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On the Haar Measure of the Quantum $SU(N)$ Group

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Abstract. We prove that the Haar state associated to the compact matrix quantum group $SU_\mu(N)$ is faithful for $\mu \in]-1, 1[$, $\mu \neq 0$, and any $N \geqq 2$.

In [Wo2] the theory of *Compact Matrix Pseudogroups* was initiated and it was shown that many of the fundamental notions of Harmonic Analysis have natural extensions in this context. Among them, probably the most important achievement is the notion of *Haar measure* and the Peter–Weyl type theorem for compact matrix pseudogroups. The next step in this direction was the Tannaka–Krein Duality theorem proved in [Wo3], which made the connection between the objects of Woronowicz and *Quantum Groups* (Drinfeld and Jimbo, see [Dr]) more clear. In fact, a compact matrix pseudogroup, or, more precisely, its algebra of "continuous functions," can be viewed as a completion of the Hopf $*$-algebra of "coefficients" of representations of a quantum group, cf. [Ro, So1, So2]. Apart from Woronowicz's approach this characterization allowed further investigations on the structure of the corresponding C^*-algebras (see [So1, VS2]).

In this paper we shall apply such a philosophy in order to obtain that the "*support of the Haar measure on $SU_\mu(N)$ is the entire space $SU_\mu(N)$*. The appropriate sense of this statement should be the *faithfulness of the Haar state on the C^*-algebra $C(SU_\mu(N))$*. Recall that a positive functional ϕ on a C^*-algebra is called *faithful* if for any element $a \neq 0$ one has $\phi(a^* a) \neq 0$. This problem was left open in [Wo2], where the faithfulness is shown only on a dense subalgebra. In particular, from our result it follows that the enveloping C^*-norm on $C(SU_\mu(N))$ can simply be given as the GNS norm associated to the Haar state.

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1. Preliminaries and Statement of Main Result

We begin by fixing throughout the entire paper a parameter $\mu \in]-1, 1[$, $\mu \neq 0$.

One defines the "algebra of continuous functions on the quantum group $SU_\mu(N)$," denoted by $C(SU_\mu(N))$ to be the *universal unital C^*-algebra* generated by N^2 elements u_{ij}, $i, j = 1, N$, subject to the following relations

$$\sum_{j=1}^{N} u_{ij}^* u_{jk} = \sum_{j=1}^{N} u_{ij} u_{jk}^* = \delta_{ik} \cdot 1 , \tag{1}$$

$$\sum_{\sigma \in S_N} (-\mu)^{I(\sigma)} u_{i(1)\sigma(1)} \dots u_{i(N)\sigma(N)} = E_{i(1)\dots i(N)} \cdot 1 , \tag{2}$$

for all n-tuples $(i(1), \dots, i(N)) \in \{1, \dots, N\}^N$, where $E_{i(1)\dots i(N)} = 0$, if there exist k, l such that $i(k) = i(l)$, and $E_{i(1)\dots i(N)} = (-\mu)^{I([i])}$ otherwise. Here by $[i]$ we denote the permutation given by $(i(1) \dots i(N))$, and, for a permutation $\sigma \in S_N$, $I(\sigma)$ stands for the number of inversions of σ.

The term "universal unital C^*-algebra" has the following sense. We take $\mathrm{Fun}(SU_\mu(N))$ to be simply the $*$-algebra given by the above generators and relations, and then let $C(SU_\mu(N))$ be the completion of $\mathrm{Fun}(SU_\mu(N))$ with respect to the norm

$$\|f\|_{\mathrm{env}} = \sup\{\|\pi(f)\| : \pi \text{ is a } *\text{-representation of } \mathrm{Fun}(SU_\mu(N))\} .$$

One can easily see that $\|f\| < \infty$ for all $f \in \mathrm{Fun}(SU_\mu(N))$.

Furthermore, $\mathrm{Fun}(SU_\mu(N))$ (resp. $C(SU_\mu(N))$) becomes a *Hopf $*$-algebra* (resp. a *Hopf C^*-algebra*) if one defines the *comultiplication*

$$\Delta : \mathrm{Fun}(SU_\mu(N)) \to \mathrm{Fun}(SU_\mu(N)) \otimes \mathrm{Fun}(SU_\mu(N)) ,$$

resp.

$$\Delta : C(SU_\mu(N)) \to C(SU_\mu(N)) \otimes C(SU_\mu(N)) ,$$

to be the $*$-homomorphisms given by

$$\Delta(u_{ik}) = \sum_{j=1}^{N} u_{ij} \otimes u_{jk} . \tag{3}$$

Throughout this paper we use the *minimal (spatial) tensor product of C^*-algebras*.

