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Topological Models on the Lattice and a Remark on String Theory Cloning

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Abstract. The addition of a topological model to the matter content of a conventional closed-string theory leads to the appearance of many perturbatively-decoupled space-time worlds. We illustrate this by classifying topological vertex models on a triangulated surface. We comment on how such worlds could have been coupled in the Planck era.

1. Many Worlds in String Theory

Topological quantum field theories [1, 2] are characterized by their invariance under local smooth deformations of the background metric. Thus adding a two-dimensional topological model to the matter content of a conventional critical closed-string theory should not affect the decoupling of the Liouville mode and hence also the theory's consistency. Could this then imply that critical string theory is not unique?

In order to address this question we must specify more precisely what we mean by topological models. One way to define them, following Atiyah [3], is through a set of axioms. The basic data is a finite-dimensional space $\mathscr{H}$ of states created by local field operators $\{\phi_1 \equiv \mathbf{I}, \phi_2, \ldots, \phi_M\}$, together with their (symmetric) two- and three-point functions on the sphere:

$$\langle \phi_a \phi_b \rangle_{\text{sph}} = \eta_{ab} \,, \qquad \langle \phi_a \phi_b \phi_e \rangle_{\text{sph}} = c_{abe} \,. \tag{1}$$

The two-point function $\eta_{ab} \equiv c_{1ab}$ must define a non-singular bilinear inner product, which identifies $\mathscr{H}$ with its dual: $(\phi_a)^* \equiv \phi^a = \eta^{ab}\phi_b$. Here and in the sequel indices are raised with the inverse metric η^{ab} and repeated indices are implicitly summed. Unitarity requires the correlation functions of self-adjoint operators to be real. Using the three-point functions and the metric, we can give $\mathscr{H}$ the structure of a *commutative* operator algebra

$$\phi_a \times \phi_b = c_{ab}{}^{e} \phi_e \,. \tag{2}$$

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Now the crucial axiom is the assumption of *factorization*, which allows us to calculate a correlation function on an arbitrary Riemann surface by first deforming the surface into a collection of three-punctured spheres connected by long thin tubes, then cutting the latter by inserting the identity operator $\mathbf{I} = |\phi_a\rangle\langle\phi^a|$. Factorizing, in particular, the four-point function on the sphere along two different channels gives the duality constraint

$$c_{ab}{}^{e} c_{ec}{}^{d} = c_{bc}{}^{e} c_{ae}{}^{d} , \tag{3}$$

which implies that the commutative operator algebra is also *associative*. Under these conditions it can be shown that there exists a self-adjoint basis, $\tilde{\phi}_a = (S^{-1})_a{}^b \phi_b$, in which the algebra is diagonalized:

$$\tilde{\eta}_{ab} = \delta_{ab} \quad \text{and} \quad \tilde{c}_{abe} = \begin{cases} \lambda_a & \text{if } a = b = e , \\ 0 & \text{otherwise} . \end{cases} \tag{4}$$

The non-vanishing correlation functions in this basis are

$$\langle \tilde{\phi}_a^n \rangle_\Gamma = \lambda_a^{2\Gamma - 2 + n} , \tag{5}$$

where Γ is the genus of the Riemann surface. When translated in the original basis these read

$$\langle \phi_{a_1} \cdots \phi_{a_n} \rangle_\Gamma = \sum_{a=1}^{M} S^a{}_{a_1} \cdots S^a{}_{a_n} \lambda_a^{2\Gamma - 2 + n} . \tag{6}$$

Note that the requirement $\langle \mathbf{I} \phi_a \phi_b \rangle_{\text{sph}} = \eta_{ab}$ implies that $\lambda_a = (S^a{}_1)^{-1}$.

Suppose now that we tensor such a topological model with the matter content of a conventional critical closed-string theory. Both the $SL(2, \mathbf{C})$ vacuum and all physical vertex operators will in this case carry an extra index $a = 1, \ldots, M$, since multiplication by ϕ_a does not change the conformal properties of fields. The string amplitudes are simply those of the conventional parent string theory, multiplied by the (constant in moduli space) correlation functions of the topological model [1]. We thus obtain M copies of the graviton, dilaton, antisymmetric tensor, etc.…, which at first sight may appear to interact. This is, however, an illusion since in the "tilde" basis these copies decouple to all orders in the string loop expansion. A similar phenomenon has been observed before, in the continuum limit of matrix models [5]. Note that these M copies differ here only in the value of the string coupling constant, i.e. the vacuum expectation value of the dilaton field, but other backgrounds can be also varied independently. For instance, modding out by a reflection of some internal coordinate times a $\mathbf{Z}_2$ symmetry of the topological model will yield a collection of string theories defined either on the circle or the orbifold.

