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Gelfand–Dikii Analysis for $N=2$ Supersymmetric KdV Equations

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Abstract. We generalize the resolvent approach of Gelfand and Dikii to the KdV equation to study the $N = 2$ supersymmetric KdV equations of Laberge and Mathieu. For the associated Lax operators, we study the coincidence limits of the resolvent kernel and its derivatives, and obtain differential equations which they satisfy. These allow us to obtain recursion relations for the analogues of the Gelfand–Dikii polynomials and to obtain a proof of Hamiltonian integrability of the supersymmetric KdV equations. We are also able to write the Lax equations for the corresponding hierarchies in terms of these polynomials.

1. Introduction

The theory of the Korteweg de Vries (KdV) equation is by now very well understood. It is one of the simplest completely integrable systems available, and plays a role in many areas of physics and mathematical physics. Recently, the importance of KdV equation in the context of conformal field theory and 2D quantum gravity has been recognized [1]. In the matrix model approach to 2D quantum gravity the Gelfand–Dikii polynomials and the recursion relations they obey play a particularly important role.

Supersymmetric generalizations of the KdV equation have also been of interest recently [2, 3]. One might expect that they will be relevant to the study of superconformal field theory and 2D supergravity. In a recent paper [5], the Gelfand–Dikii polynomials and their recursion relations for the $N = 1$ supersymmetric KdV equation have been worked out. In this paper, we consider the extension of this analysis to $N = 2$ supersymmetric KdV equations.

We will first briefly review the situation of the ordinary KdV equation. Our treatment follows that of Gelfand and Dikii [6] and is centred around the resolvent for the Lax operator. The resolvent is an important object because the KdV

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equation is intimately related to the isospectral deformation of the Lax operator. The Lax operator in question is given by

$$L = \partial^2 - u\,, \tag{1}$$

where $u \equiv u(x)$ and $\partial \equiv \frac{\partial}{\partial x}$ and the resolvent (of $-L$) is given by $R = (-L + \zeta)^{-1}$. The resolvent kernel, given by $R(x, x'; \zeta) = R\delta(x - x')$, is interesting in the coincidence limit $x' \to x$ and admits an asymptotic expansion

$$R(x;\zeta) \equiv \lim_{x' \to x} R(x, x'; \zeta) = \sum_n R_n[u]\zeta^{-n-1/2}\,. \tag{2}$$

The Gelfand–Dikii polynomials $R_n[u]$ are differential polynomials of u (i.e. polynomials of u and its derivatives) and are important for the following reason: Let $L^{n-1/2}$ be a fractional power of the Lax operator (which is a pseudodifferential operator). Then its residue (or coefficient of ∂^{-1}) is given by

$$\operatorname{res} L^{n-1/2} = 2R_n\,. \tag{3}$$

The Lax equation

$$[L, L_+^{n+1/2}] = \frac{\partial}{\partial t_n} L \tag{4}$$

is then equivalent to the non-linear pde $\frac{\partial u}{\partial t_n} = 4\partial R_{n+1}$. In particular, for $n = 1$ with $t \equiv t_1$ we have the KdV equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\{-\partial^3 u + 6u\partial u\}\,. \tag{5}$$

The kernel $R(x;\zeta)$ obeys the important partial differential equation

$$-\partial^3 R + 4(u + \zeta)\partial R + 2(\partial u)R = 0\,. \tag{6}$$

From this equation, one obtains the recursion relation

$$\partial R_{n+1} = \frac{1}{4}\mathscr{D}_2 R_n \tag{7}$$

for the Gelfand-Dikii polynomials, with the operator $\mathscr{D}_2$ being defined as $\mathscr{D}_2 \equiv \partial^3 - 4u\partial - 2(\partial u)$. This operator is a Hamiltonian operator (see, for example, [9]), in that from it one can define a Poisson bracket

$$[u(x), u(x')]_{\mathrm{PB}} = \mathscr{D}_2\delta(x - x')\,. \tag{8}$$

This Poisson structure is commonly known as the second Hamiltonian structure of the KdV equation (the KdV equation is biHamiltonian [7] and the first is defined through the Hamiltonian operator ∂) and is in fact the classical version of the Virasoro algebra [8]. With the recursion relation (7) and the functional relation

$$\frac{\delta}{\delta u}\int R\,dx = \frac{\partial R}{\partial \zeta}\,, \tag{9}$$

one can put the Lax equation (4) into Hamiltonian form:

$$\frac{\partial u}{\partial t_n} = \mathscr{D}_2 \frac{\delta H_n}{\delta u}\,, \tag{10}$$

with the Hamiltonian functionals $H_n \equiv \frac{2}{2n+1} \int R_{n+1}\, dx$. The Hamiltonians H_n are in involution with respect to the Poisson bracket (8) and furnish an infinite number of conserved quantities for the KdV equation, thus indicating complete Hamiltonian integrability.

In the rest of the paper, we will pursue the analogues of the above for $N = 2$ supersymmetric KdV equations. According to Laberge and Mathieu [3], there are two integrable $N = 2$ supersymmetric analogues of the KdV equation, related to the Lax operators

$$L_{(4)} = \partial^2 - 2\Phi D_1 D_2 + (D_2\Phi)D_1 - (D_1\Phi)D_2 - (D_1 D_2 \Phi) - \Phi^2 \,, \tag{11}$$

$$L_{(-2)} = \partial^2 + 2\Phi D_1 D_2 - (D_2\Phi)D_1 + (D_1\Phi)D_2 \,. \tag{12}$$

Here we work in $(1|2)$ superspace with coordinates $X = (x, \theta_1, \theta_2)$. Φ is an even superfield and the superderivatives D_i are defined as $D_i = \frac{\partial}{\partial\theta_i} + \theta_i \frac{\partial}{\partial x}$. The equations are given by

$$\frac{\partial\Phi}{\partial t} = \frac{1}{4}\partial\left\{ -\partial^2\Phi + 3\Phi D_1 D_2 \Phi + \frac{1}{2}(a-1) D_1 D_2 \Phi^2 + a\Phi^3 \right\}, \tag{13}$$

with $a = 4$ and $a = -2$, respectively. They were obtained by starting with a general $N = 2$ supersymmetric L, and determining the coefficients to make the Lax equation $[L, L_+^{3/2}] = \frac{\partial L}{\partial t}$ consistent. They have in common the property that they can be put into Hamiltonian form, with the "second" Hamiltonian structure being determined by the operator

$$\mathscr{D}_2 = \partial D_1 D_2 + 2\Phi\partial - (D_1\Phi)D_1 - (D_2\Phi)D_2 + 2(\partial\Phi) \,. \tag{14}$$

The Poisson bracket

$$[\Phi(X), \Phi(X')]_{\mathrm{PB}} = \mathscr{D}_2 \delta(X, X') \,, \tag{15}$$

where $\delta(X, X')$ is the $N = 2$ supersymmetric delta function, is the classical version of the $N = 2$ superconformal algebra.