Let $\tau = \tau_{SU_\mu(N)}$ be the "*Haar measure of $SU_\mu(N)$*." Its meaning is that of a *state* $\tau : C(SU_\mu(N)) \to \mathbb{C}$. Its invariance with respect to both left and right translations is given, in this setting, by

$$(\tau \otimes \mathrm{Id})(\Delta f) = (\mathrm{Id} \otimes \tau)(\Delta f) = \tau(f) \cdot 1 ,$$

for all $f \in C(SU_\mu(N))$. The existence and uniqueness of τ is proven in [Wo2; Thm. 4.2]; it is also shown there that $\tau \circ \kappa = \tau$ on $\mathrm{Fun}(SU_\mu(N))$, where $\kappa : \mathrm{Fun}(SU_\mu(N)) \to \mathrm{Fun}(SU_\mu(N))$ is the *antipode*, given as a linear antimultiplicative (unbounded in general) isomorphism satisfying

$$\kappa(\kappa(f^*)^*) = f, \quad \text{for all } f \in \mathrm{Fun}(SU_\mu(N)) ,$$

$$\kappa(u_{ij}) = u_{ji}^* .$$

All the above data say that $C(SU_\mu(N))$ is a *compact matrix pseudogroup* (*quantum group*). For $N = 2$ the corresponding algebra $C(SU_\mu(2))$ was independently found by Woronowicz ([Wo1]) and Vaksman and Soibelman ([VS1]). For arbitrary N see [Wo3, So1, VS2]. We recall now the notion of a *representation* of $SU_\mu(N)$. By this we mean a pair (v, V) with V a finite dimensional vector space and $v \in L(V) \otimes C(SU_\mu(N))$ an invertible element which satisfies the relation

$$(\mathrm{Id} \otimes \Delta)v = v \circledast v ,$$

where, in order not to complicate the notation, if we take $v = (v_{pq})_{p,q=1,\dim_V}$ to be the matrix of v in some basis of V, then $v \circledast v \in L(V) \otimes (C(SU_\mu(N)) \otimes C(SU_\mu(N)))$ is the element having, in the same basis, the matrix $(x_{pq})_{p,q=1,\dim_V}$, with

$$x_{pq} = \sum_{r=1}^{\dim_V} v_{pr} \otimes v_{rq} .$$

It is shown in [Wo2; Thms. 5.2 and 5.8] that all these representations are "smooth," that is, $v \in L(V) \otimes \mathrm{Fun}(SU_\mu(N))$. For any representation v, we shall denote the vector space $\mathrm{Span}\{v_{pq}\colon p, q = 1, \dim_V\}$ by $\mathrm{Coeff}(v)$.

The results from [Ji, Wo3] give that the theory of representations for $SU_\mu(N)$ is *the same* as that of $SU(N)$. A way of saying this (see also [Ro, So1]) is the existence of a bijection

$$\mathrm{Irr}(SU(N)) \ni v \leftrightarrow v_\mu \in \mathrm{Irr}(Su_\mu(N)) ,$$

such that

(A) it is dimension-preserving,
(B) it is self conjugate,
(C) it preserves decompositions of tensor products,
(D) the fundamental representations correspond one to the other.

A word of explanation. Let $\mathrm{Irr}(\cdot)$ be the notation for the set of equivalence classes of irreducible representations. (A) says $\dim_{V_\mu} = \dim_V$; (B) reads $(v^c)_\mu = (v_\mu)^c$, where $(\cdot)^c$ stands for the *contragradient*; (C) says that if $v' \otimes v'' = \sum w$, then $v'_\mu \otimes v'_\mu = \sum w_\mu$; finally, (D) means that the *fundamental representation of* $SU_\mu(N)$, i.e. the one on $\mathbb{C}^N$ given by the matrix $(u_{ij})_{i,j=1,N}$, has the "usual" properties of the fundamental representation of $SU(N)$ (see [Wo2]). We shall simply denote this representation by u.

After these preparations, we can state our main result:

Theorem 1.1. *For any $N \geqq 2$ and μ as before, $\tau\colon C(SU_\mu(N)) \to \mathbb{C}$ is a faithful state.*

Since the proof will be done by induction, we begin with a discussion, which essentially gives the framework for the induction step.