What we see here is another entry in the long dictionary between world-sheet and space-time properties. A *topological model* on the former translates into the appearance of *many worlds* in the latter. If these worlds were truly decoupled, this remark would have only philosophical value. However, in a theory of gravity, such worlds could have been coupled in the Planck era and/or through non-perturbative effects, as illustrated in Fig. 1. Understanding such effects in string theory is therefore intimately connected with understanding how a topological phase of two-dimensional

[1] What we are here discussing is, *a priori*, simpler than the so-called "topological strings" [2,4], obtained by coupling topological matter to *topological gravity*. Indeed, the contact algebra of topological strings does not seem to factorize into the contact algebra of pure gravity times the operator algebra of topological matter. To be sure, the structure of these models is not yet fully elucidated

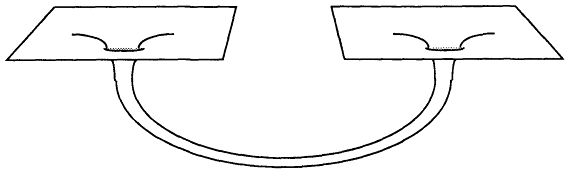

Fig. 1. A non-perturbative contribution to the functional integral for the wavefunction of two universes. Perturbative contributions would correspond to (Euclidean) space-time histories with the topology of two disjoint disks

matter can be reached. Furthermore, as has been argued by Witten, such understanding may be also relevant in the study of space-time singularities [6], as well as of a possible phase of unbroken general coordinate invariance [1, 7, 13].

To make further progress on these issues we must, however, abandon the above axiomatic definition of topological models and study instead how these arise from some local world-sheet dynamics. Several possibilities have been suggested in the literature: σ-models on a manifold with complex structure [1, 2, 8], twisted $N = 2$ supersymmetric models [2, 9], gauged WZW models [10] and finally topological models on a cut-off triangulated surface [11, 12]. The absence of spurious degrees of freedom for the metric makes the characterization of lattice topological models particularly simple and these models will be the subject of the remainder of this letter. We will, in particular, show how under some assumptions they can be completely classified, confirm the above cloning of string theories and suggest a qualitatively but intuitively appealing picture for the wormhole of Fig. 1. Though many parts of our analysis have appeared in the literature in various contexts before, putting them in a new perspective could, we hope, be a useful prelude towards addressing the aforementioned hard non-perturbative issues of string theory.

2. Topological Models on a Triangular Lattice

Consider an oriented genus-Γ surface $\mathcal{M}_\Gamma$ made out of A identical equilateral triangles. As is well known, the way of gluing these triangles together encodes all invariant information about the underlying two-dimensional metric [14]. We will consider a class of ("matter") spin models on $\mathcal{M}_\Gamma$ defined as follows: the spins, denoted by lower-case Greek letters $\{\alpha, \beta, \ldots = 1, \ldots, s\}$, live on the oriented links of the lattice. To every oriented triangular plaquette with spins α, β, and γ we assign a Boltzmann weight $P_{\alpha\beta\gamma}$, while to every link with spins α and β in the two orientations we assign a weight $l^{\alpha\beta}$. Both $P_{\alpha\beta\gamma}$ and $l^{\alpha\beta}$ must be symmetric under cyclic permutations of their indices, which corresponds to local rotations on the surface. Plaquette weights, on the other hand, need not be invariant under orientation change, so that in general $P_{\alpha\beta\gamma}$ is not equal to $P_{\alpha\gamma\beta}$.

The partition function is a product of plaquette and link weights, summed over all possible values of the spins on the oriented links of the lattice. More general correlation functions can be defined by drilling holes on the surface and fixing the values of the spins on their boundaries. As a simple illustration consider two triangles glued together to form a disk or a cylinder as shown in Fig. 2. The corresponding correlation functions, that would be equal if the plaquette weights were fully symmetric, read:

$$D_{\alpha\beta} = P_{\alpha\varrho\sigma} P_{\beta\sigma'\varrho'} l^{\sigma\sigma'} l^{\varrho\varrho'} \equiv P_{\alpha\varrho}{}^{\sigma} P_{\beta\sigma}{}^{\varrho} \tag{7}$$