Our key result will be to obtain an analogue of the differential equation (6) satisfied by the analogues of the coincidence limit of the resolvent kernel (2). We will prove this result using heat kernel techniques. From these differential equations follow recursion relations for the analogues of the Gelfand–Dikii polynomials. We also obtain functional relations which allow us to prove Hamiltonian integrability of the $N = 2$ super KdV equations. We were led to the crucial differential equations by noting relations amongst the first few Gelfand–Dikii polynomials, which were analogous to the $N = 0$ [6] and $N = 1$ cases [5]. After we completed our work, we became aware of the paper by Oevel and Popowicz [4] who also obtained these recursion relations. They noted that the $N = 2$ super KdV equations are bi-Hamiltonian (*with the meaning of Hamiltonian structure suitably generalized*) and constructed the recursion relations from the Hamiltonian structures, which they obtained by a Dirac reduction of the Gelfand–Dikii brackets defined on suitably extended Lax operators. We also describe the relation of the

Gelfand–Dikii polynomials to the formulation of the super KdV equations in terms of Lax pairs.

2. Resolvent Expansion for the Lax operator

Assuming only that the Lax operator has the form $L = \partial^2 + \dots$, we have, following Gelfand and Dikii [6], the following expansion for the resolvent of $-L$:

$$(-L+\zeta)^{-1} = \sum_{\substack{a,b \in Z \\ a+b \text{ even}}} \sum_{i,j=0}^{1} B_{a,b}^{(i,j)}[\Phi] D_1^i D_2^j \partial^a (-\partial^2 + \zeta)^{-1-\frac{a+b}{2}} . \tag{16}$$

The coefficients $B_{a,b}^{(i,j)}[\Phi]$ are to be determined, and will be found to be non-zero only for $a + b \geqq 0$. Recursion relations for $B_{a,b}^{(i,j)}[\Phi]$ will be given later on. The expansion is to be thought of as a formal one; the issue of convergence is not important as only algebraic properties are relevant to its connection with integrable systems.

Associated with the expansion (16) for the resolvent is a corresponding asymptotic expansion for the resolvent kernel $R(X, X'; \zeta)$, satisfying $(-L + \zeta)R(X, X'; \zeta) = \delta(X, X')$. The relevant asymptotic expansion is

$$R(X, X'; \zeta) = \sum_{\substack{a,b \in Z \\ a+b \text{ even}}} \sum_{i,j=0}^{1} B_{a,b}^{(i,j)}[\Phi] D_1^i D_2^j \partial^a R_0(X, X'; \zeta)^{1+\frac{a+b}{2}} , \tag{17}$$

where $R_0(X, X', \zeta)$ satisfies the equation $(-\partial^2 + \zeta)R_0(X, X'; \zeta) = \delta(X, X')$. Using the integral representation

$$\delta(X, X') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x - x' - \theta_1\theta_1' - \theta_2\theta_2')}(\theta_1 - \theta_1')(\theta_2 - \theta_2') \tag{18}$$

for the $N = 2$ supersymmetric delta function, we can explicitly evaluate $R_0(X, X'; \zeta)$. A closely related object to $R(X, X'; \zeta)$ is the heat kernel $K(X, X'; \xi)$ defined by the heat equation

$$\frac{\partial}{\partial \xi} K(X, X'; \xi) = LK(X, X'; \xi) \tag{19}$$

with the boundary condition $\lim_{X' \to X} K(X, X'; \xi) = \delta(X, X')$. The two are related by the Laplace transformation

$$R(X, X'; \zeta) = \int_0^{\infty} d\xi e^{-\xi\zeta} K(X, X'; \xi) . \tag{20}$$

The relation of the generalized Gelfand–Dikii polynomials to the formulation of the super KdV equations in terms of Lax pairs is most natural in terms of the resolvent, but we find it more convenient to use the heat kernel in proving results.

Of special interest is the "diagonal" or the coincidence limit of the resolvent,

$$R(X; \zeta) \equiv \lim_{X' \to X} R(X, X'; \xi) . \tag{21}$$

In the limit $X' \to X$, only $D_1 D_2 \partial^n R_0(X, X'; \zeta)^m$ for even n survives. We eventually find that

$$R(X; \zeta) = \frac{1}{2} \sum_{m, n \in Z} B^{(1,1)}_{2m, 2n}[\Phi] \binom{n - \frac{1}{2}}{m + n} \zeta^{-n-\frac{1}{2}} . \tag{22}$$

The Gelfand–Dikii polynomials $R_n[\Phi]$, defined by $R(X; \zeta) = \sum_{n \in Z} R_n[\Phi] \zeta^{-n-\frac{1}{2}}$ are then given by

$$R_n[\Phi] = \frac{1}{2} \sum_{m \in Z} B^{(1,1)}_{2m, 2n}[\Phi] \binom{n - \frac{1}{2}}{m + n} . \tag{23}$$

Following the work of McArthur [5] on the $N = 1$ super KdV equation, we are also interested in the following objects related to $R(X; \zeta)$:

$$R^{(1,0)}(X; \zeta) \equiv \lim_{X' \to X} D_1 R(X, X'; \zeta) , \tag{24}$$

$$R^{(0,1)}(X; \zeta) \equiv \lim_{X' \to X} D_2 R(X, X'; \zeta) , \tag{25}$$

$$R^{(1,1)}(X; \zeta) \equiv \lim_{X' \to X} D_1 D_2 R(X, X'; \zeta), \text{ etc.} \tag{26}$$

Using the same arguments which led to Eq. (23), the corresponding generalized Gelfand–Dikii polynomials defined by $R^{(i,j)}(X; \zeta) = \sum_n R^{(i,j)}_n[\Phi] \zeta^{-n-\frac{1}{2}}$ are obtained explicitly as

$$\begin{aligned} R^{(1,0)}_n[\Phi] &= D_1 R_n[\Phi] - \frac{1}{2} \sum_{m \in Z} B^{(0,1)}_{2m, 2n}[\Phi] \binom{n - \frac{1}{2}}{m + n} , \\ R^{(0,1)}_n[\Phi] &= D_2 R_n[\Phi] + \frac{1}{2} \sum_{m \in Z} B^{(1,0)}_{2m, 2n}[\Phi] \binom{n - \frac{1}{2}}{m + n} , \\ R^{(1,1)}_n[\Phi] &= - D_1 D_2 R_n[\Phi] + D_1 R^{(1,0)}_n[\Phi] - D_2 R^{(0,1)}_n[\Phi] \\ &\quad + \frac{1}{2} \sum_{m \in Z} B^{(0,0)}_{2m, 2n}[\Phi] \binom{n - \frac{1}{2}}{m + n} . \end{aligned} \tag{27}$$

3. Resolvent Analysis for the Equation Associated to $L_{(-2)}$

In this section, we have $L = L_{(-2)}$ for the Lax operator, and the relevant $N = 2$ super KdV equation is

$$\frac{\partial \Phi}{\partial t} = \frac{1}{4} \partial \{ - \partial^2 \Phi - 3(D_1 \Phi)(D_2 \Phi) - 2\Phi^3 \} . \tag{28}$$

It is easy to develop recursion relations for the coefficients $B^{(i,j)}_{a,b}$ in the expansion (16) of the resolvent of $-L$. These recursion relations are obtained by multiplying Eq. (16) by $(- L + \zeta)$ on both sides. Although cumbersome they are sufficient for our purposes. Later on, we will see that more elegant recursion relations for the generalized Gelfand–Dikii polynomials are possible.

