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BRST Model for Equivariant Cohomology and Representatives for the Equivariant Thom Class

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Abstract. In this paper the BRST formalism for topological field theories is studied in a mathematical setting. The BRST operator is obtained as a member of a one parameter family of operators connecting the Weil model and the Cartan model for equivariant cohomology. Furthermore, the BRST operator is identified as the sum of an equivariant derivation and its Fourier transform. Using this, the Mathai-Quillen representative for the Thom class of associated vector bundles is obtained as the Fourier transform of a simple BRST closed element.

1. Introduction

Recently, there has been much interest in so-called topological quantum field theories (see, e.g., [W1, W2, BS, OSvB]). Examples of these are field theories defined by a Lagrangian $\mathscr{L}$ which results in a constant (or even zero) action $\mathscr{S} = \int \mathscr{L}$. Although these theories do not contain any dynamics, they lead to very interesting expressions for topological and even differential invariants of finite dimensional manifolds. Another reason why one is interested in topological theories is because they give rise to the most simple quantum systems, namely those with finite dimensional Hilbert space.

Also, the BRST method of quantization (see, e.g., [H, KS]) is under intensive investiation, but in the case of field theories this method is far from being understood. Trying to understand the BRST method for the simplest field theories is therefore a natural thing to do.

In the first part of this paper we will show that the BRST-cohomology of certain topological models equals the equivariant cohomology of the configuration space. Although this fact is known and used for some time now, it was never shown in what context it is true. Here, we will show that equality can be proved in a finite

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dimensional context. Furthermore, we will show in detail how the BRST cohomology fits into the two well-known models for equivariant cohomology, the Weil model and the Cartan model. We will do this by introducing a one parameter family of models, containing the models mentioned above.

In the second part of this paper, we will use some ideas of Atiyah and Jeffrey [AJ], concerning the geometry behind the path integral expressions occurring in topological field theories. They showed how to obtain Witten's quantum action for Topological Yang-Mills Theory (TYMT), using a formula of Mathai and Quillen [MQ]. This formula represents the Thom class in equivariant cohomology and therefore solves a certain double complex construction. Using Fourier transform of differential forms (super Fourier transform), we will show that BRST theory solves the same double complex construction. In fact, we obtain the Mathai-Quillen formula as the Fourier transform of a simple BRST closed element of the BRST algebra. The advantage of obtained the quantum action for TYMT in this way is that it is closer to physics and that it is the beginning of an explanation why BRST theory works so well for topological theories.

The paper that we used for an algebraic description of Topological Yang-Mills Theory is [OSvB], but the same structures can be found in recent articles on 2-dimensional topological gravity. We start with a mathematical description of equivariant cohomology.

2. Equivariant Cohomology

Equivariant cohomology has been set up to calculate cohomology of quotient spaces of the form M/G, where M is some manifold and G a connected Lie group acting on M. In the case of a free and proper G-action, M/G is a manifold without singularities and one requires equivariant cohomology to coincide with the de Rham cohomology of the quotient space. We shall now define equivariant cohomology and we will see that it fulfills this requirement.

Let $EG \to BG$ be the universal G-bundle, i.e. EG is contractible and every principal G-bundle over some base space B is the pull back of the universal one by a map $B \to BG$. BG is called the classifying space of G-bundles. Every Lie group has a universal bundle that is unique up to homotopy (see, e.g., [Hu]). The standard example of a universal bundle is the inductive limit of the Hopf fibrations $S^{2n+1} \to \mathbf{CP}^n$, which is a model for the universal S^1-bundle.

Furthermore, let M be a G-manifold. We can define the associated fibre bundle $M_G := EG \times_G M$ with fibres isomorphic to M and base space BG. The equivariant cohomology of M, $M_G^*(M)$, is now defined to be the cohomology of the fibre bundle M_G:

$$H_G^*(M) := H^*(EG \times_G M)\,. \tag{1}$$

One sees immediately that in the case of a free and proper group action, M_G can be seen as a fibre bundle over M/G with fibre the contractible space EG. So we have $H^*(M_G) \cong H^*(M/G)$. On the other hand, for $M = \{x\}$ (the singleton set) we have $H_G^*(M) = H^*(BG)$ and therefore the equivariant cohomology of a point can be quite complicated (see, e.g., [AB]).

For compact connected groups G, there are two nice models for the equivariant cohomology of G-manifolds M. Note that compactness is really necessary for these models (see [AB]). The models are called the Weil model and the Cartan model. We shall describe them in some detail now.

2.1. Weil Model

The Weil model for equivariant cohomology makes use of the so-called Weil algebra $W(\mathscr{G}) := S(\mathscr{G}^*) \otimes \Lambda(\mathscr{G}^*)$. It has a $\mathbf{Z}$-gradation by giving the generaotrs ϕ^a of $S(\mathscr{G}^*)$ degree 2 and the generators ω^a of $\Lambda(\mathscr{G}^*)$ degree 1 ($a = 1, \ldots, \dim \mathscr{G}^*$). The ω^a are of course anti-commuting, the ϕ^a commuting and both sets are dual to some fixed basis $\{X_a\}$ of $\mathscr{G}$ (the Lie algebra of G).

Suppose we are given a connection on some principal G-bundle P. This gives rise to two maps, the curvature $\mathscr{G}^* \to \Omega^2(P)$ and the connection $\mathscr{G}^* \to \Omega^1(P)$. These maps generate a homomorphism

$$W(\mathscr{G}) = S(\mathscr{G}^*) \otimes \Lambda(\mathscr{G}^*) \to \Omega(P). \tag{2}$$

We shall make this map into a homomorphism of differential algebras by defining the following differential on the Weil algebra $W(\mathscr{G})$ (the f^a_{bc} are the structure constants of G with respect to the fixed basis $\{X_a\}$):

$$\begin{aligned} d_W \omega^a &= -\tfrac{1}{2} f^a_{bc} \omega^b \omega^c + \phi^a , \\ d_W \phi^a &= -f^a_{bc} \omega^b \phi^c . \end{aligned} \tag{3}$$

This definition can be extended to $W(\mathscr{G})$ using the fact that d_W is a graded derivation of degree one. Because the relations above coincide with the definitions of the curvature and Bianchi's identity, respectively, the map (2) is (more or less by definition of d_W) a homomorphism of differential algebras. However, the differential (3) does not give an interesting cohomology: $H^*(W(\mathscr{G})) \cong \mathbf{R}$ as can be seen from a shift of generators $\phi^a \to \phi^a - \frac{1}{2} f^a_{bc} \omega^b \omega^c$. It becomes interesting if we introduce two other derivations on $W(\mathscr{G})$, the interior product I_a (of degree -1) and the Lie derivative L_a (of degree zero) ($a = 1, \ldots, \dim \mathscr{G}$):

$$\begin{aligned} I_a \omega^b &= \delta^b_a , \\ I_a \phi^b &= 0 , \\ L_a &= I_a d_W + d_W I_a =: [I_a, d_W]^+ . \end{aligned} \tag{4}$$

Note that these operations are the algebraic analogues of interior product of the connection 1-form and the infinitesimal generators of the G-action, and the Lie derivative of differential forms. The action of the L_a is nothing but the natural (coadjoint) action on $W(\mathscr{G})$.

