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On the Spectral Problem for Anyons

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Abstract. We consider the spectral problem resulting from the Schrödinger equation for a quantum system of $n \geqq 2$ indistinguishable, spinless, hard-core particles on a domain in two dimensional Euclidian space. For particles obeying fractional statistics, and interacting via a repulsive hard core potential, we provide a rigorous framework for analysing the spectral problem with its multi-valued wave functions.

1. Introduction

Let $\mathscr{M}$ be a bounded domain in $\mathbb{R}^2$, with boundary $\partial\mathscr{M}$ which we assume to be smooth. The standard choice for the configuration space for a system of n indistinguishable particles constrained to the surface $\mathscr{M}$, and satisfying fractional statistics is the manifold

$$Q_n = (\mathscr{M}^n - \delta_n)/S_n \, . \tag{1.1}$$

Here $\mathscr{M}^n$ denotes the n-fold cartesian product of $\mathscr{M}$ with itself, δ_n denotes the subset of points where two or more particle coordinates coincide (the diagonal) and S_n denotes the group of permutations on n symbols. The fundamental group of Q_n, $\pi_1(Q_n)$ is the n-braid group $B_n(\mathscr{M})$ of $\mathscr{M}$.

Now let $\chi\colon \pi_1(Q_n) \to U(1)$ be a finite, one dimensional, irreducible representation; clearly such a representation is a homomorphism onto the cyclic group of the roots of unity, $U_m = \{\exp(2\pi ik/m),\ k = 0, 1, \ldots (m-1)\}$, for some $m \geqq 1$. Let $\tilde{Q}_n^{[m]}$ be the m-fold covering space of Q_n associated with the representation U_m, with $B_n(\mathscr{M})$ acting as deck transformations, and let $\pi\colon \tilde{Q}_n^{[m]} \to Q_n$, be the natural projection. It has been proposed, [10], that the space of admissible wave functions be a complex Hilbert space obtained from the class of smooth equivariant functions

$$C^\infty_{[m]}(\tilde{Q}_n^{[m]}) = \{\tilde{\psi}\colon \tilde{Q}_n^{[m]} \to \mathbb{C}\colon \tilde{\psi}(\gamma z, \gamma z^*) = \chi(\gamma)\tilde{\psi}(z, z^*), \text{ for all } \gamma \in B_n(\mathscr{M})\} \, . \tag{1.2}$$

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Here we let $z = (z_1, \ldots, z_n)$ denote a generic point of $\mathbb{C}^n$, with $z^* = (\bar{z}_1, \ldots, \bar{z}_n)$ denoting its conjugate. A map in $C^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$ is obviously multivalued on Q_n but single valued on $\tilde{Q}_n^{[m]}$. This setting is an example of what has been dubbed "quantization on multiply connected spaces," a general exposition of which is given by Dowker [4]. The "statistics" of the system is embodied in the choice of a representation, wherein the phenomenon of phase changes of the wave function on interchanging particle positions is the χ-equivariance (1.2). $m = 1$ corresponds to bosons, particles obeying Bose–Einstein statistics, $m = 2$ to fermions, particles obeying Fermi–Dirac statistics and $m \geqq 3$ corresponds to fractional statistics.

The phenomenon of fractional statistics has been explicitly demonstrated for the quasi-particles associated with the Fractional Quantum Hall Effect, [13, 21], and such systems are also conjectured to explain high temperature superconductivity, [2, 21].

A theory of particles obeying fractional statistics in two dimensions was first proposed by Leinaas and Myrheim [14]. The essence of the quantization procedure they adopt rests on the fact that the configuration space (1.1) has nontrivial topology due to the exclusion of the diagonal.

Prior to this, Laidlaw and DeWitt [12], following ideas of Schulman [17] had developed a conceptually similar quantization procedure, however applied to three dimensional space. In this case, the topology of the configuration space leads to the possibility of only two types of statistics, bosons or fermions. They did not consider however the two-dimensional case, which is the only case where fractional statistics is possible, as seen now via standard arguments, [10, 21, 22]. Fractional statistics was also discovered independently by Glodin, Menikoff and Sharp [6, 7], and by Wilczek, who actually coined the term "anyons" [19, 20]. In all the above fundamental works, obtaining fractional statistics rests on the *a*-priori exclusion of δ_n from the configuration space, (1.1). Although initially the exclusion of δ_n by Leinass and Myrheim seemed somewhat arbitrary, Goldin, Menikoff and Sharp in their framework, [6, 7], put forward a theoretical justification. In [8, 9], they also introduced the use of the braid group in the theory, a device also extensively analysed by Wu [22].

We are interested in analysing the spectral problem resulting from the Schrödinger theory for these particle systems. We assume that $\mathscr{M}$ is equipped with, for simplicity, the Euclidian metric. The Laplacian, Δ, acting on $C^{\infty}(\mathbb{C}^n)$ is given by

$$\Delta f = 4 \sum_{k=1}^{n} \partial_k \bar{\partial}_k f, \tag{1.3}$$

where $\partial_k f = \dfrac{\partial f}{\partial z_k}$, and $\bar{\partial}_k f = \dfrac{\partial f}{\partial \bar{z}_k}$. Δ lifts naturally to give a Laplacian $\tilde{\Delta}$: $C^{\infty}(\tilde{Q}_n^{[m]}) \to C^{\infty}(\tilde{Q}_n^{[m]})$. Also let $V \in C^{\infty}(Q_n)$ be a given real valued function, which will play the role of an invariant potential. V also lifts to give $\tilde{V}$: $\tilde{Q}_n^{[m]} \to \mathbb{R}$. The Schrödinger theory for the quantization procedure, with the classical Hamiltonian leads to procuring wave functions $\tilde{\psi}$: $\tilde{Q}_n^{[m]} \to \mathbb{C}$, and eigenvalues $\lambda \in \mathbb{C}$ which satisfy

$$-\tilde{\Delta}\tilde{\psi} + \tilde{V}\tilde{\psi} = \lambda\tilde{\psi} \quad \text{on } \tilde{Q}_n^{[m]}, \tag{1.4}$$

$$\tilde{\psi} = 0 \quad \text{on } \tilde{\Gamma}_n, \tag{1.5}$$

where $\tilde{\Gamma}_n$ is the lift of the boundary $\Gamma_n = \partial\mathscr{M}^n$, and the fractional statistics,

$$\tilde{\psi}(\gamma z, \gamma z^*) = \chi(\gamma)\tilde{\psi}(z, z^*), \quad \text{for all } \gamma \in B_n(\mathscr{M}) . \tag{1.6}$$

As emphasized above, the non-trivial topology of Q_n allows for fractional statistics. However, excluding δ_n also makes Q_n noncompact and non-complete, a fact which obviously renders the spectral problem for the twisted Laplacian on Q_n (1.4)–(1.6), quite intricate. The problem is further complicated by the fact that the action of the braid group has fixed points on $\tilde{\delta}_n$. We approach these problems by analysing an equivalent spectral problem on Q_n itself. To this end let $\mathscr{D}_n = \mathscr{M}^n - \delta_n$. Let φ: $\mathscr{D}_n \to \mathbb{C}$, be the function, given by $\varphi(z) = \Pi_{i<j}(z_i - z_j)$, and let φ_m: $\tilde{Q}_n^{[m]} \to \mathbb{C}$ denote any one of the m roots of the discriminant according to $\varphi_m(z) = [\varphi(z)]^{2/m}$.

Since φ^2 never vanishes on $\tilde{Q}_n^{[m]}$, given any $\tilde{\psi} \in C^\infty_{[m]}(\tilde{Q}_n^{[m]})$ we may consider the map $\tilde{f}$: $\tilde{Q}_n^{[m]} \to \mathbb{C}$ given by $\tilde{f}(z, z^*) = \tilde{\psi}(z, z^*)/\varphi_m(z)$. Clearly $\tilde{f}$ is an invariant function, $\tilde{f}(\sigma z, \sigma z^*) = \tilde{f}(z, z^*)$ for all $\sigma \in B_n(\mathscr{M})$, for $z \in \tilde{Q}_n^{[m]}$, and thus corresponds to an invariant function F: $Q_n \to \mathbb{C}, \tilde{f}(z, z^*) = (F \circ \pi)(z, z^*)$. Thus we see that (1.2) may be written

$$C^\infty_{[m]}(\tilde{Q}_n^{[m]}) = \{\tilde{\psi}: \tilde{Q}_n^{[m]} \to \mathbb{C}: \tilde{\psi} = (\varphi_m)(F \circ \pi), \text{ for some } F \in C^\infty(Q_n)\} . \tag{1.7}$$

Now setting $\alpha = 2/m$, a simple computation using (1.3) shows that $\tilde{\psi} = (\varphi_m)(F \circ \pi)$: $\tilde{Q}_n^{[m]} \to \mathbb{C}$ is an eigenfunction with eigenvalue $\lambda \in \mathbb{C}$, satisfying (1.4)–(1.6), if and only if F: $\mathscr{D}_n \to \mathbb{C}$ satisfies the eigen problem:

$$-\Delta F - \frac{4\alpha}{\varphi} \sum_{k=1}^{n} \frac{\partial \varphi}{\partial z_k} \frac{\partial F}{\partial \bar{z}_k} + VF = \lambda F \quad \text{on } \mathscr{D}_n , \tag{1.8}$$

$$F = 0 \quad \text{on } \Gamma_n . \tag{1.9}$$

$$F(\sigma z, \sigma z^*) = F(z, z^*), \quad \text{for all } \sigma \in S_n . \tag{1.10}$$

We are able to provide a set of sufficient conditions on the potential V which guarantee that (1.4)–(1.6) possesses a pure point spectrum of real positive eigenvalues, and a corresponding set of eigenfunctions contained in the space (1.2), and which form an orthonormal and complete set in an appropriate Hilbert space; Theorem 3.11.

