notice that $SL(2, \mathbb{C})$ is a non-compact group, and the space

$$\mathrm{Hom}(\pi, SL(2, \mathbb{C}))/\text{conjugation}$$

is often badly behaved, e.g. non-Hausdorff. There is, however, a natural further quotient of $\mathrm{Hom}(\pi, SL(2, \mathbb{C}))$, the *character variety,* which is an algebraic variety. Our immediate task is to show that the results for $SU(2)$ representations of Sect. 2 extend to these character varieties. A good reference for these varieties is [CS].

Recall that the character variety of a finitely presented group π into $SL(2, \mathbb{C})$ is the variety whose ring of functions is the functions on $\mathrm{Hom}(\pi, SL(2, \mathbb{C}))$ which are invariant under the conjugation action of $SL(2, \mathbb{C})$. It can be explicitly realized as follows. There is a finite set $\gamma_1, \ldots, \gamma_m$ of elements of π so that the functions

$$\tau_{\gamma_i} : \mathrm{Hom}(\pi, SL(2, \mathbb{C})) \to \mathbb{C}$$

defined by

$$\tau_{\gamma_i}(\varrho) = \mathrm{Tr}(\varrho(\gamma_i))$$

generate the ring of invariant functions. Then the image of the map

$$t : \mathrm{Hom}(\pi, SL(2,\mathbb{C})) \to \mathbb{C}^m$$

taking ϱ to the n-tuple $(\varrho(\gamma_1), \ldots, \varrho(\gamma_m))$ is closed, and is the character variety. We denote this variety by $R_{\mathbb{C}}(\pi)$ or $R_{\mathbb{C}}(X)$ when $\pi = \pi_1 X$. Two representations ϱ_1 and ϱ_2 have the same image in the character variety if and only if $\mathrm{Tr}(\varrho_1(\gamma)) = \mathrm{Tr}(\varrho_2(\gamma))$ for all $\gamma \in \pi$. In particular conjugate representations have the same character.

An $SL(2,\mathbb{C})$ representation is irreducible if the natural action on $\mathbb{C}^2$ has no invariant subspaces. If ϱ_1 and ϱ_2 have the same character and ϱ_1 is irreducible, then ϱ_1 and ϱ_2 are conjugate (see, for example [CS]. Proposition 1.5.2). Thus $R_{\mathbb{C}}(\pi)$ is a quotient of the space of conjugacy classes of representations in which certain (non-conjugate) reducible representations are identified.

Suppose ϱ_0 and ϱ_1 are representations with the same character on a closed 3-manifold. If one of them is irreducible, then they are conjugate and so their Chern-Simons invariants coincide Mod $\mathbb{Z}$. Suppose they are both reducible and ϱ_1 is diagonal. By conjugating we may assume ϱ_0 is upper triangular. Define a path ϱ_t by the formula

$$\varrho_t(\gamma) = \begin{pmatrix} a & (1-t)b \\ 0 & a^{-1} \end{pmatrix} \quad \text{whenever} \quad \varrho_0(\gamma) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

It is easy to check that this gives a path of representations from ϱ_0 to ϱ_1. Thus the Chern-Simons invariant of an $SL(2,\mathbb{C})$ representation on a closed 3-manifold depends only on the character of the representation. It therefore makes sense to try to extend the results of the previous sections to the case of $SL(2,C)$ representations, substituting character varieties for the spaces of conjugacy classes of representations. We first construct the bundle where the Chern-Simons invariants take their values.

3.1 Lemma. *The map $\mathbb{C}^2 \to R_{\mathbb{C}}(T)$ taking the pair (α,β) to the character of the representation defined by*

$$\mu \mapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix}, \quad \lambda \mapsto \begin{pmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix}$$

is the composition of an analytic branched cover with covering group G and a 1-1 and onto algebraic map.

Proof. Any representation $\varrho : \mathbb{Z}\oplus\mathbb{Z} \to SL(2,\mathbb{C})$ is reducible since $\mathbb{Z}\oplus\mathbb{Z}$ is abelian. Let $L \subset \mathbb{C}^2$ be the invariant line. By conjugation we may assume $L = \{(z,0)\}$ so that $\varrho(\gamma)$ is upper triangular for all γ.

But then the map $\varrho' : \mathbb{Z}\oplus\mathbb{Z} \to SL(2,\mathbb{C})$ defined by

$$\varrho'(\gamma) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

whenever

$$\varrho(\gamma) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

is a representation with the same character as ϱ. Thus the map is onto.

The map $t : R_{\mathbb{C}}(T) \to \mathbb{C}^3$ given by $\varrho \mapsto (\mathrm{Tr}(\varrho(\mu)), \mathrm{Tr}(\varrho(\lambda)), \mathrm{Tr}(\varrho(\mu\lambda)))$ is an algebraic embedding of $R_{\mathbb{C}}(T)$. The composite

$$\mathbb{C}^2 \to \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z},\ SL(2,\mathbb{C})) \xrightarrow{t} R_{\mathbb{C}}(T) \subset \mathbb{C}^3$$

is given by

$$(\alpha,\beta) \mapsto (2\cos(2\pi\alpha),\ 2\cos(2\pi\beta),\ 2\cos(2\pi(\alpha+\beta))$$

which is analytic. It is easy to see that the action of G (extending the action on $\mathbb{R}^2$ defined in Sect. 2) is analytic and has quotient homeomorphic to $R_{\mathbb{C}}(T)$. However, $\mathbb{C}^2/G$ and $R_{\mathbb{C}}(T)$ are not isomorphic as analytic varieties. This may be seen by comparing the dimensions of their Zariski tangent spaces at the singular points corresponding to central representations. The map $\mathbb{C}^2/G \to R_{\mathbb{C}}(T)$ is algebraic, 1-1 and onto. $\square$

Construct the bundle $E_{\mathbb{C}}(T)$ over $R_{\mathbb{C}}(T)$ by the same formulas as the definition of $E(T)$. This extends to the case when T is a union of tori. We also get an inner product

$$\langle , \rangle : E_{\mathbb{C}}(T) \times E_{\mathbb{C}}(-T) \to \mathbb{C}^*$$

defined in the same way, and a partial inner product $E(T \cup S) \times E(-S) \to E(T)$ when S and T are collection of tori.

3.2 Theorem. *The Chern-Simons invariant defines a lifting $c_X : R_{\mathbb{C}}(X) \to E_{\mathbb{C}}(\partial(X))$ of the restriction map from the character variety of X to the character variety of ∂X:*

$$\begin{tikzcd} & E_{\mathbb{C}}(\partial X) \arrow[d] \\ R_{\mathbb{C}}(X) \arrow[r] \arrow[ur] & R_{\mathbb{C}}(\partial X) \end{tikzcd}$$

such that if $\chi \in R_{\mathbb{C}}(Y)$, $Y = X_1 \cup X_2$, and χ_i denotes the restriction of χ to χ_i, then

$$\exp(2\pi i cs_Y(\chi)) = \langle c(\chi_1), c(\chi_2)\rangle.$$

Furthermore, the statement of Theorem 2.7 continues to hold with $\mathbb{C}^{2n}$ replacing $\mathbb{R}^{2n}$.

The proof will occupy the rest of this section. Our first task is to extend the notion of normal form to an $SL(2,\mathbb{C})$ connection. Since $\pi_1(T^2)$ is abelian, any representation $\varrho : \pi_1(T^2) \to SL(2,\mathbb{C})$ is conjugate to an upper triangular representation, and furthermore it is conjugate to a diagonal representation unless ϱ is non-central and $\mathrm{Tr}(\varrho(\gamma)) = \pm 2$ for all $\gamma \in \pi_1(T)$. Now let $\varrho : \pi_1(X) \to SL(2,\mathbb{C})$ be a representation. We consider the two cases when ϱ is or is not diagonalizable on the boundary. Let μ, λ be a fixed basis for $\pi_1 T$.
Case 1. If

$$\varrho(\mu) = \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix} \quad \text{and} \quad \varrho(\lambda) = \begin{pmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix}$$

for *complex* numbers α, β, then as before we define a flat connection A with holonomy conjugate to ϱ to be in *normal form* if

$$A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} dx + \begin{pmatrix} i\beta & 0 \\ 0 & -i\beta \end{pmatrix} dy$$

near the boundary. In this case, we define

$$c_X(\varrho) = [\alpha, \beta; e^{2\pi i cs_X(A)}],$$

where A is in normal form.