The operators (3) and (4) generate a $\mathbf{Z}_2$-graded Lie subalgebra of the Lie superalgebra of all graded derivations. In general, the Lie super bracket of two derivation D_1 and D_2 is defined by

$$[D_1, D_2] = D_1 D_2 - (-1)^{\deg(D_1)\deg(D_2)} D_2 D_1 .$$

In this case, the bracket is easily calculated and is given by the following formulas (for convenience, we indicate explicitly the sign of the product $D_2 D_1$, so the upper $+$ means anti-commutator, the upper $-$ means commutator of the operators).

$$\begin{aligned} [d_W, L_a]^- &= 0 , & [I_a, I_b]^+ &= 0 , \\ [d_W, I_a]^+ &= L_a , & [L_a, L_b]^- &= f^c_{ab} L_c , \\ [d_W, d_W]^+ = 2d_W^2 &= 0 , & [L_a, I_b]^- &= f^c_{ab} I_c . \end{aligned} \tag{5}$$

As in the expressions above, we sum over indices if they appear twice, one up and one down. One sees immediately that the relations (5) are independent of the choice of a basis of $\mathscr{G}$. They reflect the differential geometric situation on G-manifolds (the bracket is just the commutator), so the algebra (5) not only acts on $W(\mathscr{G})$, it also acts on the algebra of differential forms, $\Omega(M)$ and thus also on the tensor product of these two algebras. However, because we want to distinguish operators on different algebras, we use a different notation. Namely, i_a for the interior product of forms and the vertical vector field generated by X_a and $\mathscr{L}_a$ for the Lie derivative of forms in the direction of X_a.

Finally, we are able to define the Weil model for equivariant cohomology. The algebra of interest is the basic subalgebra of $W(\mathscr{G}) \otimes \Omega(M)$, denoted by $(W(\mathscr{G}) \otimes \Omega(M))_{\text{basic}}$. It consists of elements annihilated by all the $I_a \otimes 1 + 1 \otimes i_a$ and $L_a \otimes 1 + 1 \otimes \mathscr{L}_a$:

$$\begin{aligned}(W(\mathscr{G}) \otimes \Omega(M))_{\text{basic}} \\ = \left(\bigcap_{a=1}^{\dim(G)} \ker(I_a \otimes 1 + 1 \otimes i_a) \right) \cap \left(\bigcap_{b=1}^{\dim(G)} \ker(L_b \otimes 1 + 1 \otimes \mathscr{L}_b) \right). \quad (6)\end{aligned}$$

This subalgebra is stable under $d_W + d$ [as follows from (5)], so it is a differential algebra. In the sequel we shall omit the subscript W and denote this differential also by d. The Chern-Weil homomorphism (2) with $P = EG$ reduces to an isomorphism on the level of cohomology if G is compact and connected:

$$H_d^*((W(\mathscr{G}) \otimes \Omega(M))_{\text{basic}} \cong H_G^*(M). \quad (7)$$

For more details we refer to [AB].

The reason why such a sophisticated model can appear in physics is probably the same as why it appears in mathematics. E.g., in the case of TYMT, M is the affine space of connections $\mathscr{A}$ on some principal fibre bundle and G is the group of gauge transformations. The quotient is not a manifold if there are reducible connections, so one would like to work with G-invariant objects on $\mathscr{A}$. This is what the Weil model is good for.

However, the use of such models in physics asks for generalizations to infinite dimensional manifolds and (non-compact) infinite dimensional groups. We will not address these problems here.

2.2. *Cartan Model*

There is another model for equivariant cohomology which is of considerable interest. This is the Cartan model. A nice description can be found in [MQ]. We will review it here, but proofs and details are postponed until the next section.

The map $\omega^a \mapsto 0$, $W(\mathscr{G}) \otimes \Omega(M) \to S(\mathscr{G}^*) \otimes \Omega(M)$ induces an algebra isomorphism

$$(W(\mathscr{G}) \otimes \Omega(M))_{\text{basic}} \cong (S(\mathscr{G}^*) \otimes \Omega(M))^G, \quad (8)$$

where the upper G means: the (infinitesimal) G-invariant subalgebra. This isomorphism can be made into an isomorphism of differential algebras by defining the following derivation on $(S(\mathscr{G}^*) \otimes \Omega(M))^G$:

$$\tilde{d}\phi^b = 0, \qquad \tilde{d}\eta = (1 \otimes d - \phi^b \otimes i_b)\eta \quad (\eta \in \Omega(M)). \quad (9)$$

This derivation squares to zero on the space of G-invariant elements (using the fact that $\phi^b L_b \otimes 1$ acts as zero on $S(\mathscr{G}^*)$). Its cohomology equals the one of the Weil model. This model for equivariant cohomology is called the Cartan model.

3. BRST Model

In this section we explain the precise relation between the BRST differential algebra and the two models for equivariant cohomology defined earlier. If a physical system, with phase space T^*M, has a symmetry group G, then the classical BRST cohomology is just the Lie algebra cohomology of Lie(G) with values in the space of functions on T^*M (see, e.g., [KS]). In the case of a topological physical theory, the symmetry group is Diff(M). If the phase space of the topological theory is the same as the phase space of a non-topological theory with symmetry group G, then one can make a new topological theory on M/G by using the action of $G \times$ Diff(M) on M. This is the type of topological theories that we are considering. The Lie algebra of Diff(M) are the vector fields on M and the linear dual of this Lie algebra are (distribution valued) differential forms. This may be the reason why the de Rham complex comes in.

Remember that on the algebra $A = W(\mathscr{G}) \otimes \Omega(M)$ we have the following differential:

$$\begin{aligned} d\phi^a &= -\, f^a_{bc}\omega^b\phi^c\,, \\ d\omega^a &= -\tfrac{1}{2}\, f^a_{bc}\omega^b\omega^c + \phi^a\,, \\ d\eta &= \text{exterior differentiation on } \eta \in \Omega(M)\,, \end{aligned} \tag{10}$$

where the ϕ^a are generators of $S(\mathscr{G}^*)$, ω^a of $\Lambda(\mathscr{G}^*)$ and f^a_{bc} are structure constants of $\mathscr{G}$, all defined with respect to the same fixed basis of $\mathscr{G}$. Note that summation over upper and lower indices will give us automatically expressions that are basis independent. The summation convention implies that we omit the summation sign whenever there are upper and lower indices that are the same.

The (unrestricted) BRST algebra of topological models on quotient spaces is the same as above (see, e.g., [OSvB]). But the differential on it (BRST operator) differs, so we shall call it $B = W(\mathscr{G}) \otimes \Omega(M)$. The BRST operator is:

$$\delta = d + \omega^a \mathscr{L}_a - \phi^b i_b\,, \tag{11}$$

where the operators $\mathscr{L}_a$ and i_b act on differential forms only and d acts on the whole algebra. Maybe, we better write $\delta = d \otimes 1 + 1 \otimes d + \omega^a \otimes \mathscr{L}_a - \phi^b \otimes i_b$ instead of (11). The action of (11) on B coincides, e.g., with [OSvB]. It is easy to check that its square equals zero. We will show now that there is an algebra automorphism of A that carries (10) into (11).