Our strategy is to first establish that the equivalent problem (1.8)–(1.10) has a pure point spectrum of complex eigenvalues; Theorem 2.46. We obtain this result by working on weighted Sobolev spaces on $\mathscr{D}_n$, to handle simultaneously the fact that $\mathscr{D}_n$ is noncompact and non-complete, and the fact that coefficients of the elliptic operator occurring in (1.8) have singularities on δ_n, becoming unbounded on neighborhoods of $\tilde{\delta}_n$. This latter fact is a consequence, in analytical terms, of $B_n(\mathscr{M})$ having fixed points on $\tilde{\delta}_n$. In Sect. 3 we then proceed to lift the results of Theorem 2.46 to the cover $\tilde{Q}_n^{[m]}$, to obtain Theorem 3.11, the main result of the paper.

Indeed, what Theorem 3.11 demonstrates is that the lift of the Laplacian from the configuration space to the m-fold covering space $\tilde{Q}_n^{[m]}$ is a self-adjoint operator acting on an appropriate Hilbert space of functions, equivariant with respect to the action of the braid group, and which decay sufficiently rapidly to zero or neighborhoods of the lift of the diagonal. Our program obtains the results indirectly by carrying out the analysis on $\mathscr{D}_n$, and then lifting results to the cover. An explicit estimate for the rate of decay of the eigenfunctions is given in (3.18).

We also derive a lower bound, (3.22), for the first eigenvalue, the energy of the ground state for the system of anyons in terms of four basic parameters: α, n, the diameter of the surface, and the strength of the virtual force derived from the exclusion principle of the hard core potential.

One could naturally propose analysing the spectral problem for systems of anyons on more general surfaces and with more sophisticated statistics, i.e. higher dimensional and non-abelian representations of the braid group. It is known [10] that if the surface is compact with genus $g \geqq 1$, the particles must be either bosons or fermions; thus on compact surfaces, anyons can only occur if the surface is simply connected. We are thus led to considering simply connected compact surfaces, or non-compact surfaces. Our work considers the simplest case in the latter category, as a point of departure.

We note that it has been claimed by Loss and Fu [15], that without employing a repulsive hard core potential, inconsistencies arise in the theory of anyons, concerning virial coefficients. We obtain here sufficient conditions on such a potential to obtain a pure point spectrum for the classical Hamiltonian.

2. A-Priori Estimates on Weighted Sobolev Spaces

In this section we establish certain a-priori inequalities on weighted Sobolev spaces which allow us to establish the existence of a Green's operator for the elliptic operator occurring in (1.8),

$$\mathscr{L} = -\Delta - \frac{4\alpha}{\varphi} \sum_{k=1}^{n} \partial_k \varphi \bar{\partial}_k + V . \tag{2.1}$$

We show that the Green's operator on certain spaces is compact, which yields the result that $\mathscr{L}$ has a pure point spectrum, Theorem 2.46.

In general, for a subdomain $N \subset \mathscr{M}^n$ we shall let $C^\infty(N)$ denote the complex linear space of complex valued functions on N, which together with their partial derivatives of all orders, are continuous on N. $C_0^\infty(N)$ will denote the subspace of $C^\infty(N)$ whose members have compact support in N.

We define the functions, ζ_k: $\mathscr{D}_n \to \mathbb{R}$,

$$\zeta_k(z, z^*) = \sum_{\substack{j=1 \\ j \neq k}}^{n} |z_k - z_j|^{-2p}, \quad k = 1, \ldots, n , \tag{2.2}$$

where p satisfies $2p \geqq 1$, and is to be chosen specifically below. Let ζ: $\mathscr{D}_n \to \mathbb{R}$, be

$$\zeta(z, z^*) = \frac{1}{2} \sum_{k=1}^{n} \zeta_k(z, z^*) . \tag{2.3}$$

We set for any real β,

$$w^\beta(z, z^*) = \exp\{\beta\zeta(z, z^*)\}, \quad z \in \mathscr{D}_n . \tag{2.4}$$

It is obvious that

$$\zeta_k^r \leqq \left(\frac{r!}{\beta^r}\right) w^\beta, \quad \text{for all } \beta > 0, \quad \text{for all integers } r \geqq 0 . \tag{2.5}$$

We also define q: $\mathscr{D}_n \to \mathbb{R}$, by

$$q(z, z^*) = \sum_{k=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} |z_k - z_j|^{-(2p+1)} \right]^2 . \tag{2.6}$$

Obviously $|z_k - z_j| \leqq l$, where l is the diameter of $\mathscr{M}$, from which it follows that

$$q(z, z^*) \geqq n[(n-1)l^{-(2p+1)}]^2, \quad z \in \mathscr{D}_n , \tag{2.7}$$

and via Schwarz's inequality,

$$\left| \sum_{k=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} |z_k - z_j|^{-1} \right] \xi_k \right| \leqq l^{2p} [q(z, z^*)]^{\frac{1}{2}} \left(\sum_{k=1}^{n} |\xi_k|^2 \right)^{\frac{1}{2}} \tag{2.8}$$

for all $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, and all $z \in \mathscr{D}_n$.

We make a further assumption on the potential $V \in \mathbb{C}^\infty(\mathscr{D}_n)$, in addition to its invariance with respect to the action of S_n on $\mathscr{D}_n$. For constants $\tau > 0$ and $\hat{\tau} < \infty$,

$$\tau q(z, z^*) \leqq V(z, z^*) \leqq \hat{\tau} q(z, z^*), \quad \text{for all } z \in \mathscr{D}_n . \tag{2.9}$$

It will also be required that in relation to the other constants α, p and l, τ is such that

$$\tau > 1 + (3\alpha l^{2p})^2 . \tag{2.10}$$

The assumption (2.9) may be viewed as requiring the particles to exist under the influence of a hard-core potential, which dictates a type of exclusion principle with the parameter τ prescribing the strength of the virtual force resulting from the exclusion principle.

On $C_0^\infty(\mathscr{D}_n)$ we may define for each $\beta > 0$, the norm

$$\|f\|_{H_\beta}^2 = \int_{\mathscr{D}_n} \left\{ w^\beta \left(\sum_{k=1}^{n} |\partial_k f|^2 + |\bar{\partial}_k f|^2 \right) + w^\beta q |f|^2 \right\} dv_n , \tag{2.11}$$

where dv_n denotes the volume element on $\mathscr{D}_n$. $H_\beta(\mathscr{D}_n)$ is defined to be the completion of $C_0^\infty(\mathscr{D}_n)$ with respect to the norm $\|\cdot\|_{H_\beta}$. $H_\beta(\mathscr{D}_n)$ is a Hilbert space with respect to the obvious inner product yielding (2.11).

Similarly on $C^\infty(\mathscr{D}_n)$ we may define, for $\beta > 0$ the norm

$$\|g\|_{K_\beta}^2 = \int_{\mathscr{D}_n} \left\{ w^{-\beta} \left(\sum_{k=1}^{n} |\partial_k g|^2 + |\bar{\partial}_k g|^2 \right) + w^{-\beta} q |g|^2 \right\} dv_n . \tag{2.12}$$

$K_\beta(\mathscr{D}_n)$ is defined to be the completion of $C^\infty(\mathscr{D}_n)$ with respect to the norm $\|\cdot\|_{K_\beta}$. Also $K_\beta(\mathscr{D}_n)$ is a Hilbert space with the obvious inner product giving (2.12).

We write (2.1) more explicitly as

$$\mathscr{L} = -\Delta - 4\alpha \sum_{k=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j)^{-1} \right] \bar{\partial}_k + V \tag{2.13}$$

and define the sesquilinear form B: $C_0^\infty(\mathscr{D}_n) \times C^\infty(\mathscr{D}_n) \to \mathbb{C}$, given by

$$B(f, g) = \int_{\mathscr{D}_n} \left\{ 2 \sum_{k=1}^{n} [\partial_k f \bar{\partial}_k \bar{g} + \bar{\partial}_k f \partial_k \bar{g}] - 4\alpha \sum_{k=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j)^{-1} \bar{\partial}_k f \right] \bar{g} \right.$$

$$\left. + V f \bar{g} \right\} dv_n \quad \text{for } f \in C_0^\infty(\mathscr{D}_n) \text{ and } g \in C^\infty(\mathscr{D}_n) . \tag{2.14}$$

Simply integrating by parts gives from (2.13) and (2.14),

$$\int_{\mathscr{D}_n} (\mathscr{L} f) \bar{g} \, dv_n = B(f, g), \quad \text{for all } f \in C_0^\infty(\mathscr{D}_n),\ g \in C^\infty(\mathscr{D}_n) . \tag{2.15}$$

Henceforth we shall for brevity write $H_\beta(\mathscr{D}_n)$ simply as H_β, and $K_\beta(\mathscr{D}_n)$ as K_β. We now focus on some basic a-priori inequalities on the form $B(\cdot, \cdot)$; namely that it is continuous on $H_\beta \times K_\beta$ for all β, and coercive for the parameter β sufficiently small, but positive.