Case 2. If $\mathrm{Tr}(\varrho(\gamma)) = \pm 2$ for all $\gamma \in \pi_1(T)$, then

$$\varrho(\mu) = (-1)^u \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \varrho(\lambda) = (-1)^v \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some complex numbers a, b and integers u, v. We find a connection with this holonomy using developing maps, as in [KK2]. The map $D : \mathbb{R}^2 \to SL(2, \mathbb{C})$ defined by:

$$D(x,y) = \begin{pmatrix} \exp\left(\frac{i}{2}(ux+vy)\right) & -\frac{(ax+by)}{2\pi} \exp\left(\frac{i}{2}(ux+vy)\right) \\ 0 & \exp\left(-\frac{i}{2}(ux+vy)\right) \end{pmatrix}$$

satisfies

$$D(x+2\pi m, y+2\pi n) = D(x,y)\varrho(\mu^m \lambda^n)^{-1}.$$

From Proposition 2.1 of [KK2] it follows that $A = -dDD^{-1}$ is a connection 1-form with holonomy ϱ. One computes:

$$A = \begin{pmatrix} -\frac{iu}{2} & \frac{a}{2\pi} \exp(i(ux+vy)) \\ 0 & \frac{iu}{2} \end{pmatrix} dx + \begin{pmatrix} -\frac{iv}{2} & \frac{b}{2\pi} \exp(i(ux+vy)) \\ 0 & \frac{iv}{2} \end{pmatrix} dy.$$

We will say that a flat connection on T is in *normal form* if it is in the above form for some u, v, a, b. Notice that (as in the case of diagonal representations) the condition of being in normal form is independent of the choice of basis μ, λ for $\pi_1(T)$ (although the specific normal form does depend on the basis). A connection on a 3-manifold X with toral boundary will be said to be in normal form if it is in normal form near each boundary component. Write

$$A = B(u,v,a)dx + C(u,v,b)dy.$$

Then define:

$$c_X(\varrho) = \left[-\frac{u}{2}, -\frac{v}{2}; e^{2\pi i cs_X(A)} \right].$$

What must be shown is that $c_X(\varrho)$ is well-defined and depends only on the character of ϱ. We will first show that it depends only on the conjugacy class of ϱ. In what follows we will assume for notational convenience that there is only one component of the boundary. Everything extends in the obvious way to the case when the boundary has several components.

When ϱ is diagonalizable on the boundary the proof that $c_X(\varrho)$ depends only on the conjugacy class of ϱ is identical to the arguments given in Sect. 2. We will therefore consider the second case, when ϱ is parabolic on the boundary.

3.3 Lemma. *If A and A' are two gauge equivalent connections which are in normal form and equal near the boundary, then $cs(A) = cs(A') \operatorname{Mod} \mathbb{Z}$.*

Proof. There is a gauge transformation g which takes A to A'. Suppose first that g is the identity near the boundary. Then $A \cup -A$ and $A \cup -g \cdot A$ are connections on the double of X which are gauge equivalent. However $A \cup -A$ has zero Chern-Simons invariant and so A and $A' = g \cdot A$ have the same Chern-Simons invariant $\operatorname{mod} \mathbb{Z}$.

Since $g \cdot A = gAg^{-1} - dgg^{-1}$ near the boundary, it follows that $\dfrac{\partial g}{\partial r} = 0$, where r is the inward normal. Thus $g_{|T}$ is a gauge transformation of T which fixes A. A computation shows that near T, g must be of the form

$$g(e^{ix}, e^{iy}) = \pm \begin{pmatrix} 1 & ce^{i(ux+vy)} \\ 0 & 1 \end{pmatrix}$$

for some complex number c. We may assume the $+$ sign holds since if not, we may replace g by $-g$ while preserving the fact that $g \cdot A = A'$. It follows (by varying c) that there is an arc of gauge transformations on T, all fixing $A_{|T}$, joining g to 1. The rest of the proof is exactly like the proof of Theorem 2.4, so we omit it. □

Notice that conjugating an upper triangular matrix by the matrix

$$\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}$$

has the effect of multiplying the upper right-hand entry by w^2. The following lemma shows that this kind of conjugation does not affect the Chern-Simons invariant:

3.4 Lemma. *If A and A' are gauge equivalent connections such that*

$$A = B(u, v, a)dx + C(u, v, b)dy$$

and

$$A' = B(u, v, w^2 a)dy + C(u, v, w^2 b)dy$$

near ∂X then $cs(A) = cs(A') \operatorname{mod} \mathbb{Z}$.

Proof. The outline of this proof is the same as that of 2.4 and 3.4. Let $\gamma : [0, 1] \to \mathbb{C}^*$ be a path from w to 1, constant near the endpoints. Use γ to define a path of matrices g given by

$$g_t = \begin{pmatrix} \gamma(t) & 0 \\ 0 & \gamma^{-1}(t) \end{pmatrix},$$

constant near the endpoints. We extend g to be a gauge transformation $g : X \to SL(2, \mathbb{C})$ by using a collar of the boundary and extending it to be equal to 1 outside this collar. Now use g to define a path of gauge transformations g_s from $g_0 =$ the trivial gauge transformation to $g_1 = g$, as indicated in Fig. 2

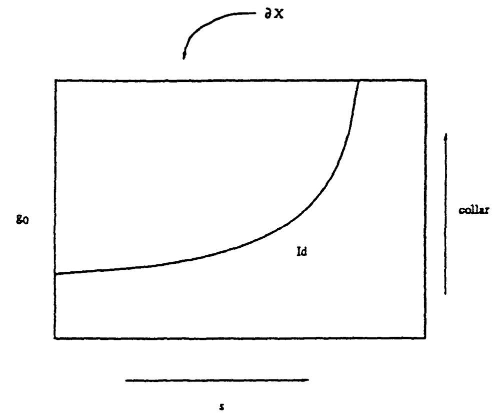

Fig. 2

Let $A_s = g_s \cdot A$. Notice that each A_s is in normal form. Since $cs(A_1) = cs(A')$ by 3.4 and $cs(A_0) = cs(A)$, it will suffice to show that $cs(A_0) = cs(A_1)$. Since these connections agree outside of a collar, we need only to compare their Chern-Simons integrals on $T^2 \times I$. Let $\mathbb{A}$ be the connection on $T^2 \times I \times I$ which is the union on the A_s. Then

$$\begin{aligned} 0 &= \frac{1}{8\pi^2} \int\limits_{T^2 \times I \times I} \operatorname{Tr}(F^{\mathbb{A}} \wedge F^{\mathbb{A}}) \\ &= cs_{T^2 \times I \times I}(A_1) - cs_{T^2 \times I \times I}(A_0) \\ &\quad - \frac{1}{8\pi^2} \int\limits_{\partial T^2 \times \{0,1\} \times I} \operatorname{Tr}\left(dA_s \wedge A_s + \frac{2}{3} A_s \wedge A_s \wedge A_s \right). \end{aligned}$$

It remains to prove that the integral vanishes. Since this is obvious over $T^2 \times 1 \times I$, we concentrate on $T^2 \times 0 \times I = \partial X \times I$. Since $A_s \wedge A_s \wedge A_s = 0$, we must show that $\int\limits_{\partial X \times I} \operatorname{Tr}(dA_s \wedge A_s)$ vanishes.

Now

$$A_s = B(u, v, \gamma^2(s)a)dx + C(u, v, \gamma^2(s)b)dy$$

near the boundary. Then $A_s \wedge A_s \wedge A_s = 0$ and

$$\begin{aligned} \operatorname{Tr}(dA_s \wedge A_s) &= \operatorname{Tr}\left(\frac{\partial B}{\partial s} C - B \frac{\partial C}{\partial s} \right) dx\, dy\, ds \\ &= \operatorname{Tr}\begin{pmatrix} 0 & (av - bu)\dfrac{\partial \gamma}{\partial s} \dfrac{i}{4\pi} e^{i(ux+vy)} \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

□

Let g_x and g_y be the gauge transformations constructed in Sect. 2. Notice that near the boundary,

$$\begin{aligned} g_x^m g_y^n \cdot (B(u, v, a)dx + C(u, v, b)dy) = B(u + 2m, v + 2n, a)dx \\ + C(u + 2m, v + 2n, b)dy. \end{aligned}$$

(Notice that conjugating by

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

takes an upper triangular matrix to a lower triangular matrix. Perhaps to include the action of g_b it would have been more elegant to consider upper and lower triangular matrices in the definition of normal form. This makes no difference in the end.)

3.5 Lemma. *If A is a flat connection with holonomy ϱ in normal form and with $c_X(A) = \left[-\frac{u}{2}, -\frac{v}{2}; z \right]$, then*

$$cs(g_x^m g_y^n \cdot A) = \left[-\frac{u}{2} + m, -\frac{v}{2} + n; ze^{\pi i(un-vm)} \right].$$

Proof. One first computes that if $A = B(u,v,a)dx + C(u,v,b)dy$ on $\partial X \times I$, then

$$g_x \cdot A = B(u-2,v,a)dx + C(u-2,v,b)dy$$

in a smaller collar of the boundary ∂X. Since g_x is the identity outside the collar $\partial X \times I$, just as in the proof of Theorem 2.5 we must show that

$$\frac{1}{8\pi^2} \int_{\partial X \times I} \mathrm{Tr}\left(dE \wedge E + \frac{2}{3} E \wedge E \wedge E \right) = v,$$

where $E = g_x \cdot A$. Write $g = g_x$, and since E is flat, $dE \wedge E + \frac{2}{3} E \wedge E \wedge E = -\frac{1}{3} E \wedge E \wedge E$. Therefore we need to compute $-\frac{1}{3} \int \mathrm{Tr}(E \wedge E \wedge E)$.

Since $E = gAg^{-1} - dgg^{-1}$, a small calculation shows

$$-\frac{1}{3} \mathrm{Tr}(E \wedge E \wedge E) = \mathrm{Tr}\left(C\left(\left[g^{-1}\frac{\partial g}{\partial r}, B - g^{-1}\frac{\partial g}{\partial x} \right] \right) \right),$$

where $B = B(u,v,a)$ and $C = C(u,v,b)$. (The fact that $\dfrac{\partial g}{\partial y} = 0$ is used here.)