The recursion relations for $B^{(i,j)}_{a,b}$ are given by

$$B^{(0,0)}_{a,b+2} = LB^{(0,0)}_{a,b} + 2\partial B^{(0,0)}_{a-1,b+1} + [2\Phi D_2 + (D_2\Phi)]B^{(1,0)}_{a-1,b+1} - [(D_1\Phi) + 2\Phi D_1]B^{(0,0)}_{a-1,b+1} - 2\Phi B^{(1,1)}_{a-2,b+2}\,, \tag{29}$$

$$B^{(0,1)}_{a,b+2} = LB^{(0,1)}_{a,b} + 2\partial B^{(0,1)}_{a-1,b+1} - [2\Phi D_2 + (D_2\Phi)]B^{(1,1)}_{a-1,b+1} + [(D_1\Phi) + 2\Phi D_1]B^{(0,0)}_{a,b} - 2\Phi B^{(1,0)}_{a-1,b+1}\,, \tag{30}$$

$$B^{(1,0)}_{a,b+2} = LB^{(1,0)}_{a,b} + 2\partial B^{(1,0)}_{a-1,b+1} - [2\Phi D_2 + (D_2\Phi)]B^{(0,0)}_{a,b} + [(D_1\Phi) + 2\Phi D_1]B^{(1,1)}_{a-1,b+1} + 2\Phi B^{(0,1)}_{a-1,b+1}\,, \tag{31}$$

$$B^{(1,1)}_{a,b+2} = LB^{(1,1)}_{a,b} + 2\partial B^{(1,1)}_{a-1,b+1} + [2\Phi D_2 + (D_2\Phi)]B^{(0,1)}_{a,b} + [(D_1\Phi) + 2\Phi D_1]B^{(1,0)}_{a,b} + 2\Phi B^{(0,0)}_{a,b}\,, \tag{32}$$

with the starting conditions $B^{(i,j)}_{a,b} = 0$ for all $a < 0$ or $b < -2$, for all i, j, and $B^{(0,0)}_{0,0} = 1$, $B^{(i,j)}_{0,0} = 0$ for $i = 1$ or $j = 1$. Using these recursion relations it is straightforward but fairly tedious to obtain the first few generalized Gelfand–Dikii polynomials:

$$R_0 = 0,\; R^{(1,1)}_0 = \frac{1}{2}\,,$$

$$R_1 = \frac{1}{2}\Phi,\quad R^{(1,0)}_0 = \frac{1}{4}D_1\Phi,\quad R^{(0,1)}_1 = \frac{1}{4}D_2\Phi,\quad R^{(1,1)}_1 = \frac{1}{4}\Phi^2\,,$$

$$R_2 = \frac{1}{8}\{\partial^2\Phi + 3(D_1\Phi)(D_2\Phi) + 2\Phi^3\}\,,$$

$$R^{(1,0)}_2 = \frac{1}{16}\{\partial^2 D_1\Phi + 3(D_2\Phi)\partial\Phi - 3(D_1\Phi)(D_1D_2\Phi) + 6(D_1\Phi)\Phi^2\}\,,$$

$$R^{(0,1)}_2 = \frac{1}{16}\{\partial^2 D_2\Phi - 3(D_1\Phi)\partial\Phi + 3(D_2\Phi)(D_1D_2\Phi) + 6(D_2\Phi)\Phi^2\}\,,$$

$$R^{(1,1)}_2 = \frac{1}{16}\{-(\partial\Phi)^2 + 2\Phi\partial^2\Phi + 3\Phi^4 + 6\Phi(D_1\Phi)(D_2\Phi) + (D_2\Phi)(\partial D_2\Phi) + (D_1\Phi)(\partial D_1\Phi)\}\,,$$

$$R_3 = \frac{1}{32}\{\partial^4\Phi + 6\Phi^5 + 10\Phi^2\partial^2\Phi + 10\Phi(\partial\Phi)^2 + 30\Phi^2(D_1\Phi)(D_2\Phi) + 5(\partial D_1\Phi)(\partial D_2\Phi) - 10(D_1\Phi)(D_2\Phi)(D_1D_2\Phi) + 5(\partial^2 D_1\Phi)(D_2\Phi) + 5(D_1\Phi)(\partial^2 D_2\Phi)\}\,. \tag{33}$$

Alternatively, these can be obtained by evaluating the coincidence limits of the heat kernel. Representing the heat kernel in the form $K(X, X', \xi) = e^{\xi L}\delta(X, X')$, using the representation (18) for the delta function and moving the exponential factor in the delta function to the left through $e^{\xi L}$, the $B_{a,b}^{(i,j)}$ are related to the coefficients of the expansion of the resulting exponential in the powers $\frac{1}{\left(\frac{a+b}{2}\right)!}\xi^{\frac{b}{2}}(ik)^a D_1^i D_2^j$.

Relationships amongst the generalized Gelfand–Dikii polynomials can already be discerned. In fact we have the following theorem:

Theorem 1. *The coincidence limits of the resolvent kernel $R(X;\zeta)$ and its derivatives $R^{(i,j)}(X;\zeta)$ defined in Eqs.* (21) *and* (24)–(26) *satisfy the following differential equations*[1]:

$$
\begin{aligned}
D_1 R - 2R^{(1,0)} &= 0\,, \\
D_2 R - 2R^{(0,1)} &= 0\,, \\
2\partial R^{(1,1)} - 2\Phi\partial R - (D_1\Phi)D_1 R - (D_2\Phi)D_2 R &= 0\,, \\
4\zeta\partial R - \mathscr{D}_2(2R^{(1,1)} - D_1 D_2 R) &= 0\,,
\end{aligned}
\tag{34}
$$

and the following relation:

$$
\frac{\delta R}{\delta\Phi} + \frac{\partial}{\partial\zeta}(2R^{(1,1)} - D_1 D_2 R) = 0\,,
$$

where $\mathscr{D}_2$ is the Hamiltonian operator (14).