Proposition 2.16. *Let β^* be the constant given by*

$$\beta^* = \frac{1}{4p} . \tag{2.17}$$

(i) *Under the conditions* (2.9), *there exists a constant C_1, depending only on α, $\hat{\tau}$, and l, such that for all $\beta > 0$,*

$$|B(f, g)| \leqq C_1 \| f \|_{H_\beta} \| g \|_{K_\beta}, \quad \text{for all } f \in H_\beta,\ g \in K_\beta . \tag{2.18}$$

(ii) *Under conditions* (2.9) *and* (2.10), *there exists a constant $C^* > \frac{1}{3}$ depending only on α, τ and l, such that for all $0 < \beta \leqq \beta^*$, and for all $f \in H_\beta$,*

$$\sup \{ |B(f, g)| : g \in K_\beta,\ \| g \|_{K_\beta} \leqq 1 \} \geqq C^* \| f \|_{H_\beta} . \tag{2.19}$$

(iii) *Under conditions* (2.9) *and* (2.10), *there exists a constant $C^{**} > \frac{2}{9}$, depending only on α, l and τ, such that for all $0 < \beta \leqq \beta^*$, and for all $g \in K_\beta$,*

$$\sup \{ |B(f, g)| : f \in H_\beta,\ \| f \|_{H_\beta} \leqq 1 \} \geqq C^{**} \| g \|_{K_\beta} . \quad \square \tag{2.20}$$

Proofs of the inequalities (2.18)–(2.20) *are given in Appendix A and B.*

To define the Green's operator for $\mathscr{L}$, we make use of a theorem of Necas [16] which is a generalization of the Lax–Milgram lemma, [5].

Lemma 2.21. ([16], p. 318) *Let H and K be complex Hilbert spaces with norms $\| \cdot \|_H$ and $\| \cdot \|_K$ respectively, and let B: $H \times K \to \mathbb{C}$ be a sesquilinear form which satisfies, for constants $C_1 < \infty$, $C_2 > 0$ and $C_3 > 0$,*

(i) $|B(f, g)| \leqq C_1 \| f \|_H \| g \|_K$, *for all* $f \in H$, $g \in K$,

(ii) $\sup \{ |B(f, g)| : g \in K,\ \| g \|_K \leqq 1 \} \geqq C_2 \| f \|_H$, *for all* $f \in H$,

(iii) $\sup \{ |B(f, g)| : f \in H,\ \| f \|_H \leqq 1 \} \geqq C_3 \| g \|_K$, *for all* $g \in K$.

Then for each linear functional $U \in (K)^*$, where $(K)^*$ is the dual of K, there exists a unique $f_U \in H$ such that

$$B(f_U, g) = \overline{U(g)}, \quad \text{for all } g \in K , \tag{2.22}$$

and moreover $\| f_U \|_H \leqq C_2^{-1} \| U \|_{(K)^*}$. □

Let $0 < \beta \leqq \beta^*$ be fixed. We define L_β to be the completion of $C_0^\infty(\mathscr{D}_n)$ with respect to the norm

$$\| f \|_{L_\beta}^2 = \int_{\mathscr{D}_n} w^\beta |f|^2 dv_n . \tag{2.23}$$

L_β is a Hilbert space with respect to the obvious inner product yielding (2.23), and because of (2.11) we have $H_\beta \subset L_\beta$.

For $u \in L_\beta$, we define the linear functional $U\colon K_\beta \to \mathbb{C}$ by $U(g) = \int_{\mathscr{D}_n} g\bar{u} dv_n$. From (2.7) and (2.12),

$$\begin{aligned} |U(g)| &= \left| \int_{\mathscr{D}_n} g\bar{u} dv_n \right| = \left| \int_{\mathscr{D}_n} [w^{\frac{-\beta}{2}} g][w^{\frac{\beta}{2}}\bar{u}] dv_n \right| \\ &\leqq [n(n-1)^2]^{-\frac{1}{2}} l^{2p+1} \int_{\mathscr{D}_n} q^{\frac{1}{2}} |w^{-\frac{\beta}{2}} g| |w^{\frac{\beta}{2}}\bar{u}| dv_n \\ &\leqq [n(n-1)^2]^{-\frac{1}{2}} l^{2p+1} \| g \|_{K_\beta} \| u \|_{L_\beta} . \end{aligned} \tag{2.24}$$

Hence with $(K_\beta)^*$ denoting the dual of K_β, from (2.24),

$$\| U \|_{K_\beta^*} \leqq [n(n-1)^2]^{-\frac{1}{2}} l^{2p+1} \| u \|_{L_\beta} = C_{n,l,p} \| u \|_{L_\beta} . \tag{2.25}$$

Applying Lemma 2.21, to the results of (2.17)–(2.20), together with (2.25) we obtain the following result.

Proposition 2.26. *For $0 < \beta \leqq \beta^*$ there exists a linear operator $G_\beta\colon L_\beta \to H_\beta$ which satisfies*

$$B(G_\beta u, g) = \int_{\mathscr{D}_n} u\bar{g} dv_n \quad \text{for all } g \in K_\beta,\ u \in L_\beta , \tag{2.27}$$

$$\text{and} \quad \| G_\beta u \|_{H_\beta} \leqq [C^*]^{-1} C_{n,l,p} \| u \|_{L_\beta} . \quad \square \tag{2.28}$$

$G_\beta\colon L_\beta \to H_\beta$ obtained above is our Green's operator for (2.1). We now establish some regularity results concerning the elliptic equation formulated weakly by (2.27).

Let $W^{k,2}(\mathscr{M}^n)$ denote the Sobolev space, [1], of complex valued functions on $\mathscr{M}^n$. For $1 \leqq k < \infty$, $W^{k,2}(\mathscr{M}^n)$ is the closure of $C^\infty(\mathscr{M}^n)$ with respect to the norm

$$\| f \|_{W^{k,2}(\mathscr{M}^n)}^2 = \int_{\mathscr{M}^n} \left\{ \sum_{|\nu| + |\mu| \leqq k} |\partial^{\nu,\mu} f|^2 \right\} dv_n, \quad \text{where for } f \in C^\infty(\mathscr{M}^n) ,$$

$\partial^{\nu,\mu} f = \partial_1^{\nu_1} \bar{\partial}_1^{\mu_1} \ldots \partial_n^{\nu_n} \bar{\partial}_n^{\mu_n} f$, and $\nu = (\nu_1, \ldots, \nu_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ are multi-indices, with $|\nu| = \nu_1 + \cdots + \nu_n$, $|\mu| = \mu_1 + \cdots + \mu_n$. Clearly $W^{0,2}(\mathscr{M}^n) = L^2(\mathscr{M}^n)$.

We let $\mathring{W}^{1,2}(\mathscr{M}^n)$ denote the closure of $C_0^\infty(\mathscr{M}^n)$ with respect to the norm, $\| \cdot \|_{W^{1,2}(\mathscr{M}^n)}$; it is a standard fact [1], that if $f \in \mathring{W}^{1,2}(\mathscr{M}^n)$, then $f = 0$ a.e. on $\partial\mathscr{M}^n$.

Since the subset δ_n of $\mathscr{M}^n$ is of real codimension 2, it follows that

$$\int_{\mathscr{D}_n} f dv_n = \int_{\mathscr{M}^n} f dv_n \quad \text{for every } f \text{ measurable on } \mathscr{M}^n \,. \tag{2.29}$$

Also since $C_0^\infty(\mathscr{D}_n)$ may be identified with a subspace of $C_0^\infty(\mathscr{M}^n)$ in the obvious manner, it follows that H_β is a subspace of $\mathring{W}^{1,2}(\mathscr{M}^n)$, and L_β is a subspace of $L^2(\mathscr{M}^n)$. We define for $\beta > 0$, $W_\beta^{k,2}(\mathscr{M}^n)$ to be the proper linear subspace of L_β,

$$W_\beta^{k,2}(\mathscr{M}^n) = \{ f \in L_\beta : \| f \|_{W_\beta^{k,2}(\mathscr{M}^n)} < \infty \}, \quad \text{where}$$

$$\| f \|^2_{W_\beta^{k,2}(\mathscr{M}^n)} = \int_{\mathscr{M}^n} \left\{ \sum_{|\nu| + |\mu| \leqq k} w^\beta |\partial^{\nu,\mu} f|^2 \right\} dv_n \,. \tag{2.30}$$

Now let $u \in L_\beta$ and set $h = G_\beta u$. Since obviously $C^\infty(\mathscr{M}^n) \subset K_\beta$, by (2.27), using (2.18) and (2.29) it follows that $h \in H_\beta \subset \mathring{W}^{1,2}(\mathscr{M}^n)$ satisfies the weak elliptic equation:

$$\int_{\mathscr{M}_n} \left\{ 2 \sum_{k=1}^n [\partial_k h \bar{\partial}_k \bar{g} + \bar{\partial}_k h \partial_k \bar{g}] - 4\alpha \sum_{k=1}^n \left[\sum_{\substack{j=1 \\ j \neq k}}^n (z_k - z_j)^{-1} \bar{\partial}_k h \right] \bar{g} + V h \bar{g} \right\} dv_n$$

$$= \int_{\mathscr{M}_n} u \bar{g} \, dv_n \tag{2.31}$$

for all $g \in C^\infty(\mathscr{M}^n)$. Integration by parts in (2.31) shows that $h \in \mathring{W}^{1,2}(\mathscr{M}^n)$ satisfies in a distributional sense

$$\mathscr{L} h = u \quad \text{in } \mathscr{M}^n \,, \tag{2.32}$$

$$h = 0 \quad \text{on } \partial \mathscr{M}^n \,, \tag{2.33}$$

where $\mathscr{L}$ is the operator (2.1).