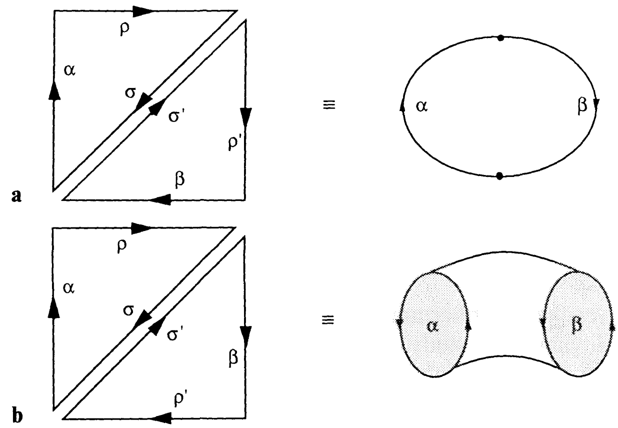

Fig. 2. a The elementary area-2 disk defining the correlator $D_{\alpha\beta}$. **b** The elementary area-2 cylinder defining the correlator $C_{\alpha\beta}$

and

$$C_{\alpha\beta} = P_{\alpha\varrho\sigma} P_{\beta\varrho'\sigma'} l^{\sigma\sigma'} l^{\varrho\varrho'} \equiv P_{\alpha\varrho}{}^{\sigma} P_{\sigma\beta}{}^{\varrho}\,, \tag{8}$$

where indices are here raised with $l^{\alpha\beta}$. All partition (correlation) functions are clearly invariant (covariant) under the similarity transformation

$$\begin{aligned} l^{\alpha\beta} &\to T^{\alpha}{}_{\gamma} T^{\beta}{}_{\delta} l^{\gamma\delta} \\ P_{\alpha\beta\gamma} &\to (T^{-1})_{\alpha}{}^{\delta} (T^{-1})_{\beta}{}^{\varepsilon} (T^{-1})_{\gamma}{}^{\zeta} P_{\delta\varepsilon\zeta}\,, \end{aligned} \tag{9}$$

with T an arbitrary invertible complex matrix. These transformations define an equivalence relation among different spin models. A further equivalence can be defined through *restriction*, if in some basis certain values of the spin never occur in the interior of $\mathscr{M}_\Gamma$. If, for example, $l^{\alpha\beta}$ were degenerate we could obtain an equivalent theory by restricting the values of the spins to those labelling a basis for the subspace of non-null eigenvectors. Without loss of generality we may therefore assume in what follows that $l^{\alpha\beta}$ has an inverse, which we denote by $l_{\alpha\beta}$. Note finally that the models considered here are the most general vertex models on the dual φ^3-graph $\mathscr{M}^*_\Gamma$. We will refer to them for brevity as *vertex models.*

Let us consider now the behaviour of correlation functions under local variations of the metric in the interior of $\mathscr{M}_\Gamma$. The key observation [11, 12, 15] is that these latter can be generated by two elementary moves: the link-flip and pyramid moves illustrated in Fig. 3a and b respectively. We may therefore *define the class of topological vertex models by imposing invariance under these two moves.* The corresponding conditions read:

$$P_{\alpha\beta}{}^{\varepsilon} P_{\varepsilon\gamma}{}^{\delta} = P_{\beta\gamma}{}^{\varepsilon} P_{\alpha\varepsilon}{}^{\delta} \tag{10a}$$

and

$$P_{\alpha\beta\gamma} = P_{\alpha\delta}{}^{\zeta} P_{\beta\varepsilon}{}^{\delta} P_{\gamma\zeta}{}^{\varepsilon}\,. \tag{10b}$$

When these are satisfied, correlation functions only depend on the genus of the surface and on the values of the spins at its boundaries. We will therefore denote them by $C^{(\Gamma)}_{\{\alpha^{(1)}\}\dots\{\alpha^{(n)}\}}$, where $\{\alpha^{(h)}\}$ are the values of the spins ordered according to the induced orientation around the h^{th} hole. A special name will be reserved for correlation