2. (Double) Coset Spaces of Compact Matrix Pseudogroups

Let G be a compact matrix pseudogroup and H a compact matrix "sub"-pseudogroup of G. This means we are given $\Phi\colon C(G) \to C(H)$ a surjective smooth

$*$-homomorphism of unital Hopf C^*-algebras, that is

(i) $$(\Phi \otimes \Phi) \circ \Delta_G = \Delta_H \circ \Phi ,$$

(ii) $$\Phi(\mathrm{Fun}(G)) = \mathrm{Fun}(H) ,$$

(iii) $$\Phi \circ \kappa_G = \kappa_H \circ \Phi \quad \text{on } \mathrm{Fun}(G) ,$$

(iv) $$\varepsilon_H \circ \Phi = \varepsilon_G \quad \text{on } \mathrm{Fun}(G) .$$

Here ε_G (resp. ε_H) is the *counit* of G (resp. H).

Then one has two actions of H on G by "translations." To give them a meaning, they will be written in terms of *coactions* of the algebra $\mathrm{Fun}(H)$ on $\mathrm{Fun}(G)$ (see, for example [BS]). We can form the "homogeneous spaces" G/H and $H \backslash G$. Again, their formal definitions will involve the algebras of "continuous functions" on these "spaces," so we put

$$C(G/H) = \{f \in C(G) \colon (\mathrm{Id} \otimes \Phi)(\Delta_G f) = f \otimes 1\} , \tag{4}$$

$$C(H \backslash G) = \{f \in C(G) \colon (\Phi \otimes \mathrm{Id})(\Delta_G f) = 1 \otimes f\} . \tag{5}$$

(These are nothing but the identifications

$$C(G/H) = C(G)^H ,$$

$$C(H \backslash G) = {}^H C(G) ,$$

of our algebras with the fixed points for the actions of H on $C(G)$.)

One can also define

$$C(H \backslash G/H) = \{f \in C(G) \colon (\Phi \otimes \mathrm{Id} \otimes \Phi)(\Delta_G \otimes \mathrm{Id})(\Delta_G f) = 1 \otimes f \otimes 1\} , \tag{6}$$

and gets the "double coset space" $H \backslash G/H$. The "canonical surjections"

$$G \to G/H , \tag{4a}$$

$$G \to H \backslash G , \tag{5a}$$

$$G \to H \backslash G/H \tag{6a}$$

have to be understood as given by the inclusions

$$C(G/H) \subsetneq C(G) , \tag{4b}$$

$$C(H \backslash G) \subsetneq C(G) , \tag{5b}$$

$$C(H \backslash G/H) \subsetneq C(G) . \tag{6b}$$

Although there are no "sections" for the "maps" (4a) (5a) (6a) (not even in the classical case), at the level of spaces of "continuous functions" one has something which is a sort of a substitute, i.e. certain left inverses for the inclusions (4b) (5b) (6b). These, of course, cannot be $*$-algebra homomorphisms, but one can still manage to have "nice" properties. In fact, these inverses can be chosen to be *conditional expectations* (see [St]). So, we have

$$E_{G/H} \colon C(G) \to C(G/H) ,$$

$$E_{H \backslash G} \colon C(G) \to C(H \backslash G) ,$$

$$E_{H \backslash G/H} \colon C(G) \to C(H \backslash G/H) ,$$

which are left inverses for the inclusions (4b) (5b) (6b), that is, using a unified notation,

$$E_M\,|_{C(M)} = \mathrm{Id}_{C(M)}\,,$$

M being one of the "spaces" G/H, $H\backslash G$ or $H\backslash G/H$. The E's are given by "integration" on H, more precisely,

$$E_{G/H} = (\mathrm{Id} \otimes (\tau_H \circ \Phi)) \circ \Delta_G\,, \tag{7}$$

$$E_{H\backslash G} = ((\tau_H \circ \Phi) \otimes \mathrm{Id}) \circ \Delta_G\,, \tag{8}$$

$$E_{H\backslash G/H} = ((\tau_H \circ \Phi) \otimes \mathrm{Id} \otimes (\tau_H \circ \Phi)) \circ (\Delta_G \otimes \mathrm{Id}) \circ \Delta_G\,. \tag{9}$$

If one takes, using the same notation, $\tau_M = \tau_G|_{C(M)}$, then by the bi-invariance of τ_G, one gets

$$\tau_G = \tau_M \circ E_M\,. \tag{10}$$

This formula will play a key role in what follows. In particular we get

Lemma 2.1. *τ_G is faithful if and only if E_M and τ_M are faithful.*

Recall that the faithfulness of E_M means

$$f \in C(G), \quad f \geqq 0, \quad f \neq 0 \Rightarrow E_M(f) \neq 0\,.$$

We shall see now that one condition in the above lemma can be checked using the following criterion.