Let $\psi : B \to A$ be the map $\psi = \exp(-\omega^a i_b) = \prod_\alpha (1 - \omega^\alpha \otimes i_\alpha)$. Note that ψ is degree preserving and that it differs in two ways from the map introduced by Mathai and Quillen [MQ, Sect. 5]. In the first place our map is an isomorphism on the whole algebra, rather than only on the basic or G-invariant subalgebra. Secondly, we discriminate between the "interior product" defined as an operation on forms and on the Weil algebra. The next theorem gives a natural setting for the algebra homomorphisms of [MQ, Sect. 5].

Theorem 3.1. *ψ is an isomorphism of differential algebras,* i.e., *the diagram*

$$\begin{tikzcd} B \arrow[r, "\psi"] \arrow[d, "\delta"'] & A \arrow[d, "d"] \\ B \arrow[r, "\psi"] & A \end{tikzcd} \quad \textit{commutes}.$$

Proof. $\psi^{-1} = \exp(\omega^a i_a)$, so it is clear that ψ is bijective. Furthermore, it is an algebra homomorphism because all the factors $(1 - \omega^\alpha \otimes i_\alpha)$ are algebra homomorphisms. This implies that $\psi^{-1} \circ d \circ \psi$ is a derivation on B, so that it would be sufficient to check that it equals δ on the generators of B. However, it is equally much work to verify directly the equivalence between the two differentials. In the sequel, we sum over roman but *not* over greek indices. We have:

$$\begin{aligned} \delta \circ \omega^\alpha i_\alpha &= -\tfrac{1}{2} f^\alpha_{bc} \omega^b \omega^c i_\alpha + \phi^\alpha i_\alpha - \omega^\alpha \delta i_\alpha\,, \\ \omega^\alpha i_\alpha \circ \delta &= \omega^\alpha i_\alpha d - \omega^\alpha \omega^a i_\alpha \mathscr{L}_a - \omega^\alpha \phi^b i_\alpha i_b\,. \end{aligned} \tag{12}$$

Subtracting these equations results in:

$$[\delta, \omega^\alpha i_\alpha] = -\tfrac{1}{2} f^\alpha_{bc} \omega^b \omega^c i_\alpha + \phi^\alpha i_\alpha - \omega^\alpha \mathscr{L}_\alpha - \omega^\alpha \omega^a [\mathscr{L}_a, i_\alpha] \tag{13}$$

from which we see that $[\delta, \omega^\alpha i_\alpha] = (1 + \omega^\alpha i_\alpha)[\delta, \omega^a i_\alpha]$. We want to commute the extra term (13) to the right of the product $\prod_\alpha (1 + \omega^\alpha i_\alpha)$. So we calculate:

$$[[\delta, \omega^\alpha i_\alpha], \omega^\beta i_\beta] = [-\omega^\alpha \mathscr{L}_\alpha, \omega^\beta i_\beta] = -f^c_{\alpha\beta} \omega^\alpha \omega^\beta i_c\,, \tag{14}$$

Finally, we get:

$$\begin{aligned} \delta \circ \psi^{-1} &= \prod_\alpha (1 + \omega^\alpha i_\alpha) \Big\{ \delta - \sum_\alpha \omega^\alpha \mathscr{L}_\alpha + \sum_\alpha \phi^\alpha i_\alpha - \tfrac{1}{2} \sum_\alpha f^\alpha_{bc} \omega^b \omega^c i_\alpha \\ &\quad + \sum_\alpha f^c_{a\alpha} \omega^a \omega^\alpha i_c - \sum_{\alpha<\beta} f^c_{\alpha\beta} \omega^\alpha \omega^\beta i_c \Big\} \\ &= \psi^{-1} \circ d\,. \quad \square \end{aligned} \tag{15}$$

As a corollary of this theorem, we get the following isomorphism:

$$H^*_\delta(B) \cong H^*_d(A) \cong H^*_{\text{de Rh.}}(M)\,, \tag{16}$$

where the last isomorphism follows from the triviality of the cohomology of $(d, W(\mathscr{G}))$.

To compute the image of the basic subalgebra of A, we need to know the images of the operators $I_a \otimes 1 + 1 \otimes i_a$ (these images already appeared in [OSvB], but now we know where they come from). It turns out that the corresponding operator on B only acts on $W(\mathscr{G})$. This follows from:

$$(I_a \otimes 1) \prod_\alpha (1 + \omega^\alpha i_\alpha) = \prod_\alpha (1 + \omega^\alpha i_\alpha)(I_a \otimes 1 + 1 \otimes i_a)\,. \tag{17}$$

Furthermore, the operators $L_a \otimes 1 + 1 \otimes \mathscr{L}_a$ commute with ψ, so the G-action on both algebras is the same. This follows also from [compare this with (5)]:

$$[d + \omega^a \mathscr{L}_a - \phi^b i_b, I_c \otimes 1] = L_c \otimes 1 + 1 \otimes \mathscr{L}_c\,. \tag{18}$$

We are now able to compute the equivalence of the Weil model, induced by ψ. The intersection of the kernels of $I_a \otimes 1$ restricts B to $S(\mathscr{G}^*) \otimes \Omega(M)$. The kernels of $L_a \otimes 1 + 1 \otimes \mathscr{L}_a$ restrict it further to the G-invariant subalgebra of $S(\mathscr{G}^*) \otimes \Omega(M)$. So the corresponding subalgebra of A_{basic} is $(S(\mathscr{G}^*) \otimes \Omega(M))^G$, the algebra of the Cartan model! Even more is true. The operator δ on this subalgebra equals the differential (9). So the following theorem arises:

Theorem 3.2. *The ω-independent, G-invariant elements of the* BRST *algebra B give the Cartan model for equivariant cohomology. We have the following commutative diagram:*

$$\begin{array}{ccc} (B,\delta) & \xrightarrow{\psi} & (A,d) \\ \uparrow & & \uparrow \\ (S(\mathscr{G}^*) \otimes \Omega(M))^G & \xrightarrow{\psi} & (W(\mathscr{G}) \otimes \Omega(M))_{\text{basic}} \,. \end{array} \tag{19}$$

The map ψ between the two restricted algebras is the same as in (8):

$$\psi^{-1}|_{A_{\text{basic}}} : \omega^a \mapsto 0 \,.$$

Proof. The commutativity follows from Eqs. (17) and (18). The last remark follows directly from the identity:

$$\psi^{-1}|_{A_{\text{basic}}} = \prod_\alpha (1 + \omega^\alpha \otimes i_\alpha)|_{A_{\text{basic}}} = \prod_\alpha (1 - \omega^\alpha I_\alpha \otimes 1)|_{A_{\text{basic}}} \,. \quad \square \tag{20}$$

We would like to point our here that the isomorphism of the bottom line in (19) was also proved by Mathai and Quillen [MQ, Sect. 5].