(These differential equations are the analogues of Eqs. (6) and (9) respectively for the ordinary KdV equation). The proof of the theorem is based on heat kernel manipulations similar to those in [5] and is provided in the appendix. The following recursion relations result from the previous theorem:

$$
\partial R_n^{(1,1)} = \frac{1}{2}(2\Phi\partial + (D_1\Phi)D_1 + (D_2\Phi)D_2)R_n\,,
\tag{35}
$$

$$
\partial R_{n+1} = \frac{1}{4}\mathscr{D}_2(2R_n^{(1,1)} - D_1 D_2 R_n)\,.
\tag{36}
$$

We also obtain the functional relation

$$
\frac{\delta R_{n+1}}{\delta\Phi} = \frac{(2n+1)}{2}(2R_n^{(1,1)} - D_1 D_2 R_n)\,.
\tag{37}
$$

The super KdV equation (28) is

$$
\frac{\partial\Phi}{\partial t} = -\,2\partial R_2\,,
$$

and is the first equation of the super KdV hierarchy (see Sect. 5)

$$
\frac{\partial\Phi}{\partial t_n} = -\,2\partial R_{n+1}\,.
\tag{38}
$$

[1] $\frac{\delta R}{\delta\Phi}$ is defined by $\delta\int dX\, R(X;\zeta) = \int dX\,\delta\Phi(X)\frac{\delta R(X;\zeta)}{\delta\Phi(X)}$.

Using (37) and the Poisson bracket (8) these equations can be put into the Hamiltonian form

$$\frac{\partial \Phi}{\partial t_n} = [\Phi, H_n]_{\mathrm{PB}} = \mathscr{D}_2 \frac{\delta H_n}{\delta \Phi}, \tag{39}$$

where H_n is given by

$$H_n = -\frac{1}{2n+1} \int dX\, R_{n+1}[\Phi] \, . \tag{40}$$

We have in fact the following theorem on the Hamiltonian integrability of the $N = 2$ super KdV equation:

Theorem 2. *The Hamiltonian functionals* (40) *are in involution with respect to the Poisson bracket* $[\,,]_{\mathrm{PB}}$, *and form an infinite number of conserved quantities for the* $N = 2$ *super* KdV *equation* (28).

To prove the theorem we note that

$$[H_n, H_m]_{\mathrm{PB}} = \int dX \frac{\delta H_n}{\delta \Phi} \mathscr{D}_2 \frac{\delta H_m}{\delta \Phi} = \int dX (2R_n^{(1,1)} - D_1 D_2 R_n) \partial R_{m+1} \, . \tag{41}$$

We also have

$$[H_{n-1}, H_{m+1}]_{\mathrm{PB}} = -\int dX (2R_{m+1}^{(1,1)} - D_1 D_2 R_{m+1}) \partial R_n \, , \tag{42}$$

using the anti-symmetry of the Poisson bracket. It is then easy to show, using the recursion relations (35) and (36), that the difference between the integrands of (41) and (42) is a total derivative, and hence (41) and (42) are equal. Iterating, we find

$$[H_n, H_m]_{\mathrm{PB}} = [H_{n-1}, H_{m+1}]_{\mathrm{PB}} = \cdots = [H_0, H_{m+n}]_{\mathrm{PB}} = 0 \, . \tag{43}$$

4. Resolvent Analysis for the Equation Associated to $L_{(4)}$

In this section we have $L = L_{(4)} = -(D_1 D_2 + \Phi)^2$, and the corresponding $N = 2$ super KdV equation is

$$\frac{\partial \Phi}{\partial t} = \frac{1}{4} \partial \{ -\partial^2 \Phi + 6\Phi D_1 D_2 \Phi + 3(D_1 \Phi)(D_2 \Phi) + 4\Phi^3 \} \, . \tag{44}$$

Recursion relations for the coefficients $B_{a,b}^{(i,j)}$ in the asymptotic expansion for the resolvent can be developed as in the previous section. Using these recursion relations it is easy to obtain the first few (generalised) Gelfand–Dikii polynomials. They are given by

$$R_1 = -\frac{1}{2}\Phi, \; R_1^{(1,0)} = -\frac{1}{4} D_1 \Phi, \; R_1^{(0,1)} = -\frac{1}{4} D_2 \Phi, \; R_1^{(1,1)} = -\frac{1}{4} D_1 D_2 \Phi \, ,$$

$$R_2 = \frac{1}{8} \{ -\partial^2 \Phi + 4\Phi^3 + 6\Phi D_1 D_2 \Phi + 3(D_1 \Phi)(D_2 \Phi) \} \, ,$$

$$R_2^{(1,0)} = \frac{1}{16}\{-\partial^2 D_1\Phi + 12\Phi^2 D_1\Phi + 3(D_1\Phi)D_1D_2\Phi + 6\Phi\partial D_2\Phi + 6(\partial\Phi)D_2\Phi\},$$

$$R_2^{(0,1)} = \frac{1}{16}\{-\partial^2 D_2\Phi + 12\Phi^2 D_2\Phi + 3(D_2\Phi)D_1D_2\Phi - 6\Phi\partial D_1\Phi - 6(\partial\Phi)D_1\Phi\},$$

$$R_2^{(1,1)} = \frac{1}{16}\{-\partial^2 D_1D_2\Phi - 6\Phi\partial^2\Phi + 3(D_1D_2\Phi)^2 - 3(\partial\Phi)^2 - 2(D_1\Phi)\partial D_1\Phi - 2(D_2\Phi)\partial D_2\Phi + 12\Phi^2 D_1D_2\Phi + 18\Phi(D_1\Phi)(D_2\Phi)\}. \tag{45}$$

We have the following analogue of Theorem (1) for the present Lax operator L:

Theorem 3. *The coincidence limits of the resolvent kernel and its derivatives satisfy the following differential equations:*

$$\begin{gathered} D_1R - 2R^{(1,0)} = 0, \\ D_2R - 2R^{(0,1)} = 0, \\ 2\partial R^{(1,1)} - \partial D_1D_2R + (D_1\Phi)D_1R + (D_2\Phi)D_2R = 0, \\ 2\zeta\partial R + \mathscr{D}_2(R^{(1,1)} + \Phi R) = 0, \end{gathered} \tag{46}$$

and the following relations:

$$\frac{\delta R}{\delta\Phi} - 2\frac{\partial}{\partial\zeta}(R^{(1,1)} + \Phi R) = 0,$$

$$\frac{\delta}{\delta\Phi}(R^{(1,1)} + \Phi R) + \left(1 + 2\zeta\frac{\partial}{\partial\zeta}\right)R = 0,$$

where $\mathscr{D}_2$ is the Hamiltonian operator (14).