We first show that $\int \mathrm{Tr}\left(C\left(\left[g^{-1}\frac{\partial g}{\partial r}, B \right] \right) \right) = 0$. The reader can check that since g is independent of y, $\mathrm{Tr}\left(C\left(\left[g^{-1}\frac{\partial g}{\partial r}, B \right] \right) \right)$ is the product of $e^{i(ux+vy)}$ with a function $h(x,r)$ which is independent of y. So

$$\int \mathrm{Tr}\left(C\left(\left[g^{-1}\frac{\partial g}{\partial r}, B \right] \right) \right) = \int_{S^1 \times I} h(x,r) e^{iux} \left(\int_{S^1} e^{ivy} dy \right) dx\, dr = 0.$$

The other term $\mathrm{Tr}\left(C\left(\left[g^{-1}\frac{\partial g}{\partial r}, g^{-1}\frac{\partial g}{\partial x} \right] \right) \right)$, is handled as follows. Write $p(x,y) = \left[g^{-1}\frac{\partial g}{\partial r}, g^{-1}\frac{\partial g}{\partial x} \right]$. We know from the proof of Theorem 2.5 that if

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

then

$$\frac{1}{8\pi^2}\int\limits_{\partial X\times I} \mathrm{Tr}(Ip(x,y))\,dx\,dy\,dr = 1.$$

Now

$$C = vI + e^{ivy}\begin{pmatrix} 0 & \frac{b}{2\pi}e^{iux} \\ 0 & 0 \end{pmatrix}.$$

So

$$\begin{aligned}
&\int \mathrm{Tr}\left(C\left(\left[g^{-1}\frac{\partial g}{\partial r}, g^{-1}\frac{\partial g}{\partial x}\right]\right)\right)dx\,dy\,dr \\
&= v\int \mathrm{Tr}(Ip)dx\,dy\,dr + \int e^{ivy}(\text{some function of } x \text{ and } r)dx\,dy\,dr \\
&= 8\pi^2 v.
\end{aligned}$$

□

This completes the proof that the map c defines a lifting from the conjugacy classes of representations of X to $E_{\mathbb{C}}(T)$. It remains then to show that the result depends only on the character of the representation. If two representations have the same character and one is irreducible, then it is shown in [CS] that they are conjugate. Thus it suffices to consider the case when two reducible representations have the same character. First notice that two diagonal representations with the same character are conjugate. Furthermore, if $\varrho : \pi_1 X \to SL(2,\mathbb{C})$ is a representation, the path of representations ϱ_s defined by

$$\varrho_s(\gamma) = \begin{pmatrix} a & (1-s)b \\ 0 & a^{-1} \end{pmatrix}$$

whenever

$$\varrho(\gamma) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

is a path of representations to a diagonal representation. Thus it suffices to show that for this path, $cs(\varrho_0) = cs(\varrho_1) \operatorname{Mod} \mathbb{Z}$.

Since the upper triangular matrices U form a subgroup of $SL(2,\mathbb{C})$, there is a developing map $D : \tilde{X} \to U$ with holonomy ϱ. But then

$$D_s(x) = \begin{pmatrix} a & (1-s)b \\ 0 & a^{-1} \end{pmatrix}$$

whenever

$$D(x) = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$$

is a path of developing maps with holonomy ϱ_s. From this it follows that we have a path of flat connections A_s in normal form with $A_s = B(u,v,(1-s)a)dx + C(u,v,(1-s)b)dy$ near ∂X. Thus

$$0 = cs(A_0) - cs(A_1) - \frac{1}{8\pi^2}\int\limits_{\partial X\times I} \mathrm{Tr}\left(dA_s\wedge A_s + \frac{2}{3}A_s\wedge A_s\wedge A_s\right).$$

Clearly $A_s \wedge A_s \wedge A_s = 0$. When we compute $dA_s \wedge A_s$, we find its diagonal entries and hence its trace to be 0, just as in the proof of 3.4. This completes the proof of Theorem 4.2. $\square$

The results of Sect. 1 and 2 apply also to $SO(3) = PSU(2)$ and $PSL(2, \mathbb{C})$ representations with some modifications. We outline one way to do this. It is perhaps not the most general way to extend the results, but it suffices for our applications.

Let $R(T, t)$ denote the conjugacy classes of $PSU(2) = SO(3)$ representations of $\mathbb{Z} \oplus \mathbb{Z}$ which have trivial second Stiefel-Whitney class, in other words so that the associated flat bundle is trivial. Thus $R(T, t)$ is the image of the $SU(2)$ representations under the map induced by the homomorphism $SU(2) \to PSU(2)$.

Let

$$H = \langle X, Y, b \mid [X, Y] = XbXb = YbYb = b^2 = 1 \rangle.$$

So H is isomorphic to G. Think of G as a subgroup of H via the map $x \mapsto X^2$, $y \mapsto Y^2$, and $b \mapsto b$. (So $(H/G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.) The action of G on $\mathbb{R}^2$ extends to H via

$$X \cdot (\alpha, \beta) = \left(\alpha + \tfrac{1}{2}, \beta\right), \quad Y \cdot (\alpha, \beta) = \left(\alpha, \beta + \tfrac{1}{2}\right), \quad b \cdot (\alpha, \beta) = (-\alpha, -\beta).$$

Extend this action to an action of H on $\mathbb{R}^2 \times S^1$ by

$$X \cdot (\alpha, \beta; z) = \left(\alpha + \tfrac{1}{2}, \beta; ze^{-4\pi\imath\beta}\right), \quad Y \cdot (\alpha, \beta; z) = \left(\alpha, \beta + \tfrac{1}{2}; ze^{4\pi i\alpha}\right),$$
$$b \cdot (\alpha, \beta; z) = (-\alpha, -\beta; z).$$

The quotient bundle $E(T, t)$ is well defined. If $R(X, t)$ denotes the $PU(2)$ representations of a 3-manifold X which have trivial w_2 on the boundary, then c_X defined a parallel lift of the restriction map $R(X, t) \to R(T, t)$. We remind the reader that $PSU(2) = SO(3)$ Chern-Simons invariants are based on the first pontrjagin class, and the map $H^4(BSO(3)) \to H^4(BSU(2))$ takes p_1 to $-4c_2$, hence the sign difference on the action of H on $\mathbb{R}^2 \times S^1$.

Similarly replacing $\mathbb{R}^2$ by $\mathbb{C}^2$ in this discussion leads to a $\mathbb{C}^*$ bundle over the character variety of $PSL(2, \mathbb{C})$ representations having trivial w_2.

Finally suppose we are given a manifold X with toral boundary and an arc ϱ_t of $SL(2, \mathbb{C})$-representations of $\pi_1(X)$, and we wish to calculate the difference between the Chern Simons invariants of $p \circ \varrho_1$ and $p \circ \varrho_0$, where $p : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$ is the quotient map. We may calculate this difference using an integral just as in Theorem 2.7, except that we must introduce a multiplicative factor of -4 because of the formula relating Pontryagin classes and Chern classes given above.

4. Chern-Simons of Manifolds with Toral Boundary

In this section we describe the representation spaces of several 3-manifolds with toral boundary in enough detail to use the results of the previous sections to compute their Chern-Simons invariants.

Our first result concerns abelian representations of X when H_1X is torsion free.

4.1 Theorem. *Let X be a 3-manifold with boundary $\partial X = T_1 \cup \cdots \cup T_n$. Assume H_1X is torsion free. Choose symplectic pairs μ_k, λ_k for H_1T_k. Let x_i, $i = 1, \ldots, m$ be a basis for H_1X and write*

$$\mu_k = \sum a_{kj} x_j \quad \text{and} \quad \lambda_k = \sum b_{kj} x_j.$$

Suppose that $\varrho : \pi_1 X \to SU(2)$ is an abelian representation and let $\gamma_j \in \mathbb{R}$ so that $\varrho(x_j) = e^{i\gamma_j}$. Then:

$$c_X(\varrho) = \left[\sum a_{1j}\gamma_j, \sum b_{1j}\gamma_j, \cdots, \sum a_{nj}\gamma_j, \sum b_{nj}\gamma_j; 1\right].$$

Proof. Since the representation is abelian, the formula:

$$\varrho(t)(x_j) = e^{i(1-t)\gamma_j}$$

defines a path of representations from ϱ to the trivial representation. On the boundary tori this path takes the form:

$$\varrho(t)(\mu_k) = \exp\left(i(1-t)\left(\sum a_{kj}\gamma_j\right)\right), \quad \varrho(t)(\lambda_k) = \exp\left(i(1-t)\left(\sum b_{kj}\gamma_j\right)\right).$$

So we can choose the paths $\alpha(t)$ and $\beta(t)$ to be:

$$\alpha_k(t) = (1-t)\left(\sum a_{kj}\gamma_j\right)$$

and

$$\beta_k(t) = (1-t)\left(\sum b_{kj}\gamma_j\right).$$

But then $\alpha_k d\beta_k - \beta_k d\alpha_k = 0$. The theorem now follows from Corollary 2.6 and Theorem 2.7. □

For example, if X is the complement of a knot in a homology sphere, let μ, λ denote the natural meridian and longitude. If $\varrho : \pi_1 X \to SU(2)$ is an abelian representation such that $\varrho(\mu) = e^{ia}$ then

$$c_X(\varrho) = [a, 0; 1].$$

The next examples we consider are 3 manifolds of the form $X = F \times S^1$ for a punctured surface F. By cutting along a torus we can consider the two cases:

1. F is a once-punctured surface of genus n.
2. F is a planar surface.

Consider first the case of a once-punctured surface F. We can describe the representation space. Write $\pi = \pi_1(F \times S^1) = \langle x_1, y_1, \ldots, x_n, y_n \rangle \times \langle \lambda \rangle$. The curves $\mu = \prod_i [x_i, y_i]$ and λ generate $\pi_1(\partial X)$. We take the orientation of λ so that μ and λ form an oriented basis. Since the centralizer of any non-abelian subgroup of $SU(2)$ is $\{\pm 1\}$, any non-abelian representation of π must send λ to ± 1. Moreover, any representation which restricts to an abelian representation of $\pi_1 F$ is abelian.

