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Non-linear Wave and Schrödinger Equations

I. Instability of Periodic and Quasiperiodic Solutions*

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Abstract. We investigate stability of periodic and quasiperiodic solutions of linear wave and Schrödinger equations under non-linear perturbations. We show in the case of the wave equations that such solutions are unstable for generic perturbations. For the Schrödinger equations periodic solutions are stable while the quasiperiodic ones are not. We extend these results to periodic solutions of non-linear equations.

1. Introduction

It is well known that the world is non-linear. However, most of our knowledge about it is derived from analysis of its linear approximations. Though non-linear perturbations are usually extremely weak, they can alter the linear behaviour qualitatively. Thus it is important to understand how the most elementary and fundamental properties of linear systems are affected by non-linear perturbations.

Consider problems concerning the time evolution. Once existence of solutions is established the next goal here is classification of the orbits (= solutions) w.r. to their localization in the configuration space of the system in question, namely, into bounded and unbounded. In the case of linear Schrödinger and wave equations the Ruelle theorem allows us to identify bounded orbits with periodic and quasiperiodic (in time) ones, produced by eigenfunctions of the Schrödinger or wave operator involved, and their linear combinations. Thus the problem: investigate stability of the (quasi) periodic solutions of the linear equations under non-linear perturbations. This problem was posed by J. Fröhlich and T. Spencer several years ago and is the subject of the present paper.

In this paper we show that periodic and quasiperiodic solutions of the linear wave equation are unstable under generic non-linear perturbations. For the Schrödinger equation some of the periodic solutions are stable while the others as well as certain quasiperiodic solutions are not.

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Now we explain the term generic used here. Our theorems contain spectral condition on the linear problem which guarantees instability. It is satisfied for an open set of the linear problems and non-linearities in a certain explicit metric. We expect the latter set to be dense and moreover its complement to be meager in some reasonable measure. However, we cannot prove this and leave it as an open problem. Instead we verify the condition in some simple cases.

Finally we also establish a condition for instability of periodic and quasiperiodic solutions of non-linear wave equation. Though this condition is expressed in terms of a linear problem it is harder to verify than in the cases listed above. Nevertheless we believe it can be done for simple equations such as the sine-Gordon equation.

In a sequel paper we address the question of what happens to those periodic and quasiperiodic solutions which disappear under non-linear perturbation. To answer it we develop the theory of resonances for non-linear wave equations. We show that the above mentioned solutions turn, under non-linear perturbations, into resonances. We estimate the life-time of the corresponding solutions.

The paper is organized as follows. In Sect. 2 we state the problem, formulate the main result and present its discussion for the case of wave equation. In Sect. 3 we discuss the genericity of the condition of our main theorem. In Sect. 4 we analyze an example (a square well potential well known in Quantum Mechanics) in which this condition is verified. In Sect. 5 we prove the main theorem modulo technical statements demonstrated in Sects. 6–8. Our main tools here are the Mourre estimate and the Fermi Golden Rule for non-elliptic and non-linear equations. In Sect. 9 we show instability of certain quasiperiodic solutions of the linear wave equations, in Sect. 10, periodic solutions of non-linear wave equations, and in Sect. 11, certain periodic and quasiperiodic solutions of linear Schrödinger equations.

2. Statement of the Problem and Results

In this section we consider a family of non-linear wave equations of the form

$$-\frac{\partial^2 u}{\partial t^2} = Hu + f_\varepsilon(u), \tag{2.1}$$

where u: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is an unknown function, H is a real, symmetric differential operator on $\mathbb{R}^n$ and f_ε: $\mathbb{R} \to \mathbb{R}$ is a family of 3 times continuously differentiable non-linear functions, once continuously differentiable in ε and obeying

$$f_\varepsilon(0) = 0. \tag{2.2}$$

We assume that all derivatives of f_ε are continuous in ε. In all sections, except of Sect. 10, we assume that

$$f_0(u) = 0. \tag{2.3}$$

We suggest thinking about H as a self-adjoint Schrödinger operator

$$H = -\Delta + V(x)$$

on $L^2(\mathbb{R}^n)$, but it could be also the operator,

$$H = c(x)^2 \varrho(x) \nabla \varrho(x)^{-1} \nabla + V(x) \tag{2.4}$$

on $L^2_{(c^2\varrho)^{-1/2}}(\mathbb{R}^n)$ arising in the wave propagation, etc. The conditions we impose are rather general. In what follows we consider H on $L^2(\mathbb{R}^n)$.

By a periodic solution to (2.1) we understand a function u, periodic in t and belonging to $H_2(0,T) \otimes D(H)$, where H_2 is the Sobolev space of order 2 and T is the period of u, which solves (2.1). The linear, ($\varepsilon = 0$)-problem has periodic solutions of the form

$$g_0(t,x) = \chi(x) \sin \omega_0 t, \tag{2.5}$$

where ω_0^2 and χ are an eigenvalue of H and the corresponding eigenfunction

$$H\chi = \omega_0^2 \chi.$$

We do not normalize χ. Because of the t-translational invariance of (2.1), all the statements below extend immediately to $\chi(x)\sin(\omega_0 t + \alpha)$ for any α. The period of g_0 is $\frac{2\pi}{\omega_0} = T_0$. Note that such periodic solutions are stable under reasonable linear perturbations (see e.g. [Kato]). Our main task is to investigate stability of such solutions under non-linear perturbations. Before proceeding note that the Froese-Herbst theory and Harnak inequality imply under conditions on H formulated below

$$|\chi(x)| \leq Ce^{-b|x|} \tag{2.6}$$

for some $b > 0$ (see [CFKS]).

Denote by D the class of functions u: $S^1 \times \mathbb{R}^n \to \mathbb{C}$ obeying

$$\sup |(x \cdot \nabla_x)^n u| < \infty$$

for $n = 0, 1, 2$. Equipped with the norm $|||u||| = \max_{n=0,1,2} \|(x \cdot \nabla_x)^n u\|_\infty$, D becomes a Banach space.

We introduce the functions

$$W_\varepsilon(u) = f_\varepsilon(u)/u \tag{2.7}$$

and

$$U_\varepsilon(u) = W_\varepsilon(u)/\varepsilon. \tag{2.8}$$

Due to the restrictions of f_ε, $U_\varepsilon(\psi)$ is twice continuously differentiable in ψ and, together with its derivatives, continuous in ε.

Definition 2.1 A periodic solution, g_0, of the ($\varepsilon = 0$) problem is said to be stable under a perturbation f_ε if for an infinite sequence of $\varepsilon \neq 0$ converging to 0, Eq. (2.1) has a periodic solution g_ε of a period $2\pi/\omega_\varepsilon$ s.t. as $\varepsilon \to 0$,

(i) $\omega_\varepsilon \to \omega_0$, where $2\pi/\omega_0$ is the period of g_0,

(ii) $g_\varepsilon\left(\frac{t}{\omega_\varepsilon}, x\right) \to g_0\left(\frac{t}{\omega_0}, x\right)$ in $e^{b|x|}L^\infty$ for some sufficiently small b and weakly in L^2,

(iii) g_ε is uniformly (in ε) bounded in D.

Now we formulate our technical restriction on H. Instead of isolating an explicit class of operators we constrain H by imposing some estimates known to be satisfied for various classes of operators. First we distinguish a compact subset of cont spec H of measure zero, which we call the *threshold set of* H. For a Schrödinger operator for which $V(\infty) = \lim_{x\to\infty} V(x)$ exists, this is $\{V(\infty)\}$. It is a standard practice in the theory of Schrödinger and related operators to avoid this set.

The next condition allows us to apply the powerful Mourre method to study embedded eigenvalues. Let

$$A = (H+1)^{-\frac{1}{2}} \frac{1}{2}(x \cdot p + p \cdot x)(H+1)^{-\frac{1}{2}}, \tag{2.9}$$

where $p = -\text{grad}_x$. We say that the Mourre estimate holds for H in an interval Δ if

$$E_\Delta(H) i[H, A] E_\Delta(H) \geq \theta E_\Delta(H)^2 + K, \tag{2.10}$$

where $\theta > 0$ and K is a compact self-adjoint operator. We say that the Mourre estimate holds for H if it holds for any compact interval $\Delta \subset \text{cont spec}\, H \setminus$ thresholds of H. The Mourre estimate is proven for a large class of Schrödinger and related operators [Mou, PSS, FHel, FHi, FS]. Self-adjoint Schrödinger operators with potentials $V(x)$ obeying

$$(x \cdot \nabla)^n V(x) \text{ is } \Delta\text{-compact} \tag{2.11}$$

for $n = 0, 1$ form such a class. There θ is any number satisfying

$$\theta < \text{dist}(\Delta, \text{thresholds of } H \text{ on the left from } \Delta)\,.$$

Moreover, K can be chosen there so as to obey

$$K \leq C(H+1)^{-\delta} \text{ for some } \delta > 0. \tag{2.12}$$

We assume in this paper that
(α) $\text{ad}_A^n(H)$ are bounded for $n = 1, 2$ (on $L^2(\mathbb{R}^n)$),
(β) the Mourre estimate with K obeying (2.12) holds for H. Moreover, there is $E_0 > 0$ s.t. $\theta > 0$ can be chosen to be independent of Δ, provided $\Delta \subset [E_0, \infty)$.