4. A Family of Models

A bit more generally we can use a parameter in the isomorphism ψ:

$$\psi_t = \exp(-t\omega^a i_a) : B \to A \,. \tag{21}$$

We can calculate what operator on B corresponds to d. It turns out that:

$$\psi_{-t} \circ d \circ \psi_t = \exp(\operatorname{ad}(t\omega^a i_a))\,(d) = d + t\omega^a \mathscr{L}_a - t\phi^b i_b + \tfrac{1}{2}\, t(1-t)\, f^c_{ab} \omega^a \omega^b i_c \,. \tag{22}$$

So, if we introduce δ_t as a notation for this differential, then $\delta_0 = d$, $\delta_1 = \delta$. Furthermore, we have

$$(I_a \otimes 1 + (1-t)\, 1 \otimes i_a) \prod_\alpha (1 + t\omega^\alpha i_\alpha) = \prod_\alpha (1 + t\omega^\alpha i_\alpha)(I_a \otimes 1 + 1 \otimes i_a) \tag{23}$$

and

$$[\delta_t, I_a \otimes 1 + (1-t)\, 1 \otimes i_a] = L_a \otimes 1 + 1 \otimes \mathscr{L}_a \,. \tag{24}$$

So we obtain a family of Lie super algebras acting on $W(\mathscr{G}) \otimes \Omega(M)$, generated by

$$\begin{gathered} I_a \otimes 1 + (1-t)\, 1 \otimes i_a \,, \\ 1 \otimes \mathscr{L}_a + L_a \otimes 1 \,, \\ \delta_t \,. \end{gathered} \tag{25}$$

Using the fact that we are only interested in elements that vanish under (25), we can make a basis transformation within the Lie algebra to alter the differential δ_t. Of course, its square is only zero on elements that vanish under the first two equations in (25). A very convenient form for the differential is:

$$d_t \otimes 1 + 1 \otimes d - \phi^b \otimes i_b \,, \tag{26}$$

where d_t acts on $W(\mathscr{G})$ as follows:

$$\begin{aligned} d_t \omega^a &= -\tfrac{1}{2}(1-t) f^a_{bc} \omega^b \omega^c \,, \\ d_t \phi &= -(1-t) f^a_{bc} \omega^b \phi^c \,. \end{aligned} \tag{27}$$

The corresponding family of subalgebras is given by the image of the map

$$\prod_\alpha (1 + t\omega^\alpha \otimes i_\alpha)|_{A_{\text{basic}}} = \prod_\alpha (1 - t\omega^\alpha I_\alpha \otimes 1)|_{A_{\text{basic}}} \,. \tag{28}$$

This map is equally well described by

$$\omega^\alpha \mapsto (1-t)\omega^\alpha \,. \tag{29}$$

From the expressions above it is obvious why the BRST model ($t = 1$) is such a special model. The algebra as well as the differential become extremely simple. I do not know whether or not the limit $t \to \infty$ can be taken and whether or not this gives a new useful model.

5. Fourier Transform of Differential Forms

In this second part of the paper we are going to describe the BRST construction of representatives for the Thom class of a vector bundle. To see why it works, we will need to extend the definition of Fourier transform from certain functions on a vector space to differential forms on that vector space. We will see that it is much more natural to take differential forms as the domain of Fourier transform, because one does not need a measure in this case.

Let V be a complex n-dimensional vector space and let $\Lambda(V)$ be its Grassmannian algebra of dimension 2^n.

Recall that Berezin integration on $\Lambda(V)$ is a linear map from $\Lambda(V)$ to $\mathbf{C}$ that is zero on elements of degrees less than n and is 1 on some fixed element $\psi^1 \wedge \ldots \wedge \psi^n \in \Lambda^n(V)$. It is called integration, because it has a lot of properties similar to ordinary integration. E.g., a linear coordinate transformation $A: V \to V$ must be compensated by a Jacobian. However, this Jacobian is $\det^{-1}(A)$ instead of $\det(A)$. (By the way, this is precisely the reason why integration of differential forms can be defined independent of coordinates: the Jacobians cancel each other!). A nice reference for this material is the book of Bryce de Witt [dW]. We need the following (trivial) extension of this integration map:

$$\int d\psi : \Lambda(V^*) \otimes \Lambda(V) \to \Lambda(V^*) \,, \tag{30}$$

where V^* is the linear dual of V. The tensor product is a tensor product between $\mathbf{Z}_2$-graded algebras. Fourier transform on Grassmann algebras can be defined in the following way.

Definition 5.1. Let $\psi^1, \ldots, \psi^n$ be generators of $\Lambda(V)$ of degree 1 such that $\int d\psi\psi^1 \wedge \ldots \wedge \psi^n = 1$ and let $\bar{\psi}_1, \ldots, \bar{\psi}_n$ be their duals in $\Lambda^1(V^*)$.

For every $\eta \in \Lambda(V)$, Fourier transform $\mathscr{F}: \Lambda(V) \to \Lambda(V^*)$ is defined by

$$\mathscr{F}(\eta) = \int d\psi(\eta \wedge e^{i\bar{\psi}_j \otimes \psi^j}), \tag{31}$$

where $\exp(i\bar{\psi}_j \otimes \psi^j) \in \Lambda(V^*) \otimes \Lambda(V)$ is given by the well known power series, which in this case stops at the n^{th} power.

Of course, as it stands it is just a copy of the definition of ordinary Fourier transform. The next proposition shows that it has also properties like ordinary Fourier transform.

Proposition 5.2. *If* $\dim(V)$ *is even, then* $\mathscr{F}^2: \Lambda(V) \to \Lambda(V)$ *equals the identity.*

Proof. $\mathscr{F}$ is a linear map, so it suffices to check the statement on homogeneous elements in $\Lambda(V)$. Let $\eta = \psi^1 \wedge \ldots \wedge \psi^k$ be an element of $\Lambda^k(V)$. The component of $\eta \cdot \exp(i\bar{\psi}_j \otimes \psi^j)$ that is in $\Lambda(V^*) \otimes \Lambda^n(V)$ is $(i)^{n-k}(-1)^{\frac{1}{2}(n+k-1)(n-k)} \bar{\psi}_{k+1} \wedge \ldots \wedge \bar{\psi}_n \otimes \psi^1 \wedge \ldots \wedge \psi^n$.

Thus,

$$\mathscr{F}(\psi^1 \wedge \ldots \wedge \psi^k) = (i)^{n^2-k^2} \bar{\psi}_{k+1} \wedge \ldots \wedge \bar{\psi}_n \in \Lambda(V^*). \tag{32}$$

Applying Fourier transform once again, we obtain

$$\mathscr{F}^2(\psi^1 \wedge \ldots \wedge \psi^k) = (i)^{n^2-k^2}(i)^{k^2} \psi^1 \wedge \ldots \wedge \psi^k. \tag{33}$$

For n even, the prefactor equals 1, so $\mathscr{F}^2$ is the identity. $\square$

Remarks. 1. Of course, so far we have just done linear algebra. Nevertheless, combining this with ordinary Fourier transform, it will turn out to be independent of the choice of a measure and therefore very useful.
2. The factor i in the exponent is only meant to give a nice prefactor when computing $\mathscr{F}^2$. It is of no importance for the convergence of the integral as it is in the ordinary case.
3. We have stressed earlier the fact that we should rather be working in an infinite dimensional context. It should not surprise me very much if this Fourier theory will turn out to be very useful in that context also. E.g., differential forms of finite codegree can easily be obtained from differential forms of finite degree, using Fourier transform.