Thus if $\varrho : \pi_1 X \to SU(2)$ is any representation, either

1. ϱ is abelian and $\varrho(\mu) = 1$; or,
2. ϱ is non-abelian and $\varrho(\lambda) = \pm 1$.

In the first case, let $\beta \in \mathbb{R}$ so that $\varrho(\lambda) = e^{2\pi i\beta}$. Then from the previous result

$$c_X(\varrho) = [0, \beta; 1].$$

The second case is slightly more complicated. First note that a homomorphism from a free group to $SU(2)$ is determined by its values on the generators and so $\mathrm{Hom}(\pi_1 F, SU(2))$ can be identified with $SU(2)^{2n}$ via the map $\varrho \mapsto (\varrho(x_1), \varrho(y_1), \ldots, \varrho(x_n), \varrho(y_n))$. This space is path connected so there exists a path $\varrho(t)$, $t \in [0,1]$ from ϱ to the trivial representation which fixes λ. We can conjugate this path so that $\varrho(t)(\mu) = e^{2\pi i\alpha(t)}$. Along this part of the path β is unchanged. We can take β to be either 0 or $\frac{1}{2}$ depending on whether $\varrho(\lambda) = 1$ or -1. The representation $\varrho(1)$ is abelian, and hence has $c_X(\varrho(1)) = [0, \beta, 1]$. Since $\varrho(1)(\mu) = 1$, $\alpha(1)$ is an integer. Therefore we can re-write

$$c_X(\varrho(1)) = [0, \beta, 1] = [\alpha(1), \beta; e^{2\pi i\beta\alpha(1)}].$$

The path $\beta(t)$ is just the constant path at β and so:

$$\int_0^1 \alpha(t)\frac{d\beta}{dt} - \beta\frac{d\alpha}{dt} = -\beta\int_0^1 \frac{d\alpha}{dt} = \beta(\alpha(0) - \alpha(1)).$$

Using Theorem 2.7 we can compare $c_X(\varrho(0))$ and $c_X(\varrho(1))$: If $c_X(\varrho(0)) = [\alpha(0), \beta; z]$, then

$$e^{2\pi i\beta\alpha(1)}z^{-1} = e^{2\pi i\beta(\alpha(0)-\alpha(1))}.$$

Write $\alpha = \alpha(0)$. Then

$$z = e^{2\pi i(-\beta\alpha + 2\beta\alpha(1))} = e^{-2\pi i\beta\alpha}$$

since $\alpha(1)$ is an integer and $\beta \in \mathbb{Z}\left[\frac{1}{2}\right]$.

Therefore:

$$c_X(\varrho) = [\alpha, \beta; e^{-2\pi i\beta\alpha}] = \begin{cases} [\alpha, 0; 1] & \text{if } \varrho(\lambda) = 1, \\ [\alpha, \frac{1}{2}; e^{-\pi i\alpha}] & \text{if } \varrho(\lambda) = -1. \end{cases}$$

We turn now to the case of a planar surface F crossed with S^1. If F is a planar surface then its fundamental group is again free. This time we write $\pi_1(F \times S^1) = \langle \mu_1, \ldots, \mu_n | \mu_1 \ldots \mu_n = 1\rangle \times \langle\lambda\rangle$, where the μ_k are loops following the k^{th} boundary component of F. The symplectic pairs μ_i, λ generate the fundamental group of the i^{th} boundary component. Let ϱ be a representation. Suppose first that the restriction of ϱ to F is abelian. Then ϱ is abelian. Choose α_k and β so that $\varrho(\lambda) = e^{2\pi i\beta}$ and $\varrho(\mu_k) = e^{2\pi i\alpha_k}$ for $k = 1, \ldots, n-1$. Since $\mu_n = (\mu_1 \ldots \mu_{n-1})^{-1})$, we can take $\alpha_n = -\sum_1^{n-1} \alpha_k$.

The generators $\mu_1, \ldots, \mu_{n-1}, \lambda$ form a basis for $H_1 X$ and so by Theorem 4.1:

$$c_X(\varrho) = \left[\alpha_1, \beta, \ldots, \alpha_{n-1}, \beta, -\sum_{i=1}^{n-1} \alpha_i, \beta; 1\right].$$

Consider now the case when ϱ is non-abelian. Choose α_k, β as before, except that now $\varrho(\mu_k)$ is only conjugate to $e^{2\pi i\alpha_k}$. Since λ is central, $\varrho(\lambda) = \pm 1$, and so $\beta \in \mathbb{Z}\left[\frac{1}{2}\right]$. Since the fundamental group of F is free we can find a path of representations ϱ_s from ϱ to a representation whose restriction to F is trivial, and so that λ is fixed along the path. We may also choose continuous functions $\alpha_k(s)$ so that for each s, $\varrho_s(\mu_k)$ is conjugate to $e^{2\pi i\alpha_k(s)}$. For this path,

$$\sum_{k=1}^{n} \int \alpha_k d\beta - d\alpha_k \beta = \beta \sum_{k=1}^{n} (\alpha_k(0) - \alpha_k(1)).$$

Notice that $\alpha_k(1) \in \mathbb{Z}$ for all k, and $\alpha_k(0) = \alpha_k$. Since ϱ_1 is abelian, and trivial on $\pi_1(F)$, Theorem 4.1 implies that

$$\begin{aligned} c_X(\varrho_1) &= [0,\beta,\dots,0,\beta;1] \\ &= \left[\alpha_1(1),\beta,\dots,\alpha_n(1),\beta;\exp\left(2\pi i\beta\sum_i \alpha_i(1)\right)\right]. \end{aligned}$$

We now apply Theorem 2.7 to the path ϱ_s. Using the fact that $2\alpha_k(1)\beta$ is an integer for each k.

$$c_X(\varrho) = \left[\alpha_1,\beta,\dots,\alpha_n,\beta;e^{-2\pi i\beta\sum_k \alpha_k}\right].$$

Let F be a k times punctured genus n surface. By decomposing F into a planar surface and a once punctured surface and applying the inner product $E(T\cup S)\times E(-S)\to E(T)$, we obtain the following result:

4.2 Theorem. *Let F be a genus n surface with k punctures and let $X = F\times S^1$. Write*

$$\pi_1 F = \left\langle x_1,y_1,\dots,x_n,y_n,\mu_1,\dots,\mu_k \;\middle|\; \prod_j [x_j,y_j]\mu_1\cdots\mu_k = 1\right\rangle.$$

Give $\pi_1\partial X$ the bases $\mu_j\times *$ and $\lambda = *\times S^1$ (pick a path from the base point to each torus in ∂X to view these in $\pi_1 X$). Let $\varrho:\pi_1 X\to SU(2)$ be a representation. Let α_j, $\beta\in\mathbb{R}$ be defined by $\varrho(\lambda) = e^{2\pi i\beta}$ and $\varrho(\mu_j) = e^{2\pi i\alpha_j}$ (i.e. for each torus $T_j\subset\partial X$ there is an element of $SU(2)$ which conjugates ϱ to this form on T_j). Then

$$c_X(\varrho) = \left[\alpha_1,\beta,\dots,\alpha_k,\beta;e^{-2\pi i\beta\sum_1^k \alpha_j}\right]. \qquad \square$$

Another class of examples for which computations are possible are the Chern-Simons invariants of certain representations of surface bundles over the circle which we called *special representations* in [KK]. D. Auckly has computed the Chern-Simons invariants of special representations of any (closed) surface bundle over S^1 in [A]. Using Auckly's computations, Theorem 2.1, and the computations for the solid torus given by 4.1 one can compute the Chern-Simons invariants of special representations of surface bundles when the surface has one boundary component. We sketch the idea, leaving the details to the reader.