Now (2.32) may be written in the convenient form

$$-\Delta h = \rho \quad \text{in } \mathscr{M}^n, \text{ where} \tag{2.34}$$

$$\rho(z, z^*) = 4\alpha \sum_{k=1}^n \sum_{\substack{j=1 \\ j \neq k}}^n (z_k - z_j)^{-1} \bar{\partial}_k h - V h + u \,, \tag{2.35}$$

allowing us to derive the following regularity result concerning solutions of (2.32), (2.33).

Proposition 2.36. *Let $0 < \theta < 1$ be any real number. Suppose that for some $r \geqq 0$ and $\beta > 0$, $h \in W_\beta^{r+1,2}(\mathscr{M}^n) \cap \mathring{W}^{1,2}(\mathscr{M}^n)$ satisfies* (2.32), (2.33), *with $u \in W_\beta^{r,2}(\mathscr{M}^n)$. Then*

$$h \in W_{\beta'}^{r+2,2}(\mathscr{M}^n) \cap \mathring{W}^{1,2}(\mathscr{M}^n), \quad \textit{where} \tag{2.37}$$

$$\beta' = \beta(1 - \theta) \,. \tag{2.38}$$

Furthermore, there exists a constant $C_{\beta,n,r}$ such that

$$\| h \|_{W_{\beta'}^{r+2,2}(\mathscr{M}^n)} \leqq C_{\beta,n,r} \{ \| h \|_{W_\beta^{r+1,2}(\mathscr{M}^n)} + \| u \|_{W_\beta^{r,2}(\mathscr{M}^n)} \} \,. \quad \square \tag{2.39}$$

A proof of Proposition 2.36 *is given in Appendix D. We also have*:

Lemma 2.40. *The embedding $H_\beta \subsetneq L_\beta$ is compact, for each $\beta > 0$.* □

A proof of Lemma 2.40 *is given in Appendix C.*

From Proposition 2.36, and Lemma 2.40, it follows that the operator G_β: $L_\beta \to L_\beta$ defined by (2.27) is compact. It follows from standard spectral theory for compact non-selfadjoint operators, [11], that G_β has a pure point spectrum consisting of complex eigenvalues. We list the nonzero eigenvalues of G_β assumed ordered by decreasing magnitude, taking account of algebraic multiplicities: $\{\mu_1, \mu_2, \ldots\} \subset \mathbb{C}$; i.e., $|\mu_1| \geqq |\mu_2| \geqq \cdots \to 0$ and

$$G_\beta u_j = \mu_j u_j, \quad j = 1, 2, \ldots \tag{2.41}$$

for eigenfunctions $u_j \in H_\beta$.

Using (2.41) in (2.31), setting $\lambda_j = \mu_j^{-1}$, we obtain via (2.32), (2.33) for the system of eigenfunctions $\{u_j: j = 1, 2, \ldots\} \subset H_\beta$,

$$\mathscr{L} u_j = \lambda_j u_j \text{ in } \mathscr{M}^n, \quad \text{with } u_j = 0 \text{ on } \partial\mathscr{M}^n, \quad j = 1, 2, \ldots . \tag{2.42}$$

Iterating the regularity results of Proposition 2.36 in (2.42) shows that the eigenfunctions u_j are smooth; one obtains easily from (2.39), $u_j \in C^\infty(\mathscr{M}^n) \cap H_{\beta^*}$.

For clarity we write the operator $\mathscr{L}$ of (2.1) as $\mathscr{L} = \mathscr{L}\left(z, z^*, \frac{\partial}{\partial z}, \frac{\partial}{\partial z^*}\right)$. It is clear from (2.1) that $\mathscr{L}$ is invariant under the action of S_n, i.e.,

$$\mathscr{L}\left(\sigma z, \sigma z^*, \sigma\frac{\partial}{\partial z}, \sigma\frac{\partial}{\partial z^*}\right) = \mathscr{L}\left(z, z^*, \frac{\partial}{\partial z}, \frac{\partial}{\partial z^*}\right), \quad \text{for all } \sigma \in S_n . \tag{2.43}$$

It is easily seen from (2.42) and (2.43) that

$$\mathscr{L}\left(z, z^*, \frac{\partial}{\partial z}, \frac{\partial}{\partial z^*}\right) u_j(\sigma z, \sigma z^*) = \lambda_j u_j(\sigma z, \sigma z^*) \quad \text{for all } \sigma \in S_n . \tag{2.44}$$

Thus, via (2.44), if we define the symmetrized eigenfunctions

$$F_j(z, z^*) = \sum_{\sigma \in S_n} u_j(\sigma z, \sigma z^*), \quad j = 1, 2, \ldots, \tag{2.45}$$

we obtain the following result, which is the primary objective of this section.

Theorem 2.46. *The operator $\mathscr{L}$: $W^{2,2}(\mathscr{M}^n) \cap H_{\beta^*} \to L_{\beta^*}$, (2.1), possesses a pure point spectrum of (possibly complex) eigenvalues $\lambda_1, \lambda_2, \ldots$, listed in increasing order of magnitude $|\lambda_1| \leqq |\lambda_2| \leqq \cdots \to \infty$, according to algebraic multiplicities, and a set of corresponding eigenfunctions $\{F_j: j = 1, 2, \ldots\}$ defined by* (2.45), (2.42) *which satisfy*:

$$\mathscr{L} F_j = \lambda_j F_j \quad \text{on } \mathscr{M}^n , \tag{2.47}$$

$$F_j = 0 \quad \text{on } \Gamma_n , \tag{2.48}$$

$$F_j(\sigma z, \sigma z^*) = F_j(z, z^*), \text{ for all } \sigma \in S_n, \text{ and } F_j \in C^\infty(\mathscr{M}^n) \cap H_{\beta^*} . \quad \square \tag{2.49}$$

We note that (2.47)–(2.49) correspond to (1.8)–(1.10) respectively. We now proceed to lift the result of Theorem 2.46 to the covering space $\tilde{Q}_n^{[m]}$, to resolve the spectral problem (1.4)–(1.6).

3. Analysis on the Covering Space

Via (1.7) $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathscr{L}$ with corresponding eigenfunction $F \in C^{\infty}(Q_n)$ satisfying (1.8)–(1.10), if and only if λ is an eigenvalue of the Schrödinger operator $-\tilde{\Delta} + \tilde{V}$ on $\tilde{Q}_n^{[m]}$, with corresponding eigenfunction $\tilde{\psi} = (\varphi^{\alpha})(F \circ \pi)$, satisfying (1.4)–(1.6).

Thus if we define the functions $\tilde{\psi}_j$: $\tilde{Q}_n^{[m]} \to \mathbb{C}$ by

$$\tilde{\psi}_j(z, z^*) = [\varphi(z)]^{\alpha} F_j(z, z^*), \quad z \in \tilde{Q}_n^{[m]}, \tag{3.1}$$

where F_j satisfies (2.47)–(2.49) it follows that $\tilde{\psi}_j$ is an eigenfunction of the Schrödinger operator on the cover $\tilde{Q}_n^{[m]}$, satisfying (1.4)–(1.6) with $\lambda = \lambda_j$.

Now let $\mathring{C}^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$ be the subspace of $C^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$ (defined by (1.2) and (1.7)) consisting of functions compactly supported in $\tilde{Q}_n^{[m]}$. On $\mathring{C}^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$ we may define the inner product:

$$((f, g))_V = \int_{\tilde{Q}_n^{[m]}} \left\{ 2 \sum_{k=1}^{n} \partial_k f \bar{\partial}_k g^* + \bar{\partial}_k f \partial_k g^* + V f g^* \right\} d\tilde{v}_n, \tag{3.2}$$

where g^* denotes the complex conjugate of g, and $d\tilde{v}_n$ is the volume element on $\tilde{Q}_n^{[m]}$.

We let $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$ be the completion of $\mathring{C}^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$ with respect to the norm resulting from (3.2); $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$ is thus a Hilbert space with inner product (3.2).

Let $\mathscr{F}(\lambda_j)$ denote the eigenspace of $\mathscr{L}$ corresponding to λ_j in (2.47)–(2.49). Clearly for the Schrödinger operator the eigenspace of λ_j, $E(\lambda_j)$, is given by $E(\lambda_j) = \{(\varphi^{\alpha})(F \circ \pi)\colon F \in \mathscr{F}(\lambda_j)\}$. Since $\mathscr{F}(\lambda_j) \subset H_{\beta*}$ it follows that $E(\lambda_j) \subset \mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$, $j = 1, 2, \ldots$. Now using Stokes' theorem, taking into account the orientation of each sheet of the cover $\tilde{Q}_n^{[m]}$, we obtain for $\tilde{g} \in \mathring{C}^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$,

$$-\int_{\tilde{Q}_n^{[m]}} \tilde{g}^* \tilde{\Delta} \tilde{\psi}_j d\tilde{v}_n = \int_{\tilde{Q}_n^{[m]}} \left\{ 2 \sum_{k=1}^{n} \partial_k \tilde{\psi}_j \bar{\partial}_k \tilde{g}^* + \bar{\partial}_k \tilde{\psi}_j \partial_k \tilde{g}^* \right\} d\tilde{v}_n, \tag{3.3}$$

where $\tilde{g}^*$ is the conjugate of $\tilde{g}$.