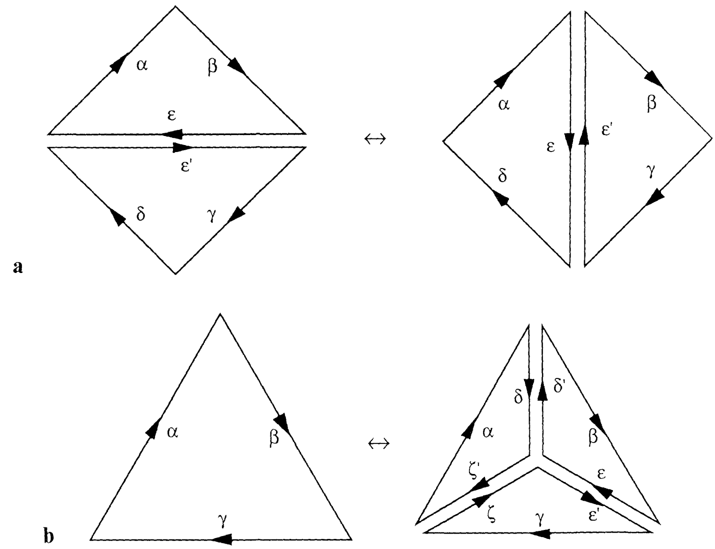

Fig. 3. **a** The link-flip move. **b** The pyramid move

functions on the sphere with n length-one boundaries, or one length-n boundary. We will refer to them for short as n-point functions on the *sphere* and *disk*, and use the already anticipated notation: $C_{\alpha_1 \ldots \alpha_n} \equiv C^{(\Gamma=0)}_{\{\alpha_1\}\ldots\{\alpha_n\}}$ and $D_{\alpha_1 \ldots \alpha_n} \equiv C^{(\Gamma=0)}_{\{\alpha_1 \ldots \alpha_n\}}$ respectively. Note that the three-point function on the disk is simply the plaquette weight: $P_{\alpha\beta\gamma} \equiv D_{\alpha\beta\gamma}$.

Some comments are in order here concerning the above definition of topological models. First, it includes as special cases all models studied in [11, 12]. Nevertheless, it could be still conceivably relaxed in a variety of ways. One may, for instance, demand topological invariance only in the continuum ($A \to \infty$) limit, or for only a subset of external (boundary) states. One may also drop the pyramid-move condition altogether. This introduces only area in addition to genus dependence, because link flips suffice by themselves to connect any two surfaces of fixed A and Γ to each other [15]. Finally one may consider continuous and/or unbounded spins. We will comment on some of these variations below, though an exhaustive study lies beyond the scope of the present letter.

The conditions for topological invariance, Eqs. (10a, b), have a simple interpretation, if we define a formal algebra $\mathcal{A}$ generated by a basis of linearly independent vectors $\{\varpi_\alpha, \alpha = 1, \ldots, s\}$, with multiplication rules

$$\varpi_\alpha \times \varpi_\beta = P_{\alpha\beta}{}^{\gamma} \varpi_\gamma \,. \tag{11}$$

The link-flip condition implies that the algebra is *associative*. In that case, the structure constants define the so-called *regular representation* where ϖ_α is represented by a matrix with entries $P_{\alpha\beta}{}^{\gamma}$, β, and γ being the column and row indices respectively. Both Eq. (7) and the pyramid-move condition (10b) are then summarized by the following elegant form for the n-point function on the disk

$$D_{\alpha_1 \ldots \alpha_n} = \mathrm{Tr}_{\mathrm{reg}}(\varpi_{\alpha_1} \times \ldots \times \varpi_{\alpha_n}) \,. \tag{12}$$

This rewriting suggests that the algebra $\mathcal{A}$ encodes all information about the underlying topological vertex model. Indeed, as we will explicitly confirm below, all correlation functions can be expressed in terms of the structure constants $P_{\alpha\beta}{}^{\gamma}$ alone. The alert reader will have, in fact, recognized that $\mathscr{A}$ plays the same role for *open* strings, as the algebra $\mathscr{H}$ did for closed ones. Indeed, the link-flip condition is the condition of *planar* duality, while the lack of commutativity reflects the fact that open-string states cannot be freely permuted on the boundary. Classifying topological vertex models turns out, therefore, to be equivalent to classifying Chan-Paton factors for open strings. This problem was analyzed some time ago by Marcus and Sagnotti [16] [2] and we will essentially repeat their argument below. Of course any open-string algebra $\mathscr{A}$ has a closed-string descendant $\mathscr{H}$, defined by the two- and three-point functions of the corresponding topological vertex model on the sphere. The precise relation between $\mathscr{A}$ and $\mathscr{H}$ will be established in Sect. 4.

3. Examples

Let us, however, first illustrate the above discussion with three concrete examples of topological vertex models. These will, in particular, give us a better understanding of how the advocated cloning of string theories occurs.

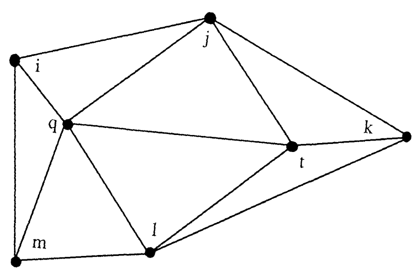

Fig. 4. Conventional spin model on the triangular lattice. This model is topological only at $T \to \infty$ or, for ferromagnetic interactions, $T = 0$