The proof is analogous to that of Theorem (1) and is also provided in the appendix. The following recursion relations for the Gelfand–Dikii polynomials follow from Theorem (3):

$$2\partial(R_n^{(1,1)} + \Phi R_n) = \mathscr{D}_2 R_n, \tag{47}$$

$$2\partial R_{n+1} = -\mathscr{D}_2(R_n^{(1,1)} + \Phi R_n), \tag{48}$$

as well as the functional relations:

$$\frac{\delta R_{n+1}}{\delta\Phi} = -(2n+1)(R_n^{(1,1)} + \Phi R_n), \tag{49}$$

$$\frac{\delta}{\delta\Phi}(R_n^{(1,1)} + \Phi R_n) = 2nR_n. \tag{50}$$

The analogue of Eq. (38) is

$$\frac{\partial\Phi}{\partial t_n} = 2\partial R_{n+1}, \tag{51}$$

and can be put into Hamiltonian form

$$\frac{\partial \Phi}{\partial t_n} = [\Phi, H_n]_{\mathrm{PB}} , \tag{52}$$

where H_n for $n = 1, 2, \ldots$ are given by

$$H_n = \frac{1}{2n+1} \int dX\, R_{n+1} . \tag{53}$$

In fact, for the $N = 2$ super KdV equation (44) there are twice as many symmetries and conservation laws as for the equation in the previous section [3]. Defining

$$H_{n-\frac{1}{2}} = \frac{1}{2n} \int dX (R_n^{(1,1)} + \Phi R_n) , \tag{54}$$

one can show, as in the proof of Theorem (2), that

$$[H_n, H_m]_{\mathrm{PB}} = [H_{n-1}, H_{m+1}]_{\mathrm{PB}} \tag{55}$$

$$[H_n, H_{m-\frac{1}{2}}]_{\mathrm{PB}} = [H_{n-1}, H_{m+\frac{1}{2}}]_{\mathrm{PB}} \tag{56}$$

$$[H_{n-\frac{1}{2}}, H_{m-\frac{1}{2}}]_{\mathrm{PB}} = [H_{n-\frac{3}{2}}, H_{m+\frac{1}{2}}]_{\mathrm{PB}} , \tag{57}$$

which on iteration establishes the following result:

Theorem 4. *The Hamiltonian functionals* (53) *and* (54) *are in involution with respect to the Poisson bracket* $[\,,]_{\mathrm{PB}}$, *and form an infinite number of conserved quantities for the* $N = 2$ *super* KdV *equation* (44).

It also follows from (55)–(57) that the flows defined by the equations

$$\frac{\partial \Phi}{\partial t_{n-\frac{1}{2}}} = [\Phi, H_{n-\frac{1}{2}}]_{\mathrm{PB}} = 2\partial(R_n^{(1,1)} + \Phi R_n) \tag{58}$$

are integrable and commute with those of (51). Together they can be regarded as forming the $N = 2$ super KdV hierarchy associated to $L_{(4)}$, which thus contains twice as many equations as the one associated to $L_{(-2)}$.

5. Relation to Lax Pairs

In this section, we relate the Gelfand–Dikii polynomials to the formulation of the super KdV hierarchy in the form of Lax equations $\frac{\partial}{\partial t_n} L = [L, L_+^{n+1/2}]$. The particular form of the expansion (16) chosen allows easy determination of fractional powers of L. In particular, for $n \in Z_+$ we have

$$L^{n-1/2} = \sum_{\substack{a,b \in Z \\ a+b \text{ even}}} \sum_{i,j=0}^{1} B_{a,b}^{(i,j)}[\Phi] \begin{pmatrix} n - \frac{1}{2} \\ \frac{a+b}{2} \end{pmatrix} D_1^i D_2^j \partial^{2n-1-b} . \tag{59}$$

Equation (59) is obtained by multiplying Eq. (16) by $\zeta^{n-1/2}$ and integrating with respect to ζ over a contour enclosing the entire spectrum of $-L$ and $-\partial^2$. It was in this context that fractional powers of operators first entered the theory of integrable systems [6]. Note that fractional powers need not be unique; recall that $L_{(4)}$ has the alternative "square root" of $i(D_1 D_2 + \Phi)$. However, Eq. (59) picks out the square root of the form $\partial + \ldots$, analogous to the ordinary (non-super) case.

The generalized Gelfand–Dikii polynomials are related to the fractional powers of L in the following way: Let $B \equiv L^{n-\frac{1}{2}} = B_+ + B_-$ be the split into differential operator and "integral operator" parts. Let the first few terms in B_- be

$$B_- = b_1 D_1 \partial^{-1} + b_2 D_2 \partial^{-1} + b_3 D_1 D_2 \partial^{-1} + b_4 \partial^{-1} + \cdots . \tag{60}$$

Then from Eqs. (59), (23) and (27) we have

$$\begin{aligned} b_1 &= 2(-D_2 R_n + R_n^{(0,1)}) , \\ b_2 &= 2(D_1 R_n - R_n^{(1,0)}) , \\ b_3 &= 2R_n , \\ b_4 &= 2(R_n^{(1,1)} + D_1 D_2 R_n - D_1 R_n^{(0,1)} + D_2 R_n^{(1,0)}) . \end{aligned} \tag{61}$$

Note that b_3 is the $N = 2$ super residue [3] of B. The relation between the super residue and R_n straightforwardly generalizes Eq. (3) for the ordinary KdV equation.

Since $[L, B_+]$ is a differential operator and $[L, B_+ + B_-] = 0$, $[L, B_+]$ is determined by the first few terms of B_-. The following lemma is readily established:

Lemma 1. *Let $B = L^{n-\frac{1}{2}}$ and let its integral operator part be given by $B_- = b_1 D_1 \partial^{-1} + b_2 D_2 \partial^{-1} + b_3 D_1 D_2 \partial^{-1} + b_4 \partial^{-1} + \cdots$. Then the Lax equation $[L, B_+] = \dfrac{\partial L}{\partial t}$ is consistent if and only if the following systems of equations are consistent:*

For $L = L_{(-2)}$,

$$\frac{\partial \Phi}{\partial t} = -\partial b_3 ,$$

$$\partial(2b_1 + D_2 b_3) = 2\Phi(D_1 b_3 - 2b_2) ,$$

$$\partial(D_1 b_3 - 2b_2) = -2\Phi(2b_1 + D_2 b_3) ,$$

$$\partial b_4 + \Phi D_2 b_1 - \Phi D_1 b_2 + (D_2 \Phi) b_1 - (D_1 \Phi) b_2 = 0 .$$