Consider the function $f_1(\chi(x) \sin t)$, where

$$f_1(u) = \frac{\partial}{\partial \varepsilon} f_\varepsilon(u) \,|_{\varepsilon=0} \equiv U_0(u) u\,.$$

Since it is periodic of the period 2π it is entirely determined by its Fourier coefficients

$$f_n(x) = \int_0^{2\pi} f_1(\chi(x) \sin t) e^{-\text{int}}\, dt. \tag{2.13}$$

The main result of this paper is the following

Theorem 2.2. *Let H have a positive, isolated eigenvalue ω_0^2 s.t. for any $n > 1$,*

$$n^2 \omega_0^2 \notin \text{disc spec}\, H \cup \text{thresholds}\, H. \tag{2.14}$$

Let $f_n = \int_0^{2\pi} f_1(\chi(x) \sin t) e^{-\text{int}}\, dt$. If there is $n > 1$ s.t. f_ε obeys

$$\langle \delta(H - n^2 \omega_0^2) f_n, f_n \rangle \neq 0, \tag{2.15}$$

then the periodic solution $g_0 = \chi \sin \omega_0 t$ of the linear, $\varepsilon = 0$, problem is unstable under non-linear perturbation f_ε.

Discussion 2.3. (i) The restriction that ω_0^2 is an isolated eigenvalue is not necessary but this is the most interesting case.

(ii) The necessary condition for (2.15) to hold is

$$n^2\omega_0^2 \in \text{cont spec } H \ \text{ for some } n > 1. \tag{2.16}$$

This relation states that "ω_0-photons" connect the eigenvalue ω_0^2 to the continuous spectrum so that the corresponding transition ("ionization") is possible. This condition is obviously satisfied if H has a continuous spectrum in a neighbourhood of $+\infty$.
(iii) Clearly, the restriction

$$n^2\omega_0^2 \notin \text{disc spec } H \ \text{ for every } n > 1 \tag{2.17}$$

can fail only in exceptional cases. The result extends to those cases if H is replaced by $H\bar{Q}_n$ and f_n by $\bar{Q}_n f_n$, where $\bar{Q}_n$ is the orthogonal projection onto $(\text{Null}(H - n^2\omega_0^2))^\perp$.
(iv) Due to assumption (2.17) the non-negative operator $\delta(H - n^2\omega_0^2)$ is well defined for $n > 1$ (see e.g. [CFKS, FS]). The finiteness of the l.h.s. of (2.15) will be established later, in the course of the proof of the theorem.
(v) In case when

$$f_n = 0 \quad \text{for all } n > 1 \text{ s.t. } \omega_0^2 n^2 \in \text{cont spec } H,$$

there is a refinement of (2.15) involving Fourier coefficients of more complicated functions (higher order perturbation theory).
(vi) The instability described in the theorem is due to the coupling of disc spec H with cont spec H, not with cont spec $\partial^2/\partial t^2$. The same phenomenon would persist if t were confined to a finite interval so that $\partial^2/\partial t^2$ would have purely discrete spectrum. On the other hand, if the resonance condition fails, i.e. if

$$\omega_0^2 n^2 \notin \text{cont spec } H \text{ for all } n > 1,$$

then periodic solutions to the linear problem are expected to be stable under non-linear perturbations.

Example. $f_\varepsilon(u) = \varepsilon u^3$. Using that

$$\sin^3\alpha = -\frac{1}{4}\sin\alpha - \frac{1}{4}\sin 3\alpha, \tag{2.16}$$

we obtain

$$f(\chi \sin t) = -\frac{1}{4}\chi^3 \sin t - \frac{1}{4}\chi^3 \sin 3t. \tag{2.17}$$

Thus we obtain

Corollary 2.4. *Let χ be an eigenfunction of H with an eigenvalue ω_0^2. If*

$$\langle \delta(H - 9\omega_0^2)\chi^3, \chi^3\rangle \neq 0, \tag{2.18}$$

then for ε sufficiently small the non-linear equation

$$-\frac{\partial^2 u}{\partial t^2} = Hu + \varepsilon u^3 \tag{2.19}$$

has no periodic solutions generated by the periodic solutions $\chi \sin\omega_0 t$ of the corresponding linear problem in the sense of Definition 2.1.

3. Genericity

To verify condition (2.15), one has to compute the spectral projector or the Green function of the linear operator H at the points $n^2\omega_0^2$, $n > 1$. Below we check this condition in special cases. We expect it to be satisfied for generic non-linearities f_ε and for generic Schrödinger operators H with continuous components in their spectra.

We define the topological space $\mathscr{F}$ of non-linearities f as follows: $f \in C^3$, $f(o) = 0$ and $f(u)/u$ maps bounded subsets into bounded subsets with the topology determined by the seminorms

$$\|f\|_L = \sup_{|u|\leq L} |f(u)/u|. \tag{3.1}$$

Denote by $\mathscr{D}$ the class of operators H obeying condition (α) of Sect. 2 with A defined in (2.9) and with the norm defined accordingly.

Theorem 3.1 *For given n and $H \in \mathscr{D}$, either* (2.15) *holds for an open and dense set of f_1's in $\mathscr{F}$ or it fails for all f_1's in $\mathscr{F}$. For given n and $f_1 \in \mathscr{F}$,* (2.15) *holds for an open set of H's in $\mathscr{D}$.*

Proof. By Eq. (2.6) and Theorem 7.1 of Sect. 7, the l.h.s. of (2.15) is continuous in $H \in \mathscr{D}$ and in $f_1 \in \mathscr{F}$. Hence the set of all H's in $\mathscr{D}$ for which (2.15) holds for fixed n and $f_1 \in \mathscr{F}$ and the set of all f_1's in $\mathscr{F}$ for which (2.15) holds for fixed n and $H \in \mathscr{D}$ are open. Now we claim that, given n and $H \in \mathscr{D}$, if there is $\widetilde{f}_1 \in \mathscr{F}$ for which (2.15) holds, then (2.15) holds for a dense set of f_1's. Indeed, let (2.15) fail for f_1. Introduce $f_{1,\delta} = f_1 + \delta\widetilde{f}_1 \in \mathscr{F}$. Then by the linearity in f_1, (2.15) holds for all $f_{1,\delta}$ with $\delta \neq 0$ and $f_{1,\delta} \to f_1$ in $\mathscr{F}$ as $\delta \to 0$. □

[AHS] show that for a wide class of Schrödinger operators (besides of $H \in \mathscr{D}$ some kind of decay at ∞ is assumed)

$$\delta(H - \lambda) \neq 0 \tag{3.2}$$

if $\lambda \in$ cont spec $H \setminus$ (thresholds $\cup$ eigenvalues). We conjecture that a similar relation holds for wave operators (2.4) and for other differential operators from the class $\mathscr{D}$. However, this relation does not suffice to show that the class of H's and f's for which (2.15) holds is sufficiently rich. We believe that the set of $H \in \mathscr{D}$ and of $f_1 \in \mathscr{F}$ for (2.15) fails is rather meager but there is no proof so far of this fact.

4. Explicit Example

In this section we verify condition (2.18) for $f(u) = u^3$ and for H, the Schrödinger operator $H = -\Delta + V(x)$ in the dimension $n = 1$ with $V(x)$, the square well potential:

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \pi \\ a & \text{for either } x < 0 \text{ or } x > \pi. \end{cases} \tag{4.1}$$

Of course, this potential does not satisfy conditions (i) and (ii) of Sect. 2. However, it will be clear that the analysis below holds for smooth versions of $V(x)$ as well as for multi-dimensional square wells and their smooth descendants.