Recall that a special representation of a surface bundle is one whose restriction to a fiber is abelian. Let $b: F\to F$ be a homeomorphism of a once punctured surface to itself which is the identity on the boundary and let $M = F\times_B S^1$ be the mapping torus. The fundamental group of M is $\pi_1 M = \langle x_1,\dots,x_{2n},t \mid tx_it^{-1} = b_*(x_i)\rangle$. Thus the boundary of M has basis

$$\mu = t$$

and

$$\lambda = \prod_{j=1}^n [x_{2j-1},x_{2j}].$$

Notice that since b is the identity on the boundary, it extends to the closed surface obtained by capping off the boundary. Moreover, the representation also extends since

$\varrho(\mu) = 1$. Let M denote the closed surface. Then $\hat{M} = M \cup X$, where $X = D^2 \times S^1$. We give the boundary of the solid torus the basis $\lambda_X = \partial D^2 \times \{1\} = \lambda$ and $\mu_X = \{1\} \times S^1 = \mu$. By Theorem 4.1,

$$c_X(\varrho) = [0, 1; 1].$$

The computation of $cs_{\hat{M}}(\varrho)$ is in [A]. We refer to that paper for the precise formula since it involves some notation which we do not repeat here. Writing $c_{\hat{M}}(\varrho) = \exp(2\pi i cs_{\hat{M}}(\varrho))$ we conclude from the inner product of Theorem 2.1 that

$$c_M(\varrho) = [0, 1; e^{2\pi i cs_{\hat{M}}(\varrho)}].$$

As an example, since j is a 4^{th} root of unity any special representation extends to a representation of the closed manifold obtained by Dehn filling so that the curve $\mu^{4k}\lambda^p$ is killed. In particular this gives the Chern-Simons invariants of some representations of $\frac{4k}{p}$ surgeries on fibered knots.

We next turn to a computation of the Chern-Simons invariants of two types of closed 3-manifolds. The first are circle bundles over oriented surfaces. These manifolds are the boundaries of tubular neighborhoods of embedded surfaces in 4-manifolds. The second type of manifold we consider are graph manifolds obtained by glueing two Seifert-fibered manifolds together along their boundary tori. This second application is needed in [KKR] in the computations of the spectral flows between flat connections on these manifolds. These examples show how to use the inner product in Theorem 2.1 to compute the Chern-Simons invariants of closed manifolds from the knowledge of the pieces.

Let F be an oriented surface with one boundary component and let $\hat{F}$ be the closed surface obtained by glueing a disc to ∂F. The circle bundle $M(n)$ over $\hat{F}$ with Euler class n is obtained by gluing $F \times S^1$ to $D^2 \times S^1$ using the map $\varphi : \partial F \times S^1 \to \partial D^2 \times S^1$ given by $(z, w) \mapsto (z, z^n w)$, where we have identified ∂F and ∂D^2. Figure 3 shows the images of $R(F \times S^1)$ and $R(D^2 \times S^1)$ in $R(T)$. The coordinates are $\partial F \times *$ and $* \times S^1$.

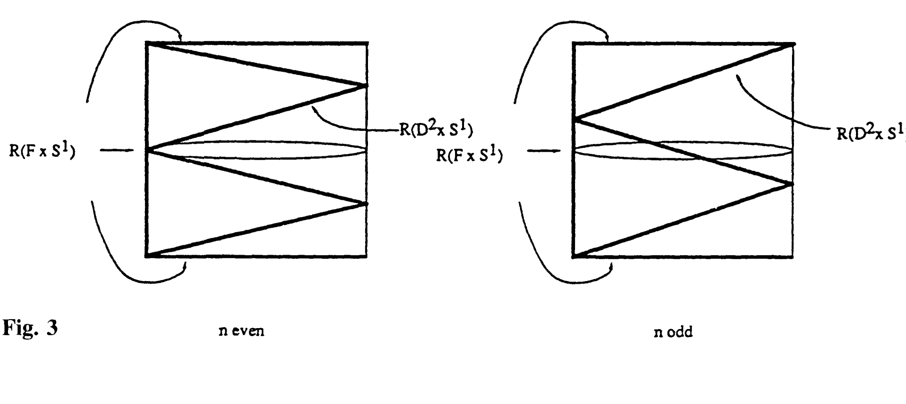

Fig. 3

(Note that if $\hat{F} = S^2$, then the only representations of $F \times S^1$ lie on the left edge of $R(T)$ since there are no non-abelian representations of $D^2 \times S^1$.)

One concludes from the figure that there are 3 types of representations: the first are representations which factor through the projection $\pi_1 M(n) \to \pi_1 \hat{F}$; these have their image in the lower left corner. The space of such representations is homeomorphic to the space of representations of $\pi_1 \hat{F}$. The second type are the representations whose restriction to $\pi_1 F$ are abelian and non-central. These correspond in the figure to the intersections along the left edge of $R(T)$. The space of such representations has $\left[\frac{n}{2}\right]$ components; each component is homeomorphic to $(S^1)^{2g}/(\mathbb{Z}/2)$, where g is the genus of F and the generator of $\mathbb{Z}/2$ acts by conjugation in each factor. The third type, corresponding to either the top left or top right corner in the figure depending on the parity of n, are representations whose restriction to $\pi_1 F$ send ∂F to -1; the space of such representations is homeomorphic to the $SO(3)$ representations of $\pi_1 \hat{F}$ which have non-trivial second Stiefel-Whitney class.

From these observations one can already conclude that the representations of the first type have zero Chern-Simons invariants, since they extend flatly over the D^2 bundle over $\hat{F}$. We can also conclude that the representations of the third type have Chern-Simons invariants in $\mathbb{Z}\left[\frac{1}{4}\right]$ since the corresponding $SO(3)$ representations extend over the D^2 bundle over $\hat{F}$.

Write $X = F \times S^1$ and let $Y = D^2 \times S^1$ and let $\mu_X = \partial F \times *$, $\lambda_X = * \times S^1$, $\mu_Y = \partial D^2 \times *$, and $\lambda_Y = * \times S^1$. The glueing map is then defined by $\mu_X \mapsto \mu_Y + n\lambda_Y$ and $\lambda_X \mapsto \lambda_Y$.

Let $\varrho : M(n) \to SU(2)$ be a representation which satisfies $\varrho(\mu_X) = 1$ and $\varrho(\lambda_X) = e^{2\pi i\left(\frac{k}{n}\right)}$ for some $k = 1, \ldots \left[\frac{n}{2}\right]$. Thus ϱ is a representation of the second type, or n is even and ϱ is a representation of the third type. In either case we may assume the restriction to $F \times S^1$ is abelian; this already holds for the second type of representations and for the third type (with n even) there is a path of representations of $\pi_1 M(n)$ from ϱ to an abelian representation.

Then by Theorem 4.2 $c_X(\varrho) = \left[0, \frac{k}{n}; 1\right]$. Notice that $\varrho(\mu_Y) = e^{-2\pi i k} = 1$ and $\varrho(\lambda_Y) = e^{2\pi i \frac{k}{n}}$. Thus $c_Y(\varrho) = \left[0, \frac{k}{n}; 1\right] = \left[-k, \frac{k}{n}; e^{-2\pi i \frac{k^2}{n}}\right]$. The last equality follows from the definition of the action of G on $\mathbb{R}^2$. We can now take the inner product since we have chosen compatible lifts. Thus $c_{M(n)}(\varrho) = 1 \cdot e^{-2\pi i \frac{k^2}{n}}$, and hence the Chern- Simons invariant is

$$cs_{M(n)}(\varrho) = -\frac{k^2}{n}.$$

This leaves the representations of type 3 when n is odd. From the discussion of $F \times S^1$ we know

$$c_X(\varrho) = \left[\frac{1}{2}, \frac{1}{2}; e^{\frac{-\pi i}{2}}\right].$$

Also, $\varrho(\mu_Y) = 1 = e^{2\pi i \frac{1-n}{2}}$ and $\varrho(\lambda_Y) = -1 = e^{2\pi i \frac{1}{2}}$. Thus

$$c_Y(\varrho) = \left[0, \frac{1}{2}; 1\right] = \left[\frac{1-n}{2}, \frac{1}{2}; e^{2\pi i \frac{1-n}{4}}\right].$$

Taking the inner product we get:

$$cs_{M(n)}(\varrho) = -\frac{1}{4} + \frac{1-n}{4} = -\frac{n}{4}.$$

We summarize:

4.3 Theorem. *Let $\varrho : \pi_1(M(n)) \to SU(2)$ be a representation of the circle bundle over a closed, oriented surface F with Euler class n. Conjugate ϱ so that the fiber is sent to $e^{2\pi i\beta}$. Then either $\beta = \frac{k}{n}$, in which case $cs(\varrho) = -\frac{k^2}{n}$, or else n is odd and $\beta = \frac{1}{2}$, in which case $cs(\varrho) = -\frac{n}{4}$.* □

We now compute the Chern-Simons invariants of certain graph manifolds. See [FS1, FS2, and KK1] for background on the representation spaces of Seifert-fibered manifolds. We consider the following situation: X and Y are two Seifert fibered manifolds over the disc with $H_1X = \mathbb{Z} = H_1Y$ and $\varphi : \partial X \to \partial Y$ is a homeomorphism. For simplicity we will assume that X and Y are the complements of regular fibers in Seifert-fibered homology spheres or, equivalently, that the Euler class of the Seifert fibration is equal to 1. This restriction is merely for convenience: the formulas are easier to write down. (These examples are the examples for which the Floer homology grading were computed in [KKR].) The general case of a manifold decomposed along a union of tori into Seifert fibered pieces can be handled similarly. Glue X to Y using φ to get the closed manifold Z_φ. We will describe the representation spaces and Chern-Simons invariants of X and Y and then apply Theorem 2.1 to calculate the representation spaces and Chern-Simons invariants of Z_φ.