For $L_{(4)}$,

$$\frac{\partial \Phi}{\partial t} = \partial b_3 ,$$

$$\partial(2b_1 + D_2 b_3) = 2\Phi(2b_2 - D_1 b_3) ,$$

$$\partial(D_1 b_3 - 2b_2) = 2\Phi(2b_1 + D_2 b_3) ,$$

$$2\partial b_4 + \partial D_1 b_1 + \partial D_2 b_2 + (D_1 \Phi) D_1 b_3 + (D_2 \Phi) D_2 b_3 = 0 .$$

Letting $B = L^{n-1/2}$, with the identifications of the coefficients b_i with the generalized Gelfand–Dikii polynomials as in Eq. (61), and using the relations of Theorems (1) and (3), one readily verifies the following result:

Theorem 5. *The Lax equation* $[L, L_+^{n+1/2}] = \frac{\partial}{\partial t_n} L$, *for* $n = 1, 2, \ldots$, *is equivalent to the equation* $\frac{\partial \Phi}{\partial t_n} = \mp 2\partial R_{n+1}$, *where* R_n *is the Gelfand–Dikii polynomial associated to the Lax operator* L *and the signs* $\mp$ *apply to* $L = L_{(-2)}$ *and* $L = L_{(4)}$ *respectively.*

For $L_{(-2)}$, the above Lax equations constitute the equations of the hierarchy corresponding to the $N = 2$ super KdV equation. For $L_{(4)}$, there is more.

Since $-L_{(4)}$ is a perfect square, we can in fact use its square root $\Lambda = D_1 D_2 + \Phi$ as the Lax operator and thus obtain a "minimal" description of the corresponding Lax hierarchy. Analogous to Lemma (1) is:

Lemma 2. *Let* C *be a power of* Λ *and let its integral operator part be given by* $C_- = c_1 D_1 \partial^{-1} + c_2 D_2 \partial^{-1} + c_3 D_1 D_2 \partial^{-1} + \cdots$. *Then the Lax equation* $[\Lambda, C_+] = \frac{\partial \Lambda}{\partial t}$ *is consistent if and only if the following system of equations is consistent*:

$$\frac{\partial \Phi}{\partial t} = \partial c_3\,, \quad D_1 c_3 = 2c_2\,, \quad D_2 c_3 = -2c_1\,. \tag{62}$$

Equivalent to Theorem (5) for $L_{(4)}$ is the following:

Theorem 6. *The Lax equation* $[\Lambda, L_{(4)+}^{n+1/2}] = \frac{\partial}{\partial t_n} \Lambda$ *for* $n = 1, 2, \ldots$ *is equivalent to the equation* $\frac{\partial \Phi}{\partial t_n} = 2\partial R_{n+1}$, *where* R_n *is the Gelfand–Dikii polynomial associated to* $L_{(4)}$.

As has already been noted, for this $N = 2$ super KdV equation, there are twice as many symmetries and conservation laws [3] as for that associated with $L_{(-2)}$. The extra flows are given by the following:

Theorem 7. *The Lax equation* $[\Lambda, \{L_{(4)}^{n-1/2} \Lambda\}_+] = \frac{\partial}{\partial t_{n-\frac{1}{2}}} \Lambda$ *is equivalent to the equation* $\frac{\partial \Phi}{\partial t_{n-\frac{1}{2}}} = 2\partial (R_n^{(1,1)} + \Phi R_n)$.

Theorem (7) is proved by applying Lemma (2) and noting that

$$(L^{n-1/2}\Lambda)_+ = -D_2(R_n^{(1,1)} + \Phi R_n) D_1 \partial^{-1} + D_1(R_n^{(1,1)} + \Phi R_n) D_2 \partial^{-1} + 2(R_n^{(1,1)} + \Phi R_n) D_1 D_2 \partial^{-1} + \cdots\,,$$

which can be established using Eqs. (59), (27) and (v) in the appendix.

6. Conclusion

In this paper, we have developed the Gelfand–Dikii formalism for the $N = 2$ supersymmetric KdV hierarchies for which Lax operators are known [3]. The recursion relations and functional relations satisfied by the generalized Gelfand–Dikii polynomials have been established, and the relationship between the Gelfand–Dikii polynomials and the Lax pair formulation of the hierarchies has been analyzed. Given the importance of the ordinary Gelfand–Dikii polynomials in the matrix model and topological approaches to two-dimensional quantum gravity [1], these results could be useful should a super-analogue of matrix models ever be found.

Appendix

Here, Theorems (1) and (3) are proved. L will denote either $L_{(-2)}$ or $L_{(4)}$. In both cases, L is self-adjoint, in that $\int dX\,\psi(X)L\chi(X) = \int dX(L\psi(X))\chi(X)$ for scalar superfields $\psi(X)$ and $\chi(X)$. We choose to work with the heat kernel $K(X, X'; \xi)$ defined by the heat equations

$$0 = \left(\frac{\partial}{\partial\xi} - L\right)K(X, X'; \xi) = \left(\frac{\partial}{\partial\xi} - L'\right)K(X, X'; \xi)$$

with the boundary condition $\lim_{X' \to X} K(X, X'; \xi) = \delta(X, X')$, where a prime on an operator denotes that it acts on the argument X' rather than on X. As already noted, the heat kernel is related to the resolvent kernel by a Laplace transform. We adopt the notation (23)–(26) with R replaced by K for the corresponding coincidence limits of the heat kernel.

Identities Common to $L_{(-2)}$ and $L_{(4)}$. The following identities rely on the self-adjointness of $L_{(-2)}$ and $L_{(4)}$ but not on their specific form:

(i) $$K^{(1,0)}(X; \xi) = \tfrac{1}{2}D_1 K(X; \xi)\,,$$

$$K^{(0,1)}(X; \xi) = \tfrac{1}{2}D_2 K(X; \xi)\,,$$

(ii) $\lim_{X' \to X} \partial K(X, X'; \xi) = \tfrac{1}{2}\partial K(X; \xi)\,,$