We will now combine this definition with ordinary Fourier transform to obtain Fourier transform of differential forms. We will take the Schwarz functions, denoted by $\mathscr{S}(V)$, as the domain for Fourier transform. Differential k-forms, $\Omega^k_s(V)$, can be seen as elements of $\mathscr{S}(V) \otimes \Lambda^k(V^*)$. Note that, although the notations are very similar, this algebra has little to do with the Weil algebra used in the previous sections. Combined Fourier transform maps this space to $\mathscr{S}(V^*) \otimes \Lambda^{n-k}(V)$:

$$\mathscr{F}(f \otimes \eta)(b) = \int_V f e^{i\langle b|\cdot\rangle} \otimes \eta e^{i\omega},$$

where $\int_V$ is integration of differential forms and ω is the canonical symplectic 2-form on $V \times V^*$. In coordinates z^i and differentials ψ^i, $\omega = \bar{\psi}_j \otimes \psi^j$, $\langle b \mid \cdot \rangle$ means $b_i z^i$

and integration of differential forms boils down to ordinary integration over the z^i and Berezin integration over the ψ^i.

A lot of properties of Fourier transform on functions extend to this combined Fourier transform. E.g., it is possible to extend the definition of the convolution product such that it is the Fourier image of the wedge product of differential forms. The convolution of a k-form and an l-form is a $(k+l-n)$-form.

Remember that convolution between two functions f and g in $\mathscr{S}(V)$ is defined by $f * g(y) = \int_V f(x)g(y-x)dx$. Therefore, it is natural to define for $\eta, \zeta \in \Lambda(V)$:

$$(\eta * \zeta)(\phi) := \int \eta(\xi) \wedge \zeta(\phi - \psi)d\psi \,, \tag{34}$$

where $\eta(\psi)$ means, η expressed in terms of generators ψ^j. The ϕ^i are just other names for the same generators, as it is the case in the definition of ordinary convolution. Here, $\zeta(\phi-\psi)$ means, substitute $\phi^i - \psi^i$ whenever a ψ^i occurs in $\zeta(\psi)$. It is not very difficult to prove that for every $\eta, \zeta \in \Lambda(V)$ we have

$$\mathscr{F}(\eta \wedge \zeta) = \mathscr{F}(\eta) * \mathscr{F}(\zeta) \,. \tag{35}$$

Of course, we can combine this convolution product with the ordinary one to obtain a convolution product on the algebra of differential forms $\mathscr{S}(V) \otimes \Lambda(V^*)$. Note that the top form $\psi^1 \wedge \ldots \wedge \psi^n$ is the unit element for the convolution product in $\Lambda(V)$, whereas for the product on $\mathscr{S}(V)$ the unit is not contained in $\mathscr{S}(V)$ (it is the Dirac distribution).

Example. Suppose $\dim(V) = 4$; $\psi^1, \ldots, \psi^4$ are generators of $\Lambda(V)$ such that $\int \psi^1\psi^2\psi^3\psi^4 d\psi = 1$. Then,

$$\mathscr{F}(\psi^1\psi^2) = \int \psi^1\psi^2 e^{i\bar{\psi}_j\psi^j} d\psi = \int \psi^1\psi^2\left(\frac{-1}{2!}\right)(\bar{\psi}_3\psi^3 + \bar{\psi}_4\psi^4)^2 d\psi = \bar{\psi}_3\bar{\psi}_4$$

and

$$\mathscr{F}(\psi^1) = -i\bar{\psi}_2\bar{\psi}_3\bar{\psi}_4 \,, \qquad \mathscr{F}(\psi^2) = i\bar{\psi}_1\bar{\psi}_3\bar{\psi}_4 \,.$$

Now, let us calculate the convolution product

$$\mathscr{F}(\psi^1) * \mathscr{F}(\psi^2) = \int \phi_2\phi_3\phi_4(\bar{\psi}_1 - \phi_1)(\bar{\psi}_3 - \phi_3)(\bar{\psi}_4 - \phi_4)d\psi = \bar{\psi}_3\bar{\psi}_4 \,.$$

So for this particular case we have verified that the Fourier image of the wedge product is the (super-) convolution product.

Another property of Fourier transform that we would like to extend is the following. If z^i are coordinates on V and b_i are the dual coordinates on V^*, then it is well known that $\mathscr{F}\left(\frac{\partial f}{\partial z^i}\right) = (-ib_i)\mathscr{F}(f)$. Thus, multiplying with b_i is a derivation for the convolution product. On the (super-) algebra $\Omega_s(V) = \mathscr{S}(V) \otimes \Lambda(V^*)$ there exists a (super-) derivation with square zero, namely the de Rham differential d. We would like to know its Fourier image.

Let us define on $\mathscr{S}(V^*) \otimes \Lambda(V)$ the Koszul differential δ as follows (don't confuse the d and δ used here with the ones of Sect. 4). δ is the derivation of degree -1 acting on generating elements by

$$\begin{aligned} \delta(f) &= 0 \qquad (f \in \mathscr{S}(V^*)) \\ \delta(f \otimes \bar{\psi}_i) &= -fb_i \otimes 1 \qquad (i = 1, \ldots, \dim(V)) \,. \end{aligned} \tag{36}$$

Here $\bar{\psi}_i$ is the image of b_i under the inclusion $V \to \Lambda(V)$. Obviously, the definition of δ does not depend on the choice of linear coordinates on V^*. We will prove now that this is the Fourier image of the de Rham differential.

Proposition 5.3. $\mathscr{F} \circ d = \delta \circ \mathscr{F}$.

Proof.

$$\begin{aligned}\delta \circ \mathscr{F}(f \otimes \eta) &= \delta(\textstyle\int f e^{i\langle b|z\rangle} \otimes \eta e^{i\bar{\psi}_j \psi^j}\, dz\, d\psi) \\ &= \textstyle\int f(z) e^{i\langle b|z\rangle}(-ib_j) \otimes \psi^j \eta e^{i\bar{\psi}_j \psi^j}\, dz\, d\psi \\ &= \int \frac{\partial f}{\partial z^j} e^{i\langle b|z\rangle} \otimes \psi^j \eta e^{i\bar{\psi}_j \psi^j}\, dz\, d\psi = \mathscr{F} \circ d(f \otimes \eta)\,. \quad \square\end{aligned}$$

Remarks. 1. One may wonder why δ depends on the linear structure of V^*, whereas d does not (δ only commutes with linear diffeomorphisms, d commutes with all diffeomorphisms). This is because Fourier transform, which carries the one into the other, depends on the linear structure.

2. It would have been very nice if we could also define the Fourier transform of differential forms on vector bundles. Because in that case we would obtain (using the next sections) representatives for the Thom class of arbitrary vector bundles. However, this does not seem to be possible without choosing a connection in the vector bundle. We will not go into this here.