Proposition 2.2. *Suppose τ_H is faithful and the counit ε_H can be extended as a character $\varepsilon_H\colon C(H) \to \mathbb{C}$. Then, in any of the three situations considered above, E_M is faithful.*

Proof. According to the three cases $M = G/H$ (resp. $M = H\backslash G$, $M = H\backslash G/H$), take Γ_M the $*$-homomorphism defined in each case, as

$$\Gamma_{G/H}\colon C(G) \to C(G) \otimes C(H)\,,$$

$$\Gamma_{H\backslash G}\colon C(G) \to C(H) \otimes C(G)\,,$$

$$\Gamma_{H\backslash G/H}\colon C(G) \to C(H) \otimes C(G) \otimes C(H),$$

given by

$$\Gamma_{G/H} = (\mathrm{Id} \otimes \Phi) \circ \Delta_G\,,$$

$$\Gamma_{H\backslash G} = (\Phi \otimes \mathrm{Id}) \circ \Delta_G\,,$$

$$\Gamma_{H\backslash G/H} = (\Phi \otimes \mathrm{Id} \otimes \Phi) \circ (\mathrm{Id} \otimes \Delta_G) \circ \Delta_G\,.$$

But if we take Ψ_M, defined as

$$\Psi_{G/H} = \mathrm{Id} \otimes \varepsilon_H\colon C(G) \otimes C(H) \to C(G)\,,$$

$$\Psi_{H\backslash G} = \varepsilon_H \otimes \mathrm{Id}\colon C(H) \otimes C(G) \to C(G)\,,$$

$$\Psi_{H\backslash G/H} = \varepsilon_H \otimes \mathrm{Id} \otimes \varepsilon_H\colon C(H) \otimes C(G) \otimes C(H) \to C(G)\,,$$

we get $\Psi_M \circ \Gamma_M = \mathrm{Id}$, so, in particular, Γ_M is *injective* (see [Wo2; p. 626]).

If we take F_M, given as

$$F_{G/H} = \mathrm{Id} \otimes \tau_H\colon C(G) \otimes C(H) \to C(G)\,,$$

$$F_{H\backslash G} = \tau_H \otimes \mathrm{Id}\colon C(H) \otimes C(G) \to C(G)\,,$$

$$F_{H\backslash G/H} = \tau_H \otimes \mathrm{Id} \otimes \tau_H\colon C(H) \otimes C(G) \otimes C(H) \to C(G),$$

one gets

$$E_M = F_M \circ \Gamma_M\,. \tag{11}$$

This factorization reduces the problem to the proof of faithfulness of F_M, which is a "standard" fact (the property we invoke is: "*Given C^*-algebras A and B, if $\varphi\colon B \to \mathbb{C}$ is a faithful state, then the completely positive map* $\mathrm{Id} \otimes \varphi\colon A \otimes B \to A$ *is faithful*," see [La]). $\square$

Remark. We can easily see that, for the Hopf C^*-algebras $C(SU_\mu(N))$ the counit ε is automatically extendable to the entire C^*-algebra (simply because ε is a *$*$-representation of* $\mathrm{Fun}(SU_\mu(N))$).

3. Proof of the Main Result

In this section we work in the following context: $G = SU_\mu(N)$, and $H = SU_\mu(N-1)$, with $N \geqq 3$. We view H as a sub-pseudogroup of G via the surjective $*$-homomorphism $\Phi\colon C(G) \to C(H)$ given by

$$\Phi(u_{iN}) = \Phi(u_{Nj}) = 0, \quad \Phi(u_{ij}) = u_{ij}, \quad \text{for } i,j < N, \quad \text{and}$$
$$\Phi(u_{NN}) = 1\,.$$

We shall try to describe the "space" $M = H\backslash G/H$, using representation theory. Let (v, V) be a representation of G. If we "restrict" it, we get a representation of H. Formally, the representation we get by "restriction" is (v_H, V), where

$$v_H = (\mathrm{Id}_{L(V)} \otimes \Phi)(v) \in L(V) \otimes C(H)\,.$$

Take P_v to the *projection onto the subspace of H-fixed vectors of V*, that is

$$P_v = (\mathrm{Id}_{L(V)} \otimes (\tau_H \circ \Phi))(v) \in L(V)\,.$$

The rank of this projection is nothing but the *multiplicity of the trivial representation of H in v_H*. We denote this number by $m_H(v)$. Take u to be the fundamental representation of G, and define the family $(v_{p,r})_{p,r\in\mathbb{N}}$ of representations of G, by

$$v_{p,r} = \underbrace{u \otimes \cdots \otimes u}_{p \text{ times}} \otimes \underbrace{u^c \otimes \cdots \otimes u^c}_{r \text{ times}}\,.$$