Let χ_{E_0} be an eigenfunction of (4.1) with an eigenvalue $0 < E_0 < a$ and ψ_E, the scattering (generalized) eigenfunction corresponding to a point $E > a$ of the continuous spectrum. Then condition (2.18) is equivalent to

$$\int_{-\infty}^{\infty} \psi_{9E_0}\chi_{E_0}^3 \, dx \neq 0. \tag{4.2}$$

Theorem 4.1. *Let* $a < \frac{1}{36}$. *Then* (4.2) *holds.*

Proof. If $a < \frac{1}{4}$, then H has only one eigenvalue E_0 obeying

$$\frac{1}{2}a < E_0 < a < \frac{1}{4}. \tag{4.3}$$

Since this is the lowest eigenvalue, $\chi_{E_0} > 0$. The scattering (generalized) eigenfunction ψ_E can be decomposed into the real and imaginary parts (i.e. two real eigenfunctions) with the real part being

$$u_E(x) = \begin{cases} \cos(\sqrt{E}x) & 0 \le x \le \pi \\ \cos(\sqrt{E-a}x) & x \le 0 \\ \cos(\sqrt{E-a}(x-\pi)) & x \ge \pi \end{cases} \tag{4.4}$$

Due to (4.3), $u_{9E_0}(x) > 0$ for $0 \le x \le \pi$, provided $a < \frac{1}{36}$. Hence for such a's

$$\int_0^{\pi} u_{9E_0}\chi_{E_0}^3 \, dx > 0.$$

The remaining part of $\int_{-\infty}^{\infty} u_{9E_0}\chi_{E_0}^3 \, dx$ can be easily computed. Since

$$\chi_{E_0} = \begin{cases} Ae^{\sqrt{a-E_0}x} & x \le 0 \\ A\dfrac{a\cos(\sqrt{E_0}\pi)}{2E_0 - a}e^{-\sqrt{a-E_0(x-\pi)}} & x \ge \pi \end{cases}$$

with $A > 0$, a normalizing constant, we compute

$$\left(\int_{-\infty}^{0} + \int_{\pi}^{\infty}\right) u_{9E_0}\chi_{E_0}^3 = \frac{3A}{8}\frac{\sqrt{a-E_0}}{a}\left[1 + \left(\frac{a\cos(\sqrt{E_0}\pi)}{2E_0 - a}\right)\right] > 0. \quad \square$$

5. Proof of Theorem 2.2

Assume (2.1) has a periodic solution with a period $T_\varepsilon = 2\pi/\omega_\varepsilon$. Then g_ε solves the "linear" equation

$$-\frac{\partial^2 u}{\partial t^2} = Hu + W_\varepsilon(g_\varepsilon)u. \tag{5.1}$$

Put differently, g_ε is an eigenfunction of the operator

$$K^{(\varepsilon)} = \frac{\partial^2}{\partial t^2} + H + W_\varepsilon(u) \quad \text{on } L^2(\omega_\varepsilon^{-1}S^1 \times \mathbb{R}^n), \tag{5.2}$$

where $\omega_\varepsilon^{-1}S^1$ is the circle of radius ω_ε^{-1}, with the eigenvalue 0:

$$K^{(\varepsilon)}g_\varepsilon = 0. \tag{5.3}$$

In order to get rid of the ε-dependence of the space on which K_ε is defined we scale the time variable as

$$t \longrightarrow \frac{t}{\omega_\varepsilon}. \tag{5.4}$$

This generates a unitary transformation under which $K^{(\varepsilon)}$ is mapped into $K_{\varepsilon,\omega_\varepsilon}$, where

$$K_{\varepsilon,\omega} = \omega^2\frac{\partial^2}{\partial t^2} + H + W_\varepsilon(\psi_\varepsilon) \quad \text{on } L^2(S^1 \times \mathbb{R}^n), \tag{5.5}$$

with

$$\begin{aligned}\psi_\varepsilon(t,x) =& g_\varepsilon\left(\frac{t}{\omega_\varepsilon}, x\right)\\ &\in L^2(S^1 \times \mathbb{R}^n).\end{aligned} \tag{5.6}$$

The scaled periodic solution ψ_ε obeys

$$K_{\varepsilon,\omega_\varepsilon}\psi_\varepsilon = 0, \tag{5.7}$$

i.e. ψ_ε is an eigenfunction of $K_{\varepsilon,\omega_\varepsilon}$ with an eigenvalue 0.

Recall that

$$W_\varepsilon(u) = \varepsilon U_\varepsilon(u).$$

Lemma 5.1. *The function U_ε, considered as a composition map $\psi \rightarrow U_\varepsilon \circ \psi$, is a bounded map from D into D, norm continuous in ε.*

Proof. Denote by U_ε' and U_ε'' the derivatives of $U_\varepsilon(\psi)$ w.r. to ψ. The statement follows from the relations

$$(x\cdot\nabla)U_\varepsilon(\psi(x)) = U_\varepsilon'(\psi(x))(x\cdot\nabla)\psi,$$

and

$$(x\cdot\nabla)^2U_\varepsilon(\psi(x)) = U_\varepsilon'(\psi(x))(x\cdot\nabla)^2\psi(x) + U_\varepsilon''(\psi(x))[(x\cdot\nabla)\psi(x)]^2,$$

and the fact that U_ε is continuous in ε uniformly in $u \in$ any compact interval of $\mathbb{R}$. □

Assume now that g_ε is born out of the periodic solution $g_0 = \chi \sin \omega_0 t$ of the ($\varepsilon = 0$)-problem in the sense of Definition 2.1. Then by Lemma 5.1, $U_\varepsilon(\psi_\varepsilon)$ is uniformly bounded in D. Moreover, Definition 2.1 implies

$$U_\varepsilon(\psi_\varepsilon) \to U_0(\psi_0) \quad \text{in } e^{b|x|}L^\infty,$$

as $\varepsilon \to 0$, where b is the same as in Definition 2.1. Remembering that by Definition 2.1 $\psi_\varepsilon \to \psi_0$ weakly in L^2, we conclude that $K_{\varepsilon,\omega}$ obeys the conditions of Theorem 8.4 of Sect. 8. The latter, applied to the case at hand, implies that

$$\delta(\overline{K}_{0,\omega_0})\overline{P}_0 f_1(\psi_0) = 0, \tag{5.8}$$

where a is the same as in Definition 2.1,

$$\psi_0(t,x) = g_0\left(\frac{t}{\omega_0}, x\right) \equiv \chi(x)\sin t,$$

$$\overline{P}_0 = \text{orthogonal projection onto } (\text{Null } K_{0,\omega_0})^\perp$$

and

$$\overline{K}_{0,\omega} = K_{0,\omega}\overline{P}_0. \tag{5.9}$$

On the direct sum of the eigenspaces of $\partial^2/\partial t^2$

$$K_{0,\omega} = \bigoplus_{n\geq 0}(H - n^2\omega^2) \tag{5.10}$$

and

$$\overline{P}_0 = \bigoplus \overline{P}_{0,n},$$

where

$$\overline{P}_{0,n} = \text{id} \quad \text{for } n \neq 1$$

and

$$\overline{P}_{0,1} = 1 \otimes \text{proj. onto } (\text{Null}(H - \omega_0^2))^\perp.$$

Consequently, on Ran $\overline{P}_0$,

$$\delta(\overline{K}_{0,\omega_0}) = \delta(H_{\omega_0} - \omega_0^2) \bigoplus_{n>1} \delta(H - n^2\omega_0^2), \tag{5.11}$$

where $H_{\omega_0} = H\overline{P}_{\omega_0}$. Expand now (remember (2.13))

$$f_1(\chi(x)\sin t) = \sum f_n(x)e^{\text{int}}.$$

Substituting this and (5.11) into (5.8) and using the orthonormality of $\{e^{\text{int}}\}$, we derive

$$\delta(H - n^2\omega_0^2)f_n = 0 \tag{5.12}$$

for all $n \neq 1$. The latter relation contradicts condition (2.15). Consequently, there are no periodic solutions in a neighbourhood of ψ_0. □

In conclusion of this section we explain the origin of relation (5.8). Consider the unperturbed, linear problem and note that $\psi_0(t,x) = \chi(x)\sin t$ is an eigenfunction of K_{0,ω_0} with the eigenvalue 0. Using the separation of variables one determines the spectrum of

$$K_{0,\omega} = \omega^2 \frac{\partial^2}{\partial t^2} + H.$$

Namely,

$$\text{point spec } K_{0,\omega} = \text{point spec } H - \{n^2\omega^2\},$$

and

$$\text{cont spec } K_{0,\omega} = \bigcup_{n \in \mathbb{Z}} (-n^2\omega^2 + \text{cont spec } H).$$

Since by resonance condition (2.16)

$$n^2\omega_0^2 \in \text{cont spec } H \quad \text{for some } n > 1,$$

the eigenvalues 0 of K_{0,ω_0} is embedded into the continuous spectrum of K_{0,ω_0} (see Fig. 1, where it is assumed that cont spec $H = [\Sigma, \infty)$).