We start first with the computation of $c_X(\varrho)$ for a non-abelian representation ϱ. Write:

$$\pi_1X = \langle x_1, \ldots, x_m, h \mid h \text{ central}, x_i^{a_i}h^{b_i} = 1\rangle.$$

Our hypothesis on the Euler class implies that the integers a_i, b_i can be chosen so that

$$a_1 \ldots a_m \sum_{i=1}^{m} \frac{b_i}{a_i} = 1.$$

Write $a = a_1 \cdots a_n$.

The pair $\mu_X = x_1 \cdots x_m$ and $\lambda_X = h$ forms a basis for $\pi_1\partial X$. Assume the orientation of X is chosen so that this basis is oriented in the induced boundary orientation. If $\varrho : \pi_1X \to SU(2)$ is a non-abelian representation, then since h is central it must be sent to ±1. The relations $x_i^{a_i}h^{b_i} = 1$ then force $\varrho(x_r)$ to be conjugate to $e^{\pi i l_r/a_r}$ for some integers l_r. Conjugate ϱ so that $\varrho(x_1 \cdots x_m) = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{R}$.

Consider the *free linkage* (cf. [KK1]) in $SU(2)$ consisting of the geodesic segments joining $\varrho(x_1 \cdots x_r)$ to $\varrho(x_1 \cdots x_{r+1})$. So the r^{th} strut of this linkage has length $\cos(\pi l_r/a_r)$. This linkage determines the representation ϱ and moreover the space of such (free) linkages maps onto $R(X)$ in the obvious way.

Consider the following 1-parameter family of free linkages L_t with L_0 corresponding to ϱ. As t increases, pull the free endpoint of the linkage along the circle e^{is} until the linkage winds around this circle in a monotone way. (That this is possible follows from the argument on p. 82 of [KK1], namely, that the distance from $1 \in SU(2)$ is a Morse function whose only critical points occur when the free linkage lies entirely on the circle.) Define the path of representations $\varrho(t)$ by leaving h fixed along the path and sending $x_1 \cdots x_r$ to the $r-1^{\text{st}}$ endpoint of L_t. Notice that the endpoint $\varrho(1)$ is abelian. Furthermore $\varrho(t)(\mu)$ is just the free endpoint of L_t. Write $e^{2\pi i\alpha(t)}$ for this endpoint. Then

$$\alpha(0) = \alpha$$

and

$$\alpha(1) = \sum_{r=1}^{m} \frac{l_r}{2a_r}.$$

Along this path, $\beta(t)$ is constant since h is fixed at ± 1. Choose β to be 0 or $\frac{1}{2}$ according to whether $\varrho(h) = 1$ or -1. Thus

$$\int_0^1 \alpha\, d\beta - \beta\, d\alpha = \beta\left(\alpha - \sum_{r=1}^{m} \frac{l_r}{2a_r}\right). \tag{$*$}$$

We next need to compute $c_X(\varrho(1))$. Since $\varrho(1)$ is abelian we can use Theorem 3.1. Notice that μ generates H_1X since (using additive notation)

$$a\mu = a\sum_r x_r = -\sum \frac{a}{a_r} b_r h = -\lambda$$

and since $H_1\partial X \to H_1X$ is onto. Using Theorem 3.1 we see that

$$c_X(\varrho(1)) = \left[\sum_r \frac{l_r}{2a_r}, -a\sum_r \frac{l_r}{2a_r}; 1\right].$$

Write

$$e = a\sum_r \frac{l_r}{a_r}.$$

We consider first the case when $\beta = 0$. Then $-a\sum_r \frac{l_r}{2a_r} \in \mathbb{Z}$ and so

$$c_X(\varrho(1)) = \left[\frac{e}{2a}, -\frac{e}{2}; 1\right] = \left[\frac{e}{2a}, 0; e^{-2\pi i\frac{e^2}{4a}}\right].$$

Applying $(*)$ (with $\beta = 0$) and Theorem 2.7 we conclude

$$c_X(\varrho) = \left[\alpha, 0; e^{-2\pi i\frac{e^2}{4a}}\right].$$

Now suppose $\beta = \frac{1}{2}$. Then $-a\sum_r \frac{l_r}{2a_r} - \frac{1}{2} \in \mathbb{Z}$ and so

$$c_X(\varrho(1)) = \left[\frac{e}{2a}, -\frac{e}{2} - \frac{1}{2} + \frac{1}{2}; 1\right] = \left[\frac{e}{2a}, \frac{1}{2}; e^{-2\pi i\left(\frac{e^2}{4a} + \frac{e}{4a}\right)}\right].$$

This time the integral in $(*)$ equals $\frac{e}{4a} - \frac{\alpha}{2}$ and so by Theorem 2.7,

$$c_X(\varrho) = \left[\alpha, \frac{1}{2}; e^{-2\pi i\left(\frac{e^2}{4a} + \frac{\alpha}{2}\right)}\right].$$

In either case we can write the exponent as $-2\pi i\left(\frac{e^2}{4a}+\beta\alpha\right)$. From the covariance of the line bundle and the fact that $2\beta \in \mathbb{Z}$ one can easily show that the equivalence class $\left[\alpha, \beta; e^{-2\pi i\left(\frac{e^2}{4a}+\beta\alpha\right)}\right]$ is independent of the choice of α and β, i.e.

$$\left[\alpha, \beta; e^{-2\pi i\left(\frac{e^2}{4a}+\beta\alpha\right)}\right] = \left[\pm\alpha + m, \pm\beta + n; e^{-2\pi i\left(\frac{e^2}{4a}+(\pm\beta+n)(\pm\alpha+m)\right)}\right].$$

Abelian representations can be handled using Theorem 3.1. Thus:

4.4 Proposition. *Let X be the complement of a regular fiber in a Seifert fibered homology sphere, with*

$$\pi_1 X = \langle x_1, \ldots, x_m, h \mid h \text{ central}, x_i^{a_i} h^{b_i} = 1\rangle$$

for integers a_i, b_i chosen such that $\sum \frac{ab_i}{a_i} = 1$, $a = a_1 \cdots a_m$. Let $\varrho : \pi_1 X \to SU(2)$ be a representation taking $x_1 \cdots x_m$ to $e^{2\pi i\alpha}$ and λ to $e^{2\pi i\beta}$.
(i) *If ϱ is non-abelian,* (so that $\beta \in \mathbb{Z}[\frac{1}{2}]$), *then*

$$c_X(\varrho) = \left[\alpha, \beta; e^{-2\pi i\left(\frac{e^2}{4a}+\beta\alpha\right)}\right]$$

with respect to the basis $\mu = x_1 \cdots x_m$ and $\lambda = h$ for $\pi_1 \partial X$.
(ii) *If ϱ is abelian, then*

$$c_X(\varrho) = [\alpha, -a\alpha; 1].$$

□

Now let Y be another Seifert-fibered space over D^2 with

$$\pi_1 Y = \langle y_1, \ldots, y_n, k \mid k \text{ central}, y_i^{c_i} h^{d_i} = 1\rangle,$$

and $\sum_i \frac{cd_i}{c_i} = 1$ where $c = c_1 \cdots c_n$. We glue X to Y using a diffeomorphism $\varphi : \partial X \to \partial Y$ to obtain Z_φ. One remark about orientations is needed: The orientation of Z_φ induces an orientation of X and Y which in turn orient their boundaries ∂X and ∂Y. Now it may happen that these orientations are not the same as the orientations given by the ordered bases $\{x_1 \cdots x_m, h\}$ and $\{y_1 \cdots y_n, k\}$.

So we assume Z_φ is given an orientation and that ε_X and ε_Y are signs so that

$$Z_\varphi = \varepsilon_X X \cup_\varphi \varepsilon_Y Y.$$

We choose the bases $(\mu_X, \lambda_X) = (x_1 \cdots x_m, h_X^\varepsilon)$ and $(\mu_Y, \lambda_Y) = (y_1 \cdots y_n, k_Y^\varepsilon)$. Let $\varphi : \partial X \to \partial Y$ be an orientation-reversing homeomorphism given in these bases by $\varphi(\mu_X) = u\mu_Y + w\lambda_Y$ and $\varphi(\lambda_X) = v\mu_Y + z\lambda_Y$. So $uz - vw = -1$.

Let $\varrho : \pi_1 Z_\varphi \to SU(2)$ be a representation. Assume that the restriction of ϱ to both X and Y is *non-abelian*. The case when one or both restrictions are abelian is handled in the same way and is easier.

The restrictions to X and Y define real numbers α_X and α_Y, half-integers β_X and β_Y, and integers e_X and e_Y as above. Specifically:

$$\varrho(x_1 \cdots x_m) = e^{2\pi i\alpha_X}, \quad \varrho(y_1 \cdots y_n) = e^{2\pi i\alpha_Y},$$
$$\varrho(h) = e^{2\pi i\beta_X}, \quad \text{and} \quad \varrho(k) = e^{2\pi i\beta_Y}.$$

Taking the signs into account:

$$c_{\varepsilon_X X}(\varrho) = \left[\alpha_X, \varepsilon_X\beta_X; e^{-2\pi i\varepsilon_X\left(\frac{e_X^2}{4a}+\beta_X\alpha_X\right)}\right]$$

and

$$c_{\varepsilon_Y Y}(\varrho) = \left[\alpha_Y, \varepsilon_Y\beta_Y; e^{-2\pi i\varepsilon_Y\left(\frac{e_Y^2}{4c}+\beta_Y\alpha_Y\right)}\right].$$

According to Theorem 2.1 $e^{2\pi i cs(\varrho)}$ is the inner product of these two. We need to evaluate this inner product. This requires fixing compatible lifts to $\mathbb{R}^2$.