For $p = r = 0$, $v_{0,0}$ will be the trivial representation. Finally, take the representation v^n given as

$$v^n = \bigoplus_{p+r \leqq n} v_{p,r}\,.$$

We know from general theory (see [Wo2; Thm. 5.7]) that

$$\operatorname{Fun}(G) = \bigcup_{n \in \mathbb{N}} \operatorname{Coeff}(v^n) .$$

In particular

$$E_M(\operatorname{Fun}(G)) = \bigcup_{n \in \mathbb{N}} E_M(\operatorname{Coeff}(v^n)) .$$

But, of course, $C(M) = \operatorname{Range} E_M = \overline{E_M(\operatorname{Fun}(G))}$, so, if we denote $E_M(\operatorname{Coeff}(v^n))$ by $\operatorname{Fun}^n(M)$, we have

$$C(M) = \overline{\bigcup_{n \in \mathbb{N}} \operatorname{Fun}^n(M)} .$$

As an intermediary step in describing $C(M)$, we shall try first to do this for the spaces $\operatorname{Fun}^n(M)$.

Proposition 3.1. *As a vector space,* $\operatorname{Fun}^n(M)$ *has dimension equal to*

$$\operatorname{Card}\{(p, r) \in \mathbb{N} \times \mathbb{N} \colon p + r \leqq n\} .$$

The above number is, of course, $(n + 1)(n + 2)/2$, *but we don't really need this.*

Proof. Let us "break" the representation v^n as a sum of irreducible ones

$$v^n = \sum_{k \in K_n}{}^{\oplus} \lambda(k) \cdot w_k ,$$

where the w_k, for $k \in K_n$, are mutually inequivalent irreducible representations of G, and the corresponding $\lambda(k)$'s are their multiplicities.

Then

$$\operatorname{Coeff}(v^n) = \sum_{k \in K_n} \operatorname{Coeff}(w_k) ,$$

and, so,

$$\operatorname{Fun}^n(M) = \sum_{k \in K_n} E_M(\operatorname{Coeff}(w_k)) .$$

Note that the spaces $\operatorname{Coeff}(w_k)$ are invariant for E_M.

On the other hand, for an arbitrary $(w, W) \in \operatorname{Irr}(G)$, the vector space $E_M(\operatorname{Coeff}(w))$ is clearly isomorphic to $(\operatorname{Range} P_{w^c}) \otimes (\operatorname{Range} P_w)$, and so we get

$$\dim E_M(\operatorname{Coeff}(w)) = m_H(w) \cdot m_H(w^c) . \tag{12}$$

By the *orthogonality relations* (see [Wo2; Thm. 5.7]) the spaces $E_M(\operatorname{Coeff}(w_k))$, $k \in K_n$ are linearly independent. So, from (12) we get

$$\dim \operatorname{Fun}^n(M) = \sum_{k \in K_n} m_H(w_k) \cdot m_H(w_k^c) . \tag{13}$$

Now, according to the general "philosophy" on representations, the *set of indices* K_n *and the numbers* $m_H(w_k)$, $k \in K_n$ *are the same as in the classical case* ($\mu = 1$), so $\dim \operatorname{Fun}^n(M)$ is a number which *doesn't depend on* μ. So, the only thing to prove is that the statement is true in the classical case. In this case we know that $\operatorname{Coeff}(v^n)$ consists of polynomial functions in u_{ij}'s and $\bar{u}_{ij}$'s of total degree at most n.

But then, by integration, we easily see that the conditional expectation sends any such polynomial to a polynomial in u_{NN} and $\bar{u}_{NN}$ again of total degree at most n. Now we are done, because, obviously, a basis in this space of polynomials is $(u^p_{NN}\bar{u}^r_{NN})_{p+r\leqq n}$. $\square$

The next step in our investigation is showing that, in fact, M is the *Quantum Disk*. We shall use the following definition. Fix a parameter $0 < q \leqq 1$. We take the "space" $\mathbb{D}_q$, to be given by means of its algebra of "continuous functions," which will be denoted $C(\mathbb{D}_q)$. It is defined as follows. Let $\mathrm{Fun}(\mathbb{D}_q)$ be the unital $*$-algebra generated by a single element z, subject to

$$1 - zz^* = q(1 - z^*z) , \tag{14}$$

and take $C(\mathbb{D}_q)$ to be the completion of $\mathrm{Fun}(\mathbb{D}_q)$ with respect to the norm

$$\|f\| = \sup\{\|\pi(f)\|: \pi = *\text{-representation of } \mathrm{Fun}(\mathbb{D}_q), \text{ with } \|\pi(z)\| \leqq 1\} .$$