Spec K_{0,ω_0}

Fig. 1

We consider now $K_{\varepsilon,\omega_\varepsilon}$ as a perturbation of K_{0,ω_0}. Extrapolating result on the Schrödinger equation (see e.g. [Sim, How, HoS, Yaj, Sig, AHS]) one expects that the eigenvalue 0 of K_{0,ω_0} disappears under the perturbation and becomes a resonance of $K_{\varepsilon,\omega_\varepsilon}$. The expansion in ε for the imaginary part of this resonance starts with ε^2 times a coefficient given by the Fermi Golden Rule and which turns out to be exactly the l.h.s. of (5.8) multiplied scalarly by $f_1(\psi_0)$. Thus (5.8) is a consequence of the assumption that the eigenvalue 0 survives the perturbation and remains to be an eigenvalue of $K_{\varepsilon,\omega_\varepsilon}$. Consequently, the imaginary part of this eigenvalue must vanish.

In the case when $f_1(\chi \sin \omega t)$ has only finite number of the Fourier coefficients, say n_0, as it happens when $f_1(u)$ is a polynomial, the leading in ε term in the imaginary part of the resonance is the ε^{2m}-term, where m is the smallest integer obeying

$$(mn_0\omega_0)^2 \in \text{cont spec } H.$$

In this paper we consider the case when $m = 1$. For $m > 1$, the leading coefficient is again given by the Fermi Golden Rule in which $f_1(\psi_0)$ on the l.h.s. of (5.8) is replaced by a more complicated function.

6. The Mourre Estimate for $K_{0,\omega}$

In this section we consider the operator

$$K_{o,\omega} = \omega^2 \frac{\partial^2}{\partial t^2} + H \tag{6.1}$$

acting on $L^2(S^1 \times \mathbb{R}^2)$. We derive the Mourre estimate for $K_{0,\omega}$ from the Mourre estimate for H. This result is used in the next section in order to obtain estimates on the resolvents of $K_{0,\omega}$ and of its perturbations, needed later.

We say that the strong Mourre estimate (SME) holds for self-adjoint operators T and A acting on the same Hilbert space and for an interval Δ if

$$E_\Delta(T) i[T, A] E_\Delta(T) \geq \theta E_\Delta(T)^2 \tag{6.2}$$

for some $\theta > 0$. Whenever it is clear which operator A is used as is the case below, we will not mention it explicitly. In this section A is given by (2.9) but on $L^2(S^1 \times \mathbb{R}^n)$ (we will not distinguish between A and $\mathbf{1} \otimes A$).

By Operator Calculus and properties of compact operators, the usual Mourre estimate (ME) on Δ and the absence of eigenvalues in Δ imply the strong Mourre estimate on a smaller interval, say Δ_1. Conversely, the strong Mourre estimate on Δ implies the absence of eigenvalues in Δ and Hölder continuity of the resolvent, acting between appropriate weighted spaces, in Δ (see e.g. [CFKS,FS]).

The main result of this section is

Proposition 6.1. *Assume H obeys the conditions of Sect. 2. Then there are an interval Δ containing 0 and $\delta > 0$ so that the strong Mourre estimate holds for $\overline{K}_{0,\omega}$, A and Δ, provided $|\omega - \omega_0| \leq \delta$.*

Proof. By non-degeneracy condition (2.14) H has no eigenvalues in some intervals around $n^2\omega_0^2$ for $n > 1$. Hence by the remark made in the paragraph before the proposition there are open intervals Δ_n containing $n^2\omega_0^2$, $n > 1$, in which the strong Mourre estimate holds. Moreover, due to condition (2.12) there is $E_1 > 0$ s.t. the strong Mourre estimate holds for H and any interval in $[E_1, \infty)$.

Let n_0 be s.t. $n_0^2\omega_0^2 \geq E_1 + 1$. Pick $\delta > 0$ so that

$$\Delta \equiv \bigcap_{\substack{n \leq n_0 \\ |\omega - \omega_0| \leq \delta}} (\Delta_n - n^2\omega^2) \neq \emptyset \tag{6.3}$$

and, for $|\omega - \omega_0| \leq \delta$ and $n \leq n_0$,

$$n^2\omega^2 \in \text{cont spec } H \leftrightarrow n^2\omega_0^2 \in \text{cont spec } H. \tag{6.4}$$

Then the strong Mourre estimate holds for H in the intervals $\Delta + n^2\omega^2$, where $n > 1$ and $|\omega - \omega_0| \leq \delta$, with a positive constant θ independent of n.

Now we expand over the eigenspaces of $\partial^2/\partial t^2$:

$$\overline{K}_{0,\omega} = (H_{\omega_0} - \omega^2) \bigoplus_{n \neq 1} (H - n^2\omega^2), \tag{6.5}$$

where, recall, $H_{\omega_0} = H\overline{P}_{\omega_0}$ with $\overline{P}_{\omega_0}$, the orthogonal projection onto (Null($H - \omega_0^2$))$^\perp$. Using this relation and the non-degeneracy condition, we get for Δ around 0 and sufficiently small

$$E_\Delta(\overline{K}_{0,\omega}) = \bigoplus_{n>1} E_\Delta(H - n^2\omega^2) = \bigoplus_{\Delta+n^2\omega^2 \subset \text{cont spec } H} E_{\Delta+n^2\omega^2}(H).$$

Using the last two relations, we derive

$$E_\Delta(\overline{K}_{0,\omega}) i[\overline{K}_{0,\omega}, A] E_\Delta(\overline{K}_{0,\omega}) = \bigoplus_{\Delta+n^2\omega^2 \subset \text{cont spec } H} E_{\Delta+n^2\omega^2}(H) i[H, A] E_{\Delta+n^2\omega^2}(H).$$

This, combined with the conclusion of the previous paragraph, implies the strong Mourre estimate for $\overline{K}_{0,\omega}$ and the interval Δ described in that paragraph. □

Since $E_\Delta(\overline{K}_{0,\omega})K_{0,\omega}[P_0, A]$ is compact, Proposition 6.1 yields

Corollary 6.2. *Under the assumptions of Proposition 6.1 the Mourre estimate holds for $K_{0,\omega}$ and Δ.*

7. Operator $K_{\varepsilon,\omega}$

In this section we study the resolvent of $\overline{K}_{\varepsilon,\omega}$. Our main tool is the Mourre estimate derived in the previous section. In fact, we consider more general self-adjoint operators of the form

$$K_\kappa = K_0 + I_\kappa \tag{7.1}$$

acting on $L^2(S^1 \times \mathbb{R}^n)$, where

$$K_0 = \omega_0^2 \frac{\partial^2}{\partial t^2} + H, \tag{7.2}$$

$$I_\kappa = \alpha \frac{\partial^2}{\partial t^2} + W \tag{7.3}$$

and

$$\kappa = (W, \alpha). \tag{7.4}$$

Here H is the same as before, but W here is the potential

$$W : S^1 \times \mathbb{R}^n \to \mathbb{R}, \tag{7.5}$$

$$\text{ad}_A^n(W) \text{ are bounded for } n = 0, 1, 2 \tag{7.6}$$

[on $L^2(S^1 \times \mathbb{R}^n)$]. Let $|||W|||$ be a norm generated by (7.6) and

$$|\kappa| = |||W||| + |\alpha|. \tag{7.7}$$

Let $\overline{P}_0$ be as before, the projection onto $(\text{Null} K_0)^\perp$, and

$$\overline{K}_\kappa = \overline{P}_0 K_\kappa \overline{P}_0. \tag{7.8}$$

The main result of this section is

Theorem 7.1. *Let K_κ be defined in* (7.1) *with W obeying* (7.6). *Then there are an interval Δ containing 0 and $\nu > 0$ depending on Δ) s.t. for $|\kappa| < \nu$ the following holds*

i) $\overline{K}_\kappa$ *has no eigenvalues in* Δ,
ii) $(\overline{K}_\kappa - z)^{-1} : \overline{P}_0 L_1^2 \to \overline{P}_0 L_{-1}^2$ *is bounded and continuous in* κ *and* z, $\operatorname{Re} z \in \Delta$.

The proof of this proposition goes along standard lines (see e.g. [AHS, CFKS, PSS, FS]) and we present here just a sketch of it.

Sketch of proof of Theorem 7.1. Let Δ_1 be the interval given in Proposition 6.1 (and denoted there by Δ): $\Delta_1 \ni 0$ and the strong Mourre estimate for $\overline{K}_{0,\omega}$ holds on Δ_1, provided $|\omega - \omega_0| \leq \delta$ for some $\delta > 0$.