Thus from (3.3), (1.4) and (1.5), we obtain

$$\int_{\tilde{Q}_n^{[m]}} \left\{ 2 \sum_{k=1}^{n} [\partial_k \tilde{\psi}_j \bar{\partial}_k \tilde{g}^* + \bar{\partial}_k \tilde{\psi}_j \partial_k \tilde{g}^*] + \tilde{V} \tilde{\psi}_j \tilde{g}^* \right\} d\tilde{v}_n = \lambda_j \int_{\tilde{Q}_n^{[m]}} \tilde{\psi}_j \tilde{g}^* d\tilde{v}_n. \tag{3.4}$$

Now let $\{\tilde{g}_{s,r}\colon r = 1, 2, \ldots\} \subset \mathring{C}^{\infty}_{[m]}(\tilde{Q}_n^{[m]})$ be such that $\lim_{r \to \infty} \| \tilde{g}_{s,r} - \tilde{\psi}_s \|_{\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})} = 0$. If we replace $\tilde{g}$ by $\tilde{g}_{s,r}$ in (3.4), then by completion by continuity we obtain

$$\int_{\tilde{Q}_n^{[m]}} \left\{ 2 \sum_{k=1}^{n} [\partial_k \tilde{\psi}_j \bar{\partial}_k \tilde{\psi}_s^* + \bar{\partial}_k \tilde{\psi}_j \partial_k \tilde{\psi}_s^*] + \tilde{V} \tilde{\psi}_j \tilde{\psi}_s^* \right\} d\tilde{v}_n = \lambda_j \int_{\tilde{Q}_n^{[m]}} \tilde{\psi}_j \tilde{\psi}_s^* d\tilde{v}_n. \tag{3.5}$$

A similar argument as that used to derive (3.5) gives also

$$\int_{\tilde{Q}_n^{[m]}} \left\{ 2 \sum_{k=1}^{n} \left[\partial_k \tilde{\psi}_s \bar{\partial}_k \tilde{\psi}_j^* + \bar{\partial}_k \tilde{\psi}_s \partial_k \tilde{\psi}_j^* \right] + \tilde{V} \tilde{\psi}_s \tilde{\psi}_j^* \right\} d\tilde{v}_n = \lambda_s \int_{\tilde{Q}_n^{[m]}} \tilde{\psi}_s \tilde{\psi}_j^* d\tilde{v}_n. \tag{3.6}$$

Hence from (3.5) and (3.6),

$$(\lambda_j - \bar{\lambda}_s) \int\limits_{\tilde{Q}_n^{[m]}} \tilde{\psi}_j \tilde{\psi}_s^* d\tilde{v}_n = 0 . \tag{3.7}$$

Choosing $j = s$ in (3.7) and (3.5) yields the fact that λ_j is real and positive for all $j \geqq 1$: (3.7) in turn yields that the eigenspaces corresponding to distinct eigenvalues are mutually orthogonal in $L^2(\tilde{Q}_n^{[m]})$ and by (3.6) hence mutually orthogonal in $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$.

We now assume that the eigenfunctions have been normalized, so that

$$\int\limits_{\tilde{Q}_n^{[m]}} \tilde{\psi}_s \tilde{\psi}_j^* d\tilde{v}_n = \delta_{sj} . \tag{3.8}$$

Now a standard argument shows that the system of eigenspaces $\{E(\lambda_j)$: $j = 1, 2, \ldots\}$ is complete in $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$: i.e.,

$$\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]}) = \overline{\bigoplus_{j \geqq 1} E(\lambda_j)} , \tag{3.9}$$

where the closure is with respect to the norm topology of $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$.

Indeed if $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]}) \neq \overline{\bigoplus_{j \geqq 1} E(\lambda_j)}$, then there exists a $u \in \mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$, $u \not\equiv 0$, such that u is orthogonal to $E(\lambda_j)$ for all j with respect to the inner product on $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$, (3.2). Without loss of generality we may choose $((u, u))_V = 1$. Using (3.2), standard variational arguments give the characterization:

$$\lambda_k = \inf \left\{ ((f, f))_V : f \in \mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]}), f \in \left(\bigoplus_{j=1}^{k-1} E(\lambda_j) \right)^{\perp}, \|f\|_{L^2(\tilde{Q}_n^{[m]})} = 1 \right\} . \tag{3.10}$$

From (3.10) we obtain immediately

$$((u, u))_V \geqq \lambda_k \quad \text{for all } k ,$$

which yields a contradiction, and hence proves (3.9).

We summarize the above results in the following.

Theorem 3.11. *There exists a sequence of real positive eigenvalues* $0 < \lambda_1 \leqq \lambda_2 \leqq \ldots \to \infty$, *with no finite accumulation point, listed according to multiplicities, and corresponding eigenfunctions* $\{\tilde{\psi}_j$: $j = 1, 2, \ldots\}$ $\subset C^\infty(\tilde{Q}_n^{[m]}) \cap \mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$ *satisfying the Schrödinger equation.*

$$-\tilde{\Delta}\tilde{\psi}_j + \tilde{V}\tilde{\psi}_j = \lambda_j \tilde{\psi}_j \quad \text{on } \tilde{Q}_n^{[m]} , \tag{3.12}$$

$$\tilde{\psi}_j = 0 \quad \text{on } \tilde{\Gamma}_n , \tag{3.13}$$

$$\tilde{\psi}_j(\gamma z, \gamma z^*) = \chi(\gamma)\tilde{\psi}_j(z, z^*), \quad \text{for all } \gamma \in B_n(\mathscr{M}),\ z \in \tilde{Q}_n^{[m]} . \tag{3.14}$$

Furthermore the system of eigenfunctions forms a complete orthonormal set in $\mathring{W}_V^{1,2}(\tilde{Q}_n^{[m]})$. *The eigenfunctions decay exponentially to zero on neighborhoods of the lift of the diagonal,* $\tilde{\delta}_n$, *on which set the eigenfunctions vanish.*

Proof. Existence of the system of eigenfunctions with the stated properties follows from the arguments above. We now simply make explicit the rate of decay of the eigenfunctions to zero on $\tilde{\delta}_n$. Equation (3.12) gives (2.47) via (3.1). Applying the estimate (2.39) to (2.47) with $u = \lambda_j F_j$ and $h = F_j$ we obtain for any $0 < \beta < \gamma \leqq \beta^*$, and $k \geqq 0$,

$$\| F_j \|_{W_\beta^{k+2}(\mathcal{M}^n)} \leqq C_{k,\beta}(1 + \lambda_j) \| F_j \|_{W_\gamma^{k+1}(\mathcal{M}^n)} . \tag{3.15}$$

Hence iterating (3.15), we obtain for $r \geqq 2$, and $0 < \beta < \beta^*$,

$$\| F_j \|_{W_\beta^{r,2}(\mathcal{M}^n)} \leqq \hat{C}_{r,\beta}(1 + \lambda_j)^{r-1} \| F_j \|_{W_{\beta^*}^{1,2}(\mathcal{M}^n)} \leqq \hat{C}_{r,\beta}(1 + \lambda_j)^{r-1} \| F_j \|_{H_{\beta^*}} , \tag{3.16}$$

where we have used (2.11), (2.30) and (2.9).

Now for $\varepsilon > 0$,. let $\mathcal{N}_\varepsilon$ be the neighborhood of the diagonal δ_n in $\mathcal{M}^n$ given by $\mathcal{N}_\varepsilon = \{z \in \mathcal{M}^n : |z_i - z_j| < \varepsilon, 1 \leqq i, j \leqq n\}$. From (2.2)–(2.4) it follows that

$$\omega^\gamma(z, z^*) \geqq \exp\left\{\frac{1}{2}\gamma n(n-1)\varepsilon^{-2p}\right\}, \quad \text{for } z \in \mathcal{N}_\varepsilon . \tag{3.17}$$

Now let $0 < \beta < \beta^*$, and set $\bar{\beta} = \frac{1}{2}(\beta + \beta^*)$. Using the Sobolev embedding theorem [1], it follows from (3.16), with $r = n + 1$, that

$$\begin{aligned} \| \omega^\beta F_j \|_{L^\infty(N_\varepsilon)} &\leqq C_n \| \omega^\beta F_j \|_{W^{n+1,2}(\mathcal{M}^n)} \\ &\leqq C_{n,\beta} \| F_j \|_{W_{\bar{\beta}}^{n+1,2}(\mathcal{M}^n)} \leqq \hat{C}_{n,\beta}(1 + \lambda_j)^n \| F_j \|_{H_{\beta^*}} . \end{aligned}$$

Combining (3.17) with this last inequality gives, for $0 < \beta < \beta^*$,

$$|F_j(z, z^*)| \leqq \tilde{C}(1 + \lambda_j)^n \| F_j \|_{H_{\beta^*}} \exp\left\{-\frac{1}{2}\beta n(n-1)\varepsilon^{-2p}\right\} \tag{3.18}$$

for all $z \in \mathcal{N}_\varepsilon$, for a constant $\tilde{C}$ depending only on n, β and l. Inequality (3.18) now implies the assertion on exponential decay. □

Finally, we give a rough lower bound for the energy of the ground state, as measured by the first eigenvalue λ_1.