We shall need only some of the properties of $\mathbb{D}_q$, which are given by the following

Proposition 3.2. *For $0 < q < 1$, the following properties hold.*

(a) *Take on $l^2(\mathbb{N})$ the operators S and Y given, in the canonical orthonormal basis, by $Se_n = e_{n+1}$; $Ye_n = q^n e_n$, $n \in \mathbb{N}$, and let $Z = S^*(1 - Y)^{1/2}$. Then the $*$-homomorphism π: $C(\mathbb{D}_q) \to \mathbb{B}(l^2(\mathbb{N}))$ defined by $\pi(z) = Z$ gives a $*$-isomorphism between $C(\mathbb{D}_q)$ and the Toeplitz algebra $\mathscr{T}$* (i.e. *the C^*-algebra generated by S*).

(b) *Any state φ: $C(\mathbb{D}_q) \to \mathbb{C}$ which satisfies:*

(i) $$\varphi(zz^*) < 1 ,$$

(ii) $$\textit{for any } x \in C(\mathbb{D}_q)\text{: } \varphi(xx^*) = 0 \Rightarrow \varphi(x^*x) = 0 ,$$

is faithful.

(c) *The set $\{z^{*p}z^r: p, r \in \mathbb{N}\}$ is linearly independent in $C(\mathbb{D}_q)$.*

Proof. Before beginning the proof, let us mention that in (b) the condition (ii) is essential (without assuming it, take for example φ: $\mathscr{T} \to \mathbb{C}$ to be defined as $\varphi(X) = \langle Xe_0 | e_0 \rangle$; note that $\varphi(ZZ^*) = 1 - q < 1$, but $\varphi(SS^*) = 0$). It is easy to see that the operator $Z = S^*(1 - Y)^{1/2}$ has norm $\leqq 1$ and satisfies relation (14). Since Y is compact, Z belongs to $\mathscr{T}$. On the other hand, let $y \in C(\mathbb{D}_q)$ be given by $y = 1 - z^*z$. Using formula (14) and

$$\mathrm{Spectrum}(1 - zz^*) \cup \{1\} = \mathrm{Spectrum}(1 - z^*z) \cup \{1\} ,$$

one can prove that

$$\mathrm{Spectrum}(y) = \{q^n: n \in \mathbb{N}\} \cup \{0\} .$$

In particular, this shows that 0 *is an isolated point in* $\mathrm{Spectrum}(z^*z)$, which implies that the polar decomposition of z *takes place in* $C(\mathbb{D}_q)$, that is, there exists a partial isometry $x \in C(\mathbb{D}_q)$ such that

$$z = x(1 - y)^{1/2}, \quad \mathrm{Range}\, x = \overline{\mathrm{Range}\, z}, \quad \mathrm{Ker}\, x = \mathrm{Ker}\, z ,$$

the last two formulas being understood in any $*$-representation of $C(\mathbb{D}_q)$. But, on the other hand, we have $zz^* = 1 - q(1 - z^*z)$. This, in particular, gives the *invertibility of* zz^*. Thus the element x is, in fact, a *coisometry*, so there exists an *isometry* $s \in C(\mathbb{D}_q)$ such that

$$z = s^*(1 - y)^{1/2} .$$

So we have

$$\operatorname{Ker} z^*z = \operatorname{Ker} z = \operatorname{Ker} x = \operatorname{Range}(1 - ss^*) .$$

But the spectral decomposition of y gives a sequence $(p_n)_{n\in\mathbb{N}}$ of mutually orthogonal projections in $C(\mathbb{D}_q)$ such that

$$y = \sum_{n\in\mathbb{N}} q^n p_n .$$

Since $\operatorname{Ker} z^*z = \operatorname{Ker}(1 - y) = \operatorname{Range}(p_0)$, we get $p_0 = 1 - ss^*$. Using now the (obvious) relation $zy = qyz$ it follows, by functional calculus, that we have $zp_n = p_{n-1}zp_n$ for all $n \geqq 1$. So z "sends" Range (p_n) into Range (p_{n-1}). But since zz^* is invertible, z will be "onto." Consequently z "sends" Range (p_n) "*onto*" Range (p_{n-1}). But then it is clear that s^* will do the same, so we get

$$p_n = s^n s^{*n} - s^{n+1} s^{*n+1}, \quad \text{for all } n \in \mathbb{N} .$$

In particular y, and consequently z, will belong to the C^*-subalgebra of $C(\mathbb{D}_q)$ generated by s. But, of course, the $*$-representation π preserves the polar decomposition. Since the operator $Z = \pi(z)$ is exactly given in polar decomposition, we infer $S = \pi(s)$, and (a) follows.