Lemma 7.2. *Given* $\Delta \subset \Delta_1$ *s.t.* $\operatorname{dist}(\partial\Delta, \partial\Delta_1) > 0$ *there is* $\nu > 0$ *s.t. the strong Mourre estimate holds for* $\overline{K}_\kappa$ *and* Δ, *provided* $|\kappa| \leq \nu$.

Proof. The result follows from the relation

$$[\overline{K}_\kappa, A] = [\overline{K}_{\kappa_0}, A] + [W, A], \tag{7.9}$$

where $\kappa_0 = (0, \alpha)$, and the estimates

$$\|[\overline{K}_{\kappa_0}, A]\| \leq C, \tag{7.10}$$

$$\|[W, A]\| \leq C|||W|||, \tag{7.11}$$

and

$$\|E_\Delta(\overline{K}_\kappa) - E_\Delta(\overline{K}_{\kappa_0})\| \leq C\|W\|. \tag{7.12}$$

The second estimate is straightforward, the first requires some simple but tedious commutator estimates. To prove the third estimate one uses the Fourier representation

$$f(T) = \int_{-\infty}^{\infty} \hat{f}(s) e^{iTs}\, ds, \tag{7.13}$$

where $\hat{f}$ is the Fourier transforms of f, and the Duhamel formula

$$e^{iT_1 s} - e^{iT_2 s} = \int_0^s e^{iT_1(s-u)}(T_1 - T_2) e^{iT_2 u}\, du. \tag{7.14}$$

Note that (7.13) holds for any self-adjoint operator T and any function f with $\int_{-\infty}^{\infty} |\hat{f}(s)|\, ds < \infty$. □

Lemma 7.2 implies that Δ contains no eigenvalues of $\overline{K}_\kappa$ for $|\kappa| \leq \nu$, and hence (i) holds, and that $(\overline{K}_\kappa - z)^{-1} : \overline{P}_0 L_1^2 \to \overline{P}_0 L_{-1}^2$ is bounded and Hölder continuous in z, $\operatorname{Re} z \in \Delta$, of the order $\frac{1}{3}$ (see the paragraph before Proposition 6.1). It remains to show that the latter map is Hölder continuous in κ. We sketch this proof. Let

$$B_\kappa = E_\Delta(\overline{K}_\kappa) i[K_\kappa, A] E_\Delta(\overline{K}_\kappa). \tag{7.15}$$

Then (7.9)–(7.12) yield

$$\|B_{\kappa'} - B_\kappa\| \leq C|||W' - W|||, \tag{7.16}$$

where $\kappa' = (\alpha, W')$ and $\kappa = (\alpha, W)$. Introduce the family

$$R_\kappa^{(s)} = (\overline{K}_\kappa - isB_\kappa - z)^{-1}. \tag{7.17}$$

This is the resolvent of the affine approximation to the family $e^{sA}K_\kappa e^{-sA}$, introduced in [Mo] (see also [PSS]). Using the second resolvent equation

$$R_{\kappa'}^{(s)} - R_\kappa^{(s)} = R_{\kappa'}^{(s)}(W' - W - isB_{\kappa'} + isB_\kappa)R_\kappa^{(s)} \tag{7.18}$$

and (7.16), one estimates on $u \in L_1^2$,

$$|\langle u, (R_{\kappa'}^{(s)} - R_\kappa^{(s)})u\rangle \le C|||W' - W|||\ \|R_{\kappa'}^{(s)*}u\|\ \|R_\kappa^{(s)}u\|. \tag{7.19}$$

Using now the Mourre estimate, we obtain

$$\begin{aligned}\|R_\kappa^{(s)}u\|^2 &\le \frac{C}{s}\|\sqrt{sB_\kappa}R_\kappa^{(s)}u\|^2 \\ &\le \frac{C}{s}|\langle u, (R_\kappa^{(s)*} - R_\kappa^{(s)})u\rangle|.\end{aligned} \tag{7.20}$$

Using that for $u \in L_1^2$ and $\operatorname{Re} z \in \Delta$ the inner product on the r.h.s. is bounded, we obtain

$$\|R_\kappa^{(s)}u\| \le \frac{C}{\sqrt{s}}. \tag{7.21}$$

This together with (7.19) yields

$$|\langle u, (R_{\kappa'}^{(s)} - R_\kappa^{(s)})u\rangle| \le \frac{C}{s}|||W' - W|||. \tag{7.22}$$

Next, as in [PSS] we have

$$|\langle u, (R_\kappa^{(s)} - R_\kappa^{(0)})u\rangle| \le C\sqrt{s}. \tag{7.23}$$

We sketch the proof of this estimate. Let $E_\Delta = E_\Delta(K_\kappa)$ and $\overline{E}_\Delta = \mathbf{1} - E_\Delta(K_\kappa)$, the complementary spectral projectors so that $E_\Delta \overline{E}_\Delta = 0$. We compute

$$\frac{d}{ds}R_\kappa^{(s)} = R_\kappa^{(s)}B_\kappa R_\kappa^{(s)} = B + C, \tag{7.24}$$

where

$$B = R_\kappa^{(s)} i[\overline{K}_\kappa, A]R_\kappa^{(s)} \tag{7.25}$$

and

$$C = -R_\kappa^{(s)} i[\overline{K}_\kappa, A]\overline{E}_\Delta R_\kappa^{(s)} - R_\kappa^{(s)}\overline{E}_\Delta i[\overline{K}_\kappa, A]E_\Delta R_\kappa^{(s)}. \tag{7.26}$$

Remembering definition (7.17) of $R_\kappa^{(s)}$, we transform

$$B = i[A, R_\kappa^{(s)}] + sR_\kappa^{(s)} i[B_\kappa, A]R_\kappa^{(s)}. \tag{7.27}$$

Hence, due to (5.21) and for $u \in L_1^2$,

$$|\langle Bu, u\rangle| \le \text{const}(s^{-\frac{1}{2}} + \|[B_\kappa, A]\|). \tag{7.28}$$

Next, using $E_\Delta \overline{E}_\Delta = 0$, we estimate

$$\begin{aligned}\|(\overline{K}_\kappa - isB_\kappa - z)u\| &\ge \|\overline{E}_\Delta(\overline{K}_\kappa - isB_\kappa - z)u\| \\ &= \|\overline{E}_\Delta(\overline{K}_\kappa - z)u\| \ge \delta\|\overline{E}_\Delta u\|,\end{aligned} \tag{7.29}$$

which implies

$$\|\overline{E}_\Delta R_\kappa^{(s)}\| \le C. \tag{7.30}$$

Equations (7.21) and (7.30) yield now that

$$|\langle Cu, u\rangle| \leq \text{const}\|[K_\kappa, A]\|/\sqrt{s}. \tag{7.31}$$

Finally, using the definition of A,

$$A = (H+1)^{-\frac{1}{2}}\frac{1}{2}(x\cdot p + p\cdot x)(H+1)^{-\frac{1}{2}}, \tag{7.32}$$

the relation

$$[K_\kappa, A] = [H + W, A] \tag{7.33}$$

and the restrictions on H and W, we obtain that

$$\|[B_\kappa, A]\| \leq \text{const}|||W||| \tag{7.34}$$

and

$$\|[K_\kappa, A]\| \leq \text{const}|||W|||. \tag{7.35}$$

Relations (7.24), (7.28), (7.31), (7.34) and (7.35) yield that

$$\left|\frac{d}{ds}\langle R_\kappa^{(s)}u, u\rangle\right| \leq \frac{C}{\sqrt{s}}, \tag{7.36}$$

which in turn implies (7.23).