Now λ_1 satisfies (3.12)–(3.14), and hence equivalently (2.47)–(2.49). Using (2.17), (2.7), (2.11) and (2.15), we obtain for any $\beta > 0$, and $g \in C^\infty(\mathcal{D}_n)$,

$$\begin{aligned} |B(F_1, g)| &= \left| \lambda_1 \int_{\mathcal{M}^n} F_1 \bar{g} dv_n \right| \\ &\leqq \lambda_1 [n(n-1)^2]^{-1} l^{2(2p+1)} \int_{\mathcal{M}^n} (w^{\frac{\beta}{2}} |F_1|) q(w^{-\frac{\beta}{2}} |g|) dv_n \\ &\leqq \lambda_1 [n(n-1)^2]^{-1} l^{2(2p+1)} \| F_1 \|_{H_\beta} \| g \|_{K_\beta} . \end{aligned} \tag{3.19}$$

Thus from (2.17) and (2.19) in (3.19), for all $\beta \leqq \dfrac{1}{4p}$,

$$C^* \| F_1 \|_{H_\beta} \leqq \lambda_1 [n(n-1)^2]^{-1} l^{2(2p+1)} \| F_1 \|_{H_\beta} . \tag{3.20}$$

Now from (B.14), (B.13) and (B.12), we have $C^* = \frac{4}{9}\min\{1, \frac{3}{4} + \alpha^2 l^{4p}\}$. Hence

$$\lambda_1 \geqq \frac{4}{9} \frac{n(n-1)^2}{l^{2(2p+1)}} \min\left\{1, \frac{3}{4} + \alpha^2 l^{4p}\right\} . \tag{3.21}$$

Introducing the parameter $d = nl^{-2}$ which measures the density of the particles on the surface $\mathscr{M}$, we may write (3.21) in the form

$$\lambda_1 \geqq \frac{3}{4}\left[\frac{2(n-1)}{3n}\right]^2 d^{2p+1} n^{2(1-p)} . \tag{3.22}$$

Choosing $p = 1$, (3.22) implies that if the density is kept constant, then the ground state energy never falls below the constant $\frac{1}{12}d^3$. Inequality (3.22) is not necessarily the best lower bound.

Appendix A. Proof of (2.18)

Let $f \in C_0^\infty(\mathscr{D}_n)$, $g \in C^\infty(\mathscr{D}_n)$. It is immediate from (2.14) that

$$|B(f,g)| \leqq 2 \int_{\mathscr{D}_n} \left\{ \sum_{k=1}^n |w^{\frac{\beta}{2}} \partial_k f| \, |w^{-\frac{\beta}{2}} \bar{\partial}_k \bar{g}| + |w^{\frac{\beta}{2}} \bar{\partial}_k f| \, |w^{\frac{\beta}{2}} \partial_k \bar{g}| \right\} dv_n$$

$$+ 4\alpha \int_{\mathscr{D}_n} \left\{ \sum_{k=1}^n \left[\sum_{\substack{j=1 \\ j \neq k}}^n |z_k - z_j|^{-1} \right] |\bar{\partial}_k f| \, |g| \right\} dv_n + \int_{\mathscr{D}_n} V |w^{\frac{\beta}{2}} f| \, |w^{-\frac{\beta}{2}} g| dv_n . \tag{A.1}$$

Now from (2.8) and Schwarz's inequality

$$\int_{\mathscr{D}_n} \left\{ \sum_{k=1}^n \left[\sum_{\substack{j=1 \\ j \neq k}}^n |z_k - z_j|^{-1} \right] |\bar{\partial}_k f| \, |g| \right\} dv_n \leqq l^{2p} \int_{\mathscr{D}_n} w^{\frac{\beta}{2}} \left(\sum_{k=1}^n |\bar{\partial}_k f|^2 \right)^{\frac{1}{2}} w^{-\frac{\beta}{2}} q^{\frac{1}{2}} \, g | dv_n$$

$$\leqq l^{2p} \left(\int_{\mathscr{D}_n} w^\beta \sum_{k=1}^n |\bar{\partial}_k f|^2 \right)^{\frac{1}{2}} \left(\int_{\mathscr{D}_n} w^{-\beta} q |g|^2 dv_n \right)^{\frac{1}{2}} \leqq l^{2p} \| f \|_{H_\beta} \| g \|_{K_\beta} .$$

Using the inequality immediately above and (2.9) in (A.1) we obtain

$$|B(f,g)| \leqq \int_{\mathscr{D}_n} \left\{ 2 \sum_{k=1}^n \left[w^{\frac{\beta}{2}} |\partial_k f| w^{-\frac{\beta}{2}} |\bar{\partial}_k \bar{g}| + w^{\frac{\beta}{2}} |\bar{\partial}_k f| w^{-\frac{\beta}{2}} |\partial_k \bar{g}| \right] + V w^{\frac{\beta}{2}} |f| w^{-\frac{\beta}{2}} |g| \right.$$

$$\left. + 4\alpha l^{2p} \| f \|_{H_\beta} \| g \|_{K_\beta} \right\} \leqq (2 + \hat{\tau} + 4\alpha l^{2p}) \| f \|_{H_\beta} \| g \|_{K_\beta} ,$$

for all $f \in C_0^\infty(\mathscr{D}_n)$, $g \in C^\infty(\mathscr{D}_n)$. The result, (2.18), now follows from this last inequality via completion by continuity. □

Appendix B. Proof of (2.19) and (2.20)

Let $\beta > 0$ and $f \in H_\beta$ be given. We choose

$$g = w^\beta f . \tag{B.1}$$

Then from (2.4) and (B.1) we have

$$\partial_k \bar{g} = w^\beta(\partial_k \bar{f} + \beta \bar{f}\, \partial_k \zeta), \quad \text{and} \quad \bar{\partial}_k \bar{g} = w^\beta(\bar{\partial}_k \bar{f} + \beta \bar{f}\, \bar{\partial}_k \zeta)\,. \tag{B.2}$$

Also from (2.3) we have

$$\partial_k \zeta(z, z^*) = -p \sum_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j)^{-(p+1)}(\bar{z}_k - \bar{z}_j)^{-p}, \quad \text{and} \tag{B.3}$$

$$\bar{\partial}_k \zeta(z, z^*) = -p \sum_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j)^{-p}(\bar{z}_k - \bar{z}_j)^{-(p+1)}\,. \tag{B.4}$$

Hence using Schwarz's inequality we obtain from (B.3), (B.4) and (2.2),

$$\sum_{k=1}^{n} [|\partial_k \zeta|^2 + |\bar{\partial}_k \zeta|^2] \leqq 2p^2 q\,. \tag{B.5}$$

Using (B.5) we obtain from (B.2),

$$\sum_{k=1}^{n} [|\partial_k g|^2 + |\bar{\partial}_k g|^2] \leqq 2w^{2\beta} \left\{ \sum_{k=1}^{n} [|\partial_k f|^2 + |\bar{\partial}_k f|^2] + \beta^2 |f|^2 \sum_{k=1}^{n} [|\partial_k \zeta|^2 + |\bar{\partial}_k \zeta|^2] \right\}$$

$$\leqq 2w^{2\beta} \left\{ \sum_{k=1}^{n} [|\partial_k f|^2 + |\bar{\partial}_k f|^2] + 2\beta^2 p^2 q |f|^2 \right\}. \tag{B.6}$$

From (B.6) and (2.12) it follows that

$$\|g\|^2_{K_\beta} \leqq \int_{\mathscr{D}_n} \left\{ 2w^\beta \sum_{k=1}^{n} [|\partial_k f|^2 + |\bar{\partial}_k f|^2] + 4p^2\beta^2 w^\beta q |f|^2 + w^{-\beta} q |w^\beta f|^2 \right\} dv_n$$

$$\leqq (2 + 4p^2\beta^2) \|f\|^2_{H_\beta}\,. \tag{B.7}$$

Now, using (B.1), (B.2) in (2.14),

$$B(f, g) = \int_{\mathscr{D}_n} \left\{ 2 \sum_{k=1}^{n} [w^\beta \partial_k f(\bar{\partial}_k \bar{f} + \beta \bar{f}\, \bar{\partial}_k \zeta) + w^\beta \bar{\partial}_k f(\partial_k \bar{f} + \beta \bar{f}\, \partial_k \zeta)] \right.$$

$$\left. - 4\alpha \sum_{k=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j)^{-1} \bar{\partial}_k f \right] w^\beta \bar{f} + w^\beta V |f|^2 \right\} dv_n\,.$$

$$= \int_{\mathscr{D}_n} \left\{ 2w^\beta \sum_{k=1}^{n} [|\partial_k f|^2 + |\bar{\partial}_k f|^2] + w^\beta V |f|^2 \right.$$

$$+ 2\beta w^\beta \bar{f} \sum_{k=1}^{n} [\partial_k f \bar{\partial}_k \zeta + \bar{\partial}_k f \partial_k \zeta]$$

$$\left. - 4\alpha \sum_{k=1}^{n} \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} (z_k - z_j)^{-1} \bar{\partial}_k f \right] w^\beta \bar{f} \right\} dv_n\,. \tag{B.8}$$