(iii) for an arbitrary operator $\mathbf{O}$ acting on the argument X

$$\lim_{X' \to X} (\partial + \partial')\mathbf{O}K(X, X'; \xi) = \partial \lim_{X' \to X} \mathbf{O}K(X, X'; \xi)\,,$$

$$\lim_{X' \to X} (D_i + D_i')\mathbf{O}K(X, X'; \xi) = D_i \lim_{X' \to X} \mathbf{O}K(X, X'; \xi)\,,$$

$$\lim_{X' \to X} (\partial - \partial')K(X, X'; \xi) = 0\,,$$

$$\lim_{X' \to X} (D_i - D_i')K(X, X'; \xi) = 0\,,$$

$$\lim_{X' \to X} (D_1 + D'_1)(D_2 + D'_2)K(X, X'; \xi) = D_1 D_2 K(X; \xi) ,$$

$$\lim_{X' \to X} (D_1 + D'_1)(D_2 - D'_2)K(X, X'; \xi) = 0 ,$$

$$\lim_{X' \to X} (D_1 - D'_1)(D_2 + D'_2)K(X, X'; \xi) = 0 ,$$

$$\lim_{X' \to X} (D_1 - D'_1)(D_2 - D'_2)K(X, X'; \xi) = 4K^{(1,1)}(X; \xi) - D_1 D_2 K(X; \xi) ,$$

$$\lim_{X' \to X} (D_i - D'_i)(D_i - D'_i)K(X, X'; \xi) = \partial K(X; \xi) \text{ (no sum on } i) ,$$

$$\lim_{X' \to X} (\partial - \partial')(D_i - D'_i)K(X, X'; \xi) = 4 \lim_{X' \to X} \partial D_i K(X, X'; \xi) - \partial D_i K(X; \xi) ,$$

$$\lim_{X' \to X} (D_1 - D'_1)(D_1 + D'_1)(D_2 - D'_2)K(X, X'; \xi) = 8 \lim_{X' \to X} \partial D_2 K(X, X'; \xi)$$
$$- 4D_1 K^{(1,1)}(X; \xi) + \partial D_2 K(X; \xi) .$$

Proof of (i) *and* (ii). We use the formal expression $K(X, X'; \xi) = e^{\xi L}\delta(X, X')$ valid in the limit $X' \to X$. Writing $\lim_{X' \to X} D_i e^{\xi L}\delta(X, X')$ in the form $\int dX \delta(X, X') D_i e^{\xi L}\delta(X, X')$, integrating by parts and using the self-adjointness of L, one can show

$$\lim_{X' \to X} D_i e^{\xi L}\delta(X, X') = - \lim_{X' \to X} e^{\xi L} D_i \delta(X, X') .$$

Thus $K^{(1,0)}(X; \xi) = \frac{1}{2} \lim_{X' \to X}[D_1, e^{\xi L}]\delta(X, X')$ (and similarly for $K^{(0,1)}(X; \xi)$). This latter quantity is $\frac{1}{2} D_1 K^{(1,0)}(X; \xi)$ as D_1 acts only on the coefficients in the expansion of the exponential and not on the delta function. The proof of (ii) is the same with D_i replaced by ∂. □

Proof of (iii). All but the last of these identities are proved by integrating the left-hand side over X against a scalar superfield $\psi(X)$, writing $\lim_{X' \to X} F(X, X')$ as $\int dX' F(X, X')\delta(X, X')$ and using repeated integration by parts. The last identity follows using $(D_1 - D'_1)(D_1 + D'_1) = 2(\partial - \partial') - (D_1 + D'_1)(D_1 - D'_1)$. □

Operator-Dependent Identities

(iv) For $L_{(-2)}$, $\lim_{X' \to X} \partial D_i K(X, X'; \xi) = \frac{1}{4} \partial D_i K(X; \xi) - \frac{1}{2} \varepsilon_{ij} D_j K^{(1,1)}(X; \xi)$.

(v) For $L_{(4)}$, $\lim_{X' \to X} \partial D_i K(X, X'; \xi) = - \frac{1}{2} \varepsilon_{ij} D_j K^{(1,1)}(X; \xi) + \frac{1}{2} \varepsilon_{ij} (D_j \Phi) K(X; \xi)$.

Proof of (iv). $D_2 K^{(1,1)}(X; \xi) = \lim_{X' \to X}[D_2, D_1 D_2 e^{\xi L_{(-2)}}]\delta(X, X')$. Using the result $D_1 e^{\xi L_{(-2)}} D_2 = - D_2 e^{\xi L_{(-2)}} D_1$ to be proved below,

$$D_2 K^{(1,1)}(X; \xi) = \lim_{X' \to X} (- 2\partial D_1 e^{\xi L_{(-2)}} + [D_1, \partial e^{\xi L_{(-2)}}])\delta(X, X')$$
$$= - 2 \lim_{X' \to X} \partial D_1 K(X, X'; \xi) + D_1 \lim_{X' \to X} \partial K(X, X'; \xi) .$$

A similar result holds with $1 \leftrightarrow 2$. To prove $D_1 e^{\xi L_{(-2)}} D_2 = - D_2 e^{\xi L_{(-2)}} D_1$, it suffices to prove $(\sigma_1)^{ij} D_i L^n_{(-2)} D_j = 0$, where σ_1 is the usual Pauli matrix. Writing $L_{(-2)} = (i\sigma_2)^{ij} D_i (\frac{1}{2} D_1 D_2 + \Phi) D_j$ and using $D_i D_j = \delta_{ij}\partial + (i\sigma_2)_{ij} D_1 D_2$, then

$(\sigma_1)^{ij} D_i L^n_{(-2)} D_j$ involves two-dimensional traces of the form $\mathrm{Tr}(\sigma_1(i\sigma_2)^m))$ for $0 \leqq m \leqq (2n+1)$. These vanish. □

Proof of (v). $\partial D_1 K(X, X'; \xi) = -D_2(D_1 D_2 + \Phi)K(X, X'; \xi) + D_2 \Phi K(X, X'; \xi)$. Since $(D_1 D_2 + \Phi)e^{\xi L_{(4)}}$ is self-adjoint, using the method of proof of (i) and (ii) it follows that

$$\lim_{X' \to X} D_2(D_1 D_2 + \Phi)K(X, X'; \xi) = \frac{1}{2} D_2 \lim_{X' \to X} (D_1 D_2 + \Phi)K(X, X'; \xi) .$$

Thus

$$\lim_{X' \to X} \partial D_1 K(X, X'; \xi) = -\frac{1}{2} D_2 K^{(1,1)}(X; \xi) + \frac{1}{2}(D_2 \Phi(X))K(X; \xi) ,$$

where (i) has been used. □

The Recursion Relations.

(vi) For $L_{(-2)}$, $\partial(2K^{(1,1)}(X; \xi) - D_1 D_2 K(X; \xi)) = (-\partial D_1 D_2 + 2\Phi\partial + (D_1\Phi)D_1 + (D_2\Phi)D_2)K(X; \xi)$.

(vii) For $L_{(-2)}$, $4\dfrac{\partial}{\partial\xi}\partial K(X; \xi) = \mathscr{D}_2(2K^{(1,1)}(X; \xi) - D_1 D_2 K(X; \xi))$.

(viii) For $L_{(4)}$, $2\partial(K^{(1,1)}(X; \xi) + \Phi K(X; \xi)) = \mathscr{D}_2 K(X; \xi)$.

(ix) For $L_{(4)}$, $2\partial \dfrac{\partial}{\partial\xi} K(X; \xi) = -\mathscr{D}_2(K^{(1,1)}(X; \xi) + \Phi K(X; \xi))$.