(a) *Ferromagnetic Potts model.* Consider Potts spins $i, j, \ldots \in \{1, \ldots, d\}$ located at the sites of the triangular lattice and interacting among nearest neighbours, as illustrated in Fig. 4. This model can be described in our language by assigning to each oriented link a pair of indices that designate the values of the spins at endpoints, $\alpha \equiv (i, j)$, and by choosing the link and plaquette weights as follows:

$$
\begin{aligned}
l^{(ij)(kl)} &= \delta^{jk}\delta^{li}\,, \\
P_{(ij)(kl)(mq)} &= \delta_{jk}\delta_{lm}\delta_{qi} W(i,j) W(k,l) W(m,q)\,,
\end{aligned}
\tag{13}
$$

so as to ensure that all spins at a common endpoint coincide. Here $W^2(i,j) = \kappa \exp \dfrac{\delta_{ij} - 1}{kT}$ is the Boltzmann weight for the corresponding link, with κ a constant. Note that, because of the fluctuating coordination number of vertices, we cannot accommodate into W a *constant external field*, a remark that will play a role in the sequel.

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[2] These same authors also suggested [17] representing certain Chan-Paton factors with boundary fermions ψ^I. These can be considered as the remnant of a topological action, $\int d^2\sigma \varepsilon^{ij} \partial_i \psi^I \partial_j \psi^I$, on the world-sheet

Simple inspection shows that the Potts model satisfies the topological invariance conditions (10a, b) only in the two extreme cases: $W(i,j) = \delta_{ij}$ or $W(i,j) = d^{-1/6}$, corresponding to a $T = 0$ or $T \to \infty$ ferromagnet. At $T = 0$ all correlation functions on a connected surface vanish unless all spins on all boundaries are aligned. Thus there is one propagating open- and one closed-string state for every value of the spin, or equivalently for every world-sheet ground state. The corresponding algebras are isomorphic to the trivial algebra of d mutually annihilating orthonormalized idempotents: $\mathscr{A} \simeq \mathscr{H} \simeq \mathbf{C}^d$. All diagonal couplings $\lambda_i = 1$, so this model cannot distinguish the genus of the surface.

At $T \to \infty$, on the other hand, $\mathscr{A} \simeq \mathrm{End}(\mathbf{C}^d)$ is isomorphic to the algebra of all $d \times d$ complex matrices. Since the high-temperature phase is, however, unique, there is no cloning of theories, i.e. there is a single propagating closed-string state. Its coupling constant is $\lambda = 1/d$, as can be read off from the partition function $Z^{(\Gamma)}(T \to \infty) = d^{n_2 - n_1 + n_0} = d^{2-2\Gamma}$, where n_2, n_1, and n_0 are the numbers of faces, edges and vertices of $\mathscr{M}_\Gamma$. This is of course the well-known counting used in the topological expansion of matrix models [14].

(b) *Lattice gauge theory:* The link variables $f, g, h, \ldots$ are in this case elements of a compact group G. The theory is defined by

$$l^{fg} = \delta(fg) \tag{14a}$$

and

$$P_{fgh} = \sum_r d_r p_r \chi_r(fgh), \tag{14b}$$

where $\delta(g)$ is the group δ-distribution, which sets its argument equal to the identity. The choice of link weights ensures that inverting orientation amounts to group inversion. The cyclically-symmetric plaquette weight, on the other hand, is an arbitrary class function of the corresponding Wilson loop, expressed in terms of its character decomposition. Here d_r is the dimension of the representation r and p_r are arbitrary coefficients. Using the orthonormality relations

$$\int_G dg\, R_{ij}^{(r)}(g) R_{kl}^{(r')}(g^{-1}) = \frac{1}{d_r} \delta^{rr'} \delta_{jk} \delta_{li}, \tag{15}$$

where dg stands for the normalized Haar measure and $R_{ij}^{(r)}(g)$ for the matrix element of g in the representation r, one can check that

$$P_{fg}{}^{h} P_{hf'g'} = P_{gf'}{}^{h} P_{fhg'} = \sum_r d_r p_r^2 \chi_r(fgf'g'), \tag{16}$$

so that the link-flip move condition is automatically satisfied. As a result, Yang-Mills in two dimensions is, modulo area dependence, a topological theory [12, 18]. Equation (15) is actually all one needs to calculate an arbitrary correlation function, with the result [12]

$$C^{(\Gamma)}_{\{g_1\}\cdots\{g_n\}} = \sum_r d_r^{2-2\Gamma-n} p_r^A \chi_r(g_1) \cdots \chi_r(g_n), \tag{17}$$

where g_h is the Wilson-loop around the h^{th} hole. To prove this statement, one deforms a surface by gluing triangular plaquettes on its boundary and integrating out the group variable(s) along the common links. Each plaquette will contribute an extra power of p_r in the character decomposition of the result. The powers of d_r in Eqs. (14b) and (15) on the other hand, will generically cancel out except when one creates or destroys a connected component of the boundary, in which case one loses or gains, respectively, an extra power of d_r. Since any Riemann surface can be constructed this way, the above result follows easily. In the case of a finite group of order $|G|$ the above analysis holds, provided we take $l^{fg} = \frac{1}{|G|}\,\delta(fg) \equiv \frac{1}{|G|}\,\delta^{fg,1}$ instead of (14a), and replace in the orthonormality relations (15) the Haar measure by the average over the group, $\frac{1}{|G|}\sum_{g\in G}$.