Again, from (B.5) we get:

$$\left|2\beta w^{\beta}\bar{f}\sum_{k=1}^{n}[\partial_k f\bar{\partial}_k\zeta+\bar{\partial}_k f\partial_k\zeta]\right|$$
$$\leqq 2\beta w^{\beta}|f|\left(\sum_{k=1}^{n}[|\partial_k f|^2+|\bar{\partial}_k f|^2]\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}[|\partial_k\zeta|^2+|\bar{\partial}_k\zeta|^2]\right)^{\frac{1}{2}}$$
$$\leqq\frac{w^{\beta}}{2}\sum_{k=1}^{n}[|\partial_k f|^2+|\bar{\partial}_k f|^2]+4\beta^2p^2w^{\beta}q|f|^2\,. \tag{B.9}$$

From (2.8),

$$4\alpha\left|\sum_{k=1}^{n}\left[\sum_{\substack{j=1\\ j\neq k}}^{n}(z_k-z_j)^{-1}\right][\bar{\partial}_k f]w^{\beta}\bar{f}\right|$$
$$\leqq 4\alpha l^{2p}\left(\sum_{k=1}^{n}|\bar{\partial}_k f|^2\right)^{\frac{1}{2}}q^{\frac{1}{2}}w^{\beta}|f|\leqq\frac{1}{2}w^{\beta}\left(\sum_{k=1}^{n}|\bar{\partial}_k f|^2\right)+8\alpha^2l^{4p}w^{\beta}q|f|^2\,. \tag{B.10}$$

Combining (B.9), (B.10) and (2.9) in (B.8) we obtain

$$B(f,g)\geqq\int_{\mathscr{D}_n}\left\{w^{\beta}\sum_{k=1}^{n}[|\partial_k f|^2+|\bar{\partial}_k f|^2]+[\tau-4\beta^2p^2-8\alpha^2l^{4p}]w^{\beta}q|f|^2\right\}dv_n\,. \tag{B.11}$$

Now under the conditions (2.10) and (2.17), for all $0<\beta\leqq\beta^*$,

$$\tau-4\beta^2p^2-8\alpha^2l^{4p}>1+\alpha^2l^{4p}-4p^2[\beta^*]^2=\frac{3}{4}+\alpha^2l^{4p}>\frac{3}{4}\,. \tag{B.12}$$

Using (B.12) in (B.11) one gets

$$B(f,g)\geqq C_2\|f\|^2_{H_\beta},\ \text{where } C_2=\min\{1,\tau-4\beta^2p^2-8\alpha^2l^{4p}\}\,. \tag{B.13}$$

Setting $g'=g/\|g\|_{K_\beta}$, we obtain from (B.13) and (B.7),

$$B(f,g')\geqq C_2\|f\|^2_{H_\beta}/\|g\|_{K_\beta}\geqq C_2(2+4p^2[\beta^*]^2)^{-1}\|f\|_{H_\beta}=C^*\|f\|_{H_\beta}\,. \tag{B.14}$$

Since from (2.17) $2+4p^2[\beta^*]^2=\frac{9}{4}$, the result (2.19) follows from (B.12), (B.13) and (B.14). □

Although the form $B(\cdot,\cdot)$, (2.14), is not symmetric, nonetheless, the proof of (2.20) is so similar to that of (2.19) that we may omit the details.

Appendix C. Proof of Lemma 2.40

Let $\{f_j\colon j=1,2\ldots\}$ be a bounded sequence in H_β; i.e. $\|f_j\|_{H_\beta}\leqq C$, for all j, for a constant C. Setting $u_j=w^{\beta/2}f_j$, we have

$$\partial_k u_j=w^{\beta/2}\left(\partial_k f_j+\frac{\beta}{2}f_j\partial_k\zeta\right), \tag{C.1}$$

and

$$\bar{\partial}_k u_j = w^{\beta/2}\left(\bar{\partial}_k f_j + \frac{\beta}{2} f_j \bar{\partial}_k \zeta\right). \tag{C.2}$$

From (C.1), (C.2), (2.7) and (2.11), and (B.5)

$$\begin{aligned}\|u_j\|^2_{W^{1,2}(\mathscr{D}_n)} &= \int_{\mathscr{D}_n} \left\{ w^\beta \sum_{k=1}^n \left[\left|\partial_k f_j + \frac{\beta}{2} f_j \partial_k \zeta\right|^2 + \right.\right. \\ &\quad \left.\left. + \left|\bar{\partial}_k f_j + \frac{\beta}{2} f_j \bar{\partial}_k \zeta\right|^2 \right] + w^\beta |f_j|^2 \right\} dv_n \\ &\leqq \int_{\mathscr{D}_n} \left\{ 2w^\beta \sum_{k=1}^n [|\partial_k f_j|^2 + |\bar{\partial}_k f_j|^2] \right. \\ &\quad \left. + 2w^\beta \left[\sum_{k=1}^n |\partial_k \zeta|^2 + |\bar{\partial}_k \zeta|^2 + \frac{1}{2} \right] |f_j|^2 \right\} dv_n \\ &\leqq 2(1 + 2p^2 + C_{n,l,p}) \|f_j\|^2_{H_\beta}, \text{ where } C_{n,l,p} = [2n(n-1)^2]^{-1} l^{2(2p+1)}. \end{aligned} \tag{C.3}$$

Thus $\{u_j\colon j = 1, 2, \ldots\}$ is a bounded sequence in $W^{1,2}(\mathscr{D}_n)$. By the Rellich lemma, [1], this sequence possesses a subsequence converging in $L^2(\mathscr{D}_n)$. Hence there exists a $u \in L^2(\mathscr{D}_n)$ such that $\lim_{k\to\infty} \|u_{j_k} - u\|_{L^2_{(\mathscr{D}_n)}} = 0$, where $\{u_{j_k}\colon k = 1, 2 \ldots\}$ denotes the subsequence. Define $f = w^{-\frac{\beta}{2}} u$ and $f_{j_k} = w^{-\frac{\beta}{2}} u_{j_k}$. Then by (2.23)

$$\begin{aligned}\|f_{j_k} - f\|_{L_\beta} &= \int_{\mathscr{D}_n} w^\beta |f_{j_k} - f|^2 dv_n \\ &= \int_{\mathscr{D}_n} |u_{j_k} - u|^2 dv_n = \|u_{j_k} - u\|_{L^2(\mathscr{D}_n)}, \text{ and similarly,}\end{aligned} \tag{C.4}$$

$$\|f\|_{L_\beta} = \|u\|_{L^2(\mathscr{D}_n)}\,. \tag{C.5}$$

The result of the lemma now follows from (C.4) and (C.5), since $\lim_{k\to\infty} \|f_{j_k} - f\|_{L_\beta} = 0$. □

Appendix D. Proof of Proposition 2.36

The result will be established by induction using the well known results on elliptic regularity as found in [5] for example. First we assume that

$$h \in W^{r+1,2}_\beta(\mathscr{M}^n) \cap \mathring{W}^{1,2}(\mathscr{M}^n)\,, \tag{D.1}$$

for some $r \geqq 0$ and some $\beta > 0$. We set $\varepsilon = \theta\beta$, and let $2\gamma + \varepsilon = \beta$, i.e.,

$$\gamma = \frac{\beta}{2}(1 - \theta)\,. \tag{D.2}$$

Our aim is thus to show that $h \in W^{r+2,2}_{2\gamma}(\mathscr{M}^n)$. We accomplish this by first establishing the intermediate result that $w^\gamma h \in W^{r+2,2}(\mathscr{M}^n)$.

To this end, we observe that from (1.3),

$$\Delta(w^\gamma h) = w^\gamma \Delta h + h\Delta(w^\gamma) + 4\sum_{k=1}^{n} \{\bar{\partial}_k(w^\gamma)\partial_k h + \bar{\partial}_k h \partial_k(w^\gamma)\} . \tag{D.3}$$

In a straightforward way we estimate the right-hand side of (D.3). Now from (2.3), (2.4), (B.3) and (B.4) of Appendix B,

$$\partial_k w^\gamma = -\gamma p w^\gamma \sum_{\substack{j=1\\ j\neq k}}^{n} (z_k - z_j)^{-(p+1)}(\bar{z}_k - \bar{z}_j)^{-p} , \tag{D.4}$$

$$\bar{\partial}_k w^\gamma = -\gamma p w^\gamma \sum_{\substack{j=1\\ j\neq k}}^{n} (\bar{z}_k - \bar{z}_j)^{-(p+1)}(z_k - z_j)^{-p}, \quad \text{and} \tag{D.5}$$

$$\Delta(w^\gamma) = \gamma^2 p^2 w^\gamma \sum_{k=1}^{n} \left| \sum_{\substack{j=1\\ j\neq k}}^{n} (z_k - z_j)^{-(p+1)}(\bar{z}_k - \bar{z}_j)^{-p} \right|^2 - \gamma p w^\gamma \sum_{k=1}^{n} \sum_{\substack{j=1\\ j\neq k}}^{n} |z_k - z_j|^{-2(p+1)} . \tag{D.6}$$