Now let $R_\kappa = R_\kappa^{(0)}$. Using the identity

$$R_{\kappa'} - R_\kappa = R_{\kappa'} - R_{\kappa'}^{(s)} + R_\kappa^{(s)} - R_\kappa + R_{\kappa'}^{(s)} - R_\kappa^{(s)}, \tag{7.37}$$

and estimates (7.22) and (7.23), we obtain

$$|\langle u, (R_{\kappa'} - R_\kappa)u\rangle| \leq Cs^{\frac{1}{2}} + \frac{C}{s}|||W' - W|||. \tag{7.38}$$

Picking now

$$s = |||W' - W|||^{\frac{2}{3}}, \tag{7.39}$$

we conclude that

$$|\langle u, (R_{\kappa'} - R_\kappa)u\rangle| \leq C|||W' - W|||^{\frac{1}{3}}. \tag{7.40}$$

Now we prove the continuity in the first entry of $\kappa : \alpha$. To this end it suffices to prove

$$\|J(R_{(\alpha',0)} - R_{(\alpha,0)})J\| \to 0 \tag{7.41}$$

as $\alpha' \to \alpha$, where $J = \langle x\rangle^{-1}$. To demonstrate this relation we expand $R_{(\alpha,0)}$ along the eigenspaces of $-\partial^2/\partial t^2$:

$$R_{(\alpha,0)} = \bigoplus_{n\geq 0} R_\alpha^{(n)} \tag{7.42}$$

[cf. (6.5)], where

$$R_\alpha^{(1)} = (H_{\omega_0} - \omega^2 - z)^{-1} \tag{7.43}$$

and

$$R_\alpha^{(n)} = (H - n^2\omega^2 - z)^{-1}, \quad n \neq 1, \tag{7.44}$$

with $\omega = \sqrt{\omega_0^2 + \alpha}$. Let accordingly $f = \oplus f_n$. Then

$$\|J(R_{(\alpha',0)} - R_{(\alpha,0)})Jf\| = \left(\sum_{n\geq 0} \|J(R_{\alpha'}^{(n)} - R_{\alpha}^{(n)})Jf_n\|^2\right)^{\frac{1}{2}} \\ \leq \sup_{n\geq 0} \|J(R_{\alpha'}^{(n)} - R_{\alpha}^{(n)})J\| \, \|f\|. \tag{7.45}$$

By a standard Mourre theory (see e.g. [GFKS, FS]) and high energy estimates

$$\sup_{n\geq 0} \|J(R_{\alpha'}^{(n)} - R_{\alpha}^{(n)})J\| \to 0 \tag{7.46}$$

as $\alpha' \to \alpha$. This together with (7.45) yields (7.41). This completes the proof of Theorem 7.1. □

Remark 7.3. For $u \in L_1^2$ with $\frac{\partial u}{\partial t} \in L_1^2$ we have

$$|\langle (R_{(\alpha,0)} - R_{(\alpha',0)})u, u\rangle| \leq C|\alpha' - \alpha|^{\frac{1}{3}}. \tag{7.47}$$

Indeed, the relation

$$R_{(\alpha',0)}^{(s)} - R_{(\alpha,0)}^{(s)} = (\alpha' - \alpha)\frac{\partial}{\partial t} R_{(\alpha',0)}^{(s)} R_{(\alpha,0)}^{(s)} \frac{\partial}{\partial t} \tag{7.48}$$

together with (7.21) yields

$$|\langle (R_{(\alpha',0)}^{(s)} - R_{(\alpha,0)}^{(s)})u, u\rangle| \leq C|\alpha' - \alpha|/s. \tag{7.49}$$

This together with (7.23) and the choice $s = |\alpha' - \alpha|^{\frac{2}{3}}$ yields (7.47).

8. Fermi Golden Rule

In this section we derive the key condition related to the Fermi Golden Rule (cf. [Sim, How, HoS, Yaj, Sig, AHS]), necessary for the eigenvalue 0 of K_0 to persist under the perturbation K_κ. First, using the Feshbach projection method we derive a convenient expression for eigenvalues of K_κ and then study asymptotic behaviour of this expression as $|\kappa| \to 0$. Both steps use the resolvent estimates of the previous section. Henceforth we adapt definitions (7.1)–(7.8) and the restrictions of Sect. 7. We let, besides,

$$P_0 = \mathbf{1} - \overline{P}_0,$$

the orthogonal projection onto NullK_0. The first result of this section is

Theorem 8.1. *Let K_κ, defined in* (7.1), *obey* (7.6). *Let λ_κ be an eigenvalue of K_κ branching out of the eigenvalue 0 of K_0, i.e. $\lambda_\kappa \to 0$ as $\kappa \to 0$. Let ψ_κ be the corresponding eigenfunction. Then $\int e^{\varepsilon|x|}|\psi_\kappa|^2 < \infty$ for some $\varepsilon > 0$ and*

$$\lambda_\kappa = \alpha - \|P_0\psi_\kappa\|^{-2}\langle (\overline{K}_\kappa - \lambda_\kappa - i0)^{-1}\varphi_\kappa, \varphi_\kappa\rangle, \tag{8.1}$$

where $(W, \alpha) = \kappa$, $\varphi_\kappa = \overline{P}_0 W P_0 \psi_\kappa$ and the inner product on the r.h.s. *is well defined due to Theorem 7.1 and restriction* (7.6) *on W.*

Proof. Since $E_{\Delta}(\overline{K}_{\kappa})K_{\kappa}[P_0, A]$ is compact, Lemma 7.2 implies:

Lemma 8.2. *Under assumption of Theorem 7.1, there are an interval Δ containing 0 (and independent of κ) and $\nu > 0$ s.t. the Mourre estimate holds for K_{κ} in Δ, provided $|\kappa| \leq \nu$.*

This lemma and the Froese-Herbst theory (see [FH, CFKS]), adapted to the operator K_{κ} imply exponential decay of the eigenfunctions of K_{κ} with eigenvalues in Δ.

Now we prove (8.1). We project the eigenequation

$$K_{\kappa}\psi_{\kappa} = \lambda_{\kappa}\psi_{\kappa} \tag{8.2}$$

onto the subspaces $\mathrm{Ran}P_0$ and $\mathrm{Ran}\overline{P}_0$:

$$(\lambda_{\kappa} - \lambda_0)P_0\psi_{\kappa} = P_0 I_{\kappa}\psi_{\kappa} \tag{8.3}$$

and

$$(\overline{K}_{\kappa} - \lambda_{\kappa})\overline{P}_0\psi_{\kappa} = \overline{P}_0 I_{\kappa} P_0\psi_{\kappa}. \tag{8.4}$$

In order to emphasize the structure of the equations we have written here λ_0 for 0. Next, we observe

$$\frac{\partial^2}{\partial t^2}P_0 = -P_0 \tag{8.5}$$

and consequently,

$$I_{\kappa}P_0 = (W - \alpha)P_0. \tag{8.6}$$

The latter relation implies in turn

$$\overline{P}_0 I_{\kappa} P_0 = \overline{P}_0 W P_0 \tag{8.7}$$

and

$$P_0 I_{\kappa} P_0 = P_0 W P_0 - \alpha P_0. \tag{8.8}$$

Equation (8.7) and Theorem 7.1 imply that (8.4) can be solved:

$$\overline{P}_0\psi_{\kappa} = -(\overline{K}_{\kappa} - \lambda_{\kappa} - i0)^{-1}\overline{P}_0 W P_0 \psi_{\kappa}. \tag{8.9}$$

Note that we could have taken here also $+i0$ instead of $-i0$. Both expressions are equal. Now writing

$$P_0 I_{\kappa}\psi_{\kappa} = P_0 I_{\kappa} P_0 \psi_{\kappa} + P_0 I_{\kappa}\overline{P}_0\psi_{\kappa}, \tag{8.10}$$

and substituting here (8.8) and (8.9) and then plugging the result into the r.h.s. of (8.3), we obtain

$$(\lambda_{\kappa} - \lambda_0 + \alpha)P_0\psi_{\kappa} = -P_0 W \overline{P}_0(\overline{K}_{\kappa} - \lambda_{\kappa} - i0)^{-1}\overline{P}_0 W P_0\psi_{\kappa}. \tag{8.11}$$

Multiplying this equation scalarly by ψ_{κ}, we derive (8.1). □

Taking the imaginary part of (8.1), we obtain

Corollary 8.3. *Under the conditions of Theorem 8.1,*

$$\delta(\overline{K}_{\kappa} - \lambda_{\kappa})\overline{P}_0 W P_0 \psi_{\kappa} = 0. \tag{8.12}$$

The main result of this section is

Theorem 8.4. *Let K_{κ}, defined in (7.1), obey (7.6). Let λ_{κ} be an eigenvalue of K_{κ} and ψ_{κ}, the corresponding eigenfunctions s.t.*

$$\lambda_{\kappa} \to 0, \tag{8.13}$$

and

$$P_0\psi_\kappa \to \psi_0 \in \mathrm{Ran}P_0 \tag{8.14}$$

as $\kappa \to 0$ *in such a way that for* b *sufficiently small,*

$$\frac{W}{|||W|||} \to U_0 \tag{8.15}$$

in $e^{b|x|}L^\infty$, *for some* $U_0 \in e^{b|x|}L^\infty$. *Then* ψ_0 *obeys*

$$\delta(\overline{K}_0)\overline{P}_0 U_0 \psi_0 = 0. \tag{8.16}$$

Proof. By Proposition 7.1 and since $\lambda_\kappa \to 0$ as $|\kappa| \to 0$, we have

$$R_\kappa = R_0 + o(|\kappa|^0), \tag{8.17}$$

as maps from $\overline{P}_0 L^2_1$ to $\overline{P}_0 L^2_{-1}$, where

$$R_\kappa = (\overline{K}_\kappa - \lambda_\kappa + i0)^{-1}. \tag{8.18}$$

Denote

$$U = \frac{W}{|||W|||}. \tag{8.19}$$

A special case ($\kappa = 0$) of the first statement of Theorem 8.1 asserts that

$$\mathrm{Ran}P_0 \subset D(e^{b|x|}) \tag{8.20}$$

for some $\varepsilon > 0$. Hence for any $\delta > 0$,

$$\|\langle x\rangle^\delta (U - U_0)P_0\| \le C\|(U - U_0)e^{b|x|}\|_\infty. \tag{8.21}$$