For the theory to be truly topological, the correlation-functions must be area-independent. This implies that for all representations, either $p_r = 0$ or $p_r = 1$ [3]. Let us assume, in particular, that $p_r = 1\ \forall r$, so that $P_{fg}{}^{h} = \delta(fgh^{-1})$. In this case $\mathscr{A}$ is the so-called *group algebra*, and the partition function measures the volume of *flat* gauge connections on $\mathscr{M}_\Gamma$, with the result: $Z^{(\Gamma)} = \sum_r d_r^{2-2\Gamma}$. Furthermore, as seen from Eq. (17), inequivalent closed-string states are in one-to-one correspondence with the *conjugacy classes* of the group. Their algebra $\mathscr{H}$, read off from their two-and three-point functions on the sphere, is isomorphic to the so-called *algebra of classes*. Comparing Eqs. (6) and (17), we see that this algebra can be diagonalized in the basis of representations $\{\tilde{\phi}_r\}$, with the diagonal structure constants $\lambda_r = 1/d_r$ [4]. We thus obtain one decoupled theory for every irreducible representation of the group. The change of basis between conjugacy classes and representations is effected, for a finite group, by the matrix $S^r{}_a = |\mathscr{C}_a|\chi_r(g_a)$, where $|\mathscr{C}_a|$ is the number of elements of the class $\mathscr{C}_a$ with representative g_a.

(c) *Gauged σ-model.* A question that arises is whether we can find a model for which $\mathscr{H}$ is the fusion algebra of group representations, equipped with the usual conjugation operation. Let us assume for definiteness that the group G is finite. Writing the fusion coefficients in the form

$$\mathscr{N}_{r_1\ldots r_n} = \sum_a \frac{|\mathscr{C}_a|}{|G|}\,\chi_{r_1}(g_a)\ldots\chi_{r_n}(g_a)\,, \tag{18}$$

shows that they are diagonalized in a basis of classes, with diagonal couplings given by $\lambda_a = \sqrt{|G|/|\mathscr{C}_a|}$. Let us therefore try to construct a spin model like the one shown in Fig. 4, but with spins taking their values in G, and with Boltzmann weights forcing them to lie in the same conjugacy class

$$W(f,h) = \sum_{g\in G}\delta(fgh^{-1}g^{-1}) = \sum_r \chi_r(f)\chi_r(h^{-1})\,. \tag{19}$$

This model is invariant under spin transformations $f \to gfg^{-1}$ with g chosen independently at every site. It resembles, in this sense, the topological gauged WZW models G/G [10], for which $\mathscr{H}$ is believed to be the fusion algebra of the corresponding quantum group representations. This is, however, where the analogy ends. Indeed, although the closed-string algebra of the above spin model *is* diagonalized in a basis of classes, its diagonal couplings, after appropriate normalizations, turn out to be $\lambda_a = 1/|\mathscr{C}_a|$, in disagreement with (18). As we will in

[3] It is amusing to observe that by choosing, for a continuous group, the coefficients p_r appropriately, one can modify the string-susceptibility exponent. We thank E. Kiritsis for bringing up this point

[4] To avoid confusion, we stress again that the diagonal structure constants are meaningful because we normalized the propagator or two-point function on the sphere to one

fact now prove, a generic fusion algebra cannot be obtained from a topological vertex model, because the inverse couplings of the latter are necessarily quantized.

4. Classification

The classification of topological vertex models is a simple corollary of some standard results on the structure of algebras [19], which we will use without proof in this section. First let us suppose that the algebra $\mathscr{A}$ associated with the topological model has a non-trivial ($\neq \{0\}$) radical $\mathscr{R}$, i.e. a non-trivial maximal nilpotent two-sided ideal. Being nilpotent, the elements of $\mathscr{R}$ have a vanishing trace in any representation of algebra. Furthermore, being an ideal, $\mathscr{R}$ is closed under multiplication by an arbitrary element of $\mathscr{A}$. Thus for any $\varrho \in \mathscr{R}$, we have $\mathrm{Tr}_{\mathrm{reg}}(\varpi_\alpha \times \varpi_\beta \times \varrho) = 0 \forall \alpha, \beta$, so that elements of the radical cannot appear on any elementary plaquette of the surface. We may therefore define an equivalent topological model by restriction: $\mathscr{A} \to \mathscr{A}/\mathscr{R}$. Put differently we can assume without loss of generality that $\mathscr{A}$ has no radical and is hence *semi-simple*.