From (D.4), (D.5), (2.3), (2.4) and (2.5) it follows that for $2p \geqq 1$ and any $\varepsilon > 0$,

$$\begin{aligned}\sum_{k=1}^{n} [|\partial_k w^\gamma|^2 + |\bar{\partial}_k w^\gamma|^2] &\leqq 2(\gamma p w^\gamma)^2 \sum_{k=1}^{n} \left\{ \sum_{\substack{j=1\\ j\neq k}}^{n} |z_k - z_j|^{-(2p+1)} \right\}^2 \\ &\leqq 2(\gamma p w^\gamma)^2 \sum_{k=1}^{n} \left\{ l^{2p-1} \sum_{\substack{j=1\\ j\neq k}}^{n} |z_k - z_j|^{-4p} \right\}^2 \leqq 2(\gamma p w^\gamma l^{2p-1})^2 \sum_{k=1}^{n} \zeta_k^2 \\ &\leqq 8(\gamma p w^\gamma l^{2p-1})^2 \zeta^2 \leqq (4\gamma p w^\gamma l^{2p-1})^2 \varepsilon^{-2} w^\varepsilon = (4\gamma p l^{2p-1}\varepsilon^{-1})^2 w^{2\gamma+\varepsilon} .\end{aligned} \tag{D.7}$$

Similarly from (D.6) we have

$$\begin{aligned}|\Delta w^\gamma| &\leqq 2(4\gamma p l^{2p-1}\varepsilon^{-1})^2 w^{\gamma+\varepsilon/2} + 8\gamma p l^{2(2p-1)} w^\gamma \zeta^3 \\ &\leqq 32[(\gamma p l^{2p-1}\varepsilon^{-1})^2 + 12\gamma p l^{2(2p-1)}\varepsilon^{-3}] w^{\gamma+\varepsilon/2} .\end{aligned} \tag{D.8}$$

From (2.34), (D.3), (D.7) and (D.8), for any $\varepsilon > 0$, and constants $C_1(\varepsilon)$, $C_2(\varepsilon)$,

$$|\Delta(w^\gamma h)|^2 \leqq 3w^{2\gamma}|\rho|^2 + C_1(\varepsilon) w^{2\gamma+\varepsilon}|h|^2 + C_2(\varepsilon) w^{2\gamma+\varepsilon}\left(\sum_{k=1}^{n} |\partial_k h|^2 + |\bar{\partial}_k h|^2 \right). \tag{D.9}$$

Also from (2.35), (2.2), (2.3) and (2.5),

$$|\rho|^2 \leqq 48\alpha^2 \left(\sum_{k=1}^{n} |\bar{\partial}_k h|^2 \right) \left(\sum_{k=1}^{n} \left[\sum_{\substack{j=1\\ j\neq k}}^{n} |z_k - z_j|^{-1} \right]^2 \right) + 3\hat{\tau}^2 \left\{ \sum_{k=1}^{n} \sum_{\substack{j=1\\ j\neq k}}^{n} |z_k - z_j|^{-(2p-1)} \right\}^2 |h|^2 + 3|u|^2$$

$$\leqq 96\alpha^2 l^{2p-1}\zeta\left(\sum_{k=1}^{n}|\bar{\partial}_k h|^2\right) + 3\hat{\tau}^2(l^{2p-1})^2(2\zeta)^2|h|^2 + 3|u|^2$$

$$\leqq 192\alpha^2 l^{2p-1}\varepsilon^{-1}w^{\varepsilon}\left(\sum_{k=1}^{n}|\bar{\partial}_k h|^2\right) + 48\hat{\tau}^2 l^{2(2p-1)}\varepsilon^{-2}w^{\varepsilon}|h|^2 + 3|u|^2 \,. \tag{D.10}$$

From (D.10), (D.9) together with the analagous estimates to (D.9), (D.10) for derivatives of $\Delta(w^\gamma h)$, it follows that if $u \in W_\beta^{r,2}$, then for $\varepsilon = \theta\beta$, and γ such that $2\gamma + \varepsilon = \beta$,

$$\Delta(w^\gamma h) \in W^{r,2}(\mathscr{M}^n), \quad \text{and} \tag{D.11}$$

$$w^\gamma h \in \mathring{W}^{1,2}(\mathscr{M}^n) \,. \tag{D.12}$$

It follows from the standard theory of elliptic regularity, [5], from (D.11) and (D.12), that $w^\gamma h \in W^{r+2,2}(\mathscr{M}^n) \cap \mathring{W}^{1,2}(\mathscr{M}^n)$, and

$$\| w^\gamma h \|_{W^{r+2}(\mathscr{M}^n)} \leqq C_{\beta,l,n,r}\{\| h \|_{W_\beta^{r+1,2}(\mathscr{M}^n)} + \| u \|_{W^{r,2}(\mathscr{M}^n)}\} \tag{D.13}$$

for a constant $C_{\beta,l,n,r}$.

Now it is easily established by induction that from (2.2)–(2.5),

$$|\partial^\nu \bar{\partial}^\mu \zeta(z, z^*)| \leqq C_{\mu,\nu} \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j\neq k}}^{n} |z_k - z_j|^{-(2p+|\nu|+|\mu|)} \tag{D.14}$$

for constants $C_{\mu\nu}$, and hence that

$$|\partial^\nu \bar{\partial}^\mu w^\gamma(z, z^*)| \leqq C'_{\mu\nu} w^\gamma \sum_{k=1}^{n} \sum_{\substack{j=1 \\ j\neq k}}^{n} |z_k - z_j|^{-(2p+|\nu|+|\mu|)}$$

$$\leqq C_{\mu,\nu,n} w^{\gamma+\frac{\varepsilon}{2}} \tag{D.15}$$

for constants $C_{\mu,\nu,n}$.

Now with the use of Leibnitz's rule together with (D.15), the assumptions (D.1), and (D.2) in (D.13), we obtain the bound (2.39) and hence the result (2.37), (2.38). □

References

1. Adams, R.: Sobolev Spaces. New York: Academic Press 1975
2. Chen, Yi-H., Wilczek, F., Witten, E., Halperin, B.I.: On anyon superconductivity. Int. J. Mod. Phys. B **3**, 1001–1067 (1989)
3. Deuschel, J., Strook, D.: Hypercontractivity and spectral gap of symmetric diffusion with applications to stochastic Ising model. J. Funct. Anal. **92**, 30–48 (1990)
4. Dowker, J.S.: Quantum mechanics and field theory on multiply connected and on homogeneous spaces. J. Phys. A **5** 936–943 (1972)
5. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Berlin, Heidelberg, New York: Springer 1983
6. Goldin, R., Menikoff, R., Sharp, D.H.: Particle statistics from induced representations of a local current group. J. Math. Phys. **21**, 650 (1980)
7. Goldin, G.A., Menikoff, R., Sharp, D.H.: Representations of a local current algebra in non-simply connected space and the Aharonov–Bohm effect. J. Math. Phys. **22**, 1664 (1981)

8. Goldin, G.A., Menikoff, R., Sharp, D.H.: Diffeomorphism groups, gauge groups and Quantum Theory. Phys. Rev. Lett. **51**, 2246 (1983)
9. Goldin, G.A., Sharp, D.H.: Rotation generators in two-dimensional space and particles obeying unusual statistics. Phys. Rev. D **28**, 830 (1983)
10. Imbo, C.S., Imbo, T.D., Sudarshan, E.C.G.: Identical particles, exotic statistics and Braid Groups. Phys. Lett. B **234**, 103–109 (1990)
11. Kato, T.: Perturbation Theory for Linear Operators. New York: Springer 1966
12. Laidlaw, M.G., DeWitt, C.M.: Feynman functional integrals for systems of indistinguishable particles. Phys. Rev D **3**, 6, 1375–1378 (1971)
13. Laughlin, R.B.: Fractional statistics in the Quantum Hall Effect. In: Wilczek, F. (ed.) Fractional Statistics and Anyon Superconductivity. pp. 262–303. Singapore. World Scientific 1990
14. Leinaas, J.M., Myrheim, J.: On the theory of identical particles. Nuovo Cimento **37B**, 1–23 (1977)
15. Loss, D., Fu, Y.: Second virial coefficient of an interacting anyon gas (to appear). Phys. Rev. Lett.
16. Necas, J.: Sur une méthode pour resoudre les equations aux derivees partielles du type elliptique voisine de la variationnelle. Ann. Scoula Normale Superiore di Pisa, Serie III **16**, 4, 305–326 (1962)
17. Schulman, L.S.: Approximate topologies. J. Math. Phys. **12**, 2, 304–308 (1971)
18. Thouless, D.J., Wu, Yong-Shi: Remarks on fractional statistics. Phys. Rev. B **31**, 2, 1191–1193 (1985)
19. Wilczek, F.: Magnetic flux, angular momentum and statistics. Phys. Rev. Lett. **48**, 1144–1146 (1982)
20. Wilczek, F.: Quantum Mechanics of fractional spin particles. Phys. Rev. Lett. **49**, 957–959 (1982)
21. Wilczek, F.: Fractional Statistics and Anyon Superconductivity. Singapore: World Scientific 1990
22. Wu, Yong-Shi: General theory for Quantum statistics in two dimensions. Phys. Rev. Lett. **52**, 24, 2103–2106 (1984)

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