3. In BRST theory of topological models, the Fourier transform will be used as follows. The algebra $\Omega(V) \otimes \Omega(V^*)$ can be given a double complex structure using the differentials $d \otimes 1$ and $(-1)^p \otimes \delta$ (p being the degree operator). The sum s of these two differentials is part of the BRST operator. Using s, we can define (extended) Fourier transform $\bar{\mathscr{F}}: \Omega_s(V) \otimes \Omega_s(V^*) \to \Omega_s(V)$ by integration over $\Omega_s(V^*)$, after multiplication with $\exp(is(z^j \bar{\psi}_j))$. From the proposition above, it follows that $\bar{\mathscr{F}} \circ s = d \circ \bar{\mathscr{F}}$. BRST theory used the map $\bar{\mathscr{F}}$ to obtain d-closed differential forms from rather simple s-closed expressions.

6. Representing Poincaré Duals with BRST Theory

In this section we show how BRST theory represents certain cohomology classes. In physics, one is interested (when quantizing via path integrals) in writing integrals over submanifolds as integrals over the whole manifold. The manifolds in this context are infinite dimensional, but for the sake of simplicity we will neglect this fact. It should be kept in mind, however.

Expressing integrals over larger manifolds asks for representatives of Poincaré duals. We shall explain this first. Let M be a smooth oriented manifold of dimension $m > 0$ and let $N \subset M$ be a smooth oriented compact submanifold of dimension $m - n$. Integration over N gives a linear map from $H^{m-n}(M)$ to $\mathbf{R}$ (assuming that N has no boundary), hence an element of $H^{m-n}(M)^*$. By Poincaré duality this element corresponds to an element of $H^n_{\mathrm{cpt}}(M)$, which is by definition the Poincaré dual η of $N \subset M$. Stated otherwise, if ω is any element of $H^{m-n}(M)$, then we have

$$\int_N i^*\omega = \int_M \omega \wedge \eta\,, \tag{37}$$

where $i: N \to M$ is the inclusion map. Obviously, the support of η may be shrunk into any open neighbourhood of N in M (see, e.g., [BT]).

One of the aims of BRST theory is to construct representatives for Poincaré duals for submanifolds of a so-called configuration space. The submanifolds are called constraint surfaces. For the sake of simplicity we assume that a constraint surface is given by the zeroes of a map $F: M \to V$, where V is a vector space. Normally, this is only true locally. In this simple case the Poincaré dual is zero, because $F^{-1}(0)$ is the boundary of another submanifold. Hence, the construction below will not give results that are very spectacular. Its only interest lies in the fact that it has an equivariant generalization, which is the topic of the next section.

The BRST construction uses the following fact. Let $\mathscr{V} \to M$ be a vector bundle with fiber the n-dimensional vector space V and let $F: M \to \mathscr{V}$ be a generic section. If τ is a representative for the Thom class of $\mathscr{V}$, then $F^*\tau$ is a representative for the Poincaré dual of the zero locus of F in M. Recall that the Thom class is represented by forms that are compactly supported in the fiber direction and that give 1 when integrated over the fibers. Representatives for the Thom class of a vector bundle are not very easy to find (for a construction, see [BT]), except for the case of a trivial bundle. In this case, the Thom class is just a normalized generator of $H^n(V)$. Using an inner product on V and an orientation, we get a volume form $dv^1 \wedge \ldots \wedge dv^n$ and the Thom class is represented by $f dv^1 \wedge \ldots \wedge dv^n$, where f is a function on V such that $\int_V f = 1$. We can use this below, since a map $F: M \to V$ can be regarded as a section of the trivial vector bundle $M \times V \to M$.

To describe the BRST construction, we introduce the BRST differential algebra. Let x^i be local coordinates on M and denote the differentials by dx^i. Locally, they generate $\Omega(M)$, the algebra of differential forms on M. Furthermore, we introduce the algebras $\Omega_s(V)$ and $\Omega_s(V^*)$ of "Schwarz" forms on V and V^*. From now on, we shall omit the subscript s. Let us assume an inner product on V so that we can choose an orthonormal basis z^i for V. Furthermore, let b_i be the dual basis and let ψ^i and $\bar{\psi}_i$ denote the associated elements in $\Omega^1(V)$ and $\Omega^1(V^*)$, respectively. The BRST algebra is the tensor product of the three algebras above. It has a **Z**-gradation defined by

$$\begin{array}{lll} \deg(x^i) = 0\,, & \deg(z^i) = 0\,, & \deg(b^i) = 0\,, \\ \deg(dx^i) = 1\,, & \deg(\psi^i) = 1\,, & \deg(\bar{\psi}_i) = -1\,. \end{array} \tag{38}$$

The BRST operator, denoted by s, is the derivation of degree 1 defined by

$$\begin{array}{lll} s(x^i) = dx^i\,, & s(b_i) = 0\,, & s(z^i) = \psi^i\,, \\ s(dx^i) = 0\,, & s(\bar{\psi}_i) = -b_i\,, & s(\psi^i) = 0\,. \end{array} \tag{39}$$

Note that its square is zero and on $\Omega(M) \otimes \Omega(V)$ it equals the de Rham differential. Furthermore, note that it does not depend on the choice of linear coordinates. Of course, the BRST cohomology in this case is just the cohomology of M, but this is what we want since we aim at constructing representatives for certain cohomology classes (namely Poincaré duals) on M.

Proposition 6.1. *Let F be a map $M \to V$ which is submersive so that $F^{-1}(0)$ is a manifold. F can be regarded as section $M \to M \times V$ of a trivial vector bundle. Let z^i and b_i be dual orthonormal coordinates on V and V^*, then*

$$\int_{V^*} e^{is\left(z^i\bar{\psi}_i - i\sum_i b_i\bar{\psi}_i\right)} db\, d\bar{\psi} \tag{40}$$

represents the Thom class of the vector bundle $M \times V$. Its pull back by $F: M \to M \times V$ is a closed form in $\Omega(M)$ representing the Poincaré dual of the submanifold given by the equations $F = 0$.

Proof. The first term in the exponent just says that we are doing a Fourier transform of the function $\exp\left(s\left(\sum_i b_i\bar{\psi}_i\right)\right) = \exp(-\langle b \mid b\rangle)$. This gives a generator of $H^n(V)$ and therefore represents the Thom class of $M \times V$ (at least if Fourier transform is normalized correctly). $\square$

Remarks. 1) Inserting a parameter in the exponent above, using $\langle z - itb \mid \bar{\psi}\rangle$ (index free notation to stress the independence of orthonormal coordinates!) instad of $\langle z - ib \mid \bar{\psi}\rangle$, we obtain a family of representatives. For $t = 0$ we obtain the Dirac distribution $\delta(z)\psi^1 \wedge \ldots \wedge \psi^n$.

2) Note that the integrand above is a differential form on $V \times V^*$. First producing a differential form on $V \times V$: and then pulling back suggests that there may be a symplectic structure involved. Indeed, the term $\langle \psi \mid \bar{\psi}\rangle$ can be interpreted as the symplectic two form on $V \times V^*$. Looking at the BRST operator s, we see that it is almost of the form $d - i_X$, used in [BV] to localize integrals to the set of zeroes of the vector field X (in this case X would be $b_i\partial_i$, the Euler vector field). The difference is that both d and i_X act on only half of the variables. One could introduce an operator acting the same on the other half. This is sometimes done in physics and is called the anti-BRST operator. However, it is not clear what one gains by doing this. Another problem is that the vector field X does not generate a circle action (for circle actions we could have used [DH]). This is the reason why we simply forget about symplectic structures and rather use the Fourier theory developed in Sect. 5.