Fix lifts α_Y and β_Y. Then

$$\alpha_X = u\alpha_Y + \varepsilon_Y w\beta_Y \tag{$*$}$$

and

$$\varepsilon_X\beta_X = v\alpha_Y + \varepsilon_Y z\beta_Y \tag{$**$}$$

are compatible lifts.

This has the consequences:

1. To decide whether or not there is a representation of $\pi_1 Z_\varphi$ which restricts to some given representations of X and Y one first checks that the $(*)$ and $(**)$ hold modulo the action of G on $\mathbb{R}^2$. Since β_X and β_Y lie in $\mathbb{Z}\left[\frac{1}{2}\right]$. It follows that α_X and α_Y are rational numbers with denominator $2v$. (If $v = 0$ then Z_φ is Seifert fibered and the computations are easier. We will assume $v \neq 0$.)

2.

$$\begin{aligned}
c_{\varepsilon_X X}(\varrho) &= \left[\alpha_X, \varepsilon_X\beta_X; \varepsilon^{-2\pi i\varepsilon_X\left(\frac{e_X^2}{4a}+\alpha_X\beta_X\right)}\right] \\
&= \Big[u\alpha_Y + \varepsilon_Y w\beta_Y, v\alpha_Y + \varepsilon_Y z\beta_Y; \\
&\qquad e^{-2\pi i\varepsilon_X\left(\frac{e_X^2}{4a}+\varepsilon_X(u\alpha_Y+\varepsilon_Y w\beta_Y)(v\alpha_Y+\varepsilon_Y z\beta_Y)\right)}\Big].
\end{aligned}$$

Write

$$\alpha_Y = \frac{p}{2v}, \quad \beta_Y = \frac{\kappa}{2}$$

for some integers p and κ. Then

$$c_{\varepsilon_Y}(\varrho) = \left[\frac{p}{2v}, \varepsilon_Y\frac{\kappa}{2}; e^{-2\pi i\varepsilon_Y\left(\frac{e_Y^2}{4c}+\frac{p}{2v}\frac{\kappa}{2}\right)}\right].$$

We can now take the inner product:

$$\begin{aligned}
cs_{Z_\varphi} &= -\varepsilon_X\left(\frac{e_X^2}{4a} + \varepsilon_X(u\alpha_Y + \varepsilon_Y w\beta_Y)(v\alpha_Y + \varepsilon_Y z\beta_Y)\right) - \varepsilon_Y\left(\frac{e_Y^2}{4c} + \alpha_Y\beta_Y\right) \\
&= -\varepsilon_X\frac{e_X^2}{4a} - \varepsilon_Y\frac{e_Y^2}{4c} - \alpha_Y\beta_Y(\varepsilon_Y(1 + uz + vw)) - \alpha_Y^2 uv - \beta_Y^2 wz \\
&= -\varepsilon_X\frac{e_X^2}{4a} - \varepsilon_Y\frac{e_Y^2}{4c} - \frac{p^2u}{4v} - \frac{wk^2}{4}(2\varepsilon_Y p + z).
\end{aligned}$$

Notice that $\frac{1}{2}\varepsilon_Y p \cong \frac{1}{2}p \operatorname{Mod}\mathbb{Z}$. We have shown that first part of:

4.5 Theorem. *Let $\varrho : \pi_1 Z_\varphi \to SU(2)$ be a representation.*
1. *If the restriction of ϱ to X and Y is non-abelian, then the Chern-Simons invariant of ϱ is*

$$-\varepsilon_X \frac{e_X^2}{4a} - \varepsilon_Y \frac{e_Y^2}{4c} - \frac{p^2 u}{4v} - \frac{wk^2}{4}(2p+z) \operatorname{Mod}\mathbb{Z}.$$

2. *If the restriction of ϱ to X is abelian, and the restriction to Y is non-abelian, then suppose $\varrho(\mu_X) = e^{2\pi i \alpha_X}$, $\varrho(\lambda_X) = e^{2\pi i \beta_X}$. Then the Chern-Simons invariant of ϱ is*

$$-\varepsilon_Y \frac{e_Y^2}{4c} + \alpha_X^2(zv - \varepsilon_X a) - 2\varepsilon_X vw\alpha_X\beta_X - \beta_X^2 wu.$$

3. *If the restriction of both X and Y is abelian, then suppose $\varrho(\mu_Y) = e^{2\pi i \alpha_Y}$, $\varrho(\lambda_Y) = e^{2\pi i \beta_Y}$. The Chern-Simons invariant of ϱ is*

$$-\alpha_Y(\varepsilon_Y c\alpha_Y + \varepsilon_X \beta_Y).$$

The proof for the second and third cases are similar to the proof of the first, using Proposition 4.4. □

The methods of Proposition 4.4 and Theorem 4.5 apply to any Seifert-fibered spaces or graph manifolds with few modifications. In light of standard results in 3-manifold theory, in particular the torus decomposition theorem, in addition to computations for Seifert-fibered 3-manifolds it would be useful to understand the representation spaces and Chern-Simons invariants of hyperbolic 3-manifolds. In the examples outlined above, the images $R(X) \to R(T)$ are lines (more precisely their preimages in $\mathbb{R}^2$ are straight line segments) but for a general 3-manifold, in particular for a hyperbolic 3-manifold with toral boundary, the image of the restriction map $R(X) \to R(T)$ can be a quite complicated curve. The polynomials of [CCGLS] are essentially the defining polynomials for the variety $\operatorname{Im}(R_{\mathbb{C}}(X) \to R_{\mathbb{C}}(T))$ for a knot complement X. In principle one can use their polynomials to parametrize the image $R_{\mathbb{C}}(X) \to R_{\mathbb{C}}(T)$ and then apply Theorem 3.2 to compute Chern-Simons invariants. To get started one needs to understand what happens at one representation and then one can apply Theorem 3.2 to see what happens at other representations. There is a natural representation on a hyperbolic 3-manifold, namely the holonomy representation of the complete hyperbolic structure.

We will show how a formula of Yoshida [Y] (see also [NZ] and [H]) relating the volume and Chern-Simons invariant of the metric connection on a hyperbolic 3-manifold relates to our cut-and-paste approach for Chern-Simons invariants of $SL(2,\mathbb{C})$ representations. From this formula and a knowledge of the Dehn-surgery space for a cusped hyperbolic 3-manifold X one can obtain information about the Chern-Simons invariants of flat connections which lie on the path component in $R_{\mathbb{C}}(X)$ which contains the holonomy of the complete hyperbolic structure. This can be useful even in computing $SU(2)$ Chern-Simons invariants since there might be paths joining $SU(2)$ representations in $SL(2,\mathbb{C})$ but not in $SU(2)$.

Let X be a complete hyperbolic 3-manifold, and let $\varrho_0 : \pi_1 X \to PSL(2,\mathbb{C})$ be the holonomy representation. In general, given a representation $\varrho : \pi_1 X \to G$ let $P(\varrho)$ denote the associated flat G bundle $P(\varrho) = \tilde{X} \times_{\pi_1 X} G$; its flat connection is induced from the trivial connection on $\tilde{X} \times G$. (Here $\tilde{X}$ denotes the universal cover of X.)

We construct a map from the oriented orthonormal frame bundle of X to $P(\varrho_0)$ as follows. Fixing a frame over a point in hyperbolic 3-space $\mathbb{H}$ identifies the frame bundle of $\mathbb{H}$ with $PSL(2,\mathbb{C})$. Let $p : PSL(2,\mathbb{C}) \to \mathbb{H}$ denote the projection. Then the map

$$p \times id : PSL(2,\mathbb{C}) \to \mathbb{H} \times PSL(2,\mathbb{C})$$

is equivariant with respect to the action of $\pi_1 X$ and descends to give a map of principal bundles $q : F(X) \to P(\varrho_0)$ covering the identity map of X. (In particular, $P(\varrho_0)$ is trivial since $F(X)$ is and if s is a section of $F(X)$, $q \circ s$ is a section of $P(\varrho_0)$.)