Equations (8.14), (8.15), (8.17) and (8.21) yield

$$\langle \delta(\overline{K}_\kappa - \lambda_\kappa)\overline{P}_0 U P_0 \psi_\kappa, P_0 U P_0 \psi_\kappa\rangle = \langle \delta(\overline{K}_0)\overline{P}_0 U_0 \psi_0, \overline{P}_0 U_0 \psi_0\rangle + o(|\kappa|^0). \tag{8.22}$$

Combining this equation with (8.12) and (8.19) we arrive at (8.16). □

9. Quasiperiodic Solutions

In this section we consider stability of quasiperiodic solutions to the linear wave equation ((2.1) with $\varepsilon = 0$). Let $g_0(t,x)$ be a quasiperiodic in t and L^2 in x solution to (2.1) with $\varepsilon = 0$. Then there are an integer $m \ge 2$, positive numbers $\omega_1, \dots, \omega_m$ and a function $G_0(t_1, \dots, t_m, x)$ periodic in $t_1, \dots, t_m$ with the period 2π and L^2 in x s.t.

$$g_0(t,x) = G_0(\omega_1 t, \dots, \omega_m t, x). \tag{9.1}$$

Moreover, G_0 solves the equation

$$-\left(\sum_{i=1}^m \omega_i \frac{\partial}{\partial t_i}\right)^2 G_0 = HG_0. \tag{9.2}$$

G_0 will be called the generating function of g_0. The last statement follows from

Lemma 9.1. *Let g_0 be a quasiperiodic solution to* (2.1), $\varepsilon = 0$. *Then it is of the form*

$$g_0 = \sum_{i=1}^{m} a_i \sin(\omega_i t + \alpha_i)\chi_i, \tag{9.2}$$

where ω_i^2 and χ_i are eigenvalues and corresponding them eigenfunctions of H:

$$H\chi_i = \omega_i^2 \chi_i. \tag{9.3}$$

Consequently, the generating function of g_0 is

$$G_0 = \sum_{i=1}^{m} a_i \sin(\omega_i t_i + \alpha_i)\chi_i$$

and it obviously obeys (9.2).

Proof. By a spectral theorem and since g_0 is a solution to (2.1) with $\varepsilon = 0$, g_0 can be written as $g_0 = g_0^{\text{point}} + g_0^{\text{cont}}$, where g_0^{point} is of form (9.3) (possibly, with $m = \infty$) while g_0^{cont} is a solution with initial conditions from the continuous spectrum subspace of H. Conditions (α) and (β) of Sect. 2 imply that H has no singular continuous spectrum (see e.g. [PSS]). The measure theory yields that $\int g_0^{\text{cont}} d\, dx \to 0$ as $|t| \to \infty$ for any $f \in L^2(\mathbb{R}^n)$. Since g_0^{point} is uniformly almost periodic, then so is g_o^{cont} as a difference of two such functions. By the property of uniformly almost periodic functions there is a sequence T_n s.t. $T_n \to \infty$ as $n \to \infty$ and $\sup_t \left| \int g_{0,T_n}^{\text{cont}} f\, dx - \int g_0^{\text{cont}} f\, dx \right| \to 0$ as $n \to \infty$, where $g_{0,T}^{\text{cont}}$ is a time shift of g_0^{cont} by T. This and the conclusion above yield that $\int g_0^{\text{cont}} f\, dx = 0$ and therefore $g_o^{\text{cont}} = 0$. So (9.3) is true, but with $m \le \infty$. However, sinced g_0 is quasiperiodic, $m < \infty$. □

Note that if H has infinite discrete spectrum, then (2.1), with $\varepsilon = 0$, has also uniformly periodic solutions. They are obtained by setting $m = \infty$ in (9.3). The stability of such solutions is an open problem.

Consider now the non-linear equation

$$-(\Omega_\varepsilon \cdot \nabla_T)^2 G = HG + f_\varepsilon(G) \tag{9.4}$$

with $\Omega = (\omega_1, \ldots, \omega_m)$, $T = (t_1, \ldots, t_m)$ and $G \in H^2(T_m) \otimes D(H)$, where T_m is the m-torus.

Definition 9.2 A solution G_0 of (9.4) with $\varepsilon = 0$ is said to be stable under a perturbation f_ε if for an infinite sequence of $\varepsilon \neq 0$ converging to 0, there are $\omega_{i,\varepsilon} > 0$ and G_ε solving (9.4) and obeying

(i) $\omega_{i,\varepsilon} \to \omega_{i,0}$,
(ii) for $G_\varepsilon \to G_0$ sufficiently small $b > 0$ in $e^{b|x|}L^\infty$ and weakly in L^2,
(iii) G_ε is uniformly bounded in D.

Definition 9.3. A (quasi-) periodic solution g_0 of the $\varepsilon = 0$ problem described above is said to be stable under the perturbation f_ε if the corresponding generating function G_0 is stable as a solution to (9.4) with $\varepsilon = 0$ under the perturbation f_ε.

Given $G_0 \in L^\infty(T_m \times \mathbb{R}^n)$, we consider the function $f(G_0)$. Since it is periodic in $t_1, \ldots, t_m$ of the period 2π it is entirely defined by its Fourier coefficients

$$f_N(x) = \int_{T_m} f_1(G_0) e^{-\imath N \cdot T} d^m T, \tag{9.5}$$

where $N=(n_1,\dots,n_m)$, $N\cdot T=\sum_{i=1}^{m} n_i t_i$. Let $|N|=\sum|n_i|$ and $(N\cdot\Omega)=\sum_{i=1}^{m} n_i^2\omega_i^2$.

Theorem 9.4. *Let H have positive isolated eigenvalues $\omega_1^2,\dots,\omega_m^2$ s.t. for any N, $|N|>1$,*

$$(N\cdot\Omega)\notin \text{disc spec } H\cup \text{thresholds } H.$$

Let g_0 be a (quasi-) periodic solution to (2.1) for $\varepsilon=0$ with a generating function G_0. If f obeys

$$\langle\delta(H-(\Omega\cdot N))f_N, f_N\rangle\neq 0, \tag{9.6}$$

where $f_N=\int f_1(G_0)e^{iN\cdot T}d^mT$, at least for one N with $|N|>1$, then g_0 is unstable under the perturbation f_ε.

Proof. The proof follows along the lines of the proof of Theorem 2.2. Definition 9.3 allows us to reduce investigation of (2.1) to that of (9.4). The latter is reduced as in the proof of Theorem 2.2 to the study of the operator

$$K_{\varepsilon,\Omega}=\left(\Omega\cdot\frac{\partial}{\partial t}\right)^2+H+W_\varepsilon(G_\varepsilon)$$

on $L^2(T_m\times\mathbb{R}^n)$ analogous to the operator $K_{\varepsilon,\omega}$ defined in (5.5). The rest of the proof repeats the corresponding part of the proof of Theorem 2.2. □

10. Perturbations of Non-linear Equations

In this section we study stability of periodic solutions of non-linear wave equations under non-linear perturbations. Thus we consider the equation

$$-\frac{\partial^2 u}{\partial t^2}=Hu+f_\varepsilon(u) \tag{10.1}$$

in which $f_0\not\equiv 0$ is a non-linear function. Here H is assumed to satisfy all the conditions of Sect. 2 and $f_\varepsilon(u)$ is supposed as before to obey (2.2) and, as a function of u and ε, to be four times continuously differentiable in u and two times in ε. In particular, f_ε might be independent of ε, in which case the problem under consideration is one of stability (bifurcation) w.r. to perturbations of the initial conditions. We define as before

$$f_1(u)=\frac{\partial}{\partial\varepsilon}f_\varepsilon(u)\mid_{\varepsilon=0}.$$

We strengthen the key Definition 2.1 of stability of periodic solutions by adding the conditions that ω_ε is differentiable at $\varepsilon=0$, that

$$g_\varepsilon\left(\frac{t}{\omega_\varepsilon},x\right)\to g_0\left(\frac{t}{\omega_0},x\right) \tag{10.2}$$

in D and that

$$\partial_t^2 g_\varepsilon\left(\frac{t}{\omega_\varepsilon},x\right)\to \partial_t^2 g_0\left(\frac{t}{\omega_0},x\right) \tag{10.3}$$

in $\langle x\rangle L^2$ as $\varepsilon\to 0$.