3) If we knew how to define Fourier transform $\Omega_{\mathrm{cv}}(\mathscr{V}) \to \Omega_{\mathrm{cv}}(\mathscr{V}^*)$, where $\mathscr{V}$ is a (non-trivial) vector bundle over M and $\Omega_{\mathrm{cv}}(\mathscr{V})$ denotes the space of differential forms on that bundle that are compatibly supported in the vertical direction, then we could obtain a representative for its Thom class in the same way. If $\mathscr{V}$ is the kernel of a bundle map between two trivial vector bundles, then there exists a similar (but more involved) formula for the Thom class. This formula is used in the case F is not submersive, but $F^{-1}(0)$ is still a manifold.

4) The summation sign occurring in the exponent indicates that we are doing something non-natural. Indeed, we are using an inner product to produce this Gaussian shaped representative.

7. Thom Classes in Equivariant Cohomology

In this section we shall describe how one can obtain representatives for the equivariant Thom class applying the "zig-zag" method of double complexes to the Cartan model. In the next section we will show that BRST theory actually gives this "zig-zag" construction.

Let V be an n-dimensional vector space on which a connected compact Lie group G acts linearly. The Cartan model of the equivariant cohomology of V is given by the differential algebra

$$((S(\mathscr{G}^*) \otimes \Omega(V))^G, 1 \otimes d - \phi^b \otimes i_b)\,. \tag{41}$$

By $\Omega(V)$ we mean differential forms on V with compact support, as is usual in the context of Thom classes.

This Cartan model can be made into a double complex by using the following grading. As a first grading we take the degree p of the polynominal in $S(\mathscr{G}^*)$. It is raised by one by $\phi^b \otimes i_b$ and invariant under $1 \otimes d$. The second degree is the sum of the previous one p and the form degree q. It is raised by one by $1 \otimes d$ and invariant under $\phi^b \otimes i_b$. The total degree is $2p + q$ and coincides with the degree introduced earlier for the Cartan model.

It is very natural in this situation to use a spectral sequence to compute the associated graded algebra of the cohomology of this double complex (a convenient reference for this material is the book of Bott and Tu [BT]). Of course we know what the answer will be. Indeed, we are computing the cohomology of a vector bundle (namely $EG \times_G V$), so what comes out should be isomorphic to $S(\mathscr{G}^*)^G$, the cohomology of the base space BG. It is nevertheless instructive to compute the spectral sequence, since we are after representatives of cohomology class. The E_1-term (computing the cohomology for $1 \otimes d$ only) of the spectral sequence is $(S(\mathscr{G}^*) \otimes H^n(V))^G$, which equals $S(\mathscr{G}^*)^G \otimes H^n(V)$ (remember that we are interested in the compact support cohomology). The G-invariance can be restricted to the first factor. This is because the G-action is homotopic to zero on the de Rham complex.

The E_2-term equals the E_1-term, because the action of $\phi^b \otimes i_b$ on E_1 is zero. This is clear from the fact that V has only cohomology in one dimension (here the difference of the two degrees comes in). In fact, the spectral sequence stops here and thus the associated graded algebra of the cohomology of this double complex equals the E_1-term. This is true in general if there is a relation between the degrees of the non-vanishing cohomology. Each G-invariant element in $S(\mathscr{G}^*) \otimes \Omega^n(V)$ can be extended to a representative for a class in the Cartan model by means of the "zig-zag" method. We shall explain this now.

Let us start with an element $\eta \in \Omega^n(V)^G$, such that $\int_V \eta = 1$. This is the top part of a representative for the Thom class in the Cartan model. Recall that the Thom class of a vector bundle is an n-form on the vector bundle, compactly supported in the direction of the fibers, such that integration over the fibers gives the class 1 in the cohomology of the basis. Therefore, the term of a Thom representative with the highest V-degree is as described above. Since η is closed, also $\phi^b \otimes i_b \eta$ is closed (the two derivations commute on G-invariant forms), hence exact, say $\phi^b \otimes d\eta_b$ (the element $\phi^b \otimes \eta_b$ must and can be taken G-invariant, using integration over G). Then $\eta + \phi^b \otimes \eta_b$ is the first two terms of the Thom representative in the Cartan model. We have

$$(1 \otimes d - \phi^b \otimes i_b)(\eta + \phi^a \otimes \eta_a) = -\phi^a \phi^b \otimes i_b \eta_b\,, \tag{42}$$

This form is again closed, hence exact, say $-\phi^a \phi^b \otimes d\eta_{ab}$. The first three terms of the Thom representative are now of course $\eta + \phi^a \otimes \eta_a + \phi^a \phi^b \otimes \eta_{ab}$ and we can go on this way. This process will stop because the form degree (i.e., the difference of the degrees of the double complex) decreases. This construction is called the "zig-zag" construction.

Actually constructing a representative this way is not so easy, however. One can try it, starting for example with

$$e^{-z^2} \psi^1 \wedge \ldots \wedge \psi^n \tag{43}$$

(this differential form does not have compact support, but L^2-cohomology and compact support cohomology are the same for vector spaces). It is the merit of [MQ] that they gave an explicit formula for this example. Their construction is a special case of the BRST construction, which we will describe now.

8. The BRST Representative of the Equivariant Thom Class

In this section we present an equivariant version of Proposition 6.1. It will turn out that the representative we get is nothing but the pull back (by F) of the Thom class representative of [MQ] and [AJ].

Let G be a compact Lie group acting freely on M and linearly on V and let F be an equivariant map from M into V. Furthermore, we assume that this map is a submersion on its zero locus (as a subset of M/G). As we promised not to dive into physics, we just present the equivariant version of the BRST operator (39). As before, let X_a denote a basis of $\mathscr{G}$ and let ϕ^a be the generators of $S(\mathscr{G}^*)$ dual to this basis. Furthermore, a $\cdot$ denotes the (linear) $\mathscr{G}$-action on V, V^* or $\Lambda^1(V^*)$. The equivariant BRST operator is defined as follows:

$$\begin{aligned} s(z^i) &= \psi^i\,, & s(\psi^i) &= -\phi^a \otimes X_a \cdot z^i\,, \\ s(x^i) &= dx^i\,, & s(b_i) &= \phi^a \otimes X_a \cdot \bar{\psi}_i\,, \\ s(\bar{\psi}_i) &= -b_i\,, & s(dx^i) &= -\phi^a \otimes X_a(x^i)\,, \\ & & s(\phi^a) &= 0\,. \end{aligned} \tag{44}$$

Hereby it is understood that s acts only on the G-invariant elements of the algebra $S(\mathscr{G}^*) \otimes \Omega(M) \otimes \Omega(V) \otimes \Omega(V^*)$. The following proposition is an equivariant version of Proposition 6.1.