Let M be a closed hyperbolic 3-manifold. Let $\mathrm{Vol}(M)$ denote the volume of M and $cs(M)$ the Chern-Simons invariant of the Levi-Civita connection on M. Then it follows essentially from Lemma 3.1 of [Y] that

$$cs_M(\varrho_0) = cs(M) - \frac{i}{\pi^2}\,\mathrm{Vol}(M).$$

Although Yoshida does not state it exactly this way, he defines a complex valued 3-form C on $PSL(2,\mathbb{C})$ with

$$C = \frac{1}{\pi^2} d\,\mathrm{vol} + ics + d\gamma,$$

where $d\,\mathrm{vol}$ is the pullback of the volume element on $\mathbb{H}$, cs is the Chern-Simons form for the Levi-Civita connection on $PSL(2,\mathbb{C}) = F(\mathbb{H})$ and $d\gamma$ is some exact form. To obtain the formula above, one takes the projection $r : \mathbb{H} \times PSL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ and computes that $r^*(C)$ is just i times the Chern-Simons form for the trivial (flat) connection in the principal bundle $\mathbb{H} \times PSL(2,\mathbb{C}) \to \mathbb{H}$. The form C is invariant with respect to translation in $PSL(2,\mathbb{C})$. Thus, if $s \in \Gamma(F(M))$, and $\hat{s} = q \circ s \in \Gamma(p(\varrho_0))$, then $\hat{s}^* r^*(C) = s^*(C)$, so that:

$$cs(\varrho_0) = -i \int_M \hat{s}^* r^*(C) = -i \int_M s^*(C) = cs(M) - \frac{i}{\pi^2}\,\mathrm{Vol}(M).$$

Now suppose X is a cusped (complete, finite volume) hyperbolic 3-manifold, and let D denote the Dehn surgery space – the space of deformations of hyperbolic structures on X (see [Th]). Let $d_0 \in D$ be the complete hyperbolic structure. For $d \in D$ let ϱ_d denote the holonomy representation of the (incomplete) hyperbolic structure d. Yoshida shows that there is an analytic function $f : D \to \mathbb{C}$ so that if $d \in D$ corresponds to an incomplete hyperbolic structure which completes to give a closed hyperbolic manifold $M(d)$ then

$$f(d) = \exp\left(\frac{2}{\pi}\,\mathrm{Vol}(M(d)) + 2\pi i cs(M(d))\right) \prod_k \exp(\mathrm{length}(\gamma_k) + i\,\mathrm{torsion}(\gamma_k)),$$

where γ_k are the geodesics added to X to complete the k^{th} cusp.

We claim that $f(d)$ is just $c_X(\varrho_d)$ (up to a constant) and that this formula is equivalent to the formula of Theorem 2.1,

$$c_{M(d)}(\varrho_d) = \langle c_X(\varrho_d), c_{D^2 \times S^1}(\varrho_d)\rangle.$$

To see this, first of all fix a meridian and longitude pair μ_k, λ_k, $k = 1, \dots, n$ for each cusp. There are coordinates $\alpha : D \hookrightarrow \mathbb{C}^n$ such that $\alpha(d_0) = 0$ and if d is an incomplete structure, $\varrho_d(\mu_k)$ is conjugate to

$$\begin{pmatrix} e^{2\pi i \alpha_k(d)} & \\ & e^{-2\pi i \alpha_k(d)} \end{pmatrix},$$

where $\alpha_k(d)$ is the k^{th} coordinate of $\alpha(d)$. (See [Th] and [NZ] for details.) The image of α is the intersection of an analytic variety with a neighborhood of 0. Let $\beta(d) = (\beta_1(d), \dots, \beta_n(d))$ be defined by

$$\varrho_d(\lambda_k) = \begin{pmatrix} e^{2\pi i \beta_k(d)} & \\ & e^{-2\pi i \beta_k(d)} \end{pmatrix}$$

together with the stipulation that $\beta_k(d_0) = 0$ for all k. Then β_k is the product of α_k and some analytic function $\tau_k : D \to \mathbb{C}$. The map taking $d \in D$ to the character of ϱ_d is an analytic embedding $D \hookrightarrow R(X, PSL(2, \mathbb{C}))$. So we can think of $(\alpha, \beta) : D \to \mathbb{C}^{2n}$ as a lift of the restriction map $R(X, PSL(2, \mathbb{C})) \to R(\partial X, PSL(2, \mathbb{C}))$, where $\mathbb{C}^{2n} \to R(\partial X, PSL(2, \mathbb{C}))$ is the map defined in Sect. 3. (See also the remark at the end of Sect. 3.) Since we have this lift, we can define a function $z : D \to C^*$ by the formula

$$c_X(\varrho_d) = [\dots, \alpha_k(d),\ \beta_k(d), \dots; z(d)].$$

Keep in mind that this is the $PSL(2, \mathbb{C})$ Chern-Simons invariant, defined using the first pontryagin class.

Let $g : D \to \mathbb{C}$ be the map

$$g(d) = \exp\left(- 8\pi i \sum_k \int_{d_0}^{d} \alpha_k d\beta_k - d\alpha_k \beta_k \right),$$

where the integral is taken over any path from d_0 to d. Since α_k and β_k are analytic, g is analytic. Moreover, by Theorems 2.7 and 3.2, if d, $d' \in D$ then

$$\frac{z(d')}{z(d)} = \frac{g(d')}{g(d)},$$

since $g(d_0) = 1$, and $z(d) = z(d_0) g(d)$.

A point $d \in D$ corresponds to a closed hyperbolic dehn filling if there are relatively prime integers (p_k, q_k) for each k so that $p_k \alpha_k + q_k \beta_k = \frac{1}{2}$.

Suppose this is the case, and let r_k, s_k be integers so that $p_k s_k - q_k r_k = 1$. Let $M(d)$ denote the closed manifold, so $M(d) = X \cup \left(\bigcup_k (D^2 \times S^1)_k \right)$, where the meridian is $\partial(D^2 \times *)_k = \mu_k^p \lambda_k^q$ and the longitude is $(* \times S^1)_k = \mu_k^r \lambda_k^s$. The representation ϱ_d is abelian and diagonalizable on each solid torus so by Theorem 4.1,

$$c_{D^2 \times S^1_k}(\varrho_d) = [\dots, 0, r_k \alpha_k + s_k \beta_k, \dots; 1]$$

with respect to the meridians and longitudes. This in turn is equal to

$$\left[\dots, \frac{1}{2}, r_k \alpha_k + s_k \beta_k, \dots; \exp\left(- 4\pi i \sum_k (r_k \alpha_k + s_k \beta_k) \right) \right]$$

using the covariance of the bundle.

Taking the inner product we have:

$$c_{M(d)}(\varrho_d) = z(d)\exp\left(-4\pi i\sum_k (r_k\alpha_k + s_k\beta_k)\right).$$

We know the left side is equal to $\exp\left(2\pi i\left(cs(M(d)) - \frac{i}{\pi^2}\operatorname{Vol}(M(d))\right)\right)$. So

$$z(d_0)g(d) = \exp\left(2\pi i\left(cs(M(d)) - \frac{i}{\pi^2}\operatorname{Vol}(M(d))\right)\right)\prod_k \exp(4\pi i(r_k\alpha_k + s_k\beta_k)).$$

The representation on the k^{th} solid torus takes the core geodesic γ_k to

$$\begin{pmatrix} e^{2\pi i(r_k\alpha_k+s_k\beta_k)} & 0 \\ 0 & e^{-2\pi i(r_k\alpha_k+s_k\beta_k)} \end{pmatrix}.$$

This isometry of hyperbolic 3-space leaves the geodesic through 0 and ∞ invariant (in the upper-half space model) and computing with the hyperbolic metric one sees that the translation length is $l_k = \operatorname{Re}(4\pi i(r_k\alpha_k + s_k\beta_k))$ and the rotation angle is $\theta_k = \operatorname{Im}(4\pi i(r_k\alpha_k + s_k\beta_k))$. Therefore:

$$e^{4\pi i(r_k\alpha_k+s_k\beta_k)} = e^{(l_k+i\theta_k)}.$$

We then get the following version of Yoshida's formula (see also Hodgson's thesis [H]):

$$\begin{aligned} z(d_0)\exp\left(-8\pi i\int_{d_0}^{d}\sum_k \alpha_k d\beta_k - \beta_k d\alpha_k\right) \\ = \exp\left(2\pi i\left(cs(M(d)) - \frac{i}{\pi^2}\operatorname{Vol}(M(d))\right)\right)\prod_k \exp(l_k + i\theta_k). \end{aligned}$$

Moreover, $z(d_0) = \exp\left(\frac{2}{\pi}\operatorname{Vol}(X) + 2\pi i cs(X)\right)$, where $cs(X)$ is defined by Meyerhoff [Me]. (This is seen as follows: first, $|z(d_0)g(d)| = \exp\left(\frac{2}{\pi}\operatorname{Vol}(M(d))\right)\prod_k \exp(l_k)$ for $d \in D$ a surgery point. As $d \to d_0$, $\operatorname{Vol}(M(d)) \to \operatorname{Vol}(X)$ and $l_k \to 0$ [Th]. Thus $|z(d_0)g(d)| \to \exp\left(\frac{2}{\pi}\operatorname{Vol}(X)\right)$ as $d \to d_0$. However, $g(d_0) = 1$, so $|z(d_0)| = \exp\left(\frac{2}{\pi}\operatorname{Vol}(X)\right)$. Also, Meyerhoff shows that $2\pi cs(M(d)) + \sum \theta_K \to 2\pi cs(X)$.)

Plugging this in and taking the log of both sides gives us:

$$\begin{aligned} -8\pi i\sum_k \int \alpha_k d\beta_k - \beta_k d\alpha_\kappa &= \frac{2}{\pi}(\operatorname{Vol}(M(d)) - \operatorname{Vol}(X)) \\ &+ \sum_k l_K + i\left(2\pi(cs(M(d)) - cs(X)) + \sum_k \theta_k\right) \bmod 2\pi i\mathbb{Z}. \end{aligned}$$

To compare this with Theorem 5.7 of Hodgson's thesis, we need to note that instead of α and β he uses $u = 4\pi i\alpha$ and $v = 4\pi i\beta$. (There is a slight error in constants in [H]; the observant reader will notice that the coefficient of the integral on the left in [H] differs from ours by a factor of i, and the coefficient of the Chern-Simons term in [H] differs from ours by a factor of 2.)

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