To prove (b), we shall work now in the "spatial" picture of $C(\mathbb{D}_q)$ given by (a). By condition (i) one gets

(i′) $$\varphi(Y) > 0 .$$

If we denote by P_n the orthogonal projection onto $\mathbb{C}e_n$, $n \in \mathbb{N}$, of course we have

$$Y = \sum_{n\in\mathbb{N}} q^n P_n ,$$

and, by (i′) there exists at least one projection P_n with $\varphi(P_n) \neq 0$. Since all the P_n's are *equivalent* (*in the sense of Murrary and von Neumann*) *in* $C(\mathbb{D}_q) = \mathscr{T}$, from (ii) it follows that

$$\varphi(P_n) \neq 0 \quad \text{for all } n \in \mathbb{N} . \tag{15}$$

To conclude the proof, take $X \in C(\mathbb{D}_q) = \mathscr{T}$ such that $\varphi(X^*X) = 0$. Because $0 \leqq X^*P_nX \leqq X^*X$, we get $\varphi(X^*P_nX) = 0$ and, by (ii), we also get

$$\varphi(P_nXX^*P_n) = 0 \quad \text{for all } n \in \mathbb{N} . \tag{16}$$

But the operators $P_nXX^*P_n$ are *rank one operators* and each of them has the form $P_nXX^*P_n = \lambda_n P_n$, for some scalars λ_n. By (15) and (16) we get $\lambda_n = 0$ for all $n \in \mathbb{N}$, which gives $P_nXX^*P_n = 0$. Consequently, $P_nX = 0$ for all $n \in \mathbb{N}$, so $X = 0$.

Finally, to prove (c) we may try to show that the $Z^{*p}Z^r$'s are linearly independent. But it turns out that this approach will require exactly the same arguments as those used in [Wo1; Thm. 1.2]. That is why we shall try to reduce our problem to that result, simply because $C(\mathbb{D}_q)$ "lives" in $C(SU_\mu(2))$, for $\mu = q^{1/2}$. More precisely, take ρ: $C(\mathbb{D}_q) \to C(SU_\mu(2))$ to be the $*$-homomorphism given by $\rho(z) = u_{11}$. Using the notations from [Wo1], $u_{11} = \alpha, u_{21} = \gamma$ (it follows that $u_{12} = -\mu\gamma^*$, $u_{22} = \alpha^*$), we know from [Wo1; Thm. 1.2] that we have a basis $(a_{mnp})_{m\in\mathbb{Z}, n, p\in\mathbb{N}}$ for $\mathrm{Fun}(SU_\mu(2))$ defined as

$$a_{mnp} = \begin{cases} \alpha^m\gamma^n\gamma^{*p}, & \text{for } m \geqq 0 \\ \alpha^{*-m}\gamma^n\gamma^{*p}, & \text{for } m < 0 \,. \end{cases}$$

Clearly we have

$$\rho(z^{*p}z^r) = \alpha^{*p}\alpha^r = \sum_{s=0}^{\min(p,r)} \beta_s a_{r-p,s,s} \,,$$

with non-zero "leading coefficient," i.e. $\beta_{\min(p,r)} \neq 0$. This easily gives the linear independence of the $\rho(z^{*p}z^r)$'s and, of course, the same will hold for the $z^{*p}z^r$'s. □

Comment. The quantum disk appears also in [Sh1]. It was also investigated in [KL], but with a different "parametrization" (which leads to another dense subalgebra instead of $\mathrm{Fun}(\mathbb{D}_q)$).

Theorem 3.3. *With the earlier notations, we have the following*:

(i) *$C(M)$ is the unital C^*-subalgebra of $C(G)$ generated by u_{NN}.*
(ii) *The map $\pi: z \mapsto u_{NN}^*$ gives rise to a unital $*$-isomorphism between $C(\mathbb{D}_{\mu^2})$ and $C(M)$.*
(iii) *$\tau_M = \tau_G|_{C(M)}$ is a faithful state on $C(M)$.*

Proof. Let us note, first, that $z_0 = u_{NN}^*$ satisfies formula (14), with $q = \mu^2$, and $\| u_{NN} \| \leqq 1$ (see, for example, [Br; formulas (4)]). So, if we denote by $\mathscr{A}$ the unital C^*-subalgebra of $C(G)$ generated by u_{NN}, from the definition, it follows that one has a surjective unital $*$-homomorphism π: $C(\mathbb{D}_{\mu^2}) \to \mathscr{A}$, given by the formula in (ii). On the other hand, it is easy to see that $u_{NN} \in C(M)$, so, $\mathscr{A} \subseteq C(M)$.