Now a semi-simple algebra has a unique decomposition into a direct sum of mutually annihilating simple components:

$$\mathscr{A} = \bigoplus_{a=1}^{M} \mathscr{A}_a \quad \text{with} \quad \mathscr{A}_a \times \mathscr{A}_b = 0 \quad \text{if} \quad a \neq b\,. \tag{20}$$

A simple algebra over the complex field, on the other hand, is always isomorphic to a complete matrix algebra

$$\mathscr{A}_a \simeq \mathrm{End}(\mathbf{C}^{d_a})\,. \tag{21}$$

Thus the most general $\mathscr{A}$ is isomorphic to an algebra of all block-diagonal matrices, with M blocks of sizes $d_a \times d_a$ ($a = 1, \ldots, M$); the dimension of $\mathscr{A}$ is $s = \sum_{a=1}^{M} d_a^2$. For instance the group algebra is isomorphic to $\bigoplus_r \mathrm{End}(\mathbf{C}^{d_r})$, while its dimension, for a finite group, is precisely the number of group elements, $\sum_r d_r^2 = |G|$.

The M components in the decomposition (20) correspond to M completely decoupled theories. Indeed, from (12) we see that the plaquette weights, and hence also all other correlation functions, vanish unless all boundary spins belong to the same irreducible component. Let us therefore concentrate on a single component or, to simplify notation, take $\mathscr{A} = \mathrm{End}(\mathbf{C}^d)$. We may choose for this algebra the basis $\{\tau_\alpha,\ \alpha = 1, \ldots, d^2\}$ of $d \times d$ matrices, such that $\tau_1 = \frac{1}{d}\mathbf{1}_{d\times d}$, while the remaining τ_α are hermitian, traceless and normalized so that $\mathrm{Tr}(\tau_\alpha \tau_\beta) = \frac{1}{d}\delta_{\alpha\beta}$. These matrices provide in fact the only non-trivial irreducible representation of the algebra, contained d times inside the regular representation. From the expression (12) for the two- and three-point functions on the disk it is then straightforward to deduce that

$$\begin{aligned} l^{\alpha\beta} &= \delta^{\alpha\beta}\,, \\ P_{\alpha\beta\gamma} &= d\,\mathrm{Tr}(\tau_\alpha \tau_\beta \tau_\gamma)\,. \end{aligned} \tag{22}$$

Note in particular that $l_{\alpha\beta}$ is nothing but the two-point function on the disk, $D_{\alpha\beta}$, which is invertible because $\mathscr{A}$ has no radical. Substituting now into expression (8) for the two-point function on the sphere, and using the orthocompleteness relation

$$\sum_{\alpha=1}^{d^2} (\tau_\alpha)_{ij}(\tau_\alpha)_{kl} = \frac{1}{d}\,\delta_{jk}\delta_{li}\ \forall i,j,k,l \in \{1,\ldots,d\}\,, \tag{23}$$

one finds

$$C_{\alpha\beta} = \mathrm{Tr}(\tau_\alpha)\,\mathrm{Tr}(\tau_\beta)\,. \tag{24}$$

Since $\mathrm{Tr}(\tau_\alpha) = \delta_{\alpha 1}$, we conclude that, out of the d^2 open-string states in $\mathscr{A}$, *only the one corresponding to the identity propagates as a closed-string state*. Allowing holes of length more than one does not, in fact, introduce any new linearly-independent states in $\mathscr{H}$. Indeed, with the help of Eqs. (22) and (23) we can compute an arbitrary correlation function with the result

$$C^{(\Gamma)}_{\{\alpha^{(1)}\}\ldots\{\alpha^{(n)}\}} = d^{2-2\Gamma-n}\prod_{h=1}^{n}\mathrm{Tr}\left(\tau_{\alpha_1^{(h)}}\cdots\tau_{\alpha_{l_h}^{(h)}}\right), \tag{25}$$

where l_h is the length of the h^{th} hole, and $\{\alpha^{(h)}\} = \{\alpha_1^{(h)},\ldots,\alpha_{l_h}^{(h)}\}$ the values of the spins ordered around it modulo cyclic permutations. Clearly any closed string state $\{\alpha^{(h)}\}$ is a simple multiple of the state $\{1\}$, the three-point coupling of this latter being equal to $1/d$. We thus conclude that the only effect of adding a topological vertex model with $\mathscr{A} = \mathrm{End}(\mathbf{C}^d)$ to the matter content of a closed string theory is to renormalize the string coupling constant. More generally, the space of propagating closed-string states is the center of $\mathscr{A}$, and contains M elements corresponding to the identities of the simple components.