Proof of (vi). From the definition of the heat kernel, $L_{(-2)}K(X, X'; \xi) = L'_{(-2)}K(X, X'; \xi)$, so $0 = \lim_{X' \to X}(D_1 - D'_1)(L_{(-2)} - L'_{(-2)})K(X, X'; \xi)$. This is equivalent to

$$\begin{aligned} 0 = \lim_{X' \to X} & (D_1 - D'_1)[(\partial + \partial')(\partial - \partial') \\ & + \frac{1}{2}(\Phi + \Phi')[(D_1 + D'_1)(D_2 - D'_2) + (D_1 - D'_1)(D_2 + D'_2)] \\ & + \frac{1}{2}(\Phi - \Phi')[(D_1 + D'_1)(D_2 + D'_2) + (D_1 - D'_1)(D_2 - D'_2)] \\ & + \frac{1}{2}((D_1\Phi) + (D'_1\Phi'))(D_2 - D'_2) + \frac{1}{2}((D_1\Phi) - (D'_1\Phi'))(D_2 + D'_2) \\ & - \frac{1}{2}((D_2\Phi) + (D'_2\Phi'))(D_1 - D'_1) \\ & - \frac{1}{2}((D_2\Phi) - (D'_2\Phi'))(D_1 + D'_1)]K(X, X'; \xi) , \end{aligned}$$

where $\Phi = \Phi(X)$ and $\Phi' = \Phi(X')$. Using the identities (i)–(iv), one obtains the desired result. □

Proof of (vii). We use

$$4(\partial+\partial')\frac{\partial}{\partial\xi}K(X,X';\xi)=2(\partial+\partial')(L_{(-2)}+L'_{(-2)})K(X,X';\xi)\,.$$

Subtracting from this the identity

$$0=(\partial-\partial')(L_{(-2)}-L'_{(-2)})K(X,X';\xi)$$

yields

$$\begin{aligned}4(\partial+\partial')\frac{\partial}{\partial\xi}K(X,X';\xi)&=[(\partial+\partial')^3+\partial D_1\Phi D_2+3\partial D'_1\Phi' D'_2\\&\quad+3\partial' D_1\Phi D_2+\partial' D'_1\Phi' D_2-(1\leftrightarrow 2)]K(X,X';\xi)\,.\end{aligned}$$

Using

$$\begin{aligned}(\partial D_1\Phi)D_2+(\partial' D'_1\Phi')D'_2&=\frac{1}{2}((\partial D_1\Phi)+(\partial' D'_1\Phi))(D_2+D'_2)\\&\quad+\frac{1}{2}((\partial D_1\Phi)-(\partial' D'_1\Phi'))(D_2-D'_2)\end{aligned}$$

and

$$\begin{aligned}&(\partial\Phi)D_1D_2+(\partial'\Phi')D'_1D'_2\\&\quad=\frac{1}{4}((\partial\Phi)+(\partial'\Phi'))[(D_1+D'_1)(D_2+D'_2)+(D_1-D'_1)(D_2-D'_2)]\\&\qquad+\frac{1}{4}((\partial\Phi)-(\partial'\Phi'))[(D_1+D'_1)(D_2-D'_2)+(D_1-D'_1)(D_2+D'_2)]\end{aligned}$$

and

$$\begin{aligned}&(D_1\Phi)\partial D_2+3(D'_1\Phi')\partial D'_2+3(D_1\Phi)\partial' D_2+(D'_1\Phi')\partial' D'_2\\&=((D_1\Phi)+(D'_1\Phi'))[(\partial+\partial')(D_2+D'_2)-\frac{1}{2}(\partial-\partial')(D_2-D'_2)]\\&\quad+((D_1\Phi)-(D'_1\Phi'))[(\partial+\partial')(D_2-D'_2)-\frac{1}{2}(\partial-\partial')(D_2+D'_2)]\end{aligned}$$

and

$$\begin{aligned}&\Phi\partial D_1D_2+3\Phi'\partial D'_1D'_2+3\Phi\partial' D_1D_2+\Phi'\partial' D'_1D'_2\\&\quad=\left[\frac{1}{2}(\Phi+\Phi')(\partial+\partial')-\frac{1}{4}(\Phi-\Phi')(\partial-\partial')\right][(D_1+D'_1)(D_2+D'_2)\\&\qquad+(D_1-D'_1)(D_2-D'_2)]+\left[\frac{1}{2}(\Phi-\Phi')(\partial+\partial')-\frac{1}{4}(\Phi+\Phi')(\partial-\partial')\right]\\&\qquad\times[(D_1+D'_1)(D_2-D'_2)+(D_1-D'_1)(D_2+D'_2)]\,,\end{aligned}$$

taking the limit $X'\to X$ and using (i)–(iv) and (vi) gives the required result. □

Proof of (viii) and (ix). These results for $L_{(4)}$ are proved in the same manner as the corresponding results (vi) and (vii) for $L_{(-2)}$; there are only minor modifications due to the different forms of the operators. □

Functional Relations. With $\frac{\delta K(X;\xi)}{\delta \Phi(X)}$ defined by $\delta \int dX K(X;\xi) = \int dX \delta\Phi(X) \times \frac{\delta K(X;\xi)}{\delta \Phi(X)}$, we have:

(x) For $L_{(-2)}$, $\frac{\delta K(X;\xi)}{\delta \Phi(X)} = \xi(2K^{(1,1)}(X;\xi) - D_1 D_2 K(X;\xi))$.

(xi) For $L_{(4)}$, $\frac{\delta K(X;\xi)}{\delta \Phi(X)} = -2\xi(K^{(1,1)}(X;\xi) + \Phi(X)K(X;\xi))$ and $\frac{\delta}{\delta \Phi(X)}(K^{(1,1)}(X;\xi) + \Phi(X)K(X;\xi)) = \left(1 + 2\xi \frac{\partial}{\partial \xi}\right) K(X;\xi)$.

Proof of (x). Writing $\int dX K(X;\xi) = \int dX \int dX' \delta(X,X')K(X,X';\xi)$ and using the representation $K(X,X';\xi) = e^{\xi L_{(-2)}}\delta(X,X')$ yields

$$\delta \int dX\, K(X;\xi) = \xi \int dX \int dX' \delta(X,X') \delta L_{(-2)} K(X,X';\xi)\,.$$

Integrating by parts gives the desired result. □

Proof of (xi). The first result is obtained using the method of proof for (x). The second result follows similarly using $K^{(1,1)}(X;\xi) + \Phi(X)K(X;\xi) = \lim_{X' \to X}(D_1 D_2 + \Phi)K(X,X';\xi)$ and $(D_1 D_2 + \Phi)e^{\xi L_{(4)}} = e^{\xi L_{(4)}}(D_1 D_2 + \Phi)$ since $L_{(4)} = -(D_1 D_2 + \Phi)^2$. □

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