Take now the state $\varphi = \tau_G \circ \pi$: $C(\mathbb{D}_{\mu^2}) \to \mathbb{C}$. We have

$$\tau_M(u_{NN}^* u_{NN}) = \tau_G(u_{NN}^* u_{NN}) < 1 \,,$$

because

$$u_{NN}^* u_{NN} = 1 - \sum_{i=1}^{N-1} u_{iN}^* u_{iN} \,,$$

and the Haar state is *faithful on* $\mathrm{Fun}(G)$ (see [Wo2; Thm. 4.2]). If we compose with π this gives $\varphi(zz^*) < 1$. But we know from [Wo2; Thm. 5.6] that the Haar state satisfies

$$\tau(f^*f) = 0 \Rightarrow \tau(ff^*) = 0 \,,$$

and, in particular, on $C(\mathbb{D}_{\mu^2})$, we have also

$$\varphi(x^*x) = 0 \Rightarrow \varphi(xx^*) = 0 \,.$$

By Proposition 3.2, it follows that φ is a *faithful state*. Consequently, we obtain two properties:

1. *π is injective*;
2. *$\tau_M|_{\mathscr{A}}$ is a faithful state on $\mathscr{A}$.*

Take $X_{pr} = u_{NN}^{p} u_{NN}^{*r} = \pi(z^{*p} z^{r})$, $p, r \in \mathbb{N}$. Since π is injective, and the $z^{*p}z^{r}$'s are linearly independent, the X_{pr}'s will be linearly independent also. But $X_{pr} \in \mathrm{Coeff}(v_{p,r})$ and, moreover, $X_{pr} \in C(M)$, so, we have, in fact, $X_{pr} \in E_M(\mathrm{Coeff}(v_{p,r}))$. In particular, $(X_{pr})_{p+r \leqq n}$ *will be a linearly independent set in* $\mathrm{Fun}^n(M)$. Using Proposition 3.1, we conclude that

$$\mathrm{Fun}^n(M) = \mathrm{Span}\{X_{pr}: p + r \leqq n\} .$$

This proves exactly that $\mathrm{Fun}^n(M) \subseteq \mathscr{A}$ for every n, so $C(M) \subseteq \mathscr{A}$, which gives $C(M) = \mathscr{A}$ and the proof is complete. □

Proof of Theorem 1.1. It is clear that Theorem 1.1 follows from Theorem 3.3 by induction, provided that the case $N = 2$ is true. But the proof for $N = 2$ is essentially contained in both [VS1] and [Wo1]. It follows from:

1. The concrete formulas for the Haar state, cf. [VS1; Thm. 5.5], [Wo1; p. 130];
2. The injectivity of the "weighted shift" representation $C(SU_\mu(2)) \to \mathscr{B}(l^2(\mathbb{N}) \otimes l^2(\mathbb{Z}))$, used in the proof of Theorem 1.2 of [Wo1]. This representation was independently defined in [VS1; Thm. 3.7] as a direct integral of all irreducible infinite dimensional representations of $C(SU_\mu(2))$. (For details, see [VS1; §5] or [Wo1; Appendix 2].) □

Comment. For the proof of case $N = 2$ one could use also $\mathbb{T}$, the "maximal torus" of $SU_\mu(2)$ (see [Wo1; Appendix 2], or [VS1; §3]) and the facts from Sect. 2 for the "space" $M = \mathbb{T} \backslash SU_\mu(2)/\mathbb{T}$. The algebra $C(M)$ is contained in the C^*-subalgebra generated by u_{11}, on which, by Proposition 3.2 (the proof of (c)), the Haar state is faithful.

Final Remark. We treated, in this paper, only the "groups" $SU_\mu(N)$, but the same method should work also for other "classical compact pseudogroups," provided that one is able to deal with the following technical problems:

A. Work with universal C^*-algebras, given explicitly by generators and relations.

B. Choose, for G, appropriate subgroups H, for which the Haar state is faithful and the double coset space $H \backslash G/H$ is a "familiar" one. For example, for $SO_\mu(N)$ one should expect to get a *quantum segment.*

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