Let g_0 be a periodic solution to (10.1) for $\varepsilon = 0$ with a period $2\pi/\omega_0$. Let

$$\psi_0(t,x) \equiv g_0\left(\frac{t}{\omega_0}, x\right). \tag{10.4}$$

Then $\psi_0 \in L^2(S^1 \times \mathbb{R}^n)$. Define

$$K_{0,\omega} = \omega^2 \frac{\partial^2}{\partial t^2} + H + \frac{\partial f_0}{\partial \psi}(\psi_0) \tag{10.5}$$

and denote by $\overline{P}_0$ the orthogonal projection onto $(\mathrm{Null} K_{0,\omega_0})^\perp$. Let $\overline{K}_{0,\omega_0} = K_{0,\omega_0}\overline{P}_0$. The main result of this section is

Theorem 10.1. *Let H and f_ε be as described in the beginning of this section. Let g_0, ω_0 and ψ_0 be as above and let for any $n > 1$,*

$$(n\omega_0)^2 \notin \text{disc spec } H \cup \text{thresholds } H. \tag{10.6}$$

If for any real α f obeys

$$\delta(\overline{K}_{0,\omega_0})\overline{P}_0\left(f_1(\psi_0) + \alpha \frac{\partial^2}{\partial t^2}\psi_0\right) \neq 0, \tag{10.7}$$

then the periodic solution g_0 of the ($\varepsilon = 0$)-problem is unstable under the perturbation f_ε.

Proof. We follow the proof of Theorem 2.2 except that we omit the last step in the latter, the connection between the conditions in terms of the function $f_1(\psi_0)$ and in terms of its Fourier coefficients. Thus we assume on the contrary, that g_0 is stable in the sense of the definition described above. Then there is a family of periodic solutions to (10.1) of periods $2\pi/\omega_\varepsilon$, described in that definition. We rescale this equation in time as $t \to t/\omega_\varepsilon$ and rewrite it as

$$\left(\omega_\varepsilon^2 \frac{\partial^2}{\partial t^2} + H\right)\psi_\varepsilon + f_\varepsilon(\psi_\varepsilon) = 0, \tag{10.8}$$

where $\psi_\varepsilon(t,x) = g_\varepsilon(t/\omega_\varepsilon, x)$. First, we give a formal derivation of (10.7), which differs from the corresponding derivation of (5.8). Differentiating Eq. (10.8) w.r. to ε at $\varepsilon = 0$, we obtain

$$K_{0,\omega_0}\psi_0' = -\alpha \frac{\partial^2}{\partial t^2}\psi_0 - f_1(\psi_0), \tag{10.9}$$

where $\alpha = \frac{\partial}{\partial \varepsilon}\omega_\varepsilon^2 \mid_{\varepsilon=0}$ and $\psi_0' = \frac{\partial}{\partial \varepsilon}\psi_\varepsilon \mid_{\varepsilon=0}$. Applying $\delta(\overline{K}_{0,\omega_0})\overline{P}_0$ to this equation, we arrive at (10.7).

To prove (10.7) rigorously one rewrites (10.9) in terms of differences rather than derivatives. This as above leads to the condition

$$\delta(\overline{K}_\varepsilon)F_\varepsilon = 0, \tag{10.10}$$

where $\overline{K}_\varepsilon = \overline{P}_0 K_\varepsilon \overline{P}_0$ with

$$K_\varepsilon = \omega_0^2 \frac{\partial^2}{\partial t^2} + H + X_\varepsilon \tag{10.11}$$

and $X_\varepsilon = (\psi_\varepsilon - \psi_0)^{-1}(f_\varepsilon(\psi_\varepsilon) - f_\varepsilon(\psi_0))$, and where

$$F_\varepsilon = \overline{P}_0\left[\frac{1}{\varepsilon}(\omega_\varepsilon^2 - \omega_0^2)\frac{\partial^2}{\partial t^2}\psi_0 + Y_\varepsilon\right]$$

with $Y_\varepsilon = \frac{1}{\varepsilon}(f_\varepsilon(\psi_0) - f_0(\psi_0))$. Next, we observe that, due to (10.2),

$$X_\varepsilon \to \frac{\partial}{\partial\psi} f_0(\psi_0) \tag{10.12}$$

in D, and that, since $f_\varepsilon(u)/u$ is C^1 in ε and $\psi_0 \in \langle x\rangle L^2$,

$$Y_\varepsilon \to f_1(\psi_0)$$

in $\langle x\rangle L^2$ as $\varepsilon \to 0$. (Weighted estimates on ψ_0 can again be obtained using an analogue of the Froese-Herbst theory). The last relation together with (10.3) yields that

$$F_\varepsilon \to \alpha\partial_t^2\psi_0 + f_1(\psi_0) \tag{10.13}$$

in $\langle x\rangle L^2$ as $\varepsilon \to 0$. Next, Eq. (10.12) and Theorem 7.1 yield

$$\langle x\rangle^{-1}\delta(\overline{K}_{\varepsilon,\omega_\varepsilon})\langle x\rangle^{-1} = \langle x\rangle^{-1}\delta(\overline{K}_{0,\omega_0})\langle x\rangle^{-1} + o(|\varepsilon|^0). \tag{10.14}$$

This together with (10.13) yields that

$$\text{l.h.s. (10.10)} = \langle\delta(\overline{K}_{0,\omega_0})\overline{P}_0 f_1(\psi_0), \overline{P}_0 f_1(\psi_0)\rangle + o(|\varepsilon|^0). \tag{10.15}$$

Comparing this with (10.10), we conclude that

$$\langle\delta(\overline{K}_{0,\omega_0})\overline{P}_0 f_1(\psi_0), \overline{P}_0 f_1(\psi_0)\rangle = 0 \tag{10.16}$$

which contradicts (10.7). □

Remark 10.2. We expect that either α in (10.7) is 0 or K_{0,ω_0} has non-trivial zero eigenfunctions, besides those generated by the symmetries of (10.1). In the latter case, $\alpha = -\langle\partial_t^2\psi_0, \varphi_0\rangle^{-1}\langle f_1(\psi_0), \varphi_0\rangle$, where φ_0 is such an eigenfunction.

11. Non-linear Schrödinger Equation

In this section we consider briefly the non-linear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi + f_\varepsilon(\psi). \tag{11.1}$$

Here ψ: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$, H is the same as in Sect. 2 and

$$f_\varepsilon(\psi) = W_\varepsilon(|\psi|)\psi \tag{11.2}$$

with $W_\varepsilon(s)$ smooth in s and ε, and obeying

$$W_0(s) \equiv 0. \tag{11.3}$$

A standard bifurcation theory shows that periodic solutions

$$g_0 = e^{-i\lambda t}\chi,$$

where λ is an isolated eigenvalue of H and χ, the corresponding eigenfunction, are stable under a non-linear perturbation f_ε. The problem in this case is reduced to the stationary bifurcation problem

$$H\psi + f_\varepsilon(\psi) = \lambda\psi.$$

(See [AF] for some interesting related results.) However, the situation changes for (quasi-) periodic solutions of the form

$$g_0 = \sum_{i=1}^{m} e^{-i\lambda_i t}\chi_i, \tag{11.4}$$

where λ_i are isolated eigenvalues of H and χ_i, the corresponding eigenfunctions, provided

$$\lambda_i - \lambda_j \notin 2\pi\mathbb{Z} \quad \text{for some } i \neq j. \tag{11.5}$$

Adapting the definitions, notation and techniques of Sect. 9, and denoting $\Lambda = (\lambda_1, \dots, \lambda_m)$, and $N \cdot \Lambda = \sum_{i=1}^{m} n_i\lambda_i$ we obtain

Theorem 11.1. *Let H have isolated eigenvalues $\lambda_1, \dots, \lambda_m$ obeying* (11.5) *and s.t. for any N, $|N| > 1$,*

$$N \cdot \Lambda \in \text{disc spec} H \cup \text{thresholds} H.$$

Let g_0 be a solution to (11.1) *with $\varepsilon = 0$ having a generating function G_0. If f_ε obeys*

$$\langle \delta(H - N \cdot \Lambda) f_N, f_N \rangle \neq 0 \tag{11.6}$$

at least for one N with $|N| > 1$, then g_0 is unstable under perturbation f_ε.

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