Both this renormalization by an inverse integer, and the cloning of theories when $\mathscr{A}$ has more one simple component, have been already illustrated by the infinite- and zero-temperature Potts ferromagnet. The point of our discussion here was to prove that an *arbitrary* topological vertex model can be reduced to the above two simple effects. We summarize this in the following

Proposition. *Inequivalent topological vertex models are in one-to-one correspondence with semi-simple algebras $\mathscr{A}$ over the complex numbers. These, in turn, are isomorphic to a direct sum of M complete matrix algebras of dimensions d_a^2 for $a = 1,\ldots,M$. The center of $\mathscr{A}$ is the commutative algebra of closed-string states, $\mathscr{H}$. The diagonalized structure constants of this latter in an orthonormal basis are $\lambda_a = 1/d_a$.*

The above analysis goes in fact through with little change, if one drops the constraint of invariance under the pyramid move of Fig. 3b. The net effect is that for every simple component of $\mathscr{A}$, $l_{\alpha\beta}$ can now be an arbitrary multiple of the identity, i.e. the two-point function $D_{\alpha\beta}$ on the disk of area two, Eq. (7). This corresponds to an independent renormalization of the world-sheet cosmological constant for every decoupled component of the theory as illustrated in the example of gauge theory with arbitrary plaquette action, Eq. (17).

More important is the fact that the above classification was based on the assumption of equivalence under the complex transformations (9). These allow us, for example, to transform lattice gauge theory into an appropriate Potts model. Nevertheless, it could happen that only a subset of these transformations are admissible. Consider for instance the coupling of topological matter to pure gravity, described by a matrix model with partition function

$$\mathcal{Z}_{\text{grav+top mat}} = \int \prod_{\alpha=1}^{s} d\Phi^\alpha e^{-\operatorname{Tr}\left(l_{\alpha\beta}\Phi^\alpha\Phi^\beta - \frac{\mu}{\sqrt{N}} P_{\alpha\beta\gamma}\Phi^\alpha\Phi^\beta\Phi^\gamma\right)} , \tag{26}$$

where Φ^α are hermitian $N \times N$ matrices, $1/N$ is the bare string coupling constant and $\ln \mu$ the world-sheet bare cosmological constant. The perturbative expansion of this integral is left invariant under arbitrary complex changes of basis, $\Phi^\alpha \to T^\alpha{}_\beta \Phi^\beta$. As a result, we conclude from our discussion that within perturbation theory the model (26) is equivalent to M decoupled models of hermitian matrices with renormalized sizes $(d_a N) \times (d_a N)$. Note, however, that only *real* transformations leave invariant the integration contours in the complex-Φ space. This could be important if one had a non-perturbative definition of the integral. Another situation in which only real transformations are allowed, is when one wants to twist by a parity that distinguishes hermitian and anti-hermitian states of the topological model. In such circumstances we need the finer classification of algebras over the field **R** of real numbers. A generic $\mathcal{A}$ is now a direct sum of complete matrix algebras over a finite extension of **R**. i.e. over the real, complex or quaternion fields. This is precisely what allows the addition of $SO(d)$, $U(d)$ or $USp(d)$ quantum numbers to open strings with Chan-Paton factors [16].

5. Conclusions

We conclude our brief tour of topological vertex models with two remarks. The first concerns the quantization of the inverse couplings of closed-string states in the diagonal ("tilde") basis. This quantization is not required by the axioms of [3], and there are examples, such as the coset G/G models [10], for which it does not hold. Such models could, to be sure, be put on the lattice if we allowed a constant external field to act on the spins in Fig. 4, or else [20] if spins could take their values in a quantum group *à la* Woronowicz [21]. Neither construction seems, however, compatible with a local, cyclically symmetric plaquette action, or equivalently with a factorizable open-string ascendant. The question of how to couple such models to a theory of both open and closed strings deserves further study.

Our second comment concerns the wormhole of Fig. 1. We may think of this as describing the approach to the simplest topological model, namely the Ising model at zero temperature. Indeed, consider a background for which the temperature T varies continuously from large values to zero as a function of Euclidean time. Then a splitting of worlds will occur precisely when T crosses the critical temperature into the ferromagnetic phase. Alternatively, we may consider a cosmological scenario in which the conformal factor, or the distance on the triangulated world-sheet, plays the role of Minkowski time [22]. If the Ising temperature were chosen below criticality at the cut-off, it would renormalize towards zero at larger scales, so that two decoupled worlds would emerge asymptotically in time. It may be possible to study this quantitatively, by considering the appropriate flows in the two-matrix-model.

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