Proposition 8.1. *The element (notation as in Sect. 6)*

$$\int\limits_{V^*} e^{is(\langle z-ib|\bar{\psi}\rangle)} db\, d\bar{\psi} = e^{-z^2} \int e^{i\langle\psi|\bar{\psi}\rangle + i\langle\phi\,\cdot\,\bar{\psi}|\bar{\psi}\rangle} d\bar{\psi} \tag{45}$$

is a closed form in $(\Omega(V) \otimes S(\mathscr{G}^*))^G$. It solves the double complex construction of Sect. 7 and it equals the equivariant Thom form of [MQ]. *Pulling back this form by F, we obtain a representative for the Poincaré dual of the zero locus of F in M/G.*

Proof. Essentially, the proof consists of showing that (45) is closed under $d - \phi^a \otimes i_a$ and that its top form part equals (43). Furthermore, Fourier Transform of Sect. 5 is used. Just as δ is the Fourier image of the de Rham differential d (see Sect. 5), we can compute the Fourier image of the equivariant operator $d - \phi^a \otimes i_a$. The equivariant BRST operator s defined above, restricted to $(S(\mathscr{G}^*) \otimes \Omega(M) \otimes \Omega(V))^G$, consists of this equivariant differential $d - \phi^a \otimes i_a$ and on $(S(\mathscr{G}^*) \otimes \Omega(V^*))^G$ it equals its Fourier image. This follows from the following calculation (we use $i_Y(\psi^j) = Y \cdot z^j$, $Y \in \mathscr{G}$). Let $f \otimes \eta$ be in $\Omega(V)$, then

$$\begin{aligned} \mathscr{F}(i_Y(f \otimes \eta)) &= \int f e^{i\langle b|z\rangle} \otimes (i_Y \eta) e^{i\bar{\psi}^j \psi^j} dz\, d\psi \\ &= -\int f e^{i\langle b|z\rangle} \otimes (-1)^{\eta} \eta (i_Y e^{i\bar{\psi}^j \psi^j}) dz\, d\psi \\ &= \int iY(z^j) f e^{i\langle b|z\rangle} \otimes \bar{\psi}_j \eta e^{i\bar{\psi}_j \psi^j} dz\, d\psi \\ &= Y\left(\frac{\partial}{\partial b_j}\right) \otimes \bar{\psi}_j \mathscr{F}(f \otimes \eta) = \left(-\frac{\partial}{\partial b_j} \otimes Y \cdot \bar{\psi}_j\right) \mathscr{F}(f \otimes \eta)\,. \end{aligned}$$

This action on the variables b and $\bar{\psi}$ coincides with the definition of s given above, since it acts as zero on $\bar{\psi}_i$ and as $-1 \otimes Y \cdot \bar{\psi}_i$ on b_i.

Next, remember the last remark of Sect. 5. The integrand above can be written as a product $e^{is\langle z|\bar{\psi}\rangle}e^{s\langle b|\bar{\psi}\rangle}$, which means that we are Fourier transforming $e^{s\langle b|\bar{\psi}\rangle}$.

Because this form is obviously s-closed, the integral itself is $d - \phi^a \otimes i_a$-closed. Furthermore, its top form part equals expression (43). This can be verified by replacing ϕ by zero.

We conclude from Sect. 7 that the BRST construction gives the equivariant Thom form of [MQ]. □

To end this section, we shall make some comments on the context in which this theorem may be used. Physicists use the expression for the Poincaré dual to compute quantum correlation functions of BRST invariant observables. We shall epxlain shortly what this means and how its works.

Denote $F^{-1}(0)/G \subset M/G$ by Z and let $i: Z \to M/G$ be the inclusion. Using the formula above, we obtain a polynomial on $H^*(M/G)$ by integrating products of closed differential forms over M/G. This polynomial, evaluated at a given product of forms, is called a quantum correlation function. BRST invariant observables for a topological quantum (field) theory are just cohomology classes on the configuration space (which equals M/G in this case). So mathematically, it is clear what the objects of interest are. They are products of classes in $H^*(M/G)$ (of total degree dim(Z)) integrated over M/G using the Poincaré dual of Z.

Two problems remain. The first is that we would like to integrate over M rather than over M/G. Secondly, we still have to apply the Weil homomorphism to get a differential form representative instead of a Cartan model representative. BRST theory solves the two problems together by introducing a dual Weil algebra, $W(\mathscr{G}^*) = S(\mathscr{G}) \otimes \Lambda(\mathscr{G})$, generated by $\bar{\phi}_a$ (degree -2) and $\bar{\omega}_a$ (degree -1) $(a = 1, \ldots, \dim(G))$. The BRST differential extends as follows

$$s\bar{\omega}_a = -f^c_{ab}\phi^b \otimes \bar{\phi}_c\,, \qquad s\bar{\phi}_a = \bar{\omega}_a\,. \tag{46}$$

Furthermore, a Riemannian metric on M and a non-degenerate invariant bilinear form on $\mathscr{G}$ is used to obtain a linear map $\nu: \mathscr{G}^* \to \Omega(M)$ from the infinitesimal action $\mathscr{G} \to \Gamma(TM)$. Because we have chosen a basis of $\mathscr{G}$ we can define the 1-forms ν^a to be the components of the map ν.

To solve the two problems mentioned above, just add the following factor to the integrand:

$$\int_{W(\mathscr{G}^*)} e^{is(\bar{\phi}^a\nu^a)} d\bar{\phi}\, d\bar{\omega} \tag{47}$$

and perform an extra integration over the ϕ^a. In [AJ] it is shown that this equivalent with multiplying by the vertical top form and substituting the "Riemannian" curvature for ϕ, thereby solving the two problems together.

Remarks. 1) We are well aware of the fact that this last part is not at all self contained. This is because we do not like rewriting parts of [AJ]. Note that we do not need to calculate the precise expression for the connection 1-form, because a change of ν can be absorbed by a transformation of the variables $\bar{\phi}$ and $\bar{\omega}$, of which the Jacobians cancel each other (as usual).

2) The final expression we obtain this way has also meaning in the case the action of G is not a free action. We still have a polynomial on cohomology classes in the Cartan model expressed as an integral over M. It is only more difficult to identify the cohomology of the cartan model with the de Rham cohomology of the quotient space,

since the latter is not well defined. However, this is a serious case to investigate, since it is very common in physics to have non-free group actions (e.g., in TYMT, if there are reducible connections).
3) Combining expressions (45) and (47), we get the quantum action for TYMT as derived in [W1].
4) Of course, expression (45) is only one of the many possibilities to construct representatives using the BRST operator. It would be interesting to know what sort of representatives one can obtain using BRST theory.

9. Conclusions

In this paper we have analyzed BRST theory as it is used in topological quantum field theories. We may conclude that the statement that the BRST cohomology equals equivariant cohomology of the configuration space can be proved rigorously in a finite dimensional context. Moreover, in this context, the algebraic description of BRST theory of [OSvB] fits nicely together with the two other models for equivariant cohomology, the Weil model and the Cartan model.

Furthermore, the expression of [MQ] for a representative of the equivariant Thom class can also be obtained using BRST theory. This agrees with results of [AJ] who used [MQ] to obtain a quantum action for TYMT.

Rather than using symplectic geometry we used Fourier transform on differential forms as a tool in the proofs. It is likely that this tool can also be used in an infinite dimensional context, but we do not have evidence for that.

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